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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

*Les mathématiques sont une science avec un S majuscule.
Car elles sont à la base de toutes les sciences,
on les retrouve absolument partout en science,
dans toutes les disciplines.
Quelqu'un a dit que les mathématiques sont comme de l'oxygène.
Je corrigerais en disant qu'elles sont comme l'air que l'on respire :
on ne se rend pas compte de leur présence,
mais lorsqu'on en manque, on étouffe !*

Emmanuel Trélat.

DÉDICACE

*A mes chers parents,
A toute ma famille,
A Samia,
A tous mes amis et amies.*

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GENERAL INTRODUCTION

The modern science is based on the application of mathematics. It is central to modern society, underpins scientific and industrial research. In general, a mathematical model is a simplified mathematical description of a real-world problem.

In this thesis, we will deal with models described by partial differential equations, namely reaction diffusion systems, They can be represented in the general form:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(t, x, u) \nabla u) + f(t, x, u, \nabla u), \quad x \in \Omega, \quad t > 0$$

where $u = u(t, x) = (u_1, \dots, u_m) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^m$ is a vector of variables. f is a linear or non-linear vector function, which is called the reaction terms, it is a regular application (at least locally Lipschitzian). $D : \mathbb{R}_+ \times \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a regular function. $\nabla \cdot (D(t, x, u) \nabla u)$ called the diffusion terms. This system is posed on an open domain $\Omega \subset \mathbb{R}^N$ (typically $N = 1, 2$ or 3), then boundary conditions have to be imposed for example, the homogeneous Dirichlet conditions ($u = 0$ on $\partial\Omega$) or the homogeneous Neumann conditions ($\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$).

Reaction-diffusion systems has its origin in chemistry, but today representatives can be found in many branches of natural sciences. models from theoretical physics, and biology...

The theory of reaction-diffusion equations appeared in the first half of the XXth century under the influence of various applications, such as heat explosion, propagation of chemical and biological waves or pattern formation. It brought together the theories of heat conduction and mass diffusion, on the one hand, and equations of chemical and biological kinetics, on the other. Diffusion theory takes its roots in atomism developed in India and in Greece beginning from the VIth century BC. The origin of chemistry goes back to ancient times with the invention of metallurgy and paints. It was a long way from the first observations and experiments to modern understanding of reaction-diffusion processes.

Mathematical models of heat explosion were introduced by Semenov and Frank-Kamenetskii in the 1930s. In order to describe temperature evolution in a closed vessel with a reacting gas, Semenov used an ordinary differential equation which took into account heat production due to a chemical reaction and heat loss through the boundaries of the vessel. It was assumed that the gas inside the vessel was well mixed and the temperature was uniformly distributed in space. Frank Kamenetskii suggested a more complete model where the temperature distribution in space was taken into account.

The theory of combustion waves began in the end of the XIXth century with the works by Mikhelson. He determined the flame structure and suggested an approximate formula for the speed of propagation. In the late 1930s, reaction diffusion waves were introduced and investigated by Kolmogorov–Petrovskii–Piskunov (KPP) and Fisher in relation with the problem of propagation of the dominant gene; Zeldovich and Frank-Kamenetskii studied them in the framework of combustion theory and Semenov for branching chain reactions.

The third application, which had an important influence on the development of the theory of reaction-diffusion equations, was related to pattern formation, Turing (1952).

There are many examples for reaction-diffusion systems in applications, we mention heat conduction, electrodynamics, dynamics of nuclear reactors, chemical reactions in distributed media and combustion theory, predator–prey models, population genetics, morphogenetic models, motion of bacteria by chemotaxis, growth of cancer tumors, model for the spatial spread of an epidemic (COVID-19, Malaria, Ebola...), models from nerve pulse and propagation neurophysiology. For the analysis of these types of problems, various methods and sophisticated techniques have been proposed, see for example Lions [30].

We mention some distinguished contemporary mathematicians such as Nouredine Alaa, Michel Pierre and Amar Youkana.

In Chapter 1, we introduce some basic definitions and collect some facts from the theory of reaction diffusion boundary-value problems.

In Chapter 2, we present the importance of mathematical modeling and some mathematical models of reaction diffusion systems and their applications in various sciences.

Finally, In Chapter 3, we show the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions of the following nonlinear parabolic system using techniques of functional analysis and potential analysis (see [52]).

$$\left\{ \begin{array}{l} -\frac{\partial u}{\partial t} + \Delta u = \lambda p(x, t)f(v) \\ -\frac{\partial v}{\partial t} + \Delta v = \mu q(x, t)g(w) \\ -\frac{\partial w}{\partial t} + \Delta w = \eta r(x, t)h(z) \\ -\frac{\partial z}{\partial t} + \Delta z = \rho e(x, t)k(u) \end{array} \right.$$

with $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and the initial conditions

$$\left\{ \begin{array}{l} u(x, 0) = \varphi(x) \quad , \quad v(x, 0) = \psi(x) \\ w(x, 0) = \gamma(x) \quad , \quad z(x, 0) = \zeta(x) \end{array} \right. , \quad \forall x \in \mathbb{R}^n$$

where $n \geq 3$, φ, ψ, γ and $\zeta : \mathbb{R}^n \rightarrow [0, \infty)$ are continuous, the constants λ, μ, η and ρ are nonnegative, f, g, h and $k : (0, \infty) \rightarrow [0, \infty)$ are nondecreasing and continuous. p, q, r and $e : \mathbb{R}^n \times (0, \infty) \rightarrow [0, \infty)$ are measurable functions and satisfying appropriate hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$.

Keywords reaction diffusion systems, parabolic systems, positive solutions, mathematical modeling.

THE BASICS OF THE REACTION DIFFUSION SYSTEMS

In this chapter, we introduce some basic definitions and collect some facts from the theory of reaction diffusion boundary-value problems.

1.1 Introduction

Reaction diffusion describes the process in which multiple participating chemicals or agents react with each other, while simultaneously diffusing or spreading through a liquid or gaseous medium. Typically, these processes are studied for their ability to produce nontrivial patterns that evolve over time. These patterns, often referred to as Turing structures or Turing patterns, are diffusion driven. In the presence of diffusion, the Turing patterns are observable, but are not present in the absence of diffusion. It is important for reaction diffusion models to replicate the behavior that is experimentally observed. That is to say that the models must be able to produce solutions with traits, such as pattern type, that are similar to experimentally observed traits. Mathematically, in this Chapter we strive to introduce the basics of reaction diffusion systems.

1.2 Terms and definitions

we introduce some basic definitions and collect some facts from the theory of reaction diffusion boundary-value problems.

Let Ω be either a bounded or an unbounded open domain in \mathbb{R}^N , and let $\partial\Omega$ be the boundary of Ω . For each $T > 0$, let $Q_T = (0, T] \times \Omega$, $\Sigma_T = (0, T] \times \partial\Omega$. Denote by $C^m(\Omega)$ the set of all continuous functions whose partial derivatives up to the m th order are continuous in Ω , and by $C^{1,m}(Q_T)$ the set of functions whose once continuously differentiable in t and m -times derivatives in x are continuous in Q_T . In particular, the set $C^{1,2}(Q_T)$ consists of all functions that are once continuously differentiable in t and twice continuously differentiable in x for all $(t, x) \in Q_T$. Similar notations are used for $C^m(\bar{\Omega})$ and $C^{1,m}(\bar{Q}_T)$, where $\bar{\Omega}$, \bar{Q}_T are the respective closures of Ω and Q_T . When $m = 0$ we denote by $C(\Omega)$, $C(\bar{\Omega})$, $C(Q_T)$, and $C(\bar{Q}_T)$ the set of continuous functions in Ω , $\bar{\Omega}$, Q_T , and \bar{Q}_T , respectively.

1.2.1 Gradient, directional derivatives, divergence, Laplacian

If $u = u(x_1, \dots, x_N)$ is a C^1 function defined on a region $\Omega \subset \mathbb{R}^N$, the gradient of u is defined by

$$\nabla u \equiv \text{grad } u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$$

If $v = (v_1, \dots, v_N)$ denotes a unit vector in \mathbb{R}^N , ($|v| = 1$), the directional derivative of u in direction v is given by

$$\frac{\partial u}{\partial v} = v \cdot \nabla u = \sum_{i=1}^N v_i \frac{\partial u}{\partial x_i}$$

If $w = (w_1(x_1, \dots, x_N), \dots, w_N(x_1, \dots, x_N))$ is a C^1 vector field on Ω , the divergence of w is defined to be the scalar field

$$\nabla \cdot w \equiv \text{div } w = \sum_{i=1}^N \frac{\partial w_i}{\partial x_i}$$

In particular, for $w = \nabla u$, we have

$$\nabla \cdot \nabla u \equiv \text{div grad } u \equiv \Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$$

The expression Δu is called the Laplacian of u

1.2.2 The divergence theorem

Let Ω be a bounded domain with piecewise smooth boundary surface S . Let n be the unit outward normal vector to S and let w be a vector field that is C^1 in Ω and C on $\Omega \cup S$. Then

$$\int_{\Omega} \nabla \cdot w d\Omega = \int_S w \cdot n dS$$

1.2.3 L^p spaces

Definition 1.1. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; we set

$$L^p(\Omega) = \{f : \Omega \rightarrow \mathbb{R}; f \text{ is measurable and } |f|^p \in L^1(\Omega)\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_{\Omega} |f(x)|^p d\mu \right]^{1/p}$$

where $\|\cdot\|_p$ is a norm

We set

$$L^\infty(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \mid \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \leq C \text{ a.e. on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C; |f(x)| \leq C \text{ a.e. on } \Omega\}$$

where $\|\cdot\|_\infty$ is a norm:

1.2.4 Test functions

Definition 1.2. Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function. The support is

$$\text{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}$$

The function is said to be of compact support on Ω if the support is a compact set contained inside Ω .

Definition 1.3. The space of test functions in Ω , denoted by $D(\Omega)$ is the space of all C^∞ functions defined on Ω which have compact supports in Ω .

$C^\infty(\Omega)$ denotes the space of all real-valued functions on Ω of class C^∞ .

$\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ is called multi-index with length $|\alpha| = \sum_{i=1}^N \alpha_i$.

