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Swelling porous-heat system with thermodiffusion effects and distributed delay

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for agreeing to review this work

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Dedication

We dedicate this thesis

To our parent for their invaluable trust in us their support for the values they instilled in me

To our supervisor Hamdi Zakaria for all the things he give us

To our sisters for their tenderness and complicity

To our brother for being by our side at all times and to our friends and all those who helped us directly or indirectly to complete this work

Abstract

This Master's thesis focuses on the study of the well-posedness and the asymptotic behavior of solutions to a swelling porous medium system with thermodiffusion effects and a distributed delay term. We establish the well-posedness of the system using the semigroup approach under suitable assumptions on the weight function of the distributed delay. Furthermore, we prove an exponential decay result by applying the energy method based on the multiplier technique, through which an appropriate Lyapunov functional is constructed.

Résumé

Ce mémoire de Master porte sur l’étude d’existence, unicité et du comportement asymptotique des solutions d'un système de milieux poreux gonflants avec des effets de thermodiffusion et un terme de retard distribué. Nous établissons le bien-posé du système en utilisant l’approche des semi-groupes, sous des hypothèses appropriées sur la fonction de poids du retard distribué. Ensuite, nous prouvons un résultat de décroissance exponentielle en appliquant la méthode de l’énergie, basée sur la technique des multiplicateurs, au moyen de laquelle nous construisons une fonctionnelle de Lyapunov appropriée.

Résumé

يتناول هذا البحث لنيل شهادة الماستر دراسة المسألة الجيدة الطرح والسلوك المُتسامي (النهائي) لحلول نظام وسط مسامي منتفخ يتضمن تأثيرات الانتشار الحراري وعبارة تأخير موزعة. نثبت جيدية الطرح لهذا النظام باستخدام منهج الزمر شبه التحليلية، وذلك تحت افتراضات مناسبة على دالة الوزن الخاصة بالتأخير الموزع. بعد ذلك، نحصل على نتيجة التناقص الأسي للطاقة باستخدام طريقة الطاقة المعتمدة على تقنية المضاعفات، والتي نقوم من خلالها ببناء دالة ليابونوف مناسبة.

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Chapitre 1

Introduction

In 1972, Goodman and Cowin [12] established in the theory of elastic solids with voids, the notion of a continuum theory of granular materials with interstitial voids which provided the relation between the elasticity theory and the porous media theory which has attracted the attention of many researchers, check [4, 7, 9, 10, 11, 16] for further information about porous-elastic theory.

The original field equations of the onedimensional porous materials theory is given mathematically by the following two basic evolution equations (see [17])

$$\rho_1 u_{tt} = T_{1x} - P_1 + F_1,$$
$$\rho_2 \varphi_{tt} = T_{2x} + P_2 + F_2.$$

The functions (T_1, T_2) represent the partial tensions, (P_1, P_2) the internal body forces and (F_1, F_2) the external forces associated with the dependent variables u and φ respectively. The duo positive constant coefficients ρ_1 and ρ_2 are the density of each constituent, where the partial tensions (T_1, T_2) are given by

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} u_x \\ \varphi_x \end{pmatrix},$$

where $\mathcal{M}$ is the positive definite matrix $\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$, i.e.,

$$a_2^2 < a_1 a_3, \text{and } P_1 = P_2 = 0$$

Finally we chose

$$F_1 = 0 \quad \text{and} \quad F_2 = \gamma_1 \theta_x + \gamma_2 P_x - \mu_1 \varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s)\, \varphi_t(x, t-s)\, ds.$$

The asymptotically stable system may be almost destabilized under the effect of time delay which has become an active area of research, as its effects appear in a wide range of applications including biology, chemistry, physics, engineering and many other modelling of the phenomena (see [1, 5, 22]). Part of the system's past history should be included in the most realistic model, where typically delay differential equations (DDEs), which depend on the values of certain unknown functions at previous times to determine the derivatives of certain unknown functions at this moment, are models that contain past history.

Many researchers have looked at the impact of the delay term on the asymptotic behavior of solutions, that may lead to a wild-behaved system instead of a well one. For example in 2003, Quintanilla [21] examined the following porous elastic system with damping through porous-viscosity $(\alpha\varphi_t)$

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad x \in (0,1)\,, \quad t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha\phi_t = 0, \quad x \in (0,1)\,, \quad t > 0, \end{cases}$$

and he demonstrated that, with these complementary controls, the solutions do not decay exponentially, then in 2006 Magańa and Quintanilla [18] established that the viscoelasticity $(-\gamma u_{txx})$ in the porous-elastic system is insufficient to cause an exponential decrease in the solutions. Recently, in [2] Apalara considered the following one-dimensional porous-elastic system with finite memory and showed a general decay of the energy for the case of equal speed of wave propagation, for more discussions (see [3, 6, 8, 13, 14, 15] and the references therein)

Due to the fact that many phenomena rely on their past, the distributed delay term is important and appears in many different works. Furthermore, its impact on the asymptotic behavior of the solution for various problem types (see [8, 7]).

In this work, we consider the following swelling porous-heat system with thermodiffusion effects and distributed delay

$$\begin{cases} \rho_1 u_{tt} - a_1 u_{xx} - a_2\varphi_{xx} = 0, \\ \rho_2\varphi_{tt} - a_3\varphi_{xx} - a_2 u_{xx} - \gamma_1\theta_x - \gamma_2 P_x + \mu_1\varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\,\varphi_t(x, t-s)\,ds = 0, \\ c\theta_t + dP_t - k\theta_{xx} - \gamma_1\varphi_{xt} = 0, \\ d\theta_t + rP_t - hP_{xx} - \gamma_2\varphi_{xt} = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$ with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x)\,, \; u_t(x,0) = u_1(x)\,, & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x)\,, \; \varphi_t(x,0) = \varphi_1(x)\,, & x \in (0,1), \\ \theta(x,0) = \theta_0(x)\,, \quad P(x,0) = P_0(x)\,, & x \in (0,1), \\ \varphi_t(x,-t) = f_0(x,t)\,, & t \in (0,\tau), \\ u(0,t) = \varphi(0,t) = \theta(0,t) = P(0,t) = 0, & \forall t \geq 0, \\ u_x(1,t) = \varphi_x(1,t) = \theta(1,t) = P(1,t) = 0, & \forall t \geq 0, \end{cases} \tag{1.2}$$

where u is the displacement of the fluid and φ is the elastic solid material, ρ_1 and ρ_2 are the densities of u and φ, respectively, θ is the temperature difference and P is the chemical potential, k is the heat and h is the mass diffusion conductivity coefficients. The coefficients a_1 and a_3 are positive constants and $a_2 \neq 0$ is a real number such that $a_1a_3 > a_2^2$. The coefficients μ_1 is positive constant and such that is a real number such that

$$\int_{\tau_1}^{\tau_2} \mu_2(s)\,ds < \mu_1. \tag{1.3}$$

The physical positive constants γ_1, γ_2, r, c and d satisfying

$$\lambda = rc - d^2 > 0. \tag{1.4}$$

In this work, we study a swelling porous-heat system incorporating thermodiffusion effects and a distributed delay term, which distinguishes the system (1.1)-(1.2) from others in the literature. Specifically, the inclusion of these terms makes the system significant in modeling complex physical phenomena.

