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Mémoire

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Existence et unicité des solutions périodiques d'une classe d'équations différentielles itératives avec la méthode de point fixe

Option : AFA

Par :

1. Bouchema Wiame
2. Neghra Khaoula

Encadré par : Benhadri Mimia — MCA U. SKIKDA

Devant le jury :

Présidente : Loulmi Soumia — MCB U. SKIKDA
Examinatrice: Foughali Fouzia — MCB U. SKIKDA

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CONTENTS

Abstract in English

Fixed point theory has a long history of being used in nonlinear differential equations, in order to prove existence, uniqueness, or other qualitative properties of solutions. However, using the fixed point theorem for stability and periodicity of solutions have a more recent appearance. In this dissertation, we focus on the study of the quantitative and qualitative properties of a particular case of state-dependent delayed differential equations, called iterative differential equations. We start by giving some fixed point theorems. After we introduce results for delay differential equations and necessary relevant definitions. Finally, we derive some sufficient conditions for the existence of positive periodic solution of the iterative fonctional differential equations, by using the Krasnoselskii fixed-point theorem in the cone of a Banach space. Also by using the contraction mapping principle, the uniqueness of the solution is also analyzed. An Example is also given to illustrate the claims established.

Keywords: Fixed points theorem; Green functions; Krasnoselskii's theorem; Periodicity; Positivity; Delay differential equations; iterative differential equation; Integral equation; Variable delays.

Abstract in French

La théorie du point fixe est utilisée depuis longtemps dans les équations différentielles non linéaires, afin de prouver l'existence, l'unicité ou d'autres propriétés qualitatives. Cependant, l'utilisation du théorème du point fixe pour la stabilité et la périodicité des solutions a une apparition plus récente. Nous nous concentrons sur l'étude des propriétés quantitatives et qualitatives d'une càs particulier d'équations différentielles non linéaires à retard dépendantes du temps et de l'état, appelées équations différentielles itératives. Nous commençons par donner quelques théorèmes de point fixe. Ensuite, nous donnons des notions préliminaires sur les équations différentielles à retard qui sont necéssaires par la suite. A la fin, en utilisant le théorème du point fixe de la théorie du cône nous étudions l'existence des solutions périodiques positives pour une équation différentielle itérative. De plus nous étudions l'unicité de la solution periodic par l'application du contraction de Banach. Finalement, un exemple est également donné pour illustrer l'éfficacité des résults établies.

Mots clés: Points fixe; Green function; Theorème de Krasnoselskii; Solutions périodiques positives; Equations à retard; Equations différentielle itérative; Equation intégrale; Retards.

Abstract in Arabic

نظرية النقطة الثابتة تستعمل منذ سنوات لدراسة المعادلات التفاضلية غير الخطية، لإثبات الوجود والوحدانية وخواص أخرى نوعية وكمية. لكن استعمالها لإثبات الاستقرار و الاستقرار المقارب و الحلول الدورية الموجبة للمعادلات ذات التأخر تعتبر حديثة الظهور. في مذكراتنا ركزنا مع الخواص النوعية والكمية لبعض أنواع المعادلات التفاضلية ذات تأخر. بدأنا ببعض نظريات النقطة الثابتة، بعدها أساسيات حول المعادلات التفاضلية المتأخرة وأخيرا وباستعمال نظرية النقطة الثابتة لكراسنوسكي المعرفة على مخروط من فضاء بناخ درسنا وجود على الأقل حل وحيد و حل مضاعف وباستعمال مبدأ التقليص لبناخ لنقطة ثابتة أثبتنا وحدانية الحل الدوري لمعادلات غير خطية تكرارية. في النهاية قدمنا مثال توضيحي يؤكد نجاعة النتائج المدروسة.

Dedication

I dedicate this humble work

To my family, especially for

My dear parents, my father Rabah

And mother Djamila for their patience, love,

Support,encouragement and interest.

To my sisters Hana and Houda, my brothers Isslem and Amar

My Allah protect them

Everyone who had an impact on my life.

Bouchema Wiame

Dedication

I dedicate this humble work

To my family, especially for

my mother and father

My Allah protect them

my husband for their

support of me,my sisters and

My friends. Without forgetting

All the teachers. Thank you

For everything

Neghra Khaoula

We dedicate this work to all our loved ones.

Thank you our parents for believing in our abilities.

Thank you our teachers for all the support you have provided at the end.

Thank you our families, all our friends, for everything

Khaoula and Wiame

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After praising God Almighty and thanking him for ending this dissertation, we extend our sincere thanks and great gratitude to the distinguished **Dr. Benhadri Mimia** for the continuous knowledge and guidance, she gave us and for her continuous effort, advice and directions from the beginning of the research phase until the completion of this dissertation, and whatever phrases and sentences. We wrote, the words of thanks we remain unable to fulfill her right, so may God reward her with the best reward for us and make that in the balance of her good deeds.

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Acronyms

Abbreviation	**Meaning**
ODEs	Ordinary Differential Equations
FDEs	Functional Differential Equations
DDEs	Delay Differential Equations
NDDEs	Neutral Delay Differential Equations

List of Symbols

Here we state some conventions regarding mathematical notation that we will use in this dissertation.

$\mathbb{R}^n$	Euclidean space of n- dimensions;
$\mathbb{R}^+ = [0, \infty)$	set of positive real numbers;
(X, d)	this is called metric space;
(a, b)	open interval from a to b;
$\mathbb{N}$	set of natural numbers;
$\frac{d}{dt}$	first derivative with respect to t;

$[a, b]$	closed interval from a to b;
$\mathbb{C}([a, b], \mathbb{R})$	space of continuous functions mapping from the interval $[a, b]$ to $\mathbb{R}$;
ω	period;

$\mathbb{C}([-\tau, 0]; \mathbb{R}^n)$	the space of all continuous $\mathbb{R}^n -$ valued functions φ defined on $[-\tau, 0]$;
	with a norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \|\varphi(\theta)\|$;
$f : A \to B$	the mapping f from A to B;
$x^{[n]}$	it is the n iterate of the state x;
$d(x, y)$	this is called the distance between x and y;

$\|.\|_\infty$ uniform norm;

$\mathbb{C}_\omega$ set of periodic functions;

$|x|$ the Euclidean norm of a vector x;

$\sum\limits_{i=0}^{n}$ summation from index $i = 0$ to $i = n$.

Other notations will be explained when they first appear.

Introduction

The fixed point theory is at the heart of the nonlinear analysis then that it provides the necessary tools to have theorems of existence in many different non-linear problems. It uses its tools of analysis and topology. Advancements in fixed point theory enrich many scientific fields such as biology, chemistry, computer science, economics and game theory. Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory, (see [2], [50]).

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known the contraction mapping theorem or the contraction mapping principle). It was first stated by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space, and provides a constructive method to find those fixed points.

Concerning the topological branch, results are obtained using topological properties of the set X. The main result is Schauder fixed point theorem which stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem.

In 1955, Krasnoselskii studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name. The reader is referred to the classical textbook on fixed point [51]. This is a captivating result and it has a number of interesting applications. The proof of this result combines the Banach contraction principle and Schauder fixed point theorem and thus it is a blend of the two branches.

Delay differential equations are a class of differential equations where the derivatives at the current time depending on the solution at previous times. Strictly speaking, delay differential equations are a specific example of functional differential equations, in which the functional part of the differential equations is the evaluation of a functional on the past of the process. Mathematical modeling involving delay differential equations is widely used for analysis and predictions in various areas of the life sciences, for example, population dynamics, epidemiology, immunology, physiology, neural networks, (see for example, [4, 12]), and the references cited therein. The delays can represent gestation times, incubation periods, or transport delays. In many cases time delays can be substantial such as gestation, forestation, deforestation and maturation or can represent little lags such as acceleration and deceleration in physical processes. Due to their importance in numerous applications have been devoted to the existence of periodic solutions of several different types of delay differential equations. For some specific work concerning the existence of periodic solutions to periodic population models which were carried out using the fixed point theory, the reader is referred to see ([35], [21], [24], [43]), amongst others, and the references therein.

The state-dependent delay dierential equations fascinate their readers as they occur widely in many applications such as classical electrodynamics, population models, mechanical models, the dynamics of economic systems, the transmission of infectious diseases, refer to ([20],[37], [52]). Iterative differential equations have distinctive characteristics as a particular type of state-dependent delay-differential equations.

Now, intensive scientific work has been carried out on various dynamical aspects of iterative differential equations and has been reported. In particular, a number of existence results are investigated by many authors, (see, [58] , [53]). In this dissertation, we have been interested in the use of fixed point theory to problem of periodicity and positivity for iterative differential equations. In this work we present a collection of results to some problems of iterative differential equations by using fixed point theory.

This dissertation consists of three chapters

Chapter 1 : The first chapter is devoted to point out the tools which are needed in the next chapters. The aim of this chapter is to introduce the basic concepts, notations, and elementary results which will be used throughout this work. We recall some classical results from functional analysis such as the Ascoli-Arzela theorem. The main part of the chapter is dedicated to the presentation of fixed point theorems, the Schauder's and Banach fixed point theorem and that of Krasnoselskii. Moreover, we analyze some examples to illustrate how to apply these theorems to some specific differential equations.

Chaptre 2 : This chapter is devoted to DDEs analysis, we present some basic preliminaries and we discuss the existence and uniqueness theorem for the solution and properties of them. The authors provide the appropriate mathematical tools which will be needed to understand the concepts that will be developed in this dissertation for the study of the periodicity of delay differential equations (DDEs). Significant and interesting model of delay equations emanating from biology is given in the beginning of this chapter.

Chaptre 3 : This chapiter exposes results published in [41], we will investigate the existence of positive periodic solutions for iterative differential equations. In the process we convert the given system into an equivalent integral equation. Then we construct appropriate mappings and use Krasnoselskii's fixed point theorem in a cone of a Banach space to show the existence of a positive periodic solution of the equation. Easily verifiable sufficient conditions are established. Also, by using the contraction mapping principle, the uniqueness of the solution are established. In particular, the results improve some previous ones in the literature. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

CHAPTER 1

Fixed point Theorems

Contents

This chapter we will mention (and complete) some important theorems in the theory of the fixed point, as well as some tools of the functional analysis necessary for the rest of this dissertation. In particular, the Ascoli-Arzela theorem is an element of it is very useful in this work to prove compactness in function spaces challenged on compact sets and not necessarily compact sets. The theorems of Banach, Schauder and Krasnoselskii will be presented in this chapter. These elements of the analysis have been presented. Taken from a few books chosen as (see [15], [50], [54]).

1.1 Notation and priliminaries

This section contains an elementary set of definitions, theorems, and examples which were motivated by the examples in the last section and were formulated to aid us in deciding which fixed point theorem to use and which stability properties to prove.

1.1.1 Normed and Banach space

Definition 1.1 *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ a function. Then d is called.a metric on X if the following properties hold.*

1) $d(x, y) = 0$ *if and only if* $x = y$ *for some* $x, y \in X$;
2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called metric space.

Definition 1.2 The metric space (X, d) is complete if every Cauchy sequence in (X, d) has a limit in that space. Asequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $d(x_n, x_m) < \varepsilon$.