Let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. We write $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$ and it acts on the space $C^\infty(\Omega)$. Thus, for $f \in C^\infty(\Omega)$, $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$ is it partial derivatives of order $|\alpha|$

1.2.5 Sobolev spaces

Definition 1.4. Let $1 \leq p < \infty$ and k be a non-negative integer. Then, Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall 0 \leq |\alpha| \leq k\}$$

The space is equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{1/p}$$

$W_0^{k,p}(\Omega) = \overline{D(\Omega)} \Big|_{W^{k,p}(\Omega)}$ i.e., $W_0^{k,p}(\Omega)$ is the closure of $D(\Omega)$ with respect to the $\|\cdot\|_{W^{k,p}(\Omega)}$.

When $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ and there are real Hilbert spaces with the following inner product

$$(u, v)_{H^k(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \int_{\Omega} D^\alpha u D^\alpha v dx$$

and the norm

$$\|u\|_{H^k(\Omega)} := \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_2^2 \right)^{1/2}$$

For, $k = 0$

$$W^{0,p}(\Omega) = L^p(\Omega)$$

$W^{k,p}(\Omega)$ are Banach spaces.

Given that Ω is smooth, then

$$W_0^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) : u = Du = \dots = D^{k-1}u = 0 \text{ on } \partial\Omega \right\}$$

For $p = 2$, we have

$$H_0^k(\Omega) := \left\{ u \in H^k(\Omega) : u = Du = \dots = D^{k-1}u = 0 \text{ on } \partial\Omega \right\}$$

For $p = 2$, and $k = 1$, we have

$$W_0^{1,2}(\Omega) := \left\{ u \in W^{1,2}(\Omega) = H^1(\Omega) : u = 0 \text{ on } \partial\Omega \right\}$$

and we denote it by $H_0^1(\Omega)$

For $p = 2, k = 2$, we write

$$W^{2,2}(\Omega) = H^2(\Omega).$$

Theorem 1.1. *Let Ω be smooth and $u \in L^2(\Omega)$ such that $\Delta u \in L^2(\Omega)$, then $u \in H^2(\Omega)$.*

1.2.6 Green's formula

Theorem 1.2. *Let Ω be bounded and smooth. Let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ then*

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_{\Omega} v \Delta u \, dx$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative, if $u = v$, then

$$\int_{\Omega} \|\nabla u\|^2 \, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, ds - \int_{\Omega} u \Delta u \, ds = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, ds + \int_{\Omega} u (-\Delta u) \, ds$$

Then

$$\int_{\Omega} (-\Delta u) u \, dx = \int_{\Omega} \|\nabla u\|^2 \, dx - \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, ds$$

Theorem 1.3 (Lax-Milgram). *Let $a : V \times V \rightarrow \mathbb{R}$ be a bilinear, continuous, and coercive functional. Then, for each $f \in V^* \exists! u^* \in V$:*

$$a(u^*, v) = (f, v), \text{ for all } v \in V$$

1.2.7 Poincare's inequality

Proposition 1.1. *Suppose Ω is a bounded set, then there exists a constant $C(\Omega) > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)}, \text{ for all } u \in W_0^{1,2}(\Omega)$$

1.2.8 Basic reproduction number

In epidemiology, the basic reproduction number, or basic reproductive number (sometimes called basic reproduction ratio or basic reproductive rate), denoted R_0 (pronounced R nought or R zero), of an infection can be thought of as the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. The definition describes the state where no other individuals are infected or immunized (naturally or through vaccination). Some definitions, such as that of the Australian Department of Health, add absence of "any deliberate intervention in disease transmission".

R_0 is not a biological constant for a pathogen as it is also affected by other factors such as environmental conditions and the behavior of the infected population. Furthermore R_0 values are usually estimated from mathematical models, and the estimated values are dependent on the model used and values of other parameters. Thus values given in the literature only make sense in the given context and it is recommended not to use obsolete values or compare values based on different models. R_0 does not by itself give an estimate of how fast an infection spreads in the population.

The most important uses of R_0 are determining if an emerging infectious disease can spread in a population and determining what proportion of the population should be immunized through vaccination to eradicate a disease. In commonly used infection models, when $R_0 > 1$ the infection will be able to start spreading in a population, but not if $R_0 < 1$. Generally, the larger the value of R_0 , the harder it is to control the epidemic. For simple models, the proportion of the population that needs to be effectively immunized (meaning not susceptible to infection) to prevent sustained spread of the infection has to be larger than $1 - \frac{1}{R_0}$. Conversely, the proportion of the population that remains susceptible to infection in the endemic equilibrium is $\frac{1}{R_0}$.

The basic reproduction number is affected by several factors, including the duration of infectivity of affected people, the infectiousness of the microorganism, and the number of susceptible people in the population that the infected people contact.

1.3 Functional analysis

In this section we recall some basic facts concerning unbounded linear operators acting in a real Hilbert space and in particular *dissipative* and *m-dissipative* operators. If H is a real Hilbert space we denote by (\cdot, \cdot) the inner product in H , by $|\cdot|$ the corresponding

norm in H and by $\|\cdot\|$ the associated norm in the space of linear bounded operators from H to H , denoted by $\mathcal{L}(H)$

Proposition 1.2. *Let $(H, (\cdot, \cdot))$ be an inner product space, then for all $u, v \in H$*

$$|(u, v)| \leq |u| |v| \quad (\text{Cauchy-Schwarz inequality})$$

Definition 1.5. A linear operator on H is a pair $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a linear manifold of H and A is a linear map from $\mathcal{D}(A)$ to H . $\mathcal{D}(A)$ is called the domain of A . One says that the operator $(A, \mathcal{D}(A))$ is densely defined if $\mathcal{D}(A)$ is dense in H .

Note that one often just says the linear operator A , but that its domain $\mathcal{D}(A)$ is implicitly taken into account. For such an operator, its range $\text{rang}(A)$ is defined by

$$\text{rang}(A) := A\mathcal{D}(A) = \{f \in H \mid f = Ag \text{ for some } g \in \mathcal{D}(A)\}$$

In addition, one defines the kernel $\text{Ker}(A)$ of A by

$$\text{Ker}(A) := \{f \in \mathcal{D}(A) \mid Af = 0\}$$

Let us also stress that the sum $A + B$ for two linear operators is a priori only defined on the subspace $\mathcal{D}(A) \cap \mathcal{D}(B)$, and that the product AB is a priori defined only on the subspace $\{f \in \mathcal{D}(B) \mid Bf \in \mathcal{D}(A)\}$. These two sets can be very small.

Definition 1.6. For any pair of linear operators $(A, \mathcal{D}(A))$ and $(B, \mathcal{D}(B))$ satisfying $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Af = Bf$ for all $f \in \mathcal{D}(A)$, one says that $(B, \mathcal{D}(B))$ is an extension of $(A, \mathcal{D}(A))$ to $\mathcal{D}(B)$, or that $(A, \mathcal{D}(A))$ is the restriction of $(B, \mathcal{D}(B))$ to $\mathcal{D}(A)$, we write $A \subset B$.

Definition 1.7. The unbounded operator $A : \mathcal{D}(A) \rightarrow H$ is closed if its graph $G(A) = \{(f, Af) \mid f \in \mathcal{D}(A)\}$ is closed in $H \times H$.

Remark 1.1. *A is closed if and only if for any sequence $(f_n) \subset \mathcal{D}(A)$ such that $f_n \rightarrow f$ and $Af_n \rightarrow g$ we have $f \in \mathcal{D}(A)$ and $Af = g$.*

Definition 1.8. We call A is closable if it has a closed extension. We denote the smallest closed extension of A by \bar{A} .

Remark 1.2. The closure \bar{A} of a closable operator $A : \mathcal{D}(A) \rightarrow H$ is uniquely defined through

$$\begin{aligned}\mathcal{D}(\bar{A}) &= \{f \in H \mid \exists (f_n) \subset \mathcal{D}(A) \text{ such that } f_n \rightarrow f \text{ and } (Af_n) \text{ converges}\} \\ \bar{A}f &= \lim_{n \rightarrow \infty} Af_n.\end{aligned}$$

1.3.1 Maximal dissipative operator

Definition 1.9. The unbounded operator $A : \mathcal{D}(A) \rightarrow H$ is called *dissipative* if

$$(Af, f) \leq 0, \quad \forall f \in \mathcal{D}(A)$$

The *dissipative* operator A is called *m-dissipative* if $\text{rang}(I - A) = H$, i.e

$$\forall g \in H, \exists f \in \mathcal{D}(A) \text{ such that } f - Af = g$$

Notice first that *dissipative* operators can be characterized as follows :

Proposition 1.3. Let A be an *m-dissipative* operator. Then

- (a) $\mathcal{D}(A)$ is dense in H ,
- (b) A is closed,
- (c) For all $\lambda > 0$ the operator $\lambda I - A$ is an isomorphism from $\mathcal{D}(A)$ onto H . Moreover $(\lambda I - A)^{-1}$ is a linear bounded operator such that $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$.

For the proof, see Brézis [13].

Example 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth. Consider the Laplace operator $\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ defined by

$$\Delta u := \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

Δ is *m-dissipative* operator.

$H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$, using Green's formula, we have

$$\begin{aligned} (\Delta u, u) &= \int_{\Omega} (\Delta u)u \\ &= - \int_{\Omega} \|\nabla u\|^2 + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} u d\eta \\ &= - \int_{\Omega} \|\nabla u\|^2 \\ &\leq 0. \end{aligned}$$

Then Δ is dissipative operator.

Next, we show that $\text{rang}(I - \Delta) = L^2(\Omega)$. i.e., for any $f \in L^2(\Omega)$, $\exists u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $u - \Delta u = f$.

Let $E = H_0^1(\Omega)$ with the following inner product

$$(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx \text{ for all } u, v \in H_0^1(\Omega),$$

We show that E is a Hilbert space.

$H_0^1(\Omega)$ is closed, since $H_0^1(\Omega) = \overline{D(\Omega)}\big|_{H^1(\Omega)}$, where Ω is bounded.