In Chapter 2, using semigroup theory, we establish the well-posedness of the system, demonstrating the existence and uniqueness of solutions under appropriate assumptions. In Chapter 3, by employing the energy method, we prove the exponential stability of the system, irrespective of the wave propagation speeds. These results highlight the impact of the distributed delay and thermodiffusion effects on the system. Further details regarding the techniques employed can be found in [7, 24].

Chapitre 2

Preliminaries and analytical tools

2.1 Functionel spaces

Here we recall the essential notions about sobolev spaces semigroups some useful theoreme and some important inequalities which we will use in this thesis. Let Ω be an open subset of $\mathbb{R}^N$, $n \in N$ supplied with the lebesgue musure dx

Definition 1 (Complet space). *A normed space $(E, \| . \|)$ is complete if every cauchy sequence of E is converge in E*

Definition 2 (Banach space). *A complete normed vector space is called banach space*

Definition 3 (Hilbert space). *A Hilbert space is a vector space endowed with scaler product$< u, v >$and wich is complete for the norme*

$$\|u\| \equiv \langle u, u\rangle^{\frac{!}{2}}$$

Definition 4 (L^p space). *Let $1 \leq p \leq \infty$ define the standard Lebesgue space $L^p(\Omega)$by*

$$L^p(\Omega)\,\{u\dot{:}\Omega \to R\}\, u is\ mesurable\ and \quad \int\limits_{\Omega} |u|\, dx < \infty$$

the functionnal $\| . \|_{L^P}$ defined by

$$\|u\|_{L^p} \equiv \left(\int\limits_{\Omega} |u|^p\right)^{1\div p}$$

is a norm on $L^p(\Omega)$

Definition 5 (L$^\infty$ space). *Let (E, T, m) be a mesured space and f mesurable function from Eto R we say that f is essentialy bounded on that $f \in L^\infty \equiv L^\infty_R(E, T, m)$ if ther existec $\in$ Rsuch that $|f| \leq c$ if $f \in L^\infty$ we set $\|f\|_\infty \equiv \inf\{c \in R_+, |f| \leq c\}$ if $f \in L^\infty$ we set $\|f\|_\infty \equiv \dot{+}\infty$*

Let $I =]a, b[$ be a bounded or unbounded interval and let $p \in \mathbb{R}$ with $1 \leq p \leq \dot{+}\infty$

Definition 6 ($W^{1.P}$ Space). *The sobolev space $W^{1.P}$ is defined by*

$$W^{1,P} \equiv \left\{u \in L^P(I), \exists g \in L^p(I) \int u\acute{\varphi} = \int g\varphi \forall\varphi \in c^1_c(I)\right\}$$

we set $H^1(I) = W^{1,2}(I)$ for $u \in W^{1,p}(I)$ and denote by $\acute{u} = g$

The space $W^{1,p}$ is equipped with the norme $\|u\|_{w^{1,p}} = \|u\|_{L^P} \dotplus \|u\acute{}\|_{L^p}$.

On some times if $1 < p < \infty$ of the equivalent norme $\left(\|u\|_{L^p}^p \dotplus \|u\acute{}\|_{L^p}^p\right)^{1 \div p}$ the space H^1 is equipped with the scaler product $< u, v >_{H^1} = < u, v >_{L^2} \dotplus < u\acute{}, v\acute{} >_{L^2}$

the associated norme $\|u\|_{H^1} = \left(\|u\|_{L^2}^2 \dotplus \|u\acute{}\|_{L^2}^2\right)^{1 \div 2}$ is equivalente to the norme of$W^{1,2}$

Proposition 1. *The space $W^{1,P}$ is Banach space for $1 \le p \le +\infty$.*

The space $W^{1,p}$ is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.

The space $H^1$is a separable hilbert space

Definition 7 ($W_0^{1,p}$ Space)**.** *Given $1 \le p < \infty$ we denote by$W_0^{1.p}$the closure of $c_0^1(I)$ in $W^{1,p}(I)$ we denote $H_0^1(I) = W_0^{1,2}(I)$ the space$W_0^{1,p}$is equipped with the incluste norme by$W^{1,p}$ the space$H_0^1$is equipped with the scaler product induced byH^1*

Proposition 2. *The space $W_0^{1,p}$ is a separable banach space it is also reflective for $1 < p < \infty$.*

The space $H_0^1$is a separable hilbert space

Definition 8 ($W^{m,p}$ Space)**.** *Given an integer $m \ge 2$ and a real number 1$\le p \le \infty$ we define by induction space $W^{m,p}(I) = \left\{u \in W^{m-1,p}(I), u\acute{} \in W^{m-1,p}(I)\right\}$ we set $H^m(I) = W^{m.2}(I)$.*

Remark 1. *We easily verify that $u \in W^{m,p}$if and only if there existe in functions $g_1 ... g_m$such that $-\int uD^j\varphi = (-1)^j \int g_j\varphi, \forall\varphi \in c_c^\infty(I)\, \forall j \in 1..m$ $D^j\varphi$ denotes the derivative to order j of φ we can therfore consider then$u \in W^{m,p}(I)$ to successive derivatives $u\acute{} = g_1$ $(u\acute{})\acute{} = g_2$ the order m we denote the $Du...D^m u$ where $W^{m,p}$ space is equipped with the norm*

$$\|u\|_{W^{m,p}} = \|u\|_{L^P} \dotplus \sum_{\alpha=1}^{m} \|D^\alpha u\|_{L^p}$$

Definition 9 (Bochner space)**.** *Let X be a banach space $1 \le p \le \infty$,-$\infty < a < b < \infty$ then L^p (a, b, x) denotes the space of L^p functions from (a, b) into X it is a banach space for the norme*

$$\|u\|_{L^P([a,b],X)} = \left(\int_a^b \|u\|^p \, dt\right)^{\frac{1}{p}}$$

wher the integrale is understood in the bochner sense

Ther existe a constant c depending only on $I \le \infty$ such that$\|u\|_{L^\infty} \le c\,\|u\|_{W^{1,p}}(I)\, \forall 1 \le p \le \infty$ more over when is bounded we have

the injection $W^{1,P}(I) \subset C(I)$is compact for $1 < p \le \infty$

the injection $W^{1,P}(I) \subset L^q(I)$is compact for $1 \le q < \infty$

Theorem 3 (Lax Milgram)**.** *Let $a(u, v)$ be a bilinear continuous and coarsive form then for any$\varphi \in H^1$ther existe aunique $u\in H$ such that $a(u, v) = (\varphi, v)\, \forall v \in H$ more over if a is symetric then u is characterized by the property $u \in H$ end$1 \diagdown 2a(u, u) - (\varphi, u) = \min_{V \in H} \{1 \diagdown 2a(v, v) - (\varphi, v)\}$*

Theorem 4 (Fubini). *Assume that* $F\in L^1(\Omega_1\times\Omega_2)$ *for almost any* $x\in\Omega_1 F(x,y)\in L^1_y(\Omega_2)$ *and* $\int_{\Omega_2} F(x,y)\,dy\in L^{1()}_x(\Omega_1)$ *similary for almost any* $y\in(\Omega_2), F(x,y)\in L^1_x(\Omega_1)$ *and* $\int_{\Omega_1} F(x,y)\,dx\in L^1_y(\Omega_2)$ *and morover we have*

$$\int_{\Omega_1} dx\int_{\Omega_2} F(x,y)\,dy=\int_{\Omega_2} dy\int_{\Omega_1} F(x,y)\,dx=\int\int_{\Omega_1\times\Omega_2} F(x,y)\,dxdy$$