Definition 1.3 *A* vector space $(\mathcal{V}, +, \cdot)$ is a normed space if for each $x, y \in \mathcal{V}$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(i) $\|x\| = 0$ *if and only if* $x = 0$,
(ii) $\|\alpha x\| = |\alpha| \|x\|$ *for each* $\alpha \in \mathbb{R}$, *and*
(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Remark 1.1 A normed space is a vector space and it is a metric space with $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 1.4 A Banach space is a complete normed space.

We list two examples, (see Burton, [15]).

Example 1.1 a) The space $\mathbb{C}([a, b], \mathbb{R}^n)$ consisting of all continuous function $f : [a, b] \to \mathbb{R}^n$ is a vector space over the reals.

b) If $\|f\| = \max_{a \le t \le b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then $(\mathbb{C}, \|.\|)$ is a Banach space.

Example 1.2 Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:

i) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$;

ii) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $p \in (1, \infty)$;

iii) $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition 1.5 *A* sequence $\{x_n\}$ in a normed space X is said to be Cauchy if $\lim_{m,n \to \infty} \|x_m - x_n\| = 0$, *i.e.*, for $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \ge n_0$.

Definition 1.6 *A* normed space $(X, \|.\|)$ is said to be complete if it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.

Definition 1.7 *A* complete normed space is called a Banach space.

Example 1.3 The linear space $\mathbb{C}([a,b])$ of continuous functions on the closed and bounded interval $[a,b]$ is a Banach space with the uniform convergence norm $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$.

Definition 1.8 Every finite- dimensional normed space is a Banach space.

Definition 1.9 *A* closed subspace of a Banach space is a Banach space.

Proposition 1.1 *Let X be a metric space. Then the following are equivalent*

i) X is a compact.

ii) Every sequence in X has a convergent subsequence.

iii) X is comlpete and totally bounded.

Proposition 1.2 *Let X be a subset of a complete metric space X. Then we have the following:*

a) $\mathcal{M}$ is compact if and only if $\mathcal{M}$ is closed and totally bounded.

b) $\overline{\mathcal{M}}$ is compact if and only if $\mathcal{M}$ is totally bounded.

A subset $\mathcal{M}$ of a topological space is said to be relatively compact if its closure is compact, i.e., $\overline{\mathcal{M}}$ is compact. In particular, we have an interesting result.

Proposition 1.3 *Let $\mathcal{M}$ be a closed subset of a complete metric space. Then $\mathcal{M}$ is compact if and only if it is relatively compact.*

Definition 1.10 ([15]) Let $\{f_n\}$ be a sequence of real valued functions with $f_n : [a,b] \to \mathbb{R}$.

a) $\{f_n\}$ is uniformly bounded on $[a,b]$ if there exists $M > 0$ such that $|f_n(t)| \le M$ for all n and all $t \in [a,b]$.

b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply

$$|f_n(t_1) - f_n(t_2)| < \epsilon$$

for all n.

The following results gives the main method of proving compactness in the spaces in which we are interested.

Definition 1.11 (Ascoli-Arzela, [15]) If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.

Remark 1.2 Let k_1 and k_2 be two strictly positive real numbers. The subset F of continuous real functions on $[a, b]$, differentiable on $]a, b[$ which satisfy

$$|f(t)| \leq k_1 \text{ and } \sup_{c \in]a,b[} |f'(c)| \leq k_2$$

for all $t \in [a, b]$,is relatively compact in $C([a, b], \mathbb{R})$.

Indeed, for all $f \in F$; the finite increment theorem proves that for all $t_0, t \in [a, b]$ there exists $c \in]t_0, t[$ such that

$$|f(t) - f(t_0)| = |f'(c)| |t - t_0|,$$

so $|f(t) - f(t_0)| \leq k_2 |t - t_0|$ let's fixe $t_0 \in [a, b]$. Let $\varepsilon > 0$ and by taking $\eta = \dfrac{\varepsilon}{k_2}$, we have

$$\forall t \in [a, b], |t - t_0| \leq \eta \Rightarrow |f(t) - f(t_0)| \leq \varepsilon.$$

F is said to be "uniformly bounded" if and only if

$$\forall x \in [a, b], \{f(x), f \in F\} \text{ is bounded,}$$

which exactly is the equicontinuity of F at t_0. As we can take for t_0 any point of $[a;b]$, we deduce that F is equicontinuous. We have $t \in [a,b]\,, |f\,(t)| \leq k_1$ for all $f \in F$ wich implies that $\|f\|_\infty \leq k_1$, so we conclude that

$$\forall f \in F, B'\,(0,k_1)\,,$$

i,e

$$F \subset B'\,(0,k_1)\,,$$

hence the boundedness of F. Finally, as F is bounded and equicontinuous, then the Ascoli-Arzela's theorem ensures that F is relatively compact

But if we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([15], Theorem 1.2.2 p. 20) and is as follows.

Theorem 1.1 *([15]) Let $\mathbb{R}^+ = [0,\infty)$ and let $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k(t)\}$ is an equicontinuous sequence of $\mathbb{R}^d$-valued functions on $\mathbb{R}^+$ with $|\phi_k(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^d$.*

Proof. It is clear that set of functions $\{\phi_k(t)\}$ is uniformly bounded on $\mathbb{R}^+$. Thus, considering intervals $[0,n]$, n a positive integer, and using a diagonalization process there is a subsequence, say $\{\phi_k(t)\}$ again, converging uniformly on any compact subset of $\mathbb{R}^+$ to some continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$. Because $\phi(t) \to 0$ as $t \to \infty$, it will now be possible to show that $\|\phi_k - \phi\| \to 0$ as $k \to \infty$, where $\|.\|$ denotes the supremum metric on $\mathbb{R}^+$. From the definition of $q(t)$, for any $\varepsilon > 0$ there is a $T > 0$ with $q(t) < \varepsilon/2$ if $t \geq T$, which yields

$$|\phi_k(t) - \phi(t)| \leq 2q(t) < \varepsilon \text{ if } k \in \mathbb{N} \text{ and } t \geq T, \qquad (**)$$

where $\mathbb{N}$ denote the set of positive integers. On the other hand, since $\{\phi_k(t)\}$ converges to $\phi(t)$ uniformly on $[0, T]$ as $k \to \infty$, for the ε there is a $\kappa \in \mathbb{N}$ with

$$|\phi_k(t) - \phi(t)| < \varepsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq T,$$

which together with (**), implies that $\|\phi_k - \phi\| < \varepsilon$ if $k \geq \kappa$. This shows that

$$\|\phi_k - \phi\| \to 0 \text{ as } k \to \infty.$$

Let us give an example for compact set: (see Burton, [15]). ■

1.2 Fixed point theorems

Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory.

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known the contraction mapping theorem or the contraction mapping principle). It was first stated by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space, and provides a constructive method to find those fixed points. Concerning the topological branch, results are obtained using topological properties of the set X. The main result is Schauder fixed point theorem which stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem. Although historically the two branches of the fixed point theory had separate development. In 1958, Krasnoselskii established that the sum of two operators $\mathcal{A} + \mathcal{B}$ has a fixed point in a nonempty closed convex subset $\mathcal{M}$ of a Banach space X, where one of them is a contraction and the another one is compact (see below).

1.2.1 Banach fixed point

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem, the famous contraction principle, which is one of the most important results of analysis. It is the most widely applied fixed point result in different areas of mathematics and applications. It requires the structure of a complete metric space with contractive condition on the map which is easy to test in many situations. It has been generalized in many different directions. Moreover, the proof of the Banach contraction principle gives a sequence of approximate solutions and useful information as regards the rate of convergence toward the fixed point. This is very important since a fundamental principle both in mathematics and computer science is iteration. Particularly, fixed point iterations and monotone iterative techniques are the core methods when solving a large class of abstract and applied mathematical problems and play an important role in many algorithms.

Let describe this theorem.

Definition 1.12 Let f be a mapping in the set M. we call fixed point of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.

Theorem 1.2 *(Contraction Mapping Principle, Smart* [50] *). Let* $(\mathcal{S}, \rho)$ *be a complete metric space and let* $\mathcal{P} : \mathcal{S} \to \mathcal{S}$*. If there is a constant* $0 \leq \alpha < 1$ *such that for each pair* $\phi_1, \phi_2 \in \mathcal{S}$*, we have*

$$\rho(\mathcal{P}\phi_1, \mathcal{P}\phi_2) \leq \alpha\rho(\phi_1, \phi_2),$$

then there is one and only one point $\phi \in \mathcal{S}$ with $\mathcal{P}\phi = \phi$.

Example 1.4 Let $f(t,x)$ be a continuous real-valued function defined for t in the interval $[0,T]$, and x in $\mathbb{R}$. The Cauchy initial value problem is the problem of finding a continuously differentiable function x on $[0,T]$ satisfying the differential equation

$$\begin{cases} x'(t) = f(t, x(t)), \; t \in [0,T] \\ x(0) = \zeta. \end{cases} \tag{1.1}$$

Proof. Consider the space $C([0,T])$ of continuous real-valued functions with standard supremum norm and f is $L-$lipschitzian with respect to x. Integrating both sides of *(1.1)* we obtain

$$x(t) = \zeta + \int_0^t f(s,x,(s))ds.$$

We denote the function defined by the right side of the above by $\mathcal{P}x$. Precisely,

$$(\mathcal{P}x)(t) = \zeta + \int_0^t f(s,x(s))ds.$$

Thus $\mathcal{P} : \mathbb{C}([0,T] \to \mathbb{C}([0,T])$, and a solution to *(1.1)* corresponds to a fixed point x of $\mathcal{P}$. Observe that for any $x,y \in [0;T]$,

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &= \left| \int_0^t f(s,x(s))ds - \int_0^t f(s,y(s))ds \right| \\ &\leq \int_0^t |f(s,x(s)) - f(s,y(s))| \, ds \\ &\leq L \int_0^t |x(s) - y(s)| \, ds \\ &= Lt \, \|x - y\| \, . \end{aligned}$$

It follows that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq LT \, \|x - y\| \, .$$

If $LT < 1$, then the result is immediate via the Banach Contraction Principle.

■

1.2.2 Schauder's fixed point theorem

Definition 1.13 Let (X, d) be a generalized metric space.Asubset C of X is called compact if every open cover of C has a finite subcover A subset C of X is sequentially compact if every sequence in C contains a convergent subsequence with limit in C.

Definition 1.14 A set C of topological space is said relatively compact if its closure is compact, i.e., $\overline{C}$ is compact. The set C is sequentially relatively compact if every sequence in C contains a convergent subsequence (the limit need not be an element of C), i.e., $\overline{C}$ is sequentially compact.

Definition 1.15 Let X ,Y be two generalized metrics spaces $K \subset X$ and $f : K \to Y$ be a an open operator. Then f is called:

(i) compact, if for any bounded subset $A \subset K$ we have $f(A)$ is relatively compact or $\overline{f(A)}$ is compact;

(ii) Complete continuous, if f is continuous and compact;

(iii) with relatively compact range, if f is continuous and $f(K)$ is relatively compact or $\overline{f(K)}$ is compact.