Thus, $H_0^1(\Omega)$ is a Hilbert space provided with the inner product of $H^1(\Omega)$,

We now prove that $\|u\|_{H^1(\Omega)} \sim \left(\int_{\Omega} \|\nabla u\|^2 dx\right)^{\frac{1}{2}}$,

Here, we use Poincare's inequality

$$\int_{\Omega} u^2 \leq c \int_{\Omega} \|\nabla u\|^2,$$

So,

$$\int_{\Omega} \|\nabla u\|^2 dx + \int_{\Omega} u^2 dx \leq (1+c) \int_{\Omega} \|\nabla u\|^2 dx,$$

But,

$$(1+c) \int_{\Omega} \|\nabla u\|^2 dx \leq (1+c) \|u\|_{H^1(\Omega)}^2,$$

Then,

$$\|u\|_{H^1(\Omega)} \sim \left(\int_{\Omega} \|\nabla u\|^2 dx\right)^{\frac{1}{2}},$$

Therefore, $H_0^1(\Omega)$ provided with the new norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_{\Omega} \|\nabla u\|^2\right)^{\frac{1}{2}},$$

Is complete, but this norm comes from the inner product on $H_0^1(\Omega)$ which is

$$(u, v)_{H_0^1(\Omega)} = \int_{\Omega} \nabla u \cdot \nabla v dx,$$

Let $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ be given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_0^1(\Omega),$$

since $H_0^1(\Omega)$ provided with the inner product $(.,.)_{H_0^1(\Omega)}$ is a Hilbert space. Then $a(.,.)$ is continuous, bilinear and coercive. By Theorem 1.3, we deduce that $\exists! u \in H_0^1(\Omega)$ such that $a(u, v) = (f, v)$ for all $v \in H_0^1(\Omega)$. Take φ in $D(\Omega)$, it implies $a(u, v) = (f, v)$. Hence $-\Delta u + u = f$ in distribution. since $\Delta u = f - u \in L^2(\Omega)$ and from Theorem 1.1, we have that $u \in H^2(\Omega)$ and conclude thus, for all $f \in L^2(\Omega)$, $\exists! u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $-\Delta u + u = f$. This means that $\text{rang}(I - \Delta) = L^2(\Omega)$.

Then Δ is m -dissipative operator.

Definition 1.10. Let A be an m -dissipative operator. For every $\lambda > 0$, set

$$R(\lambda; A) = (\lambda I - A)^{-1}, A_{\lambda} = \lambda A R(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I$$

$R(\lambda; A)$ is called the resolvent of A and A_{λ} is called the Yosida approximation of A . Keep in mind that $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$.

Theorem 1.4. Let A be an m -dissipative operator. Then

- (a) $A_{\lambda} v = \lambda R(\lambda; A) A v, \quad \forall (v, \lambda) \in \mathcal{D}(A) \times (0, \infty)$
- (b) $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda; A) v = v, \quad \forall v \in H$
- (c) $\lim_{\lambda \rightarrow \infty} A_{\lambda} v = A v, \quad \forall v \in \mathcal{D}(A)$
- (d) $(A_{\lambda} v, v) \leq 0, \quad \forall (v, \lambda) \in H \times (0, \infty)$
- (e) $|A_{\lambda} v| \leq \lambda |v|, \quad \forall (v, \lambda) \in H \times (0, \infty)$

For the proof, see Brézis [13]

Theorem 1.5. *Let $A : D(A) \subset H \rightarrow H$ be a m -dissipative operator on a real Hilbert space H . Then given any $u_0 \in D(A)$ there exists a unique solution*

$$u \in C^1([0, \infty), H) \cap C([0, \infty), D(A))$$

satisfying

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - Au = 0 & \text{on } [0, \infty) \\ u(0) = u_0 \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0| \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(t) \right| = |Au(t)| \leq |Au_0|, \quad \forall t \geq 0$$

For the proof, see Brézis [13].

Remark 1.3. *Let A be a m -dissipative operator and let $a \in \mathbb{R}$. The problem*

$$\begin{cases} \frac{du}{dt} - Au + au = 0 & \text{on } [0, +\infty) \\ u(0) = u_0 \end{cases}$$

reduces to problem (1.1) using the following change. Set

$$U(t) = e^{\lambda t} u(t)$$

Then U satisfies

$$\begin{cases} \frac{\partial U}{\partial t} - AU = 0 & \text{on } [0, \infty) \\ U(0) = U_0 \end{cases}$$

1.3.2 Adjoint operators

Definition 1.11. Let $A : \mathcal{D}(A) \rightarrow H$ be a densely defined operator. We recall that the domain of the adjoint A^* of A is defined by

$$\mathcal{D}(A^*) = \{f \in H \mid \exists! f^* \in H \text{ with } (f^*, g) = (f, Ag), \forall g \in \mathcal{D}(A)\}$$

and $A^* f := f^*$ for all $f \in \mathcal{D}(A^*)$.

Definition 1.12. Let $A : \mathcal{D}(A) \rightarrow H$ be a densely defined operator.

The operator A is called symmetric if $A \subset A^*$. Equivalently,

$$(Au, v) = (u, Av), \quad \forall u, v \in \mathcal{D}(A)$$

The operator A is called self-adjoint if $A = A^*$. That is, if and only if

$$A \subset A^* \text{ and } \mathcal{D}(A^*) \subset \mathcal{D}(A)$$

The operator A is called essentially self-adjoint if its closure \bar{A} is self-adjoint.

An important class of m -dissipative operators is given by the result below.

Theorem 1.6. *Let A be a m -dissipative, symmetric operator. Then A is self adjoint.*

For the proof, see Brézis [13].

Example 1.2. *The Laplace operator $\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is self-adjoint*

By Example 1.1 Δ is m -dissipative operator. In view of Theorem 1.6, it suffices to verify that Δ is symmetric.

Now, $u, v \in D(\Delta)$. Then

$$(\Delta u, v)_{L^2} = \int_{\Omega} (\Delta u)v = - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial n} v dr$$

But $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ and $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$ since, $v \in H_0^1(\Omega)$, we have that

$$(\Delta u, v)_{L^2} = - \int_{\Omega} \nabla u \cdot \nabla v$$

Also,

$$(u, \Delta v)_{L^2} = \int_{\Omega} u(\Delta v) = - \int_{\Omega} \nabla v \cdot \nabla u = - \int_{\Omega} \nabla u \cdot \nabla v$$

So,

$$(\Delta u, v) = (u, \Delta v), \quad \forall u, v \in \mathcal{D}(\Delta)$$

Hence, Δ is self-adjoint.

Theorem 1.7. *Let A be a self-adjoint m -dissipative operator. Then for every $u_0 \in H$ there exists a unique function*

$$u \in C([0, +\infty); H) \cap C^1((0, +\infty); H) \cap C((0, +\infty); D(A))$$

such that

$$\begin{cases} \frac{\partial u}{\partial t} - Au = 0 & \text{on } (0, +\infty) \\ u(0) = u_0 \end{cases}$$

Moreover, we have

$$|u(t)| \leq |u_0| \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(t) \right| = |Au(t)| \leq \frac{1}{t} |u_0| \quad \forall t > 0$$

For the proof, see Brézis [13].

1.3.3 The Hille-Yosida theorem in Banach spaces

The Hille-Yosida theorem extends to Banach spaces. The precise statement is the following.

Definition 1.13. Let X be a Banach space. The unbounded linear operator $A : D(A) \subset X \rightarrow X$ is called *dissipative* if

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \quad \text{for all } \lambda > 0 \text{ and all } x \in D(A)$$

The *dissipative* operator A is called *m -dissipative* if for all $\lambda > 0$ the operator $\lambda I - A$ is surjective.

Theorem 1.8. *Let A be m -dissipative. Then given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); X) \cap C([0, +\infty); D(A))$$

such that

$$\begin{cases} \frac{\partial u}{\partial t} - Au = 0 & \text{on } [0, +\infty) \\ u(0) = u_0 \end{cases}$$

Moreover,

$$\|u(t)\| \leq \|u_0\| \quad \text{and} \quad \left\| \frac{\partial u}{\partial t}(t) \right\| = \|Au(t)\| \leq \|Au_0\|, \quad \forall t \geq 0$$

For the proof, see, e.g., [49], [61].

1.3.4 Green functions

Definition 1.14. Suppose that Ω is an open set. A function $u \in C^2(\Omega)$ is harmonic on Ω if $\Delta u \equiv 0$ (The set of all harmonic functions on Ω is denoted by $\mathcal{H}(\Omega)$); superharmonic on Ω if $\Delta u \leq 0$ (The set of all superharmonic functions on Ω is denoted by $\mathcal{U}(\Omega)$); subharmonic on Ω if $\Delta u \geq 0$ (The set of all subharmonic functions on Ω is denoted by $\mathcal{S}(\Omega)$).

Remark 1.4. A function u is superharmonic if and only if $-u$ is subharmonic, and a function is harmonic if and only if it is both subharmonic and superharmonic.

We recall that, if $y \in \mathbb{R}^N$, then the function defined by

$$U_y(x) = \begin{cases} -\log \|x - y\| & (x \neq y; N = 2) \\ \|x - y\|^{2-N} & (x \neq y; N \geq 3) \\ +\infty & (x = y) \end{cases}$$

is superharmonic on \mathbb{R}^N and harmonic on $\mathbb{R}^N \setminus \{y\}$.

Definition 1.15. Let Ω be an open subset of \mathbb{R}^N . A function $G_\Omega : \Omega \times \Omega \rightarrow [0, +\infty]$, called the Green function of Ω , will be defined so that $G_\Omega(\cdot, y) = U_y - h_y$, where h_y is the greatest harmonic minorant of U_y on Ω . (To ensure the existence of h_y and hence G_Ω , a mild restriction on Ω will be required when $N = 2$.) It will follow that, if μ is a measure on Ω , then the equation

$$G_\Omega \mu(x) = \int_\Omega G_\Omega(x, y) d\mu(y), \quad (x \in \Omega)$$

defines a non-negative superharmonic function on Ω , provided only that $G_\Omega \mu$ is not identically $+\infty$ on any component of Ω . A superharmonic function of this form will be called a (Green) potential.

Definition 1.16. An open set Ω in \mathbb{R}^N is said to be Greenian if, for each y in Ω , the function U_y has a subharmonic minorant on Ω .

Theorem 1.9. (i) All open sets in \mathbb{R}^N ($N \geq 3$) are Greenian.

(ii) Any open subset of a Greenian open set is Greenian.

(iii) The set \mathbb{R}^2 is not Greenian.

(iv) If $\Omega \subseteq \mathbb{R}^2$ and there exists z in Ω such that U_z has a subharmonic minorant on Ω , then Ω is Greenian.

(v) If $\Omega \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \setminus \partial\Omega$ is not connected, then Ω is Greenian. In particular, all bounded open sets in \mathbb{R}^2 are Greenian.

The proof of this Theorem is given in [7].