Theorem 5. *Let* A *be infinitisimal generator of a* c_0*semi group* $s(t)$ *on*X *if* $F[0,\infty[\times X\rightarrow X$*is continously differntiable from*$[t_0,T]\times X\rightarrow X$ *then the weak solution of system is a strong solution*

2.2 Some important inequalities

Cauchy−schwartz inequality

let H be a vector cpace scaler product $<u,v>$is bilear form of $H\times H$ in R symmetric positive definite $(u,v)\geq 0\forall u\in H$ and $(u,v)>0$ if $u\neq o$ reccall that a scaler product satisfies the cauchy-schwarz inequality

$$|<u,v>|\leq <u,u>^{1\div 2}<v,v>^{1\div 2}\ \forall u,v\in H$$

Young inequality

let p and q two real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$ then $\forall(f,g)\in L^P(\Omega)\times L^q(\Omega)\,\forall\varepsilon>0$

$$\int_\Omega |f.g|\,dx\leq\frac{\varepsilon}{p}\int_\Omega |f^p|\,dx+\frac{1}{q\varepsilon^{\frac{q}{p}}}\int_\Omega |g^q|\,dx$$

Poincare inequality

we assume that I is bounded then ther exists a conctant c dependent of I such that:

$$\|u\|_{w1,p}\leq c\,\|u\|_{L^p}\,\forall u\in W_0^{1,p}$$

in other words on $\mathrm{W}_0^{1,p}(I)$ the quantity $\|u\|^p_{L^P}$is a norm equivalent to the norm of $\mathrm{W}^{1,p}(I)$

holder inequality:

let $1\leq p\leq\infty$ assume that f$\in L^p(\Omega)$ and g$\in L^q(\Omega)$whit$\frac{1}{p}+\frac{1}{q}=1$ then f.g$\in L^1(\Omega)$and

$$\begin{aligned}\int_\Omega |f.g|\,dx &\leq \left(\int_\Omega |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_\Omega |g|^q\,dx\right)^{\frac{1}{q}}\\ \|f.g\|_{L^1} &\leq \|f\|_{L^p}\cdot\|g\|_{L^q}\end{aligned}$$

Remark 2. *caushy shwarts inequality is a speacial case of holden inequality with p=q=2*

2.3 Semigroup Method

The semigroup method is a powerful tool for solving evolution equations. It can be used to deal with many initial value problems or initial boundary value problems for both linear and nonlinear evolution equations.

To introduce the concept of semigroups of linear contraction operators, let us first look at a simple example. Consider the following initial boundary value problem for the heat equation :

$$\begin{cases} u_t - u_{xx} = 0 \\ u \mid_{x=0} = u \mid_{x=\pi} = 0 \\ u \mid_{t=0} = u_0(x) \in L^2(0,\pi) \end{cases} \tag{3}$$

appling the method of separate variable it is easy to see that the solution to the above probleme can be express as a fourier series

$$u(x,t) = \sum_{k=1}^{\infty} a_k \exp\left(-k^2 t\right) \sin\left(kx\right)$$

.

$$a_k = \frac{2}{\pi} \int_0^{\pi} u_0\left(x\right) \sin kx dx$$

since it is asumed that $u_0 \in l^2$by the parsevale equality then

$$\sum_{k=1}^{\infty} a_k^2 = \frac{2}{\pi} \int_0^{\pi} u_0^2\left(x\right) dx$$

since the series given by has afast decay factoreexp $\left(-k^2 t\right)$for $t > 0$ it can be easily proved that $u\left(x,t\right)$given bythis series is infinitly time defferential with respect to x and t for $t > 0$and it satisfies the equation and boundary condition for 3 morover we have

$$\int_0^{\pi} u\left(x,t\right) - u_0\left(x\right)^2 dx = \int_0^{\pi} \sum_{k=1}^{\infty} a_k \left(\exp\left(-k^2 t\right) - 1\right)^2 dx < \frac{\pi}{2} \sum_{k=1}^{\infty} a_k^2 \left(\exp\left(-k^2 t\right) - 1\right) \to 0$$

as $t \to 0$in other words as $t \to 0 u\left(x,t\right)$ converg to $u_0\left(x\right)$in$L^2\left(0,\pi\right)$since it can be easily seen from the energy methode that solution$u\left(x,t\right)$ is uniquely determined by u_0 thas we can view$u\left(x,t\right)$ as the image of $u_0\left(x\right)$ under amapping $s\left(t\right)$

$$u\left(x,t\right) = s\left(t\right) u_0\left(x\right) = \sum_{k=1}^{\infty} a_k \exp\left(-k^2 t\right) \sin kx$$

we infen from this definition that for any $t \in [0,\infty[$ $s\left(t\right)$ is a linear operator from $L^2\left(0,\pi\right)$to$L^2\left(0,\pi\right)$no we investigate the properties of $s\left(t\right)$ firstly by definition

1 $s\left(0\right) = I$

indeed $s\left(t\right) u_0 = u\left(x,t\right) \Longrightarrow s\left(0\right) u_0 = u\left(x,0\right) = u_0 \Longrightarrow s\left(0\right) = I$

2 $s\left(t_1 \dotplus t_2\right) = s\left(t_1\right) s\left(t_2\right)$

indeed : for any $t_1, t_2 \geq 0$

$$\begin{aligned} s(t_2)(s(t_1)u_0(x)) &= s(t_2)u(x,t_1) = s(t_2)\sum_{k=1}^{\infty} a_k \exp\left(-k^2 t_1\right)\sin(kx) \\ &= \sum_{k=1}^{\infty} a_k \exp\left(-k^2 t_1\right)(s(t_2)\sin(kx)) \\ &= \sum_{k=1}^{\infty} a_k \exp\left(-k^2(t_1 \dotplus t_2)\right)\sin(kx) = s(t_1 \dotplus t_2)u_0(x) \end{aligned}$$

3 $\|s(t)\| \leq 1$
indeed : by perseval enequality we have for all $t \geq 0$

$$\|u(.,t)\|^2 = \frac{\pi}{2}\sum_{k=1}^{\infty} a_k \exp(-2kt) \leq \frac{\pi}{2}\exp(-2t)\sum_{k=1}^{\infty} a_k = \exp(-2t)\|u_0\|^2$$

4 as in 2,4 we can also show that for any $u_0 \in L^2(0.,\pi)$ $s(t)u_0 \in C\left([0,\dotplus\infty[\,,L^2(0,\pi)\right)$ for $t \in [0,\dotplus\infty[$ $u(x,t)$ can be viewed as an abstract continous function valued in $L^2(0,\pi)$

Theorem 6. *lumer —phillips*

Let $A : D(A) \longrightarrow H$ be a dense domaine limate operator in H then A is the infinitisimal generator of a co—semigroup of contraction if and only if