Theorem 1.3 *(Schauder's first point theorem, Burton* [15] *) Let $\mathcal{M}$ be a nonempty compact convex subset of a Banach space and let $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ be continuous. Then $\mathcal{P}$ has a fixed point in $\mathcal{M}$.*

Theorem 1.4 *(Schauder's second fixed point theorem, Burton* [15] *) Let $\mathcal{M}$ be a nonempty convex subset of a normed space and let $\mathcal{P} : \mathcal{M} \to \mathcal{K}$ where $\mathcal{K}$ is a compact subset of $\mathcal{M}$. Then $\mathcal{P}$ has a fixed point in $\mathcal{K}$.*

1.2.3 Krasnoselskii fixed point

The fixed point theorem of Krasnoselskii combined the two main fixed point theorems, Banach contraction mapping principle and Schauder fixed point theorem into the following result. Firstly, we recall the theorem of Schauder. Schauder's theorems which require compactness instead of completeness and which yield a, possibly non unique, fixed point. More precisely, Schauder's fixed point theorem shows us that a continuous map on a compact convex subset of a Banach space has a fixed point. The next two results are found in Smart [50] and Burton [15].

Krasnoselskii's theorem may be combined with Banach and Schauder's fixed point theorems. In a certain sense, we can interpret this as follows: if a compact operator has the fixed point property, under a small perturbation, then this property can be inherited. The theorem is useful in establishing the existence results for perturbed operator equations. It also has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of periodic solutions.Thus the existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. (see [50] Smart, 1980;p.31).

Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result:

Theorem 1.5 ***(Krasnoselskii,*** *see Burton* [13] ***).*** *Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathcal{M}$ into X such that the following conditions hold*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$, $\forall x, y \in \mathcal{M}$;

(ii) *$\mathcal{A}$ is continuous and $\mathcal{A}\mathcal{M}$ is contained in a compact set;*

(iii) $\mathcal{B}$ *is a contraction with* $\alpha < 1$;

then there is a $z \in \mathcal{M}$, *with* $z = \mathcal{A}z + \mathcal{B}z$.

Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-\mathcal{B})x-(I-\mathcal{B})y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\leq \|x-y\|+\|\mathcal{B}x-\mathcal{B}y\| \\ &\leq \|x-y\|+\alpha\|x-y\| \\ &= (1+\alpha)\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} \|(I-\mathcal{B})x-(I-\mathcal{B})y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\geq \|x-y\|-\|\mathcal{B}x-\mathcal{B}y\| \\ &\geq \|x-y\|-\alpha\|x-y\| \\ &= (1-\alpha)\|x-y\|. \end{aligned}$$

In short

$$(1-\alpha)\|x-y\| \leq \|(I-\mathcal{B})x-(I-\mathcal{B})y\| \leq (1+\alpha)\|x-y\|.$$

This inequality shows that $(I-\mathcal{B}) : \mathcal{M} \to (I-\mathcal{B})\mathcal{M}$ is continuous and one to one. Thus, $(I-\mathcal{B})^{-1}$ exist and is continuous. Let us pose $U := (I-\mathcal{B})^{-1}\mathcal{A}$ It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathcal{M} \text{ such that } (I-\mathcal{B})^{-1}\mathcal{A}z{=}z.$$

This is equivalent to $z{=}\ \mathcal{A}z{+}\mathcal{B}z$. ■

Remark 1.3 If $\mathcal{A} = 0$, the theorem can be summed up in Banach's theorem and if $\mathcal{B} = 0$ then the theorem is none other than Schauder's theorem.

In addition, an example is eventually analyzed to illustrate the effectiveness of the proved results of Krasnoselskii theorem.

Example 1.5 For better understanding this observation, now, we analyze an example to illustrate the application of Krasnoselskii fixed point theorem for proving the existence of $\omega-$periodic solutions of the following differential equation,

$$x'(t) = -a(t)x(t) - g(t,x), \tag{1.2}$$

where $a(t) = a(t+\omega)$, and the function $g(t,x)$ is periodic in t of period ω.

Proof. We can transform this equation in another form while writing, formally

$$x'(t)e^{-\int_0^t a(s)ds} = -a(t)e^{-\int_0^t a(s)ds}x(t) - g(t,x)e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)e^{-\int_0^t a(s)ds} + a(t)e^{-\int_0^t a(s)ds}x(t) = -g(t,x)e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)e^{-\int_0^t a(s)ds}\right)' = -g(t,x)e^{-\int_0^t a(s)ds},$$

then integrating from $t-\omega$ to t, we obtain

$$\int_{t-\omega}^{t}\left(x(u)e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-\omega}^{t} g(u,x)e^{-\int_0^u a(s)ds}du,$$

which gives

$$x(t)e^{-\int_0^t a(s)ds} - x(\omega-t)e^{-\int_0^{\omega-t} a(s)ds} = -\int_{t-\omega}^{t} g(u,x)e^{-\int_0^u a(s)ds}du,$$

or

$$x(t) = x(\omega-t)e^{-\int_{\omega-t}^t a(s)ds} - \int_{t-\omega}^{t} g(u,x)e^{-\int_t^u a(s)ds}du. \tag{1.3}$$

If we suppose that $e^{-\int_{\omega-t}^t a(s)ds} := \alpha$ and if $(X, \|.\|)$ is a Banach space of functions $\varphi : \mathbb{R} \to X$ continuous and $\omega-$ periodic, then the equation *(1.3)* can be written as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (\mathcal{H}\varphi)(t),$$

where $\mathcal{B}$ is contraction provides that the constant $\alpha < 1$ and $\mathcal{A}$ is compact mapping. ■

This example shows the birth of the mapping $\mathcal{H}\varphi := \mathcal{B}\varphi + \mathcal{A}\varphi$ which is identified with a sum of a contraction and a compact mapping.

Finally, let us indicate a very useful Theorem 1.6 when trying to prove that the existence and positivity of a solution to a periodic problem.

Definition 1.16 *(see, [29] , [24]) Let X be a Banach space and let Ω be a closed, nonempty subset of X. Ω is a cone if*

i) $\alpha u + \beta v \in \Omega$ for all $u, v \in \Omega$ and all $\alpha, \beta \geq 0$;

ii) $u, -u \in \Omega$ imply $u = 0$.

The proof of Krasnoselskii's fixed point theorem stated below can be found in [27]

Theorem 1.6 *(Guo-Krasnoselskii, [27]) Let X be a Banach space, and let $\Omega \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{P} : \Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \Omega,$$

be a completely continuous operator such that either

i) $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Omega \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Omega \cap \partial\Omega_2$; or

ii) $\|\mathcal{P}u\| \geq \|u\|$ for $u \in \Omega \cap \partial\Omega_1$ and $\|\mathcal{P}u\| \leq \|u\|$ for $u \in \Omega \cap \partial\Omega_2$.

Then $\mathcal{P}$ has a fixed point in $\Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

CHAPTER 2

Retarded differential equations with applications

Contents

This chapter provides background material necessary for the rest of the dissertation. Some preliminaries and basic definitions are given for delay differential equations. Strictly speaking a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process. Like ordinary differential equations, delay differential equations have several features which make their analysis more complicated. The survey of the theory related to delay differential equations can be found for example in the books, [4], [15], [5], [19], [21] – [32], [43] – [39].

2.1 Basic concepts of delay differential equations

Motivation

The questions have been asked by many researchers "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. To clarify more, we give a biological system in which the present rate of change of some unknown function depends upon past values of the same function.

Real example of delay differential equation

To have a better understanding and reading of this section, we will focus on a simple real example. The goal is to help the reader to understand the most relevant aspects of delay differential equations. The following is an example presented in [4]. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t = \alpha\rho$. Then the dynamics of N can be described by the linear delay differential equation

$$\frac{d}{dt}N(t) = -\mu N(t) - rN(t-h), \tag{2.1}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Equation (2.1) describes the changes in N.

With deeper study and understanding of population dynamics, people started to consider introducing state-dependent delay into population models, as was pointed out in Arino et al. [4].

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population.

Mathematical point of view:

To determine a solution past time t_0, we need to prescribe the value of $N(t_0-h)$. Suppose we have the initial value $N(t_0-h)$. Once we advance, say to $N(\varepsilon)$, with $t_0<\varepsilon<t_0+h$ small, notice that to calculate the derivative at $t=\varepsilon$ so that we can advance the next step, we need to know

$$\frac{d}{dt}N(\varepsilon)=-\mu N(\varepsilon)-rN(\varepsilon-h),$$

where $\varepsilon-h\in(t_0-h,t_0)$. In this manner, we realize that we need to know the values of $N(.)$ on the whole interval $[t_0-h,t_0]$. If we do not specify these values, we obtain an unsatisfactory notion of uniqueness, as the following example

$$x'(t)=-\frac{\pi}{2}x(t-1)\,,x(0)=\frac{1}{\sqrt{2}}.$$

Here

$$x_1(t)=\sin\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right] \text{ and } x_2(t)=\cos\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right],$$

are both solutions to the above equation at $t_0=0$. But if we specify the initial behavior on the interval $[-1,0]$, we obtain that only one solution exists to each delay differential equations, by the existence-uniqueness result in Theorem 2.1 that we give below.

Clearly, to begin with, an initial value problem requires more information than an analogous problem for a system without delays. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - h, t_0]$. Each such initial function determines a unique solution to the delay differential equation. If we require that initial functions be continuous, then the space of solutions has the same dimensionality as $C([t_0 - h, t_0], \mathbb{R})$.

In the next section, with the previous discussion as a guide, let us now define the DDEs problem for a given initial function.

2.1.1 A general initial value problem

Suppose $\tau > 0$ is a given real number $\tau > 0$, denote $C([a,b], \mathbb{R}^n)$, the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We will denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ as $|x|$ from now on in order to avoid confusion with another norm we shall use. If $[a,b] = [-\tau, 0]$, we let C=$C\left([-\tau, 0], \mathbb{R}^n\right)$ and designate the norm of an element φ in C by

$$\|\varphi\|_\tau := \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)| \,.$$

Let $\sigma \in \mathbb{R}, A > 0$ and $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$, be defined by

$$x_t = x\,(t + \theta) \;\text{ for }\; -\tau \leq \theta \leq 0.$$

Definition 2.1 ([32]) *If Ω is a subset of $\mathbb{R}\times C$, Let $f : \mathbb{R} \times C \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the* *relation*

$$\begin{cases} x'(t) = f(t, x_t), t \geq \sigma, \\ \textit{and } x_\sigma = \varphi, \end{cases} \tag{2.2}$$

is a retarded functional differential equation and we will denote this equation by DDEs. The quantitie $\tau \geq 0$, is called the delay.

Definition 2.2 ([32]) *A function x is said to be a solution of (2.2) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, and x satisfies (2.2) for $t \in [\sigma, \sigma + A]$. In such a case we say that x is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (2.2) with initial value at σ or simply a solution of (2.2) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (2.2) is a very general type of equation and includes ordinary differential equations ($\tau = 0$). Although the structure of these equations is similar to ordinary differential equations, the crucial difference is that a delay differential equation (or a system of equations) is an infinite dimensional problem and the corresponding phase space is a functional space usually the space of continuous functions is considered.