Definition 1.17. Let Ω be Greenian. Then, for each y in Ω , the function U_y has a greatest harmonic minorant h_y on Ω . The function $G_\Omega : \Omega \times \Omega \rightarrow [0, +\infty]$, defined by

$$(1.2) \quad G_\Omega(x, y) = U_y(x) - h_y(x)$$

is called the Green function for Ω . Clearly $G_\Omega(x, x) = +\infty$ for any x in Ω . In the case where $\Omega = \mathbb{R}^N$ ($N \geq 3$) we simply write G for the Green function. The formula (1.2) has no meaning for open sets Ω which are not Greenian. We say that such sets Ω do not possess a Green function.

Theorem 1.10. The Green function for \mathbb{R}^N ($N \geq 3$) is given by

$$G(x, y) = U_y(x) \quad (x, y \in \mathbb{R}^N)$$

Proof. Let $y \in \mathbb{R}^N$. If h is any harmonic minorant of U_y , then $h \leq r^{2-N}$ on $B(y, r)$ by the maximum principle. Since r can be arbitrarily large, $h \leq 0$ on \mathbb{R}^N . Hence 0 is the greatest harmonic minorant of U_y on \mathbb{R}^N and the Green function for \mathbb{R}^N is as stated. ■

Theorem 1.11. Let Ω be Greenian. Then:

(i) $G_\Omega(x, y) = G_\Omega(y, x)$ for any x and y in Ω

(ii) the function $(x, y) \mapsto G_\Omega(x, y)$ is continuous on $\Omega \times \Omega$ (in the extended sense)

(iii) the function $(x, y) \mapsto G_\Omega(x, y)$ is superharmonic on $\Omega \times \Omega$.

The proof of this Theorem is given in [7].

1.4 Reaction diffusion systems

1.4.1 Derivation of reaction diffusion systems

A reaction diffusion system consists of several coupled partial differential equations that describe how the concentration of one or more substances are affected by reaction and diffusion processes, that come from a balance or conservation law.

A conservation law is just a mathematical formulation of the basic fact that the rate at which a quantity changes in a given domain must equal the rate at which the quantity flows across the boundary plus the rate at which the quantity is created, or destroyed, within the domain. For example, consider a population of a certain animal species in a fixed geographical region. The rate of change of the animal population must equal the rate at which animals migrate into the region, minus the rate at which they migrate out, plus the birth rate, minus the death rate. Such a statement is a verbal expression of a balance or conservation law. One can make similar kinds of statements for many quantities, heat energy, the concentration of a chemical, the density of automobiles on a freeway, and so on.

To derive the reaction diffusion system, let us first consider a m participating species are assumed whose concentrations are given by the vector $u(t, x) = (u_1, \dots, u_m) \in \mathbb{R}^m$. The idea is to write a mass conservation equation that describes the reaction diffusion process. Suppose that the reaction diffusion process is occurring in a volume $\Omega \subset \mathbb{R}^N$ (typically $N = 1, 2$ or 3) that is enclosed by an surface $\partial\Omega = S$. The total amount of material in Ω is given as

$$\int_{\Omega} u(t, x) dx$$

The assumption is made such that there are only two processes that affect the concentration u . When material moves across the boundary S , there is a change in u . Denote the flux of material across S as Φ . For classical diffusion,

$$\Phi = -D\nabla u$$

where D may be a function of t, x and u . The minus sign indicates that the diffusion transports matter from high to low levels of concentration. By integrating over the surface S , we get that the total flow of material across S is

$$-\int_S \Phi \cdot ds = \int_S D\nabla u \cdot ds$$

The remaining source of change in the concentration u comes from the creation of new material from the reactions. If we denote this as $f = (f_1(t, x, u), \dots, f_m(t, x, u))$, the total

contribution of material from the reaction is

$$\int_{\Omega} f dx$$

The observation that the rate of change of the total concentration must equal to the rate of flow of material across S plus the material created in Ω gives the general mass conservation equation

$$\frac{\partial}{\partial t} \int_{\Omega} u dx = \int_S D \nabla u \cdot ds + \int_{\Omega} f dx$$

By the Divergence Theorem, we have

$$\int_{\Omega} \left[\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) - f \right] dx = 0$$

since Ω is arbitrary, we have

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) + f(t, x, u)$$

when $D = \text{diag}(d_1, \dots, d_m)$ is a constant diagonal matrix whose entries represent the coefficients of diffusion for each species, we have

$$\frac{\partial u}{\partial t} = D \Delta u + f(t, x, u)$$

This derivation can be found in [45].

1.4.2 Boundary conditions

When the diffusion medium Ω is a bounded domain in \mathbb{R}^n the reaction diffusion equations are supplemented by suitable boundary conditions on the boundary surface $\partial\Omega$. The appropriate condition on the boundary depends on the physical mechanism surrounding the diffusion medium. Often the condition on the boundary depends on the material properties both inside and outside the diffusion medium. If the outside environment is known there are the following three basic types of boundary conditions.

(1) Dirichlet boundary condition

When the density function is specified on the boundary surface, the boundary condition is given in the form

$$u = h(t, x) \quad (t > 0, x \in \partial\Omega)$$

where h is the density of a reservoir with which the diffusion medium is in contact. This condition is called the Dirichlet or first boundary condition, when $h \equiv 0$ we say the homogeneous Dirichlet boundary condition.

(2) Neumann boundary condition

When the flux across the boundary surface is prescribed, the boundary condition becomes

$$\frac{\partial u}{\partial \nu} = h(t, x) \quad (t > 0, x \in \partial\Omega)$$

where h represents the rate of flow of the density and $\frac{\partial u}{\partial \nu}$ is the directional derivative of u in the direction ν .

This condition is called the Neumann or second boundary condition, when $h \equiv 0$ we says the homogeneous Neumann boundary condition.

(3) Robin boundary condition

In many physical problems such as heat conduction the flow across the boundary surface is proportional to the difference between the surrounding density and the density just inside Ω . Consider $\frac{\partial u}{\partial \nu}$ as the outward normal derivative of u on $\partial\Omega$ and denote the surrounding density by $h(t, x)$. Then

$$\frac{\partial u}{\partial \nu} + \beta u = h(t, x) \quad (t > 0, x \in \partial\Omega)$$

where β is a proportionality constant, which can vary from point to point on $\partial\Omega$. This condition is called the Robin or third boundary condition, when $h \equiv 0$ we says the homogeneous Robin boundary condition.

1.4.3 Reaction diffusion equations

A reaction diffusion equation comprises a reaction term and a diffusion term

1.4.3.1 Reaction

A "reaction" often refers to an interaction of some sort between two or more objects. A simple example of a decaying reaction corresponds to $f(u) = -u$, and leads to the system,

$$\frac{du}{dt} = -u,$$

which has many applications including radioactive decay, and protein degradation. As a result, a more precise interpretation of the reactive term, $f(u)$, is that it describes the change in concentration of u depending on its local value

It follows that the reactive term $f(u)$ for more than one chemical species will describe the local changes in concentrations that are due to the production and decay of each chemical in addition to the interactions between chemicals. Some examples of such

reactions are chemical reactions, and population dynamics where two or more species are competing for resources.

Example 1.3 (System). *Imagine that there is an isolated ecosystem that only contains two species. One species feeds on the vegetation found in the ecosystem, and the second species is a predator that will feed on the first species. In this system, the reactions will be the birth and death rates of each species, and the rate of predation. The most famous predator-prey system is the Lotka-Volterra model Lotka (1925). Letting u be the number of prey and v be the number of the predator, his model was,*

$$(1.3) \quad \begin{aligned} \frac{du}{dt} &= a_1u - a_2uv \\ \frac{dv}{dt} &= b_1uv - b_2v \end{aligned}$$

where $a_i, b_i, i \in \{1, 2\}$ are growth and decay constants. We describe the role of each of the terms in 1.3. The a_1u term indicates that the prey population will grow with rate a_1 (i.e. the rate of reproduction will be proportional to the current population size). The $-a_2uv$ term corresponds to a decrease in prey population due to predation which has rate a_2 . The b_1uv term indicates that if the predators have food, then they will reproduce. And finally, the $-b_2v$ term indicates that if the predators do not have food, the predator population will decrease.

Mathematically, the reaction equation is a linear or nonlinear partial differential equation of first order.

1.4.3.2 Diffusion

The word diffusion comes from the Latin *diffundere* meaning "to pour out", Oxford (2014). Long before it was used to describe a physical phenomenon, "diffusion" was used to describe the dissemination of knowledge, branching out, and a general abstract concept of spreading out. It was not until the 1800s that the term diffusion was used in physics to describe how particles of gases, liquids, and solids intermingle and move from areas of high concentration to low concentration without chemical combination as a result of each particle's kinetic energy. Adolf Fick described diffusion mathematically, and arrived

at this result using a macroscopic approach. In fact, the diffusion equation

$$\frac{\partial u}{\partial t} - D\Delta u = 0$$

Example 1.4 (Heat Equation). *In this example, we will investigate the Heat equation that describes the distribution of heat (or variation in temperature) over time in a fixed region of space. First we will establish some notations. Here we take $\Omega \subseteq \mathbb{R}^N$, an open set with boundary $\partial\Omega$. We are concerned with the problem of finding a function $u(t, x) : [0, \infty) \times \bar{\Omega} \rightarrow \mathbb{R}$ that satisfies:*

$$(1.4) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{on } (0, \infty) \times \Omega \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega \\ u(x, 0) = u_0(x) & \text{on } \Omega \end{cases}$$

where $u(x, t)$ is the temperature as a function of the spatial variables x_i and time t , $\frac{\partial u}{\partial t}$ denotes the rate of change of temperature at a point over time. The heat equation follows from Fick's second law. The heat equation and its variations also occur in other diffusion phenomena.

In order to apply the Hille-Yosida theorem, we will consider $u(t)$ as the function $x \mapsto u(x, t)$ that belongs to a space of functions depending only on x , say for example, a function space $H = L^2(\Omega)$ or $H = H_0^1(\Omega)$. Also, for simplicity, we will assume Ω is of class C^∞ , and $\Gamma = \partial\Omega$ is bounded. Now we will state and prove some results regarding the existence, uniqueness of solutions to (1.4).