A is a descriptive ther exists an $\lambda > 0$ such that $\operatorname{Im}(\lambda I - A) = H$

2.4 Energy Method

The study of stability for evolving system is often linked to the construction of lyapanov funtionale the general method for constructing lyapanov functionproposed by Vkolmanovski and Lshaikhet already successfuly used for differential equations discrete time difference equation and continneous time difference equation is used ther to study the stability of evolving delay equation in particular partial differential equation

Definition 10. *Letu and H be two separable hilbert space such that $u \subset H \equiv H^*$ $\subset u^*$ where the injections are continuous and dense*

Let $\|.\|$ and$\|.\|_*$be the norme of U,H and H* respectively$< .,. >$and $< .,. >$ be the scalar product of u and H respectively and $< .,. >$ be the scaler product between u end u^*

suppose that $\quad |u| \leq \beta \|u\|$

Let $C(-h,0,H)$ be the banach space containing all continuous functions on $[-h,0]$ inH such that $x_t \in C(-h,0,H)$ for each $t \in [0,\infty]$the function defined by $x_t(s) = x(t \dotplus s)$for all $s \in [-h,0]$

the space $C(-h,0,u)$ is defined similary

Let $A(t,0) : u \to u^* f_1(t,.) : C(-h,0,H) \to u^*$and $f_2(t,.) : C(-h,0,u) \to u^*$ are the famillies of non lineare operator defined for $t > 0 A(t,0) = 0 f_1(t,0) = 0 f_2(t,0) = 0$

we have the equation

$$\begin{aligned} \frac{du(t)}{dt} &= A(t, u(t)) + f_1(t, u(t)), t > 0 \\ u(s) &= \Psi(s), s \in [-h, 0] \end{aligned}$$

Let us denote by $u(t, \Psi)$the solution to equation coresponding to the initial condition Ψ

Definition 11. *The triviale solution to equation 1,23 is said to be stable if for all* $\varepsilon > 0$ *ther exist*$\delta > 0$ *such that we for* $|u(t, \Psi)| < \varepsilon$ *for all* $t \geq 0$ *if* $|\Psi|_{CH} \equiv \sup |\Psi(s)| < \delta s \in [-h, 0]$

Definition 12. *.The triviale solution to equation 1,23 is said to be exponentially stable if it is stable and ther existe a positive constant* λ *such that for all* $\Psi \in C(-h, 0, u)$ *exists c such that*

$|u(t, \Psi)| \leq c \exp(-\lambda t)$ for $t > 0$

Chapitre 3

Well-posedness

3.1 Reformulation of the System and Functional Framework

In this Chapter, we prove the existence and uniqueness of solutions for (1.1)-(1.2). As in [19], we introduce the new variable

$$z(x,\rho,s,t)=\varphi_t\left(x,t-\rho s\right),\ x\in\left(0,1\right),\ \rho\in\left(0,1\right),s\in\left(\tau_1,\tau_2\right)\ t>0. \tag{3.1}$$

Then the above variable z satisfies

$$sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0,x\in\left(0,1\right),\rho\in\left(0,1\right),s\in\left(\tau_1,\tau_2\right),t>0.$$

Therefore, problem (1.1) takes the form

$$\begin{cases}\rho_1u_{tt}-a_1u_{xx}-a_2\varphi_{xx}=0,\\ \rho_2\varphi_{tt}-a_3\varphi_{xx}-a_2u_{xx}-\gamma_1\theta_x-\gamma_2P_x+\mu_1\varphi_t+\displaystyle\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)z\left(x,1,s,t\right)ds=0,\\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0,\\ c\theta_t+dP_t-k\theta_{xx}-\gamma_1\varphi_{xt}=0,\\ d\theta_t+rP_t-hP_{xx}-\gamma_2\varphi_{xt}=0,\end{cases} \tag{3.2}$$

with the following initial and boundary conditions :

$$\begin{cases}u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right),\\ \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right),\\ \theta\left(x,0\right)=\theta_0\left(x\right),\ \ P\left(x,0\right)=P_0\left(x\right),\\ z(x,\rho,s,0)=f_0\left(x,s\rho\right),\\ z(x,0,s,t)=\varphi_t\left(x,t\right),\\ u\left(0,t\right)=\varphi\left(0,t\right)=\theta\left(0,t\right)=P\left(0,t\right)=0,\\ u_x\left(1,t\right)=\varphi_x\left(1,t\right)=\theta\left(1,t\right)=P\left(1,t\right)=0,\end{cases} \tag{3.3}$$

Introducing the vector function $U=\left(u,u_t,\varphi,\varphi_t,z,\theta,P\right)^T$, system (3.2)-(3.3) can be written as

$$\begin{cases}U'\left(t\right)=\mathcal{A}U\left(t\right),\ \ t>0,\\ U\left(x,0\right)=U_0\left(x\right)=\left(u_0,u_1,\varphi_0,\varphi_1,f_0,\theta_0,P_0\right)^T,\end{cases} \tag{3.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ u_t \\ \varphi \\ \varphi_t \\ z \\ \theta \\ P \end{pmatrix} = \begin{pmatrix} u_t \\ \frac{1}{\rho_1}\left[a_1 u_{xx} + a_2\varphi_{xx}\right] \\ \varphi_t \\ \frac{1}{\rho_2}\left[a_3\varphi_{xx} + a_2 u_{xx} + \gamma_1\theta_x + \gamma_2 P_x - \mu_1\varphi_t - \int_{\tau_1}^{\tau_2}\mu_2(s)\, z(x,1,s,t)\, ds\right] \\ -\frac{1}{s} z_\rho(x,\rho,,s,t) \\ \left(\frac{rk}{\lambda}\right)\theta_{xx} - \left(\frac{hd}{\lambda}\right)P_{xx} + \left(\frac{r\gamma_1 - d\gamma_2}{\lambda}\right)\varphi_{tx} \\ \left(\frac{ch}{\lambda}\right)P_{xx} - \left(\frac{kd}{\lambda}\right)\theta_{xx} + \left(\frac{c\gamma_2 - d\gamma_1}{\lambda}\right)\varphi_{tx} \end{pmatrix}.$$

The energy space is defined as :

$$\begin{aligned}\mathcal{H} &= H_*^1(0,1) \times L_*^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \\ &\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \times L^2(0,1) \times L^2(0,1),\end{aligned}$$

where

$$L_*^2(0,1) = \left\{\varphi \in L^2(0,1) : \int_0^1 \varphi(x)\, dx = 0\right\}$$

and

$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1).$$

Let

$$U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T, \bar{U} = \left(\bar{u}, \bar{u}_t, \bar{\varphi}, \bar{\varphi}_t, \bar{z}, \bar{\theta}, \bar{P}\right)^T.$$