Definition 2.3 *Equation (2.2) is called*:

i*) linear if $f(t, \varphi) = L(t, \varphi)$, where L is linear in φ.*

ii) nonhomogeneous if $f(t, \varphi) = L(t, \varphi) + h(t)$, where $h(t) \neq 0$, it is called homogeneous if $h = 0$.

iii) autonomous if $f(t, \varphi) = g(\varphi)$, where g does not depend on t.

Equation (2.2) is a very general type of equation and includes differential-difference equations. To be more explicit we give some classes of equations that can be expressed by (2.2), we have equations with a fixed delay (the simplest possible case) such as

$$x'(t) = f(t, x(t), x(t-\tau)),$$

or nonlinear nonautonomous differential equations with multiple time varying delays on the same state x

$$x'(t) = f(t, x(t), x(t-\tau_1(t)), ..., x(t-\tau_p(t))),$$

with $0 \leq \tau_i(t) \leq \tau$ for all $i = 1, ..., p$.
We also have integrodifferential equations with a distributed delay

$$x'(t) = \int_{-\tau}^{0} g(t, x(t+\theta))\, d\theta,$$

where we see how in the integration process we need to know the values of x in $[t-\tau, t]$ for each t where the vector field is defined.

The delay also may be functions $\tau_i(t, x(t)) > 0, (i = 1, 2, ..., p)$ (state-dependent delays), the following class of functional differential equations with state and time-varying delays,

$$x'(t) = f(t, x(t), x(t-\tau_1(t, x(t))), ..., x(t-\tau_p(t, x(t)))),$$

Iterative differential equations have distinctive characteristics as a particular type of state-dependent delay-differential equations

$$x'(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

where the iterate $x^{[n]}(t)$ stands for x composed with itself n times, i.e. $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, $x^{[3]}(t) = x^{[2]}(x(t))$, ..., $x^{[n]}(t) = x^{[n-1]}(x(t))$ are the iterates of the state x. In recent years, the iterative differential equations have been the subject of several investigations (see, [52] , [37] and the references cited therein).

2.1.2 Existence and uniqueness theory

The existence and uniqueness theory for delay equations can be derived from the more general theory of functional differential equations. Since we intend to consider only equations of the form (2.2) we will not make use of the full generality available. Nevertheless, the more general theory leads to a presentation that is simpler and also benefits from an analogy with similar results in the theory of ordinary differential equations.

We now state the basic theory of DDEs.

Lemma 2.1 *([32]) Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (2.2) through (σ, φ) is equivalent to solving the integral equation:*

$$x(t) = \varphi(\sigma) + \int_{\sigma}^{t} f(s, x_s)\, ds \text{ for } t \geq \sigma, \text{ and } x_\sigma = \varphi.$$

Lemma 2.2 *([32]) If $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma - \tau, \sigma + A]$.*

Proof. Since x is continuous on $[\sigma-\tau, \sigma+A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon, \forall \theta \in [-\tau, 0]$. ■

The existence and uniqueness of the solutions of DDEs are given by the following Theorems.

Theorem 2.1 *(Local existence, [32]) Suppose $\overline{\Omega}$ is an open subset in $R \times C$ and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous. For any $(\sigma, \varphi) \in \overline{\Omega}$, there exists a solution of equation (2.2) through (σ, φ).*

Definition 2.4 (Lipschitzian, [32]). *We say $f(t, \varphi)$ is Lipschitz in φ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \varphi_i) \in K, i = 1, 2$,*

$$|f(t, \varphi_1) - f(t, \varphi_2)| < k\,|\varphi_1 - \varphi_2|\,.$$

Theorem 2.2 *(Existence and uniqueness,* [42] *) Suppose $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$, $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in $\overline{\Omega}$. If $(t_0, \varphi) \in \overline{\Omega}$, then there is a unique solution of Eq. (2.2) through (t_0, φ).*

Proposition 2.1 . *If f is at most affine i.e. $f(t, \varphi) \leq a + b\,|\varphi|$, with $a, b > 0$, then there exists a global solution of the equation (2.2) i.e. $\forall\varphi$, the solution $x(\sigma, \varphi)$ is defined on $[A, \infty[$.*

In the following we also require continuous dependence of solutions on initial conditions, for which the following theorem gives a result analogous to that for ordinary differential equations.

Theorem 2.3 . *Suppose x is a solution through (t_0, φ) of the equation (2.2) and that it is unique on $[t_0-\tau, \beta]$. If $\{(t_n, \varphi_n)\} \subset \mathbb{R}\times C$ is a sequence such that $(t_n, \varphi_n) \to (t_0, \varphi)$ as $n \to \infty$, then for all sufficiently large n every solution x_n through φ_n exists on $[t_n - \tau, \beta]$, and $x_n \to x$ uniformly on $[t_0 - \tau, \beta]$.*

2.2 Neutral delay differential equations

Now are ready to give the definition of an other class of delay differential equations so-called the Neutral delay diffferential equations (NDDEs).

Definition 2.5 [32] *Suppose that $\mathbb{R} \times C$ is open with elements (t, φ). A function $D : \overline{\Omega} \to \mathbb{R}^n$ is said to be atomic at β on $\overline{\Omega}$ if D is continuous* *together with its first and second Fréchet derivatives with respect to* φ*; and* D_φ*, the derivative with respect to* φ*, is atomic at* β *on* $\overline{\Omega}$*.*

Definition 2.6 [32] *Suppose that* $\overline{\Omega} \subseteq \mathbb{R} \times C$ *is open,* $f : \overline{\Omega} \to \mathbb{R}^n$*,* $D : \overline{\Omega} \to \mathbb{R}^n$ *are given continuous functions with* D *atomic at zero. The equation*

$$\frac{dD}{dt}(t, x_t) = f(t, x_t), \tag{2.3}$$

is called the neutral delay differential equation NDDE (D, f)*.*

If the delayed argument occurs in the highest order derivative of the state we call it neutral functional differential equation.

The following equations are some examples of neutral differential equations

Example 2.1 [32] *If* $\tau > 0$*,* B *is an* $n \times n$ *constant matrix,* $D(\varphi) = \varphi(0) - B\varphi(-\tau)$*, and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - Bx(t-\tau)] = f(t, x_t).$$

Example 2.2 [32] *If* $\tau > 0$*,* x *is a scalar,* $D(\varphi) = \varphi(0) - \sin(-\tau)$*, and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - \sin x(t-\tau)] = f(t, x_t). \tag{2.4}$$

Remark 2.1 *Note that when* x *is continuous differentiable, (2.4) is equivalent to*

$$x'(t) - (\cos x(t-\tau))\, x'(t-\tau) = f(t, x_t).$$

Definition 2.7 [32] *A function* x *is said to be a solution of (2.3) on* $[\sigma - \tau, \sigma + A]$ *if there are* $\sigma \in \mathbb{R}$ *and* $A > 0$ *such that*

$$x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n), \;\; (t, x_t) \in \overline{\Omega}, \;\; t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (2.3) on $[\sigma, \sigma + A]$. For a given $t_0 \in \mathbb{R}, \varphi \in C$, and $(\sigma, \varphi) \in \overline{\Omega}$, we say $x(t, \sigma, \varphi)$ is a solution of equation (2.3) with initial value φ at σ or simply a solution through (σ, φ) if there is an $A > 0$ such that $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$; we say $x(t, \sigma, \varphi)$ is a solution of (2.3) on $[\sigma - \tau, \infty)$, if for every $A > 0$, $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.

Theorem 2.4 *(Existence,* [42] *) If $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$ and $(t_0, \varphi) \in \overline{\Omega}$, then there exists a solution of the NDDE (L, f) through (t_0, φ).*

2.3 Method of steps

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solve the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solution (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance. The method of steps was first proposed by Bellman and Cooke [5]. This approach, furnishes a method of finding explicit solutions. The desired solution is found on successive intervals by solving ordinary differential equations without delays in each interval. As an illustration to this approach, consider the DDE:

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), t \geq t_0 \\ x(t) = \varphi_0(t), \ t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.5}$$

For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, therefore one can obtain the solution in the next step interval $[t_0, t_0+\tau]$ by solving the following ordinary differential equation:

$$x'(t) = f(t, x(t), \varphi_0(t-\tau)) = g_0(t, x(t), \text{ for } t_0 \leq t \leq t_0 + \tau.$$

Under suitable hypotheses on g_0, existence and uniqueness of a solution of this equation (hence a solution of (2.5)) on $[t_0 - \tau, t_0]$ can be established. Denoting this solution by $\varphi_1(t)$ and restricting equation (2.5) to the interval $[t_0 + \tau, t_0 + 2\tau]$, we find the ordinary differential equations

$$x'(t) = f(t, x(t), \varphi_1(t-\tau)) = g_1(t, x(t) \text{ for } t_0 + \tau \leq t \leq t_0 + 2\tau,$$

with the initial condition $x(t_0+\tau) = \varphi_1(t_0+\tau)$, for which we can again establish existence and uniqueness of a solution φ_2. Thus we have now extended the solution x to the interval $[t_0 + \tau, t_0 + 2\tau]$, and we now have a formula for $x(t)$ when $t \in [t_0 - \tau, t_0 + 2\tau]$.

In general, by assuming that $\varphi_{k-1}(t), \forall(k = 1, 2, ...)$ is defined on the interval $[t_0 + (k-2)\tau, t_0 + (k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$x'(t) = f(t, x(t), \varphi_{k-1}(t-\tau)), \text{ for } t_0 + (k-1)\tau \leq t \leq t_0 + k\tau,$$

with the initial condition $x(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$. We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[t_0 - \tau, \infty)$.

Remark 2.2 *The method of steps can be extended to differential equations with other types of delays, such as multiple delays, variable delay and even state dependent delay or for neutral systems. The difficulty is to locate the primary discontinuities.*

2.3.1 Primary discontinuity of delay differential equation

Definition 2.8 [21] *If the solution of a DDE and its derivatives of order* μ *are continuous at some point in the time interval, but the derivative of order* $\mu+1$ *is not, then such a point is called a primary discontinuity of the given problem.*

Theorem 2.5 [21] *The points* $\xi_\mu := \mu\tau$ *the primary discontinuities of problem (2.5). More precisely,* $x^{(\mu)}$ *is continuous at* ξ_μ *but* $x^{(\mu+1)}$ *is, in general, not, even if the functions* φ *and* f *have continuous derivatives of all orders.*

Proof. See [21]. Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t=0$, the first derivative $x'(t)$ has a primary discontinuity, since the integrable equation

$$x'(t) = f(t, x(t), \varphi(t-\tau)), t \in [0, \tau],$$

may satisfy the condition $x(0) = \varphi(0)$, but it is unlikely to satisfy the additional condition $x'(0^+) = \varphi'(0^-)$. Only for special choices of the initial function $\varphi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\varphi'(0^-) = f(0, \varphi(0), \varphi(-\tau))$. ■

Example 2.3 *We illustrale this method by using the special cases of equation (2.5), the following is an example presented in ([32]), canada 1998, Let*

$$\begin{aligned} x'(t) &= ax(t-\tau), t \in [0, +\infty) \\ x(t) &= 1, t \in [-\tau, 0], \end{aligned}$$

where a is positive constant. Using the method of steps, it is easy to see that the solution $x(t)$ *is a piecewise polynomial. On each subinterval* $[i\tau, (i+1)\tau]$,

$x(t)$ is an $(i+1)$-th. order polynomial, i.e.,

$$x(t) = \sum_{j=1}^{i+1} \frac{a^j}{j!} (t - (j-1)\tau)^j, i \in \mathbb{N}.$$

It is also clear that integer multiples of τ are primary discontinuities for this particular problem.