Theorem 1.12. *Suppose $u_0 \in L^2(\Omega)$. Then the problem (1.4) has a unique solution, $u(x, t)$ satisfying*

$$\begin{aligned} u &\in C([0, \infty); L^2(\Omega)) \cap C((0, \infty); H^2(\Omega) \cap H_0^1(\Omega)), \\ u &\in C^1((0, \infty); L^2(\Omega)) \end{aligned}$$

Furthermore,

$$|u(t)|_{L^2(\Omega)} \leq |u_0|_{L^2(\Omega)} \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(t) \right|_{L^2(\Omega)} = |\Delta u(t)|_{L^2(\Omega)} \leq \frac{1}{t} |u_0|_{L^2(\Omega)}, \quad \forall t > 0$$

Proof. By Example 1.1 Δ is m -dissipative operator and by Example 1.2 Δ is self-adjoint. Then we apply the theorem 1.7. ■

Mathematically, the diffusion equation is a linear or nonlinear partial differential equation of second order.

1.4.4 Monotone systems

1.4.4.1 Linear systems

Consider the linear parabolic system of equations

$$(1.5) \quad \frac{\partial u}{\partial t} = a(x, t)\Delta u + \sum_{i=1}^N b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u$$

in a domain $\Omega \subset \mathbb{R}^n$ which can be bounded or unbounded. Here $x = (x_1, \dots, x_N)$ $u = (u_1, \dots, u_m)$, $a(x, t)$, $b_i(x, t)$ and $c(x, t)$ are continuous square matrices, which satisfy the following conditions:

- $a(x, t)$ is a diagonal matrix with positive diagonal elements, $a_i(x, t) \geq a_0 > 0$ for some a_0 and for all x and t ,
- $b_i(x, t)$ are diagonal matrices,
- $c(x, t)$ has non-negative off-diagonal elements.

If these conditions are satisfied, system (1.5) is called a monotone system. This is a class of systems for which positiveness and comparison theorems similar to those formulate below for the scalar equation are applicable. The conditions on the matrices a , b_i and c are essential and they cannot be omitted.

Consider the operator

$$Lu = a(x, t)\Delta u + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t}$$

Lemma 1.1. *Let Ω be a bounded domain. If $Lu \leq 0$ in $\Omega \times (0, T]$, $u(x, 0) \geq 0$ in Ω and $u(x, t) \geq 0$ at the boundary $\partial\Omega$, then $u(x, t) \geq 0$ in $\Omega \times [0, T]$.*

The proof of this and of the following assertions are given in [58] in the onedimensional case ($n = 1$). The proofs are similar in the multi-dimensional case. Here and everywhere below inequality between vectors is understood componentwise.

Lemma 1.2. *Let Ω be an unbounded domain. Suppose that the sum of the elements in each row of the matrix $c(x, t)$ is bounded from above independently of x and t . If $Lu \leq 0$ in $\Omega \times (0, T]$, $u(x, 0) \geq 0$ in Ω , $u(x, t) \geq 0$ for $x \in \partial\Omega$, and*

$$\liminf_{|x| \rightarrow \infty} u(x, t) \geq 0$$

uniformly in $t \in [0, T]$, then $u(x, t) \geq 0$ in $\Omega \times [0, T]$.

Theorem 1.13. *Let Ω be an unbounded domain. Suppose that the matrices a, b_i and c satisfy the following conditions in $\Omega \times [0, T]$:*

$$|a_j(x, t)| \leq M, \quad |b_{ij}(x, t)| \leq M(|x| + 1), \quad c_{ij}(x, t) \leq M(|x|^2 + 1)$$

where a_j are diagonal elements of the matrix a , b_{ij} are diagonal elements of the matrix b_i , c_{ij} are elements of the matrix c , M is a positive constant. If $Lu \leq 0$ in $\Omega \times (0, T]$

$$u_i(x, t) \geq -Be^{\beta|x|^2}, \quad x \in \Omega, \quad i = 1, \dots, m$$

$u(x, t) \geq 0$ for $x \in \partial\Omega, t \in [0, T]$ and $u(x, 0) \geq 0$ for $x \in \Omega$, then $u(x, t) \geq 0$ for $x \in \Omega, t \in [0, T]$.

Let us recall the notion of functional irreducibility [58]. A bounded continuous matrix $B(x)$ is called functionally irreducible if the numerical matrix, formed by the C -norms of the elements of the matrix F , is irreducible.

Theorem 1.14 (Positiveness theorem). *Let domain Ω be bounded or unbounded. Suppose that the matrix $c(x, 0)$ is functionally irreducible. If $Lu \leq 0$ in $\Omega \times (0, T]$, $u(x, t) \geq 0$ for $x \in \partial\Omega, t \in [0, T]$, $u(x, 0) \geq 0$ for $x \in \Omega$ and $u(x, 0) \neq 0$, then $u(x, t) > 0$ for $x \in \Omega, t \in (0, T]$.*

1.4.4.2 Semi-linear systems

Consider the semi-linear parabolic system of equations

$$(1.6) \quad \frac{\partial u}{\partial t} = a(x, t)\Delta u + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + F(u, x, t)$$

in a domain $\Omega \subset \mathbb{R}^N$ which can be bounded or unbounded. The matrices $a(x, t)$ and $b_i(x, t)$ satisfy the same conditions as in the preceding section. Moreover we assume that the

function $F = (F_1, \dots, F_m)$ satisfies the condition

$$\frac{\partial F_i}{\partial u_j} \geq 0, \quad i \neq j, \quad i, j = 1, \dots, m$$

for all u, x, t . If these conditions are satisfied, then we say that system (1.6) is monotone.

Theorem 1.15 (Comparison theorem). *Suppose that matrices a, b_i and $\frac{\partial F_i}{\partial u_j}$ are bounded and continuous for $x \in \Omega, t \in [0, T]$ and u in an arbitrary bounded domain in \mathbb{R}^p . Let $u^1(x, t)$ and $u^2(x, t)$ be classical solutions of system (1.6) in a bounded or unbounded domain Ω . If $u^1(x, 0) \geq u^2(x, 0)$ for $x \in \Omega$, and for $x \in \partial\Omega, t \in [0, T]$, then this inequality also holds for all $x \in \Omega, t \in [0, T]$.*

The proof of this theorem follows from Theorem 1.13 applied for the difference $v = u^1 - u^2$.

1.4.5 Importance of reaction diffusion problems

The importance of reaction diffusion systems appears in their use in several areas, including them [25]:

(i) In chemical engineering: Theoretical models of steady and unsteady state reaction diffusion problems have been developed to obtain the substrate and product concentrations for enzymes immobilized within particles. Reaction diffusion models are characterized by carbon monoxide poisoning, nitrogen oxide removal, oscillating chemical reactions, pulse splitting and shedding, Rayleigh-Benard convection, and kinetics of methylene blue adsorption (film-pore diffusion model). A theoretical model based on the Michaelis-Menten enzymatic conversion of the substrate and the diffusion of the substrate was created. They also describe the steady-state oxygen diffusion in a spherical cell and equilibrium of isothermal gas sphere, flame propagation, autocatalytic chemical reactions, and neutron population in a nuclear response and branching.

(ii) In Biological and medical sciences: A few important applications of reaction diffusion equations include population dynamics models, gene propagation models, ecological invasions, a spread of epidemics, tumor growth, and wound healing, distribution of heat sources in a human head, transmission of pulses in nerves, and neurophysiology.

(iii) In mechanical engineering: A simplified kinematical description of a rigidly rotating spiral induced in a general two component reaction diffusion medium is elaborated by application of a free boundary approach. The potential energy generated by

an external force as a result of a deformation is propagated among mass points by the principle of reaction and diffusion.

(iv) In civil engineering: A theoretical model based on fundamental reaction diffusion principles has been formulated to describe the process of concrete carbonation. It is a major time-limiting factor for the durability of reinforced concrete.

MATHEMATICAL MODELING AND MODELS

In this chapter, we present the importance of mathematical modeling and some mathematical models of reaction diffusion systems and their applications in various sciences.

2.1 Mathematical modeling

Over the last several decades, mathematical modeling has been playing a major role in understanding and solving many real-life problems, under certain conditions. Most mathematical models have been like individual works of art that reflected the scientific views and personal characteristics of the modeler. However, many attempts are being made to unify the mathematical models in order to provide a standardized and reliable method of investigation accessible for every scientist. Modeling is a multistep process involving the following [57]:

- i. Identifying the problem.
- ii. Constructing or selecting the appropriate model.
- iii. Figuring out what data need to be collected.
- iv. Deciding the number of variables and predictors to be chosen for greater accuracy.
- v. Analytically or numerically computing the solution and testing the validity of models.
- vi. Implementing the models in real-world situations.

Usually, modeling is an iterative process in which we start from a crude model and gradually refine it until it is suitable for solving the problem, and modeling enables us to gain insight into the original situation. The purpose of the model is to understand the underlying phenomenon and perhaps to make predictions about future behavior. If the predictions do not compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

2.1.1 What is mathematical modeling?

We begin with a dictionary definition of the word model:

Definition 2.1. model (n) a miniature representation of something, a pattern of something to be made, an example for imitation or emulation, a description or analogy used to help visualize something (e.g. an cell, an atom) that cannot be directly observed, a system of postulates, data and inferences presented as a mathematical description of an entity or state of affairs.

This definition suggests that modeling is an activity, a cognitive activity in which we think about and make models to describe how devices or objects of interest behave.

There are many ways in which devices and behaviors can be described. We can use words, drawings or sketches, physical models, computer programs, or mathematical formulas. In other words, the modeling activity can be done in several languages, often simultaneously. Since we are particularly interested in using the language of mathematics to make models, we will refine the definition just given:

Definition 2.2. mathematical model (n) a representation in mathematical terms of the behavior of real devices and objects. Mathematical modeling is the application of mathematics to describe real-world problems and investigating important questions that arise from it. Using mathematical tools, the real-world problem is translated to a mathematical problem, which mimics the real-world problem. A solution to the mathematical problem is obtained, which is interpreted in the language of real-world problem to make predictions about the real world.

By real-world problems, I mean problems from biology, chemistry, engineering, ecology, environment, physics, social sciences, statistics, wildlife management and so on. Mathematical modeling can be described as an activity which allows a mathematician to be biologist, chemist, ecologist, economist depending on the problem that he/she is tackling. The primary aim of a modeler is to undertake experiments on the mathematical representation of a real-world problem, instead of undertaking experiments in the real world.

We want to know how to make or generate mathematical representations or models, how to validate them, how to use them, and how and when their use is limited. But before delving into these important issues, it is worth talking about why we do mathematical modeling.