Concerning the weight of the delay, we only assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| < \mu_1, \tag{3.5}$$

we define the inner product in $\mathcal{H}$ as follows

$$\begin{aligned}\left\langle U, \bar{U}\right\rangle_{\mathcal{H}} &= \int_0^1 \left[\rho_1 u_t\bar{u}_t + a_1 u_x\bar{u}_x + \rho_2\varphi_t\bar{\varphi}_t + a_3\varphi_x\bar{\varphi}_x + a_2\left(u_x\bar{\varphi}_x + \varphi_x\bar{u}_x\right)\right] dx \\ &+ \int_0^1 \left[c\theta\bar{\theta} + d\left(P\bar{\theta} + \theta\bar{P}\right) + rP\bar{P}\right] dx \\ &+ \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z(x,\rho,s,t)\, \bar{z}(x,\rho,s,t)\, ds d\rho dx. \end{aligned} \tag{3.6}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{\begin{array}{c} U \in \mathcal{H} \mid u \in H_*^2(0,1) \cap H_*^1(0,1),\ \varphi \in H^2(0,1) \cap H_0^1(0,1), \\ u_t \in H_*^1(0,1), \varphi_t \in H_0^1(0,1) \\ \theta,\ P \in H_0^1(0,1),\ z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ z(x,0,s,t) = \varphi_t \end{array}\right\},$$

where

$$H_*^2(0,1) = \left\{f \in H^2(0,1);\ f_x(0) = f_x(1) = 0\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Then we state the main result :

Theorem 7. *Assume that $U_0 \in \mathcal{H}$ and (1.4) holds, then problem (3.1)-(3.2) admits a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

3.2 Proof of Well-posedness via Semigroup Theory

To obtain the above result, we prove that the operator $\mathcal{A}$ is maximal monotone. For this, we proceed in two steps.

Step 1 : Dissipativity of $\mathcal{A}$

For any $U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T \in D(\mathcal{A})$, by using the inner product (3.6) and integrating by parts, we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - \mu_1 \int_0^1 \varphi_t^2 dx \\ &- \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z_\rho(x,\rho,s,t) z(x,\rho,s,t) ds d\rho dx \\ &- \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) ds dx.\end{aligned} \tag{3.7}$$

For the last term of the right-hand side of (3.7)), we have

$$\begin{aligned}-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z(x,\rho,s,t)\, z_\rho d\rho ds dx &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,s,t)\, d\rho ds dx \\ &= -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds dx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,0,s,t)\, ds dx\end{aligned} \tag{3.8}$$

Using the fact that $z(x,0,s,t) = \varphi_t(x,t)$ we deduce that

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - \left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx \\ &- \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t) ds dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) ds dx.\end{aligned} \tag{3.9}$$

Using Young's inequality, we arrive at

$$\begin{aligned}-\int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) ds dx &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 \varphi_t^2 dx \\ &+ \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t) ds dx\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9)

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx$$

Consequently, from the assumption (3.5), $\mathcal{A}$ is a dissipative operator.

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0.$$

Step 2 : Surjectivity of $\mathrm{Id} - \mathcal{A}$

To prove that the operator $Id - \mathcal{A}$ is surjective, that is, for any $F = (f_1, ..., f_7)^T \in \mathcal{H}$, there exists $U = (u, u_t, \varphi, \varphi_t, z, \theta, P)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})\, U = F, \tag{3.11}$$

which is equivalent to

$$\begin{cases} u - u_t = f_1, \\ \rho_1 u_t - a_1 u_{xx} - a_2 \varphi_{xx} = \rho_1 f_2, \\ \varphi - \varphi_t = f_3, \\ \rho_2 \varphi_t - a_3 \varphi_{xx} - a_2 u_{xx} - \gamma_1 \theta_x - \gamma_2 P_x + \mu_1 \varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x, 1, s, t) ds = \rho_2 f_4, \\ sz + z_\rho = s f_5, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\, \varphi_{tx} = \lambda f_6, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\, \varphi_{tx} = \lambda f_7. \end{cases} . \tag{3.12}$$

Suppose that we have found u and φ with the appropriate regularity. Therefore, the first and the third equations in (3.12) give

$$\begin{cases} u_t = u - f_1, \\ \varphi_t = \varphi - f_3. \end{cases} \tag{3.13}$$

It is clear that $u_t \in H_*^1(0,1)$ and $\varphi_t \in H_*^1(0, L)$. We note that the fifth equation in (3.12) with $z(x, 0, s, t) = \varphi_t(x, t)$, has a unique solution given by

$$z(x, \rho, s) = \varphi(x) e^{-s\rho} - f_3(x)\, e^{-s\rho} + s e^{-s\rho} \int_0^\rho e^{s\sigma} f_5(x, \sigma) d\sigma. \tag{3.14}$$

Clearly, $z, z_\rho \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$. By using (3.12), (3.13) and (3.14) the functions (u, φ, θ, P) satisfy the following system

$$\begin{cases} \rho_1 u - a_1 u_{xx} - a_2 \varphi_{xx} = g_1, \\ \eta\varphi - a_3 \varphi_{xx} - a_2 u_{xx} - \gamma_1 \theta_x - \gamma_2 P_x = g_2, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\, \varphi_x = g_3, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\, \varphi_x = g_4, \end{cases} . \tag{3.15}$$

where

$$\begin{aligned} \eta &= \rho_2 + \mu_1 + \int_{\tau_1}^{\tau_2} e^{-s} \mu_2(s)\, ds, \\ g_1 &= \rho_1 f_1 + \rho_1 f_2, \\ g_2 &= \rho_2 f_4 + \eta f_3 - \int_{\tau_1}^{\tau_2} s e^{-s} \mu_2(s) \int_0^1 e^{\sigma s} f_5(x, \sigma) d\sigma ds, \\ g_3 &= \lambda f_6 - (r\gamma_1 - d\gamma_2)\, f_{3x}, \\ g_4 &= \lambda f_7 - (c\gamma_2 - d\gamma_1)\, f_{3x}. \end{aligned}$$

We multiply $(3.15)_1$ by $\tilde{u}$, $(3.15)_2$ by $\tilde{\varphi}$, $(3.15)_3$ by $\frac{c}{\lambda}\tilde{\theta}$, $(3.15)_4$ by $\frac{r}{\lambda}\tilde{P}$, $(3.15)_3$ by $\frac{d}{\lambda}\tilde{P}$ and $(3.15)_4$ by $\frac{d}{\lambda}\tilde{\theta}$ and integrate their sum over $(0,1)$ to find the following variational formulation

$$\mathcal{B}\left((u, \varphi, \theta, P)^T, \left(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{P}\right)^T \right) = \mathcal{G}\left(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{P}\right)^T, \tag{3.16}$$

where $\mathcal{B} : \left[H_*^1(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2(0,1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned}\mathcal{B}\left((u,\varphi,\theta,P)^T, \left(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\right)^T\right) &= \rho_1 \int_0^1 u\tilde{u}dx + a_1 \int_0^1 u_x\tilde{u}_x dx + a_2 \int_0^1 (\varphi_x\tilde{u}_x + u_x\tilde{\varphi}_x)\, dx \\ &+ \eta \int_0^1 \varphi\tilde{\varphi}dx + a_3 \int_0^1 \varphi_x\tilde{\varphi}_x dx + c\int_0^1 \theta\tilde{\theta}dx + k\int_0^1 \theta_x\tilde{\theta}_x dx \\ &+ r\int_0^1 P\tilde{P}dx + h\int_0^1 P_x\tilde{P}_x dx + d\int_0^1 \left(\theta\tilde{P} + P\tilde{\theta}\right) dx \\ &+ \gamma_1 \int_0^1 \left(\theta\tilde{\varphi}_x - \varphi_x\tilde{\theta}\right) dx + \gamma_2 \int_0^1 \left(P\tilde{\varphi}_x - \varphi_x\tilde{P}\right) dx,\end{aligned}$$