As a generalization of (2.5), we consider

$$x'(t) = f(t, x(t), x(t - \tau(t)), x'(t - \tau(t))), t \in [0, \overline{a}],$$

where $t-\ \tau(t)$ is a strictly increasing function and

$$0 < \tau(t) \leq t,\ \ \overline{a} = \inf_{t \geq 0} \{t - \tau(t)\}.$$

Remark 2.3 *The method of steps can be extended to delay differential equations with additive noise term (see chapitre 3).*

CHAPTER 3

Positive periodic solutions of iterative functional differential equations

Contents

The goal of this chapter is to present a recent work published in [41], namely,

H. C. Koyuncuoglu and M. Adivar, positive periodic solutions of iterative functional differential equations, *Bull. Int. Math. Virtual Inst, 12 (2) (2022), 227-235.*

By using the fixed point method, some sufficient conditions are obtained on the existence of positive periodic solutions for iterative functional differential equations with a positive parameter λ. In the processus we invert the given equation into an equivalent integral equation. Then, we construct an appropriate mapping and use Krasnoselskii's fixed point theorem in a cone of a Banach space to show the existence of positive periodic solutions of the given equation. Easily verifiable sufficient conditions are established. Also, by using the contraction mapping principle, the uniqueness of the solution is obtained. For this type of equations, we combine different techniques to prove new results.

3.1 Short biographies of iterative differential equations

The state-dependent delay differential equations fascinate their readers as they occur widely in many applications such as , in models of hematopoiesis, the delay whish represents the cell cycle duration depends, in fact, on both the time and the density of mature cells. In the models of insect population dynamics, the delay descibes the life cycle periodic and depend also on the time and the population size. This dependence on the state variable is due to the competition among larval for habitat and food during larval stage, refer to ([20],[37],[49],[52]). Iterative differential equations have distinctive characteristics as a particular type of state-dependent delay-differential equations. The origin of this type of equation dates back to the early 19th century, and to the best of our knowledge, their study began with a simple example but a very important one dealt with by Babbage[14] in 1815, and after Babbage's equation, the theory slowly evolved (see, [14]). Now, intensive scientific work has been carried out on various dynamical aspects of iterative differential equations and has been reported. In particular, a number of existence results, periodicity and continuous dependence ([8],[10],[58]) are investigated by many authors.

Qualitative properties of differential equation have taken great interest due to the importance of understanding the continuous phenomena in application of mathematics. Existence, uniqueness, stability, periodicity, and oscillation of solution of differential equations became topic of many studies in the existing literature. Iterative differential equation that belong to the class of functional equations are visited by several researchers because of their complicated dynamics. We refer to [20], [37] and the recent studies [53], [58], [57] as inspiring studies on this topic. For example, in [20] the equation

$$x'(t) = x(x(t)), \tag{3.1}$$

is considered and existence uniqueness andanalyticity of solutions are studied.

Afterwards, the iterative differential equation of the form

$$x'(t) = f(x(x(t))), \tag{3.2}$$

is handled by [37].

Recently, intensive scientific work has been carried out in various dynamical aspects of iterative differential equations, and many results have been reported in the literature. For instance, stability, uniqueness of periodic solutions and some other fundamental properties of solutions of certain delay differential equations have been discussed in [8] , [9] , [10] , [11] , [48] , [49] , [58] . For example, in 2017, Zhao and Liu, [58] analyzed the following iterative functional differential equations with variable coefficients,

$$y^{'}(t) = c_0(t)y^{[0]}(t) + c_1(t)y^{[1]}(t) + c_2(t)y^{[2]}(t) + ... + c_n(t)y^{[n]}(t) + F(t), \tag{3.3}$$

where $c_0(t) = 0$, and $y^{[0]}(t) = t$, $y^{[1]}(t) = y(t)$, $y^{[2]}(t) = y(y(t))$, $y^{[3]}(t) = y^{[2]}(y(t)), ..., y^{[n]}(t) = y^{[n-1]}(y(t))$ are the iterates of the state y. By using

Krasnoselskii's fixed point theorem, the authors derived some verifiable sufficient conditions for the existence and uniqueness of periodic solutions to (3.3).

In recent years, the study of iterative differential equations has becom more popular, and we refer the papers [8],[9] [10], [11], [26], [36], [48], [49], [52] to readers as rapidly growing literature on the differential equations with iterative terms.

3.2 Statement of the problem and preliminaries

As we mentioned previously, one of our main objectives in this part is to discuss the work carried out in [41]. More precisely, very recently, the authors focused on a specific type of differential equations with state-dependent delay with a positive parameter $\lambda > 0$, given in the following form:

$$x'(t) = a(t)x(t) - \lambda f(t, x(t), x(t - \tau(t, x(t)))), t \in \mathbb{R}, \tag{3.4}$$

where $\tau(t, x(t)) = t - x(t)$, and the functions $a : \mathbb{R} \to (0, \infty)$, $f : \mathbb{R} \times (0, \infty) \times (0, \infty) \to (0, \infty)$ are continuous. By employing a cone theoretical fixed point theorem, they established some sufficient conditions for the existence of positive periodic solutions of the iterative functional differential equation (3.4). Their approch allowed them to obtain som results that are not covered by the existing literature.

For $\omega > 0$, we define the set P_ω as a set of all continuous scalar functions $x(t)$, periodic in t of period ω, i.e., $x(t + \omega) = x(t)$ for all $t \in \mathbb{R}$. Then

$(P_\omega, \|.\|)$ is a Banach space when it is endowed by the supremum norm

$$\|x\| := \sup_{t\epsilon\mathbb{R}} |x(t)| = \sup_{t\epsilon[0,\omega]} |x(t)| \,.$$

Since we are searching for the existence of periodic solutions for equation (3.4), it is natural to intoduce the following assumption:

(H1) For a positive integer ω, the functions a and f are ω- periodic in t, that is

$$a(t+\omega) = a(t), \;\; f(t+\omega, x, y) = f(t, x, y),$$

for all $t \in \mathbb{R}$, and $(x, y) \in (0, \infty) \times (0, \infty)$.

Before starting the main result of this paper, we establish the equivalent formulation for the solution of equation (3.4).

The following lemma is fundamental to our results.

Lemma 3.1 *Assume (H1) holds. If $x \in P_\omega$, then $x(t)$ is a solution of equation (3.4) if and only if*

$$x(t) := \lambda \int_t^{t+\omega} G(t,u) f(u, x(u), x(x(u))) du, \tag{3.5}$$

where

$$G(t,u) := \frac{\exp(\int_u^t a(s)ds)}{1 - \exp(-\int_0^\omega a(s)ds)}. \tag{3.6}$$

Proof. Suppose (H1) holds and let $x \in P_\omega$ be a solution of (3.4). Multiplying both sides of (3.4) by $\exp(-\int_0^t a(s)ds)$ and then integrate from t to $t+\omega$, we obtain

$$\begin{aligned} & x(t+\omega)\exp\left(-\int_0^{t+\omega} a(s)ds\right) - x(t)\exp\left(-\int_0^t a(s)ds\right) \\ = \; & \lambda \int_t^{t+\omega} \exp(-\int_0^u a(s)ds) f(u, x(u), x(x(u))) du, \end{aligned}$$

and the fact that $x(t)=x(t+\omega)$ and (H1), we deduce

$$\begin{aligned} & x(t)\exp\left(-\int_0^t a(s)ds\right) \qquad (3.7) \\ = \; & \frac{\lambda}{1-\exp(-\int_0^t a(s)ds)}\int_t^{t+\omega}\exp(-\int_0^u a(s)ds)f(u,x(u),x(x(u)))du. \end{aligned}$$

Dividing both sides of the above equation by $\exp\left(-\int_0^t a(s)ds\right)$, we obtain

$$x(t)=\lambda\int_t^{t+\omega}G(t,u)f(u,x(u),x(x(u)))du,$$

where G is as in (3.6). Conversely, by using Leibniz's rule, we may easily verify that

$$x(t)=\lambda\int_t^{t+T}G(t,u)f(u,x(u),x(x(u)))du,$$

solves (3.4).

We use (3.5) to define the operator $\Phi : P_\omega \to P_\omega$ by

$$(\Phi x)(t)=\lambda\int_t^{t+\omega}G(t,u)f(u,x(u),x(x(u)))du, \qquad (3.8)$$

where G is as in (3.6).

It is easy to see that for any $u\in[t,t+\omega]$, thanks to (3.6), we have

$$G(t+\omega,u+\omega)=G(t,u),$$

and

$$0<\frac{\sigma}{1-\sigma}\le G(t,u)<\frac{1}{1-\sigma},\ \text{for all } u\in[t,t+\omega], \qquad (3.9)$$

where

$$\sigma:=\exp(-\int_0^\omega a(s)ds). \qquad (3.10)$$

Let' s define the following subset of P_ω

$$P_{\omega,M}^L:=\{x\in P_\omega:\|x\|\le L, |x(t_2)-x(t_1)|\le M|t_2-t_1|,\ L,M\ge 0\}. \qquad (3.11)$$

Lemma 3.2 *(see, [58]) For any $x, y \in P^L_{\omega,M}$, we have*

$$\left\|x^{[m]} - y^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \left\|x - y\right\|, m = 1, 2, ... \tag{3.12}$$

Proof. *Take $x, y \in P^L_{\omega,M}$, for $m = 1$ we have*

$$\left\|x - y\right\| \leq \left\|x - y\right\|.$$

Suppose (3.12) is satisfied for $m = k$, and we show that it is true for $m = k + 1$,

$$\begin{aligned}
\left\|x^{[k+1]} - y^{^{[k+1]}}\right\| &\leq \left|x(x^{[k]}) - x(y^{[k]})\right| + \left|x(y^{[k]}) - y(y^{[k]})\right| \\
&\leq M\left|x^{[k]} - y^{[k]}\right| + \left|x(y^{[k]}) - y(y^{[k]})\right|,
\end{aligned}$$

which gives

$$\begin{aligned}
\left\|x^{[k+1]} - y^{^{[k+1]}}\right\| &\leq M\left\|x^{[k]} - y^{[k]}\right\| + \left\|x - y\right\| \\
&\leq M\sum_{j=0}^{k-1} M^j \left\|x - y\right\| + \left\|x - y\right\| \\
&\leq (\sum_{j=0}^{k-1} M^{j+1} + 1)\left\|x - y\right\| \\
&\leq \sum_{j=0}^{k} M^j \left\|x - y\right\|.
\end{aligned}$$

Acoording to proof by induction - Rrecurrence relations

$$\left\|x^{[m]} - y^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \left\|x - y\right\|, \forall \in \mathbb{N}.$$

A special case of the previous Lemma 3.2, when $m = 2$, we may obtain the following inequality

$$\left\|x(x) - y(y)\right\| \leq (1 + M)\left\|x - y\right\|, \tag{3.13}$$

for any $x, y \in P^L_{\omega,M}$. ∎

■

Before proceeding, we firstly introduce the following assumption to be imposed later on.