2.1.2 Why do we do mathematical modeling?

Since the modeling of devices and phenomena is essential to both engineering and science, engineers and scientists have very practical reasons for doing mathematical modeling. In addition, engineers, scientists, and mathematicians want to experience the sheer joy of formulating and solving mathematical problems.

2.1.2.1 Mathematical modeling and the scientific method

We need to identify a "real world " and a "conceptual world." The external world is the one we call real; here we observe various phenomena and behaviors, whether natural in origin or produced by artifacts. The conceptual world is the world of the mind-where we live when we try to understand what is going on in that real, external world. The conceptual world can be viewed as having three stages: observation, modeling, and prediction. In the observation part of the scientific method we measure what is happening in the real world. Here we gather empirical evidence and "facts on the ground." Observations may be direct, as when we use our senses, or indirect, in which case some measurements are taken to indicate through some other reading that an event has taken place. For example, we often know a chemical reaction has taken place only by measuring the product of that reaction. In this elementary view of how science is done, the modeling part is concerned with analyzing the above observations for one of (at least) three reasons. These rationales are about developing: models that describe the behavior or results observed; models that explain why that behavior and results occurred as they did; or models that allow us to predict future behaviors or results that are as yet unseen or unmeasured. In the

prediction part of the scientific method we exercise our models to tell us what will happen in a yet to be conducted experiment or in an anticipated set of events in the real world. These predictions are then followed by observations that serve either to validate the model or to suggest reasons that the model is inadequate. The modeling is central to all of the conceptual phases in the elementary model of the scientific method. We build models and use them to predict events that can confirm or deny the models. In addition, we can also improve our gathering of empirical data when we use a model to obtain guidance about where to look.

2.1.2.2 Mathematical modeling and the practice of engineering

Engineers are interested in designing devices and processes and systems. That is, beyond observing how the world works, engineers are interested in creating artifacts that have not yet come to life. As noted by Herbert A. Simon (in *The Sciences of the Artificial*), "Design is the distinguishing activity of engineering." Thus, engineers must be able to describe and analyze objects and devices into order to predict their behavior to see if that behavior is what the engineers want. In short, engineers need to model devices and processes if they are going to design those devices and processes. While the scientific method and engineering design have much in common, there are differences in motivation and approach that are worth mentioning. In the practices of science and of engineering design, models are often applied to predict what will happen in a future situation. In engineering design, however, the predictions are used in ways that have far different consequences than simply anticipating the outcome of an experiment. Every new building or airplane, for example, represents a model-based prediction that the building will stand or the airplane will fly without dire, unanticipated consequences. Thus, beyond simply validating a model, prediction in engineering design assumes that resources of time, imagination, and money can be invested with confidence because the predicted outcome will be a good one.

2.1.3 Principles of mathematical modeling

Mathematical modeling is a principled activity that has both principles behind it and methods that can be successfully applied. The principles are over-arching or meta-principles phrased as questions about the intentions and purposes of mathematical modeling. These meta-principles are almost philosophical in nature. We will now outline the principles, and in the next section we will briefly review some of the methods [17].

- **Why?** What are we looking for? Identify the need for the model.
- **Find?** What do we want to know? List the data we are seeking.
- **Given?** What do we know? Identify the available relevant data.
- **Assume?** What can we assume? Identify the circumstances that apply.
- **How?** How should we look at this model? Identify the governing physical principles.
- **Predict?** What will our model predict? Identify the equations that will be used, the calculations that will be made, and the answers that will result.
- **Valid?** Are the predictions valid? Identify tests that can be made to validate the model, i.e., is it consistent with its principles and assumptions?
- **Verified?** Are the predictions good? Identify tests that can be made to verify the model, i.e., is it useful in terms of the initial reason it was done?
- **Improve?** Can we improve the model? Identify parameter values that are not adequately known, variables that should have been included, and/or assumptions/restrictions that could be lifted. Implement the iterative loop that we can call “model-validate-verify-improve-predict.”
- **Use?** How will we exercise the model? What will we do with the model?

This list of questions and instructions is not an algorithm for building a good mathematical model. However, the underlying ideas are key to mathematical modeling, as they are key to problem formulation generally. Thus, we should expect the individual questions to recur often during the modeling process, and we should regard this list as a fairly general approach to ways of thinking about mathematical modeling.

Having a clear picture of why the model is wanted or needed is of prime importance to the model-building enterprise. Suppose we want to estimate how much power could be generated by a dam on a large river. For a first estimate of the available power, we wouldn't need to model the dam's thickness or the strength of its foundation. Its height, on the other hand, would be an essential parameter of a power model, as would some model and estimates of river flow quantities. If, on the other hand, we want to design the actual dam, we would need a model that incorporates all of the dam's physical characteristics (e.g., dimensions, materials, foundations) and relates them to the dam site and the river flow conditions. Thus, defining the task is the first essential step in model formulation.

We then should list what we know for example, river flow quantities and desired power levels as a basis for listing the variables or parameters that are as yet unknown. We should also list any relevant assumptions. For example, levels of desired power may be linked to demographic or economic data, so any assumptions made about population

and economic growth should be spelled out. Assumptions about the consistency of river flows and the statistics of flooding should also be spelled out.

Which physical principles apply to this model? The mass of the river's water must be conserved, as must its momentum, as the river flows, and energy is both dissipated and redirected as water is allowed to flow through turbines in the dam (and hopefully not spill over the top!). And mass must be conserved, within some undefined system boundary, because dams do accumulate water mass from flowing rivers. There are well-known equations that correspond to these physical principles. They could be used to develop an estimate of dam height as a function of power desired. We can validate the model by ensuring that our equations and calculated results have the proper dimensions, and we can exercise the model against data from existing hydroelectric dams to get empirical data and validation.

If we find that our model is inadequate or that it fails in some way, we then enter an iterative loop in which we cycle back to an earlier stage of the model building and re-examine our assumptions, our known parameter values, the principles chosen, the equations used, the means of calculation, and so on. This iterative process is essential because it is the only way that models can be improved, corrected, and validated.

2.1.4 Some methods of mathematical modeling

Now we will review some of the mathematical techniques we can use to help answer the philosophical questions posed in Section 1.2. These mathematical principles include: dimensional homogeneity, abstraction and scaling, conservation and balance principles, and consequences of linearity.

2.1.4.1 Dimensional homogeneity and consistency

Dimensions are a measurable extent of some base quantities, such as length (L), time (T), mass (M) etc. Units are labels for different dimensions and combination of dimensions, such as kilograms (kg)...

Dimensional homogeneity is the concept where the dimensions of variables on both sides of an equation are the same. If the units are the same, the dimensions of each term must be the same, for example

$$Length = \frac{Length}{Time} \times Time$$

is a dimensionally homogenous equation.

This idea that is central to mathematical modeling. It is quite logical that every term in an energy equation has total dimensions of energy, and that every term in a balance of mass should have the dimensions of mass. This statement provides the basis for a technique called dimensional analysis "a method by which we deduce information about a phenomenon from the single premise that the phenomenon can be described by a dimensionally correct equation among certain variables".

2.1.4.2 Abstraction and Scaling

An important decision in modeling is choosing an appropriate level of detail for the problem at hand, and thus knowing what level of detail is prescribed for the attendant model. This process is called abstraction and it typically requires a thoughtful approach to identifying those phenomena on which we want to focus, that is, to answering the fundamental question about why a model is being sought or developed. For example, a linear elastic spring can be used to model more than just the relation between force and relative extension of a simple coiled spring, as in an old-fashioned butcher's scale or an automobile spring. It can also be used to model the static and dynamic behavior of a tall building, perhaps to model wind loading, perhaps as part of analyzing how the building would respond to an earthquake. In these examples, we can use a very abstract model by subsuming various details within the parameters of that model. In addition, as we talk about finding the right level of abstraction or the right level of detail, we are simultaneously talking about finding the right scale for the model we are developing. For example, the spring can be used at a much smaller, micro scale to model atomic bonds, in contrast with the macro level for buildings. The notion of scaling includes several ideas, including the effects of geometry on scale, the relationship of function to scale, and the role of size in determining limits all of which are needed to choose the right scale for a model in relation to the "reality " we want to capture.

2.1.4.3 Conservation and balance principles

When we develop mathematical models, we often start with statements that indicate that some property of an object or system is being conserved. For example, we could analyze the motion of a body moving on an ideal, frictionless path by noting that its energy is conserved. Sometimes, as when we model the population of an animal colony or the volume of a river flow, we must balance quantities, of individual animals or water volumes, that cross a defined boundary. We will apply balance or conservation principles

to assess the effect of maintaining or conserving levels of important physical properties. Conservation and balance equations are related in fact, conservation laws are special cases of balance laws.

2.1.4.4 Constructing linear models

Linearity is one of the most important concepts in mathematical modeling. Models of devices or systems are said to be linear when their basic equations whether algebraic, differential, or integral are such that the magnitude of their behavior or response produced is directly proportional to the excitation or input that drives them. Even when devices are more fully described by nonlinear models, their behavior can often be approximated by linearized or perturbed models, in which cases the mathematics of linear systems can be successfully applied. We apply linearity when we model the behavior of a device or system that is forced or pushed by a complex set of inputs or excitations. We obtain the response of that device or system to the sum of the individual inputs by adding or superposing the separate responses of the system to each individual input. This important result is called the principle of superposition. Engineers use this principle to predict the response of a system to a complicated input by decomposing or breaking down that input into a set of simpler inputs that produce known system responses or behaviors.

2.1.5 Characteristics of mathematical models

A good model should be as simple as possible. A model helps one to go beyond the surface of a phenomenon to an understanding of the mechanisms and relationships.

The characteristics and requirements of mathematical models include [27]:

- i. **Realism of models:** A mathematical model should represent the reality as closely as possible, but at the same time it should be mathematically tractable.
- ii. **Robustness of models:** A mathematical model is said to be robust if small changes in the parameters lead to small changes in the behavior of the model. Any dynamically behavioral is considered to be robust when it is detected in a dense set formed by key parameters of model systems. The decision is made by using sensitivity analysis of the models.
- iii. **Non-uniqueness of models:** A particular situation need not have only one mathematical model. We can search for better and different models.

iv. Hierarchy of models: For every situation, we can obtain a hierarchy of models. Models obtained at each stage should be more realistic and better than the previous one.

v. Self-consistency of models: A mathematical model that involves equations and inequations should be self-consistent.

vi. Estimation of parameters: Every mathematical model contains some parameters which control the dynamics of the model system, and these need to be estimated. Optimal control theory is one of the methods for estimating these vital parameters. For example, it is desirable to estimate the proportion of a population of voters who will vote for a particular candidate. That proportion is the unobservable parameter; the estimate is based on a small random sample of voters.

vii. Generality and applicability of models: Some models are used in a large number of situations and there are others which are applicable to some specific situations only. For example, logistic maps and the Laplace equation are used in a wide variety of situations.

viii. Imperfections of models and the cost of modeling: No model is complete in all respects. There is always a chance for improvement. However, there is an inherent cost involved for each improvement.

ix. Transferability of mathematical models: If a single model is applicable to many fields, then the model is very useful and this is known as the transferability of the model. For example, the Laplace equation model is used in many fields such as irrotational flows, electrostatic potential, gravitational potential, and so on.

x. Criteria for successful models: A successful model (*a*) gives good agreement between observations and predictions, and (*b*) has simplicity in the model equations.