and $\mathcal{G} : \left[H_*^1(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2(0,1)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$\begin{aligned}\mathcal{G}\left(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\right)^T &= \int_0^1 g_1\tilde{u}dx + \int_0^1 g_2\tilde{\varphi}dx + \frac{c}{\lambda}\int_0^1 g_3\tilde{\theta}dx \\ &+ \frac{r}{\lambda}\int_0^1 g_4\tilde{P}dx + \frac{d}{\lambda}\int_0^1 g_3\tilde{P}dx + \frac{d}{\lambda}\int_0^1 g_4\tilde{\theta}dx.\end{aligned}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{G}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $\left(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{P}\right) \in H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\times L^2(0,1)$, problem (3.16) admits a unique solution $(u,\varphi,\theta,P) \in H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\times L^2(0,1)$. The application of the classical regularity theory, it follows from (3.15) that $(u,\varphi,\theta,P) \in H_*^2(0,1)\times H_*^2(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$. Hence, the operator $Id - \mathcal{A}$ is surjective. Consequently, the operator $\mathcal{A}$ is maximal monotone. By the **Lumer-Phillips Theorem** (see [20]), $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$. This concludes the proof of Theorem 7. □

Chapitre 4

Exponential Stability

4.1 Energy Functional and Preliminary Estimates

In this section, we define the energy associated with problem (3.2)â€“(3.3) and prove some key differential inequalities.

We define the energy functional $E(t)$ as :

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + a_1 u_x^2 + \rho_2 \varphi_t^2 + a_3 \varphi_x^2 + 2a_2 u_x \varphi_x + c\theta^2 + 2d\theta P + rP^2\right] dx$$
$$+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx. \tag{4.1}$$

From (1.4), we deduce :

$$c\theta^2 + 2d\theta P + rP^2 = \frac{\lambda}{r}\theta^2 + \left(\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right)^2 > 0, \quad \text{for } \theta, P \neq 0.$$

Hence, the energy functional $E(t)$ is strictly positive.

The stability result reads as follows.

Theorem 8. *Let $(u, z, \varphi, \theta, P)$ be the solution of (3.2)-(3.3) and (1.4) holds. Then there exist two positive constants k_0 and k_1, such that*

$$E(t) \leq k_0 e^{-k_1 t}, \ \forall t \geq 0. \tag{4.2}$$

We now recall a series of lemmas that will be crucial for the exponential decay result.

Lemma 9. *Let $(u, z, \varphi, \theta, P)$ be the solution of (3.1)-(3.2) and (1.4) holds. Then, the energy functional, defined by equation (4.1), satisfies*

$$\frac{d}{dt}E(t) \leq -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - C\int_0^1 \varphi_t^2 dx \leq 0. \tag{4.3}$$

where

$$C = \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)$$

Démonstration. Multiplying $(3.2)_1$, $(3.2)_2$, $(3.2)_4$ and $(3.2)_5$ by u_t, φ_t, θ and P, respectively, and integrating over $(0,1)$ with respect to x, using integration by parts and the boundary conditions, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^1 \left(\rho_1 u_t^2 + a_1 u_x^2 + \rho_2\varphi_t^2 + a_3\varphi_x^2 + 2a_2 u_x\varphi_x + c\theta^2 + 2d\theta P + rP^2\right)dx\right]$$
$$= -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - \mu_1\int_0^1 \varphi_t^2 dx$$
$$-\int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)dsdx \tag{4.4}$$

On the other hand, multiplying $(3.2)_3$ by $|\mu_2(s)|\, z$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2(x,\rho,s,t)\, dsd\rho dx = -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, dsdx$$
$$+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,0,s,t)\, dsdx. \tag{4.5}$$

Recalling $z(x,0,s,t) = \varphi_t$, a combination of (4.4) and (4.5) gives

$$\frac{d}{dt}E(t) = -k\int_0^1 \theta_x^2 dx - h\int_0^1 P_x^2 dx - \left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx$$
$$-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)dsdx - \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)\, z(x,1,s,t)dsdx \tag{4.6}$$

Now, estimate the last term of the right-hand side of (4.6) as follows :
Using Young's inequality, we arrive at

$$-\int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)\, z(x,1,s,t)dsdx \le \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\int_0^1 \varphi_t^2 dx$$
$$+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)dsdx \tag{4.7}$$

Substituting (4.7) into (4.6), and using (3.5), we obtain (4.3), which completes the proof. □

Lemma 10. *Let (u,φ,z,θ,P) be the solution of (3.2)-(3.3). Then the functional*

$$L_1(t) = -\rho_1\int_0^1 u u_t dx,$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$L_1'(t) \le -\rho_1\int_0^1 u_t^2 dx + \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right)\int_0^1 u_x^2 dx + \varepsilon_1\int_0^1 \varphi_x^2 dx. \tag{4.8}$$

Démonstration. By differentiating $L_1(t)$ with respect to t we obtain

$$L_1'(t) = -\rho_1\int_0^1 \left[u_t^2 + u_{tt}u\right]dx$$

Using $(3.2)_1$ we deduce that

$$u_{tt} = \frac{1}{\rho_1}\left[a_1 u_{xx} + a_2\varphi_{xx}\right]$$

We replace u_{tt} in $L_1'(t)$ we find

$$L_1'(t) = -\rho_1 \int_0^1 \left[u_t^2 + \left(\frac{1}{\rho_1}\left(a_1 u_{xx} + a_2\varphi_{xx}\right)\right) u\right] dx$$

And integrating by parts, we obtain

$$L_1'(t) = -\rho_1 \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 u_x \varphi_x dx,$$

By Young's inequality we have

$$\int_0^1 \varphi_x\left(a_2 u_x\right) dx \le \varepsilon_1 \int_0^1 \varphi_x^2 dx + \frac{a_2^2}{4\varepsilon_1}\int_0^1 u_x^2 dx.$$

We obtain

$$L_1'(t) \le -\rho_1 \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 \varphi_x^2 dx + \frac{a_2^2}{4\varepsilon_1}\int_0^1 u_x^2 dx.$$

So, the final result is

$$L_1'(t) \le -\rho_1 \int_0^1 u_t^2 dx + \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right)\int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 \varphi_x^2 dx.$$

□

Lemma 11. *Let $(u, z, \varphi, \theta, P)$ be the solution of (3.2)-(3.3). Then the functional*

$$L_2(t) = a_1\rho_2 \int_0^1 \varphi\varphi_t dx - a_2\rho_1 \int_0^1 \varphi u_t dx,$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$L_2'(t) \le -\frac{a}{2}\int_0^1 \varphi_x^2 dx + C_1(\varepsilon_2)\int_0^1 \varphi_t^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx + \frac{2a_1^2\gamma_1^2}{a}\int_0^1 \theta_x^2 dx$$
$$+ \frac{2a_1^2\gamma_2^2}{a}\int_0^1 P_x^2 dx + \frac{2a_1^2\mu_1}{a}\int_0^1\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z^2(x,1,s,t) ds dx, \tag{4.9}$$

where

$$a = a_1 a_3 - a_2^2 > 0, \qquad C_1(\varepsilon_2) = a_1\rho_2 + \frac{2a_1^2\mu_1^2}{a} + \frac{a_2^2\rho_1^2}{4\varepsilon_2}.$$