(H2) There exist positive constants α_1, α_2, such that for all $t \in \mathbb{R}, x, y, \widehat{x}, \widehat{y} \in \mathbb{R}$,

$$\|f(t,x,y) - f(t,\widehat{x},\widehat{y})\| \le \alpha_1 \|x - \widehat{x}\| + \alpha_2 \|y - \widehat{y}\| . \tag{3.14}$$

Next, we define a cone K as a subset of P_ω by

$$K := \{x \in P^L_{\omega,M} : x(t) \ge \sigma \|x\| \text{ for all } t \in [0,\omega]\}, \tag{3.15}$$

where σ is as in (3.10).

Now, we present the second auxiliary result.

Lemma 3.3 *Assume (H1), and (H2), that the inequality*

$$\frac{\lambda}{1-\sigma}\left(\alpha_1 L + \alpha_2(1+M)L + \rho_1\right)(2 + \omega \|a\|) \le M, \tag{3.16}$$

and

$$\frac{\lambda}{1-\sigma}\left(\alpha_1 L + \alpha_2(1+M)L + \rho_1\right) \le L. \tag{3.17}$$

hold, where M is as in (3.11) and σ is as in (3.10), and $\rho_1 = \max_{u\in[0,\omega]} |f(u,0,0)|$. Then the mapping Φ defind by (3.8) is continuous on K and $\Phi(K) \subset K$.

Proof. Let $x \in K$, it is easy to show that $(\Phi x)(t+\omega) = (\Phi x)(t)$ for all $t \in \mathbb{R}$. Moreover, we get by (3.9) that

$$\begin{aligned}
(\Phi x)(t) &= \lambda \int_t^{t+\omega} G(t,u) f(u,x(u),x(x(u)))du \\
&\ge \lambda \int_0^{\omega} \frac{\sigma}{1-\sigma} f(u,x(u),x(x(u)))du \\
&= \sigma\lambda \int_0^{\omega} \frac{1}{1-\sigma} f(u,x(u),x(x(u)))du \\
&\ge \sigma \|\Phi x\| ,
\end{aligned}$$

where σ is as in (3.10). ■

Now we prove that the mapping Φ is continuous. Let the sequence $\{x_n\} \in K$ such that $x_n \to x$ in K. Since the function f satisfies (3.14), we consider

$$\begin{aligned}
& \|\Phi x_n - \Phi x\| \\
= \;& \max_{t\in[0,\omega]} \left| \lambda \int_t^{t+\omega} G(t,u) \left(f(u, x_n(u), x_n(x_n(u))) - f(u, x(u), x(x(u))) \right) du \right| \\
\leq \;& \frac{\lambda}{1-\sigma} \max_{t\in[0,\omega]} \int_t^{t+\omega} \left| f(u, x_n(u), x_n(x_n(u))) - f(u, x(u), x(x(u))) \right| du,
\end{aligned}$$

which converges to 0 as $n \to \infty$ by Lebesgue convergence theorem.

Let $x \in K$. For having $\Phi(x) \in K$, we will show that $|(\Phi x)(t_2) - (\Phi x)(t_1)| \leq M\,|t_2 - t_1|\,, \forall t_1, t_2 \in \mathbb{R}$. Let $t_1, t_2 \in [0,\omega]$ with $t_1 < t_2$, we have

$$\begin{aligned}
& |(\Phi x)(t_2) - (\Phi x)(t_1)| \\
= \;& \lambda \left| \int_{t_2}^{t_2+\omega} G(t_2,u) f(u, x(u), x(x(u))) du - \int_{t_1}^{t_1+\omega} G(t_1,u) f(u, x(u), x(x(u))) du \right| \\
= \;& \lambda \left| \int_{t_2}^{t_1} G(t_2,u) f(u, x(u), x(x(u)) du + \int_{t_1}^{t_1+\omega} G(t_2,u) f(u, x(u), x(x(u))) du \right. \\
& \left. + \int_{t_1+\omega}^{t_2+\omega} G(t_2,u) f(u, x(u), x(x(u))) du - \int_{t_1}^{t_1+\omega} G(t_1,u) f(u, x(u), x(x(u))) du \right| \\
\leq \;& \lambda \left| \int_{t_2}^{t_1} G(t_2,u) f(u, x(u), x(x(u)) du \right| + \lambda \left| \int_{t_1+\omega}^{t_2+\omega} G(t_2,u) f(u, x(u), x(x(u))) du \right| \\
& + \lambda \left| \int_{t_1}^{t_1+\omega} \left(G(t_2,u) - G(t_1,u) \right) f(u, x(u), x(x(u))) du \right| \\
\leq \;& \lambda \int_{t_2}^{t_1} |G(t_2,u)| \, |f(u, x(u), x(x(u))| \, du + \lambda \int_{t_1+\omega}^{t_2+\omega} |G(t_2,u)| \, |f(u, x(u), x(x(u)))| \, du \\
& + \frac{\lambda}{1 - \exp(-\int_0^\omega a(s)ds)} \int_{t_1}^{t_1+\omega} |f(u, x(u), x(x(u)))| \\
& \times \left| \exp(\int_u^{t_2} a(s)ds) - \exp(\int_u^{t_1} a(s)ds) \right| du.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \int_{t_1}^{t_1+\omega} \left| \exp(\int_u^{t_2} a(s)ds) - \exp(\int_u^{t_1} a(s)ds) \right| du \\ = & \int_{t_1}^{t_1+\omega} \exp(\int_u^{t_2} a(s)ds) \left| 1 - \exp(\int_{t_2}^{t_1} a(s)ds) \right| du \\ \leq & \ \omega \|a\| \, |t_2 - t_1| \exp(-\int_0^{\omega} a(s)ds). \end{aligned}$$

In view of conditions (3.13) and (3.14), we get

$$\begin{aligned} |f(u,x,y)| &= |f(u,x,y) - f(u,0,0) + f(u,0,0)| \\ &\leq |f(u,x,y) - f(u,0,0)| + |f(u,0,0)| \\ &\leq \alpha_1 \|x\| + \alpha_2 \|y\| + |f(u,0,0)| \\ &\leq \alpha_1 \|x\| + \alpha_2 \|x(x)\| + \max_{u\in[0,\omega]} |f(u,0,0)| \\ &\leq \alpha_1 \|x\| + \alpha_2 \sum_{j=0}^{2-1} M^j \|x\| + \rho_1 \\ &\leq \alpha_1 \|x\| + \alpha_2 (1+M) \|x\| + \rho_1. \end{aligned}$$

From above, we obtain

$$\begin{aligned} & |(\Phi x)(t_2) - (\Phi x)(t_1)| \\ \leq & \ \frac{2\lambda}{1-\sigma} (\alpha_1 \|x\| + \alpha_2(1+M) \|x\| + \rho_1) |t_2 - t_1| \\ & + \lambda \frac{\exp(-\int_0^{\omega} a(s)ds)}{1 - \exp(-\int_0^{\omega} a(s)ds)} (\alpha_1 \|x\| + \alpha_2(1+M) \|x\| + \rho_1) \times \omega \|a\| \, |t_2 - t_1| \\ \leq & \ \frac{2\lambda}{1-\sigma} (\alpha_1 L + \alpha_2 (1+M) L + \rho_1) |t_2 - t_1| \\ & + \frac{\lambda\omega}{1-\sigma} \|a\| (\alpha_1 L + \alpha_2(1+M)L + \rho_1) |t_2 - t_1| \\ \leq & \ M |t_2 - t_1|. \end{aligned}$$

From the above discussion, we can easly get,

$$|(\Phi x)(t_2) - (\Phi x)(t_1)| \leq M |t_2 - t_1|.$$

For any $x \in K$, it follows from (3.9) and (3.13),(3.14),(3.17) that

$$\begin{aligned}
|(\Phi x)(t)| &= \left|\lambda \int_t^{t+\omega} G(t,u) f(u, x(u), x(x(u))) du\right| \\
&\leq \frac{\lambda}{1-\sigma} \int_t^{t+\omega} |f(u, x(u), x(x(u)))| \, du \\
&\leq \frac{\lambda}{1-\sigma} \left(\alpha_1 \|x\| + \alpha_2 (1+M) \|x\| + \rho_1\right) \\
&\leq \frac{\lambda}{1-\sigma} \left(\alpha_1 L + \alpha_2 L(1+M) + \rho_1\right) \\
&\leq L.
\end{aligned}$$

Therefore $\|\Phi x\| = \sup_{t \in [0,\omega]} |(\Phi x)(t)| \leq L$ This shows that $\Phi(x)$ is uniformly bounded.

From (H2) and (3.16) we obtain $\Phi(K) \subset K$. Since K is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, \omega]$, we can apply the Arzela-Ascoli theorem to confirm that K is a compact subset from this space. Also, and since any continuous operator maps compact sets K into compact sets K, then we have proved that Φ is a compact operator.

3.3 Existence of positive periodic solutions

A quick review of the literature can lead to a vast literature using Schauder's Fixed Point Theorem to prove the existence of periodic solutions to the advanced functional differential equation, for example (see, [15], [13]). However, we will point out that we present some existence results for positive periodic solutions of a very general form of advanced functional differential equations, which may cover some special equations already treated, and the use of a fixed point theorem of Krasnoselskii in a cone of a Banach space rather than Schauder's theorem distinguishes our work from the already established literature, since our results also imply the existence of multiple periodic solutions in special cases. It is well known that obtaining existence results for multiple solutions of differential equations is an exhausting task in the qualitative theory of differential equations.

3.3.1 Multiple positive periodic solutions of equation (3.4)

In this part, we will use the Krasnoselskii in a cone of a Banach space Theorem 1.6 (see, chapter 1) to prove the existence of multiple periodic solutions of the equation (3.4).

Throughout this section, we assume the conditions (H1), (H2), and (3.16) is satisfied. For convenience in the following discussion, we introduce the following limits:

- $\underline{f_0} \;:=\; \lim_{x\to 0^+} \min_{t\in[0,\omega]} \dfrac{f(t,x,y)}{x};$
- $\overline{f_0} \;:=\; \lim_{x\to 0^+} \max_{t\in[0,\omega]} \dfrac{f(t,x,y)}{x};$
- $\underline{f_\infty} \;:=\; \lim_{x\to \infty} \min_{t\in[0,\omega]} \dfrac{f(t,x,y)}{x};$
- $\overline{f_\infty} \;:=\; \lim_{x\to \infty} \max_{t\in[0,\omega]} \dfrac{f(t,x,y)}{x}.$

Theorem 3.1 *If $\underline{f_\infty} = \infty$ and $\overline{f_0} = 0$ (or $\underline{f_0} = \infty$ and $\overline{f_\infty} = 0$) for each $y \in (0,\infty)$, then the equation (3.4) has a positive ω -periodic solution for any λ satisfying*

$$\frac{1-\sigma}{\omega} \le \lambda \le \frac{1-\sigma}{\sigma\omega}, \tag{3.18}$$

where σ is as in (3.10).