2.2 Reaction diffusion systems and their applications

Turing (1952) [56], published a paper, in which a revolutionary idea about the mechanism of morphogenesis (the development of structures in an organism during its life) has been proposed. From the mathematical point of view Turing's idea immediately leads to the construction of reaction diffusion system exhibiting so-called Turing instability (see, e.g [44]). Nowadays nonlinear reaction diffusion system are basic equations for many well-known nonlinear models used to describe a wide range of processes in physics, biology, medicine, chemistry, ecology, etc.

This section is mostly devoted to the investigation of two-component reaction diffusion system of the form

$$(2.1) \quad \begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (d_1(u)\nabla u) + f(u, v) \\ \frac{\partial v}{\partial t} &= \nabla \cdot (d_2(v)\nabla v) + g(u, v) \end{aligned}$$

where $u = u(t, x)$ and $v = v(t, x)$ are two unknown functions representing the densities of populations (cells, tumours, chemicals), $f(u, v)$ and $g(u, v)$ are the given smooth functions describing the interaction between them and the environment, the functions $d_1(u)$ and $d_2(v)$ are the relevant diffusivities (hereafter they are positive smooth functions) and the subscripts t and x denote differentiation with respect to these variables. The class of reaction diffusion system (2.1) generalizes a number of nonlinear models describing various processes in biology, medicine and ecology (see, e.g., the books ([46], [44], [48], [14]).

Usually the diffusivities $d_k (k = 1, 2)$ are taken to be positive constants D^k . An important subclass of reaction diffusion system of the form (2.1) consists of those, which satisfy the well-known requirements leading to Turing instability, hence they can be used for description of the chemical basis of morphogenesis [46]. The classical examples of such reaction diffusion system are the Gierer-Meinhardt system, the Schnakenberg system, etc. The two component diffusive Lotka-Volterra system (DLVS)

$$(2.2) \quad \begin{aligned} \frac{\partial u}{\partial t} &= D^1 u_{xx} + u(a_1 + b_1 u + c_1 v) \\ \frac{\partial v}{\partial t} &= D^2 v_{xx} + v(a_2 + b_2 u + c_2 v) \end{aligned}$$

is another common reaction diffusion system of the form (2.1) ([46], [44], [48], [14]. System (2.2) is the standard generalization of the classical Lotka-Volterra system that takes into account the diffusion process for interacting species (see the terms $D^1 u_{xx}$ and $D^2 v_{xx}$). Although the classical Lotka-Volterra system was independently introduced by Lotka and Volterra about 90 years ago, its different generalizations are widely studied at present because of their importance for mathematical modelling of a wide range of processes in biology, ecology, economics, etc.

It is well known that DLVS (2.2) models several types of interaction between two populations of species. Three common types are predator-prey interaction, the competition (for food, space, etc.) of species and mutualism. Each type of species interaction is defined by the signs of coefficients in DLVS (2.2). For example, the coefficients

$$a_k > 0, b_k \leq 0, c_k \leq 0, k = 1, 2$$

are used in order to describe the competition, while the cases $c_1 b_2 < 0$ and $c_1 > 0, b_2 > 0$ model predator-prey interaction and mutualism, respectively (see [45] for details).

A separate subclass of the reaction diffusion system class (2.1) is formed by so-called $\lambda - \omega$ systems, which possess spiral wave solutions. Spiral waves occur naturally in a wide variety of biological, physiological and chemical effects (see [44] and the references therein). Typically the $\lambda - \omega$ systems have a complicated structure involving the nonlinearities $\lambda(u^2 + v^2)$ and $\omega(u^2 + v^2)$, hence their analysis is rather complicated. The most widely known are spiral waves occurring in the Belousov-Zhabotinskii reaction ???. In contrast to the $\lambda - \omega$ systems, the corresponding mathematical model is simple and it is the Belousov-Zhabotinskii system

$$\begin{aligned} \frac{\partial u}{\partial t} &= D^1 u_{xx} + u(a_1 - b_1 u - c_1 v) + r v \\ \frac{\partial v}{\partial t} &= D^2 v_{xx} - v(a_2 + b_2 u) \end{aligned}$$

where all the parameters should be nonnegative. As noted above, typically the diffusivities $d_k (k = 1, 2)$ in reaction diffusion system of the form (2.1) are taken to be positive constant, however, in certain insect dispersal models they depend on the densities u and v . For example, a power dependence is adopted in diffusion models, when there is an increase in diffusion due to population pressure [45], [44].

2.3 Some reaction diffusion models

2.3.1 Mathematical modeling of COVID-19 transmission

dynamics

Coronavirus disease 2019 (COVID-19) is an infectious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). The disease was first identified December 2019 in Wuhan, China, and has since spread globally, resulting in the ongoing 2020 pandemic outbreak. The COVID-19 pandemic is considered as the biggest global threat world-wide because of millions of confirmed infections, accompanied by hundreds of thousands deaths over the world. Notice, by July 16, 2020, report 13.787.830 confirmed cumulative cases with 588.810 deaths. according to the report by the World Health Organization. The global problem of the outbreak has attracted the interest of researchers of different areas, giving rise to a number of proposals to analyze and predict the evolution of the pandemic. In this example we selected the model [47]. They proposed

a compartmental mathematical model for the spread of the COVID-19 disease with special focus on the transmissibility of super-spreaders individuals. They computed the basic reproduction number threshold, They studied the local stability of the disease free equilibrium in terms of the basic reproduction number, and they investigated the sensitivity of the model with respect to the variation of each one of its parameters.

They proposed a model that takes into account the super-spreading phenomenon of some individuals. Moreover, they considered a fatality compartment, related to death due to the virus infection. In doing so, the constant total population size N is subdivided into eight epidemiological classes: susceptible class (S), exposed class (E), symptomatic and infectious class (I), super-spreaders class (P), infectious but asymptomatic class (A), hospitalized (H), recovery class (R), and fatality class (F). The model takes the following form:

$$\left\{ \begin{array}{l} \frac{dS}{dt} = -\beta \frac{1}{N} S - l\beta \frac{H}{N} S - \beta' \frac{P}{N} S \\ \frac{dE}{dt} = \beta \frac{1}{N} S + l\beta \frac{H}{N} S + \beta' \frac{P}{N} S - \kappa E \\ \frac{dI}{dt} = \kappa \rho_1 E - (\gamma_a + \gamma_i) I - \delta_i I \\ \frac{dP}{dt} = \kappa \rho_2 E - (\gamma_a + \gamma_i) P - \delta_p P \\ \frac{dA}{dt} = \kappa (1 - \rho_1 - \rho_2) E \\ \frac{dH}{dt} = \gamma_a (I + P) - \gamma_r H - \delta_h H \\ \frac{dR}{dt} = \gamma_i (I + P) + \gamma_r H \\ \frac{dF}{dt} = \delta_i I + \delta_p P + \delta_h H \end{array} \right.$$

with β quantifying the human-to-human transmission coefficient per unit time (days) per person, β' quantifies a high transmission coefficient due to super-spreaders, and l quantifies the relative transmissibility of hospitalized patients. Here κ is the rate at which an individual leaves the exposed class by becoming infectious (symptomatic, super-spreaders or asymptomatic); ρ_1 is the proportion of progression from exposed class E to symptomatic infectious class I ; ρ_2 is a relative very low rate at which exposed individuals become super-spreaders while $1 - \rho_1 - \rho_2$ is the progression from exposed to asymptomatic class; γ_a is the average rate at which symptomatic and super-spreaders individuals become hospitalized; γ_i is the recovery rate without being hospitalized; γ_r is the recovery rate of hospitalized patients; and δ_i, δ_p , and δ_h are the disease induced death rates due to infected, super-spreaders, and hospitalized individuals, respectively.

2.3.2 A reaction diffusion malaria model with incubation period in the vector population

Malaria is one of the most important parasitic infections in humans and more than two billion people are at risk every year. To understand how the spatial heterogeneity and extrinsic incubation period (EIP) of the parasite within the mosquito affect the dynamics of malaria epidemiology, [31] proposed a nonlocal and time-delayed reaction diffusion model. They defined the basic reproduction ratio R_0 and show that R_0 serves as a threshold parameter that predicts whether malaria will spread. Furthermore, a sufficient condition is obtained to guarantee that the disease will stabilize at a positive steady state eventually in the case where all the parameters are spatially independent. Numerically, they showed that the use of the spatially averaged system may highly underestimate the malaria risk. The spatially heterogeneous framework in this example can be used to design the spatial allocation of control resources.

The model is based on monitoring the temporal and spatial dynamics of host (the human) and vector (the adult female mosquito) densities. They developed a spatial model for malaria infection by ignoring superinfection, immunity and clinical death. The human population is divided into two epidemiological classes: susceptible (S_h) and infectious classes (I_h). Assume that the density of total human population $N_h(t, x) = S_h(t, x) + I_h(t, x)$ is described by a logistic population growth law, and that all populations perform an unbiased random walk. Thus, we have the following reaction diffusion equation,

$$\frac{\partial N_h(t, x)}{\partial t} = D_h \Delta N_h(t, x) + b_h N_h(t, x) \left[1 - \frac{N_h(t, x)}{K(x)} \right]$$

The diffusion coefficient D_h and the natural growth rate b_h for humans are supposed to be positive constants while the carrying capacity $K(x)$ is a positive function of location x , allowing for the diversity in habitats. Let Ω be a spatial habitat with smooth boundary $\partial\Omega$. We assume that all populations remain confined to the region Ω for all time, and supplement the Neumann boundary condition to the above equation

$$\frac{\partial N_h}{\partial n} = 0, \quad \forall t > 0, \quad x \in \partial\Omega$$

For more detail see [31].