Démonstration. By differentiating $L_2(t)$ with respect to t we obtain

$$L_2'(t) = a_1\rho_2 \int_0^1 \left[\varphi_t^2 + \varphi_{tt}\varphi\right] dx - a_2\rho_1 \int_0^1 \left[\varphi_t u_t + u_{tt}\varphi\right].$$

Using the equations $(3.2)_1$ and $(3.2)_2$, we deduce that

$$u_{tt} = \frac{1}{\rho_1}\left[a_1 u_{xx} + a_2\varphi_{xx}\right]$$
$$\varphi_{tt} = \frac{1}{\rho_2}\left[a_3\varphi_{xx} + a_2 u_{xx} + \gamma_1\theta_x + \gamma_2 P_x - \mu_1\varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\right],$$

we replace u_{tt}, φ_{tt} in $L_2'(t)$ and integrating by parts, we find

$$L_2'(t) = -a\int_0^1 \varphi_x^2 dx + a_1\rho_2\int_0^1 \varphi_t^2 dx - a_2\rho_1\int_0^1 \varphi_t u_t dx + \gamma_1 a_1\int_0^1 \varphi\theta_x dx \quad (4.10)$$
$$+ \gamma_2 a_1\int_0^1 \varphi P_x dx - \mu_1 a_1\int_0^1 \varphi\varphi_t dx - a_1\int_0^1 \varphi\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx,$$

where

$$a = a_3 a_1 - a_2^2 > 0.$$

By using Young's and Poincaré inequalities, the subsequent inequality holds true for any positive constant β

$$\gamma_1 a_1\int_0^1 \varphi\theta_x dx \le \frac{1}{4\beta}\int_0^1 \varphi^2 dx + \beta\gamma_1^2 a_1^2\int_0^1 \theta_x^2 dx$$
$$\le \frac{C_p}{4\beta}\int_0^1 \varphi_x^2 dx + \beta\gamma_1^2 a_1^2\int_0^1 \theta_x^2 dx$$

where C_p is the Poincaré constant, given that $C_p < 1$ in our case. If we set $\beta = \frac{2}{a}$, we obtain

$$\gamma_1 a_1\int_0^1 \varphi\theta_x dx \le \frac{a}{8}\int_0^1 \varphi_x^2 dx + \frac{2\gamma_1^2 a_1^2}{a}\int_0^1 \theta_x^2 dx.$$

Using the same method we have

$$\gamma_2 a_1\int_0^1 \varphi P_x dx \le \frac{a}{8}\int_0^1 \varphi_x^2 dx + \frac{2\gamma_2^2 a_1^2}{a}\int_0^1 P_x^2 dx$$

and

$$-\mu_1 a_1\int_0^1 \varphi\varphi_t dx \le \frac{a}{8}\int_0^1 \varphi_x^2 dx + \frac{2\mu_1^2 a_1^2}{a}\int_0^1 \varphi_t^2 dx.$$

Clearly, for any positive ε_2, we have

$$-a_2\rho_1\int_0^1 \varphi_t u_t dx \le \frac{a_2^2\rho_1^2}{4\varepsilon_2}\int_0^1 \varphi_t^2 dx + \varepsilon_2\int_0^1 u_t^2 dx.$$

By using Young's and Cauchy–Schwarz and Poincaré inequalities and 3.5, we have

$$-a_1\int_0^1 \varphi\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx \le \frac{a}{8}\int_0^1 \varphi_x^2 dx + \frac{2a_1^2\mu_1}{a}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)$$

Finally, by integrating these inequalities into (4.10), we deduce the result presented in our lemma □

Lemma 12. *Let $(u, z, \varphi, \theta, P)$ be the solution of (3.2)-(3.3) and (1.3) holds. Then the functional*

$$L_3(t) = \frac{a_1\rho_2}{a_2}\int_0^1 \varphi_t u dx - \frac{a_3\rho_1}{a_2}\int_0^1 u_t\varphi dx,$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$L_3'(t) \le -\frac{a_1}{2}\int_0^1 u_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{2a_1\gamma_1^2}{a_2^2}\int_0^1 \theta_x^2 dx + \frac{2a_1\gamma_2^2}{a_2^2}\int_0^1 P_x^2 dx \quad (4.11)$$
$$+ C_2(\varepsilon_3)\int_0^1 \varphi_t^2 dx + \frac{2a_1\mu_1}{a_2^2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx + \varepsilon_3\int_0^1 u_t^2 dx,$$

where

$$C_2(\varepsilon_3) = \frac{2a_1\mu_1^2}{a_2^2} + \frac{1}{4\varepsilon_3}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2.$$

Démonstration. By differentiating $L_3(t)$ with respect to t we find

$$L_3'(t) = \frac{a_1\rho_2}{a_2}\left[\int_0^1 \varphi_{tt}u + u_t\varphi_t\right]dx - \frac{a_3\rho_1}{a_2}\int_0^1 [u_{tt}\varphi + u_t\varphi_t]\,dx.$$

Using the equations $(3.2)_1$ and $(3.2)_2$, and integrating by parts, we obtain

$$\begin{aligned} L_3'(t) &= -a_1\int_0^1 u_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{\gamma_1 a_1}{a_2}\int_0^1 \theta_x u dx \qquad (4.12)\\ &+ \frac{\gamma_2 a_1}{a_2}\int_0^1 P_x u dx - \frac{\mu_1 a_1}{a_2}\int_0^1 \varphi_t u dx\\ &- \frac{a_1}{a_2}\int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,dsdx + \left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)\int_0^1 \varphi_t u_t dx. \end{aligned}$$

Using Young's inequality, the following inequality holds true for any positive constant β

$$\frac{\gamma_1 a_1}{a_2}\int_0^1 \theta_x u dx \le \beta\frac{a_1\gamma_1^2}{a_2^2}\int_0^1 \theta_x^2 dx + \frac{a_1}{4\beta}\int_0^1 u^2 dx,$$

then by using Poincaré inequality we get

$$\frac{\gamma_1 a_1}{a_2}\int_0^1 \theta_x u dx \le \beta\frac{a_1\gamma_1^2}{a_2^2}\int_0^1 \theta_x^2 dx + \frac{a_1 C_p}{4\beta}\int_0^1 u_x^2 dx.$$

Now, if we put $\beta = 2$ we obtain

$$\frac{\gamma_1 a_1}{a_2}\int_0^1 \theta_x u dx \le \frac{2a_1\gamma_1^2}{a_2^2}\int_0^1 \theta_x^2 dx + \frac{a_1}{8}\int_0^1 u_x^2 dx.$$

With the same method, we can derive

$$\frac{\gamma_2 a_1}{a_2}\int_0^1 P_x u dx \le \frac{2a_1\gamma_2^2}{a_2^2}\int_0^1 P_x^2 dx + \frac{a_1}{8}\int_0^1 u_x^2 dx$$

and

$$\frac{\mu_1 a_1}{a_2}\int_0^1 \varphi_t u dx \le \frac{2a_1\mu_1^2}{a_2^2}\int_0^1 \varphi_t^2 dx + \frac{a_1}{8}\int_0^1 u_x^2 dx.$$