Proof. *Assume* $\underline{f_\infty} = \infty$. *Then there exist constants* $R > 0$ *and* $c_1 \geq \dfrac{1}{(1-\sigma)\sigma}$ *such that* $f(t,x,y) \geq c_1 x$ *for each* $(t,y) \in [0,\omega] \times (0,\infty)$ *and all* $x \geq R$. *We define*

$$\Omega_1 := \{x \in P^L_{\omega,M} : x < \frac{R}{1-\sigma}\}$$

as an open, bounded subset of the cone K. *Obviously, if* $x \in \partial\Omega_1 \cap K$, *then* $\|x\| = \dfrac{R}{1-\sigma}$ *and*

$$R \leq x(t) \leq \frac{R}{1-\sigma} \text{ for all } t \in [0,\omega]. \tag{3.19}$$

Then, combining (3.9), (3.18) and (3.19), we have

$$\begin{aligned} \|\Phi x\| &\geq \frac{\lambda\sigma}{1-\sigma}\int_0^\omega f(u,x(u),x(x(u)))du \geq \frac{\lambda\sigma}{1-\sigma}\int_0^\omega c_1 x(u)du \\ &\geq \frac{\sigma}{\omega}\int_0^\omega c_1 x(u)du \geq \frac{R}{1-\sigma} = \|x\|. \end{aligned}$$

This means $\|\Phi x\| \geq \|x\|$ *for all* $x \in \partial\Omega_1 \cap K$. *On the other* $\overline{f_0} = 0$ *and choose* $\widetilde{R} > \dfrac{R}{1-\sigma} > 0$ *large enough so that* $f(t,x,y) \leq c_2 x$ *for all* $0 \leq x \leq \widetilde{R}$, $(t,y) \in [0,\omega] \times (0,\infty)$, *and* $0 < c_2 \leq \sigma$. *we introduce the ball*

$$\Omega_2 := \{x \in P^L_{\omega,M} : \|x\| < \widetilde{R}\} \subset K,$$

and if $x \in \partial\Omega_2 \cap K$, *then one may get*

$$\|\Phi x\| \leq \frac{\lambda}{1-\sigma}\int_0^\omega f(u,x(u),x(x(u)))du \leq \frac{\lambda}{1-\sigma}\int_0^\omega c_2 x(u)du \leq \widetilde{R} = \|x\|,$$

by using (3.9), and (3.18). Thus, $\|\Phi x\| \leq \|x\|$ *for all* $x \in \partial\Omega_2 \cap K$ *and Theorem 3.1 implies the mapping* Φ *has a fixed point for* $x \in K \cap (\overline{\Omega_2}\backslash\Omega_1)$. *Consequently, the advanced differntial equation (3.4) has a positive* ω *-periodic solution.* ■

In the second part of the proof we assume $\underline{f_0} = \infty$. Then there exists $S > 0$ such that $f(t,x,y) \geq c_3 x$ for all $0 < x < S$ and $(t,y) \in [0,\omega] \times (0,\infty)$,where $c_3 \geq \dfrac{1}{\sigma^2}$. We define

$$\Upsilon_1 := \{x \in P^L_{\omega,M} : x < S\},$$

and if $x \in \partial\Upsilon_1 \cap K$, then we have $\|x\| = S$. By using the inequalities (3.9) and (3.18) and the defing cone (3.15), we deduce $\sigma S \leq x \leq S$ when $x \in \partial\Upsilon_1 \cap K$. In the light

$$\|\Phi x\| \geq \frac{\lambda\sigma}{1-\sigma}\int_0^\omega f(u, x(u), x(x(u))du \geq \frac{1-\sigma}{\omega}\frac{\sigma}{1-\sigma}\int_0^\omega c_3 x(u)du \geq S = \|x\|.$$

This proves $\|\Phi x\| \geq \|x\|$ for $x \in \partial\Omega_2 \cap K$. Futhemore, if $\overline{f_\infty} = 0$, then there exists $\widetilde{S} > 0$ such that $f(t, x, y) \leq c_4 x$ for all $x \geq \widetilde{S}, (t, y) \in [0, \omega] \times (0, \infty)$,where $0 < c_4 \leq \sigma$. We set $S_{\max} \geq \max\{2S, \frac{1}{\sigma}\widetilde{S}\}$ and define

$$\Upsilon_2 := \{x \in P^L_{\omega,M} : x < S_{\max}\}.$$

Obviously, if $x \in \partial\Upsilon_2$, then $\|x\| = S_{\max}$ and $\widetilde{S} \leq \sigma\|x\| \leq x(t)$ for $t \in [0, \omega]$.

Assume $x \in \partial\Upsilon_2 \cap K$, and by inequalities (3.9) and (3.18) concider

$$\begin{aligned} \|\Phi x\| &\leq \frac{\lambda}{1-\sigma}\int_0^\omega f(u, x(u), x(x(u)))du \\ &\leq \frac{\lambda}{1-\sigma}\int_0^T c_4 x(u)du \leq \frac{1}{\sigma\omega}\int_0^\omega c_4 x(u)du \leq S_{\max} = \|x\|. \end{aligned}$$

This implies $\|\Phi x\| \leq \|x\|$ for $x \in \partial\Upsilon_2 \cap K$ and from Theorem 3.1, the mapping Φ has a fixed point x in $K \cap (\overline{\Upsilon_2}\backslash\Upsilon_1)$. As a result, (3.4) has a positive ω-periodic solution. The proof is complete.

Theorem 3.2 *Assume $\underline{f_0} = \underline{f_\infty} = \infty$ for each $y \in (0, \infty)$, and there exists a*

$$f(t, x, y) \leq A\sigma, \tag{3.20}$$

for all $x \in (0, A]$, and $(t, y) \in [0, \omega] \times (0, \infty)$. Then (3.4) has two positive periodic solutions for any λ satisfying

$$\frac{1-\sigma}{\omega} \leq \lambda \leq \frac{1-\sigma}{\sigma\omega}, \tag{3.21}$$

where σ is as in (3.10).

Proof. If $\underline{f_0} = \infty$ holds, then there exists $0 < \underline{A} < A$ such that $f(t, x, y) \geq c_5 x$ for $(t, y) \in [0, \omega] \times (0, \infty)$ and all $0 < x \leq \underline{A}$, where $c_5 \geq \dfrac{1}{\sigma^2}$. We define

$$\Lambda := \{x \in P^L_{\omega,M} : \|x\| < \underline{A}\}.$$

Then for $x \in \partial\Lambda_1 \cap K$, we get

$$\begin{aligned} \|\Phi x\| &\geq \frac{\lambda\sigma}{1-\sigma}\int_0^\omega f(u, x(u), x(x(u)))du \geq \frac{\lambda\sigma}{1-\sigma}\int_0^\omega c_5 x(u)du \\ &\geq \frac{\sigma}{\omega}\int_0^\omega c_5 x(u)du \geq \frac{\sigma}{\omega}\frac{1}{\sigma^2}\omega\underline{A}\sigma = \underline{A} = \|x\|, \end{aligned}$$

due to (3.9), (3.21), and the cone (3.15). This means $\|\Phi x\| \geq \|x\|$ for $x \in \partial\Lambda_1 \cap K$. Moreover, if $\underline{f_\infty} = \infty$, then there exists $\widehat{A} > A$ such that $f(t, x, y) \geq c_6 x$ for all $x \geq \widehat{A}$, $c_6 \geq \dfrac{1}{\sigma^2}$ and $(t, y) \in [0, \omega] \times (0, \infty)$. Set

$$\Lambda_2 := \{x \in P^L_{\omega,M} : \|x\| < \widehat{A}\}.$$

Then for $x \in \partial\Lambda_2 \cap K$, we obtain the result $\|\Phi x\| \geq \|x\|$ by applying the same procedure given above. Moreover, we introduce

$$\Lambda_3 := \{x \in P^L_{\omega,M} : \|x\| < A\}.$$

In this case, when $x \in \partial\Lambda_3 \cap K$ we have the following

$$\|\Phi x\| \leq \frac{\lambda}{1-\sigma}\int_0^\omega f(u, x(u), x(x(u)))du \leq \frac{1}{\sigma\omega}\int_0^\omega A\sigma du = A = \|x\|,$$

by (3.9), (3.20), and (3.21). Since $\underline{A} < A < \widehat{A}$, by Theorem 3.1 the mapping Φ has two fixed points in $K \cap (\overline{\Lambda_3}\backslash\Lambda_1)$ and $K \cap (\overline{\Lambda_2}\backslash\Lambda_3)$. This is equivalent to existence of at least two positive periodic solution of (3.4). The proof is complete. ∎

The following result can be proven similar to proof of Theorem 3.3. Therefore, we omit its proof.

Theorem 3.3 *Suppose* $\overline{f_0} = \overline{f_\infty} = 0$ *hold and there exists a constant* $Y > 0$ *such that*

$$f(t, x, y) \geq \frac{Y}{\sigma},$$

for all $x \in [\sigma Y, Y]$ *and* $(t, y) \in [0, \omega] \times (0, \infty)$*. Then (3.4) has two positive periodic solutions for*

$$\frac{1-\sigma}{\omega} \leq \lambda \leq \frac{1-\sigma}{\sigma\omega}. \tag{3.22}$$

As an implementation of our results, we provide the following example.

3.3.2 Uniqueness of periodic solution of equation (3.4)

In this part we use the principle of contraction mapping to prove the uniqueness of the solution of the equation (3.4).

Theorem 3.4 *Suppose (H1) and (H2) hold. If*

$$\frac{\lambda}{1-\sigma}\omega(\alpha_1 + \alpha_2(1+M)) < 1, \tag{3.23}$$

then equation (3.4) has a unique solution.

Proof. Similarly as in the proof of lemma 3.3 and by using (3.23). For $x, y \in P^L_{\omega,M}$, we have

$$|(\Phi x)(t) - (\Phi y)(t)| \leq \frac{\lambda}{1-\sigma}\omega(\alpha_1 + \alpha_2(1+M))\,|x-y|\,,$$

By the principle of contraction mapping, Φ has a unique xed point on $P^L_{\omega,M}$ and in view of Lemma 3.1 this fixed point is a solution of equation (3.4). ■

An Example

In this section, we analyze an example to show the effectiveness of our results.

Example 3.1 We consider the equation

$$x'(t) = a(t)x(t) - \lambda f(t, x(t), x(t - \tau(t, x(t)))), t \in \mathbb{R}.$$

Corresponding to equation (3.4), we let $a(t) = 2 + \sin(10\pi t)$, $f(t, x, y) = \frac{1}{8}\exp(-x - y)$, $\lambda = 2$. At first, the function a is $\frac{1}{5}$ -periodic, and the initial condition (H1) holds. Furthermore, the condition given in (H2) can be easily verified for the function f with constants $\alpha_1 = \frac{1}{8}$ and $\alpha_2 = \frac{1}{8}$. Again by straightforward computations, the limit results $\underline{f_0} = \infty$ and $\overline{f_\infty} = 0$ hold. Then, the iterative differential equation is written as follows

$$x'(t) = (2 + \sin(10\pi t))x(t) - \frac{1}{4}\exp(-x(t) - x(x(t))). \tag{3.24}$$

As the next task, we focus on the crucial inequality (3.16) in Lemma 3.2, and we observe it holds for any $M \geq 1$, where $\sigma = \exp(-\frac{2}{5})$ (see (3.10). Besides, it should be highlighted that $\lambda = 2$ satisfies the inequality (3.22). It is straightforward to show that all conditions of Theorem 3.2 are fulfilled, and consequentially the the iterated differential equation (3.24) has a positive $\frac{1}{5}$-periodic solution.

Conclusion

This dissertation studies the positivity of periodic solutions of iterative differential equations. The approach used in our project is based on the fixed point technique. This method relies mainly on necessary arguments: an elementary variation of parameter formula, a non-empty cone subset of a Banach space and a fixed point application. The benefit of this approach is that the fixed point arguments can yield existence, and two positive periodic solutions for such an equation in one step. The main difficulty of this approach is to define a cone K of a Banach space and a suitable mapping, when the mapping is compact on K. The results in this work extend and improve some exist results in the literature in some ways. Recently, Benhadri et al. in [**6**] have addressed this technique to investigate the periodicity of another class of impulsive delay differential equations. However, there are many problems to be solved for iterative fractional differential equations and other variants, persistence, and so on. We leave these for our -future work.

BIBLIOGRAPHY

[1] **M. Adıvar, H. C. Koyuncuo glu and Y. N. Raffoul**, Existence of periodic solutions in shifts $\delta\pm$ for neutral nonlinear dynamic systems on time scales, *Appl. Math. Comput., 242 (2014), 328-339.*

[2] **R. P. Agarwal, D. O.Regan and D. R. Sahu**, Fixed point theory for Lipschitzian -type mappings with applications, *Springer Dordrecht Heidelberg London New York.*

[3] **S. Althubiti, H. A. Makhzoum and Y. N. Raffoul**, Periodic solution and stability in nonlinear neutral system with infinite delay, *Applied Mathematical Sciences, 7(136) (2013), 6749–6764.*

[4] **O. Arino , M. L. Hbid and E. Ait Dads**, Delay differential equations and applications, *Published by Springer, P. O. Box 17 (2002), 3300 A. A Dordrecht, The Netherlands.*

[5] **R. Bellman and K. L. Cooke**, Differential-difference equations, Academic Press, *New York-London, (1963).*

[6] **M. Benhadri, T. Caraballo**, Necessary and sufficient conditions for the existence of positive periodic solutions for neural networks with time varying and distributed delays, Differential Equations and Dynamical Systems (2022).

[7] **E. Beretta, F. Solimano, and Y. Takeuchi**, A mathematical model for drug administration by using the phagocytosis of red blood cells*, J Math Biol. 35(1), 1996 Nov; 1–19.*

[8] **A. Bouakkaz, A. Ardjouni, and A. Djoudi,** Periodic solutions for a nonlinear iterative functional differential equation, Electron. J. Math. Anal appl, 7 (1) (2019), 156-166.

[9] **A. Bouakkaz, A. Ardjouni, and A. Djoudi,** Periodic solutions for a second order nonlinear functional differential equation with iterative terms by Schauder's fixed point theorem, Acta Math. Univ. Comenianae, 87 (2), (2018), 223-235.

[10] **A. Bouakkaz, A. Ardjouni, R. Khemis, and A. Djoudi,** Periodic solutions of a class of third order functional differential equations with iterative source terms, Bol. Soc. Mat. Mex, 26 (2020), 443-458.

[11] **A. Bouakkaz, and R. Khemis,** Positive periodic solutions for a class of second-order differential equations with state-dependent delays, Turk. J. Math, 44 (4) (2020), 1412-1426.

[12] **R. K. Brayton**, Bifurcation of periodic solutions in a nonlinear difference -differential equation of neutral type*, Quart. Appl. Math., 24 (1966), 215-224.*

[13] **T. A. Burton**, Krasnoselskii's inversion principle and fixed points*, Nonlinear Analysis, 30 (1997), 3975-3986.*

[14] **C. Babbage,** An essay towards the calculs of functions. Phislophical Transactions of the Royal Society of London, 105: (1815) 389-423.

[15] **T. A. Burton,** Stability by Fixed Point theory for functional differential equations,*Department of mathematics southern illinois university Carbondale,illnois, Dover Publications,I N C Mineola New York, (2006)*.

[16] **Y. Chen**, New results on positive periodic solutions of a periodic integro-differential competition system, Appl. Math. Comput., 153 (2) (2004), 557-565.

[17] **F. D. Chen, F. X. Lin, and X. X. Chen**, Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models with feedback control, *Appl. Math. Comput., 158 (1) (2004), 45-68.*

[18] **F. D. Chen**, Periodic solution and almost periodic solution for a delay multispecies logarithmic population model, *Appl. Math. Comput.*, 171 (2005), 760-770.

[19] **R. D. Driver**, Ordinary and delay differential equation, *Springer Verlag, New York, (1977).*

[20] **E. Eder,** The functional differential equation $x'(t) = x(x(t))$, J. Differential Equation, 54 (1984), 390-400.

[21] **L. E. El'sgol'ts and S. B**. Norkin, *Introduction to the theory and application of differential equations with deviating arguments, Academic Press, New York, (1973).*

[22] **M. Fan and K. Wang**, Global periodic solutions of a generalized n-species Gilpin-Ayala competition model, *Comput. Math. Appl., 40 (2000), 1141-1151.*

[23] **M. Fan, P. Y. Wong, and R. P. Agarwal**, Periodicity and stability in a periodic n-species Lotka-Volterra competition system with feedback controls and deviating arguments, *Acta Math. Sinica, 19 (4) (2003), 801-822.*

[24] **E. Fridman**, Introduction to time-delay system: Analysis and control. Springer, (2014).

[25] **M. E. Gilpin and F. J. Ayala**, Global models of growth and competition, *Proc. Natl. Acad. Sci., USA 70 (1973) 3590-3593.*

[26] **A. Guerfi, A. Ardjouni,** Periodic solutions for totally nonlinear iterative differential equations Bull. Int. Math. Virtual Inst. 12 (1) (2022), 69-82.

[27] **D. Guo, V. Lakshmikantham**, *Nonlinear Problems in Abstract Cones. Academic Press, New York (1988).*

[28] **D. J. Guo**, Positive solutions to nonlinear operator equations and their applications to nonlinear integral equations, *Advances in Mathematics*, 13 (4) (1984), 294-310, (Chinese).

[29] **D. J. Guo**, Nonlinear functional analysis, Shandong science and technology press, Shandong, China, (2001), (in Chinese).

[30] **J. K. Hale**, Theory of functional differential equations, Springer Verlag, NY, (1977).

[31] **J. K. Hale and K .R. Meyer**, A class of functional equations of neutral type, Mem. Am. Math. Soc. 76 (1967), 1-65.

[32] **J. K. Hale and S. M. Verduyn Lunel,** Introduction to functional differential equations, Ser. Applied Mathematical Sciences. New York: Springer-Verlag, vol. 99 (1993).

[33] **J. K. Hale, S.M.V. Lunel**, Introduction to Functional Differential Equations, Applied Mathematical Sciences, vol. 99. Springer, New York (1993).

[34] **M. N. Islam and Y. N. Raffoul**, Periodic solutions of neutral nonlinear system of differentiale quations with functional delay, J. Math. Anal. Appl, 331 (2007), 1175-1186.

[35] **E. R. Kaufman and Y. N. Raffoul,** Periodic solutions for a neutral nonlinear dynamical equation on a time scale,J. Math. Anal. Appl, 319 (2006), 315-325.

[36] **E. R. Kaufiman,** Existence and uniqueness of solutions for a second-order iterative boundary-value problem, Electron. J. Differential Equations, 150 (2018), 1-6.

[37] **W. Ke,** On the equation $x'(t) = f(x(x(t)))$, Funkcial. Ekvac, 33 (1990), 405-425.

[38] **V. Kolmanovskii and A. Myshkis**, Appied theory of functional differential equations. Kluwer Academic, Publishers, (1992).

[39] **V. Kolmanovskii and A. Myshkis**, Introduction to the theory and applications of functional. Differential Equations, 463 (1999).

[40] **V. B. Kolmanovskii, A. D. Myshkis**, Applied Theory of Functional Differential Equations. Kluwer Academic, Dordrecht (1992).

[41] **H. C. Koyuncuoglu and M. Adivar,** Positive periodic solutions of iterative functional differential equations, Bull. Int. Math. Virtual Inst, 12 (2) (2022), 227-235.

[42] **M. A. Krasnoselskii,** Positive solutions of operator equations, Noordhoff, Groningen, (1964).

[43] **Y. Kuang**, Delay differential equation with application in population dynamics, Academic Press, Boston, (1933).

[44] **G. Lin, Y. Hong**, Periodic solution in nonautonomous predator-prey system with delays, *Nonlinear Anal. Real World Appl.*, 10 (2009), 1589-1600.

[45] **Y.K. Li**, On a periodic delay logistic type population model, Ann. Differential Equations 14 (1998), 29-36.

[46] **Z. Li and X. Wang**, Existence of positive periodic solutions for neutral functional differential equations, Electron. J. Diff. Equa., 2006 (34), (2006), 1-8.

[47] **B. Lisena**, Global attractivity in nonautonomous logistic equations with delay. Nonlinear Analysis: RWA, 9 (2008), 53-63.

[48] **B. Mansouri, A. Ardjouni and A. Djoudi,** Periodicity and continuous dependence in iterative differential equtions, Rend. Circ. Mat. Palermo, 69 (2) (2020), 561-576.

[49] **J. G. Si and X. P. Wang,** Analytic solutions of an iterative functional-differential equtions, J. Math. Anal. App, 262 (2001), 490-498.

[50] **D. R. Smart**, Fixed point theorems, Cambridge tracts in mathematics,. Cambridge University Press, London-New York, no. 66, (1974).

[51] **H. L. Smith**, An introduction to Delay differential equations with applications to the Life Sciences, Texts in Applied Mathematics, vol. 57 (2011), Springer, New York, Dordrecht, Heidelberg, London.

[52] **S. Stanek,** On global properties of solutions of functional differential equations $x'(t) = x(x(t)) + x(t)$, Dynamic Systems and Applications, 4 (1995), 263-278.

[53] **W. Wang,** Positive pseudo almost periodic solutions for a class of differential iterative equations with biological background, App. Math. Lett, 46 (2015), 106-110.

[54] **E. Zeidler**, Applied functional analysis, Springer-Verlag, New York, (1995).

[55] **W. Zhang** , Numerical analysis of delay differential and integrodifferential equations, A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Doctor of Philosophy, *Canada (1998).*

[56] **G. Zhang, S. S. Cheng**, Positive periodic solutions of coupled delay differential systems depending on two parameters, *Taiwan. Math. J.*, 8 (2004), 639-652.

[57] **H. Y. Zhao and M. Feckan,** Periodic solutions for a class of differential equations with delays depending on state, Math. Commun, 23 (2018), 29-42.

[58] **H. Y. Zhao and J. Liu,** Periodic solutions of an iterative functional differential equation with variable coefficients, Math. Methods App. Sci, 40 (1) (2017), 286-292.

[59] **H. Y. Zhao, L. Sun**, Periodic oscillatory and global attractivity for chemostat model involving distributed delays, *Nonlinear Anal. Real World Appl.*, 7 (2006), 385-394.