By successively applying Young's, Cauchy–Schwarz, and Poincaré inequalities, we can write

$$\begin{aligned} \frac{-a_1}{a_2}\int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,dsdx &\le \frac{a_1}{4\beta}\int_0^1 u^2 dx + \frac{a_1}{a_2^2}\beta\int_0^1\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z(x,1,s,t)\,ds\right)^2 dsdx\\ &\le \frac{a_1}{4\beta}\int_0^1 u^2 dx + \frac{a_1}{a_2^2}\beta\int_{\tau_1}^{\tau_2}|\mu_2|\,ds\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,ds\\ &\le \frac{a_1 C_p}{4\beta}\int_0^1 u_x^2 dx + \frac{a_1}{a_2^2}\beta\int_{\tau_1}^{\tau_2}|\mu_2|\,ds\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t) \end{aligned}$$

Taking $\beta = 2$ and 3.5 we get

$$\frac{-a_1}{a_2}\int_0^1 u\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)\,dsdx \le \frac{a_1}{8}\int_0^1 u_x^2 dx + \frac{2a_1\mu_1}{a_2^2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)\,dsdx.$$

Likewise, from Young's inequality, we also obtain :

$$\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)\int_0^1 \varphi_t u_t dx \le \varepsilon_3\int_0^1 u_t^2 dx + \frac{1}{4\varepsilon_3}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2\int_0^1 \varphi_t^2 dx.$$

Finally, by integrating these inequalities into (4.12), we deduce the result presented in our lemma. □

Lemma 13. *Let $(u, z, \varphi, \theta, P)$ be the solution of (3.2)–(3.3). Then the functional*

$$L_4(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho dx$$

satisfies , for $m > 0$, the estimate

$$\begin{aligned} L_4'(t) \le & -m \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t)\, ds dx \\ & - m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho dx + \mu_1 \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{4.13}$$

Démonstration. Differentiating $L_4(t)$, and using $(3.2)_3$, we obtain,

$$\begin{aligned} L_4'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z(x, \rho, s, t) z_\rho(x, \rho, s, t)\, ds d\rho dx \\ &= - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| \frac{d}{d\rho} z^2(x, \rho, s, t)\, ds d\rho dx. \end{aligned}$$

Integration by parts gives,

$$\begin{aligned} L_4'(t) =& - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \frac{d}{d\rho} \left[e^{-s\rho} z^2(x, \rho, s, t)\right] ds d\rho dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho dx \\ =& - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho} \int_{\tau_1}^{\tau_2} \left(e^{-s\rho} z^2(x, \rho, s, t)\right) ds d\rho dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho \\ =& \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} \left[z^2(x, 1, s, t) - z^2(x, 0, s, t)\right] ds dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| e^{-s\rho} z^2(x, \rho, s, t)\, ds d\rho. \end{aligned}$$

Using the fact that $z(x, 0, s, t) = \varphi_t$ and $e^{-s} \le e^{-s\rho}$, we get for all $\rho \in [0, 1]$

$$\begin{aligned} L_4'(t) \le & - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t)\, ds dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_0^1 \varphi_t^2 dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\sigma dx. \\ \le & - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t)\, ds dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_0^1 \varphi_t^2 dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\sigma dx. \end{aligned}$$

Since $-e^{-s}$ is an increasing function , we have $-e^{-s} \le -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$.
Finally, setting $m = e^{-\tau_2}$, recalling (3.5), we obtain (4.14). □

4.2 Construction of the Lyapunov Functional and Proof of Exponential Decay

Now, we turn to prove our main result in this section.

Démonstration. (**of Theorem** 8) We define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{4} N_i L_i\left(t\right),$$

where N and N_i $(i = 1, 2, 3, 4)$ are positive constants that will be chosen later.

By differentiating $\mathcal{L}(t)$, exploiting (4.3) and (4.8)-(4.13), we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\rho_1 N_1 - \varepsilon_2 N_2 - \varepsilon_3 N_3\right] \int_0^1 u_t^2 dx \\
& - \left[\frac{a_1}{2} N_3 - \left(a_1 + \frac{a_2^2}{4\varepsilon_1}\right) N_1\right] \int_0^1 u_x^2 dx \\
& - \left[CN - C_1\left(\varepsilon_2\right) N_2 - C_2\left(\varepsilon_3\right) N_3 - \mu_1 N_4\right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{a}{2} N_2 - a_3 N_3 - \varepsilon_1 N_1\right] \int_0^1 \varphi_x^2 dx \\
& - \left[kN - \frac{2a_1^2\gamma_1^2}{a} N_2 - \frac{2a_1\gamma_1^2}{a_2^2} N_3\right] \int_0^1 \theta_x^2 dx \\
& - \left[hN - \frac{2a_1^2\gamma_2^2}{a} N_2 - \frac{2a_1\gamma_2^2}{a_2^2} N_3\right] \int_0^1 P_x^2 dx \\
& - \left[N_4 m - \frac{2a_1^2}{a} N_2 \mu_1 - \frac{2a_1}{a_2^2} N_3 \mu_1\right] \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2\left(s\right)\right| z^2(x, 1, s, t) ds dx \\
& - N_4 m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2\left(s\right)\right| z^2(x, \rho, s, t) ds d\rho dx.
\end{aligned}$$

Primarily, we set

$$\varepsilon_i = \frac{1}{N_i}, \; i = 1, 2, 3,$$

then we choose N_1 large enough so that

$$N_1 > \frac{2}{\rho_1},$$

Now, we can select N_3 large enough so that

$$\frac{a_1}{2} N_3 > \left(a_1 + \frac{a_2^2 N_1}{4}\right) N_1$$

and choose N_2, N_4 large enough such that

$$\frac{a}{2} N_2 > a_3 N_3 + 1,$$

$$N_4 m > \frac{2a_1^2\mu_1}{a} N_2 + \frac{2a_1\mu_1}{a_2^2} N_3.$$

Finally, we select N large enough so that

$$\begin{aligned} CN &> C_1(\varepsilon_2) N_2 + C_2(\varepsilon_3) N_3 + \mu_1 N_4, \\ kN &> \frac{2a_1^2\gamma_1^2}{a} N_2 + \frac{2a_1\gamma_1^2}{a_2^2} N_3, \\ hN &> \frac{2a_1^2\gamma_2^2}{a} N_2 + \frac{2a_1\gamma_2^2}{a_2^2} N_3, \end{aligned}$$

Consequently, from the above, we deduce that there exist a positive constant α_0 such that

$$\mathcal{L}'(t) \leq -\alpha_0 E(t). \tag{4.14}$$

On the hand, it is not hard to see that $\mathcal{L}(t) \sim E(t)$, i.e. there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain that

$$\mathcal{L}'(t) \leq -k_1 \mathcal{L}(t), \quad \forall t \geq 0, \tag{4.16}$$

where $k_1 = \dfrac{\alpha_0}{\alpha_2}$. A simple integration of (4.16) over $(0,t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_1 t}, \quad \forall t \geq 0.$$

It gives the desired result of Theorem 8 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. □

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The authors declare there is no conflict of interest.

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