2.3.3 Threshold dynamics of reaction diffusion model of Ebola virus disease (EVD)

Yamazaki [60], he studied the reaction diffusion Ebola PDE model that consists of equations that govern the evolution of susceptible, infected, recovered and deceased human individuals, as well as Ebola virus pathogens in the environment.

We consider Ω that is a bounded, open and connected subset of $\mathbb{R}^n, n \geq 2$, with a smooth boundary $\partial\Omega$. The system of equations of our concern is the following, which hereafter will be referred to as the Ebola PDE model:

$$(2.3) \quad \begin{cases} \frac{\partial S}{\partial t} = \bar{D}_1 \Delta S + \pi - (\beta_1 I + \beta_2 D + \lambda P) S - \mu S \\ \frac{\partial I}{\partial t} = \bar{D}_2 \Delta I + (\beta_1 I + \beta_2 D + \lambda P) S - (\mu + \delta + \gamma) I \\ \frac{\partial R}{\partial t} = \bar{D}_3 \Delta R + \gamma I - \mu R \\ \frac{\partial P}{\partial t} = \bar{D}_4 \Delta P + \sigma + \xi I + \alpha D - \eta P \\ \frac{\partial D}{\partial t} = \delta I - bD \end{cases}$$

subjected to the initial conditions of

$$(S, I, R, P, D)(x, 0) = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x)$$

and standard Neumann boundary conditions. Here $S = S(x, t)$, $I = I(x, t)$, $R = R(x, t)$, $P = P(x, t)$, $D = D(x, t)$ represent the number of susceptible, infectious, recovered human individuals, the Ebola virus pathogens in the environment, and the number of human individuals who deceased due to EVD, respectively, at location $x \in \Omega$, time $t \geq 0$.

With the description of the model parameters

Parnmeter	Definition
$\bar{D}_i, i = 1, 2, 3, 4$	Diffusion coefficients of S, I, R, P , respectively
π	Recruitment rate of susceptible human individuals
η	Decay rate of Ebola virus in the environment
ξ	Shedding rate of infectious human individuals
α	Shedding rate of deceased human individuals
δ	Disease-induced death rate of human individuals
β_1	Effective contact rate of infectious human individuals
β_2	Effective contact rate of deceased human individuals
λ	Effective contact rate of Ebola virus
γ	Recovery rate of infectious human individuals
μ	Natural death rate of human individuals
$\frac{1}{b}$	Mean caring duration of deceased human individuals
σ	Recruitment rate of Ebola virus in the environment

2.3.4 A reaction diffusion within-host HIV model with cell-to-cell transmission

a reaction diffusion within-host HIV model is proposed. It incorporates cell mobility, spatial heterogeneity and cell-to-cell transmission, which depends on the diffusion ability of the infected cells. In the case of a bounded domain, the basic reproduction number R_0 is established and shown as a threshold: the virus free steady state is globally asymptotically stable if $R_0 < 1$ and the virus is uniformly persistent if $R_0 > 1$. The explicit formula for R_0 and the global asymptotic stability of the constant positive steady state are obtained for the case of homogeneous space. In the case of an unbounded domain and $R_0 > 1$.

In this example [53] the following general reaction diffusion HIV infection model with spatial heterogeneity:

$$\begin{cases} T_t = \nabla \cdot (d_1(x)\nabla T) + \lambda(x) - d(x)T - \beta_1(x)TV - \beta_2(x)TT^*, & t > 0, x \in \Omega \\ T_t^* = \nabla \cdot (d_2(x)\nabla T^*) + \beta_1(x)TV + \beta_2(x)TT^* - r(x)T^*, & t > 0, x \in \Omega \\ V_t = \nabla \cdot (d_3(x)\nabla V) + N(x)T^* - e(x)V, & t > 0, x \in \Omega \\ T(0, x) = T^0(x) \geq 0, T^*(0, x) = T^{*0}(x) \geq 0, V(0, x) = V^0(x) \geq 0, & x \in \Omega \end{cases}$$

with

$$\frac{\partial T(t, x)}{\partial v} = \frac{\partial T^*(t, x)}{\partial v} = \frac{\partial V(t, x)}{\partial v} = 0, \quad t > 0, x \in \partial\Omega$$

where Ω is the spatial domain and v is the outward normal vector to $\partial\Omega$. since the target tissues architecture and composition and the physiological environments withinhost are too complicated to describe, we take Ω as the target tissues and assume that model parameters depend on space. Here, $T(t,x)$, $T^*(t,x)$ and $V(t,x)$ represent the concentrations of healthy cells (CD4 T cells), infected cells and virions at time t and location x , respectively. Assume that the initial data are continuous and bounded functions on $\bar{\Omega}$, and the mobility of healthy cells, infected cells and virions take the standard Fick diffusion or Brownian motion for simplification. $\lambda(x)$ is recruitment rate of CD4 cells, $d(x)$ death rate of healthy cells, $r(x)$ death rate of infected cells, $e(x)$ death rate of virions, $N(x)$ virus production rate, $\beta_1(x)$ virus infection rate, $\beta_2(x)$ cell-to-cell infection rate, $d_1(x)$ diffusion rate of healthy cells, $d_2(x)$ diffusion rate of infected cells, $d_3(x)$ diffusion rate of virions. We assume that $\beta_2(x)$ is a continuous nonincreasing function of $d_2(x)$, and $\beta_2(x)$ is small enough if $d_2(x)$ is sufficiently large. since we mainly focus on the early stage of infection, for simplification, the Neumann boundary conditions are adopted in the case of bounded domain.

In this example, we assume that all the location-dependent parameters are continuous, strictly positive and uniformly bounded functions on $\bar{\Omega}$, and denote the maximum value of $a(x)$ over $\bar{\Omega}$ as \bar{a} and the minimum value of $a(x)$ over $\bar{\Omega}$ as \underline{a} , where $a = \lambda, \beta_1, \beta_2, d_1, d_2, d_3, d, r, e, N$.

2.3.5 Reaction diffusion waves of blood coagulation

The main function of the coagulation system is terminating bleeding, caused by the vessel wall damage by covering the injury site with a fibrin clot. The reaction of fibrin polymerization appears at the final stage of the proteolytic enzymatic cascade where the activated clotting factors act as catalysts for activation of the others. The key enzyme of the coagulation cascade is thrombin as it catalyzes fibrinogen conversion to fibrin and the distribution of the thrombin concentration in blood plasma has crucial influence on the kinetics of the clot formation . Formation of thrombin appears due to the prothrombin activation in the coagulation cascade. The process can be launched by the tissue factor expressed to the blood flow in case of the endothelium rupture (ex- trinsic pathway), or through the activation of factor *XII* which triggers activation of factor *XI* in case of the contact with the foreign surface (intrinsic pathway). Both pathways lead to the activation of factor *X* that contributes to the prothrombin conversion to thrombin. Once the thrombin concentration reaches the threshold value, further prothrombin activation

takes place due to the positive feedback loops of the coagulation cascade We consider the simplified model of the propagation stage of coagulation cascade. can be modeled with the following system of the reaction diffusion equations [18]:

$$\begin{aligned}\frac{\partial T}{\partial t} &= D\Delta T + \left(k_2 U_{10} + \frac{\overline{k}_2}{\overline{h}_{510}} U_{10} U_5\right) \left(1 - \frac{T}{T_0}\right) - h_2 T, \\ \frac{\partial U_5}{\partial t} &= D\Delta U_5 + k_5 T - h_5 U_5, \\ \frac{\partial U_8}{\partial t} &= D\Delta U_8 + k_8 T - h_8 U_8, \\ \frac{\partial U_9}{\partial t} &= D\Delta U_9 + k_9 U_{11} - h_9 U_9, \\ \frac{\partial U_{10}}{\partial t} &= D\Delta U_{10} + k_{10} U_9 + \frac{\overline{k}_{89}}{\overline{h}_{89}} U_9 U_8 - h_{10} U_{10}, \\ \frac{\partial U_{11}}{\partial t} &= D\Delta U_{11} + k_{11} T - h_{11} U_{11}.\end{aligned}$$

Here T , U_i denote the concentrations of thrombin and activated forms of the i th factor respectively, T_0 denotes the initial prothrombin concentration. First term of each equation corresponds to the diffusion of the factors in blood plasma while other terms describe chemical reactions of the coagulation cascade. Since concentrations of precursors are rather high we consider activation reactions as quasi-first order reactions with rate constants k_i , \overline{k}_i . Similarly, we assume concentration of plasma inhibitors such as antithrombin to be high enough and take the rates of inhibition reactions to be constant and equal h_i .

2.3.6 Block effect on HCV infection by HMGB1 released from virus-infected cells

The nuclear protein high-mobility group box 1 (HMGB1) can have an active role in deoxyribonucleic acid (DNA) organization and the regulation of transcription. Based on the new findings from a recent experimental study, the blocking effect on HCV infection by HMGB1 released from virus-infected cells is investigated using a diffusive model for viral infection dynamics. In the model, the diffusion of the virus depends not only on its concentration gradient, but also on the concentration of HMGB1. The basic reproduction number, threshold dynamics, stability properties of the steady states, travelling wave solutions, and spreading speed for the proposed model are studied. We show that the HMGB1-induced blocking of HCV infection slows the spread of virus compared with random diffusion only. Numerically, it is shown that a high concentration of HMGB1 can

block the spread of virus and this confirms, not only qualitatively but also quantitatively, the experimental result [59]

$$(2.4) \quad \begin{cases} \frac{\partial U(x,t)}{\partial t} = D_0 \Delta U + \xi(x) - \frac{\beta(x)U(x,t)\omega(x,t)}{1+a(x)\omega(x,t)+b(x)U(x,t)} - d_U U(x,t) \\ \frac{\partial V(x,t)}{\partial t} = D_0 \Delta V + \frac{\beta(x)U(x,t)\omega(x,t)}{1+a(x)\omega(x,t)+b(x)U(x,t)} - d_V V(x,t) - \alpha_1(x)V(x,t) \\ \frac{\partial M(x,t)}{\partial t} = D_1 \Delta M + \alpha_2(x)V(x,t) - d_M M(x,t) \\ \frac{\partial \omega(x,t)}{\partial t} = \nabla \cdot (D_\omega(M)\nabla \omega) + \frac{k(x)V(x,t)}{1+\gamma(x)M(x,t)} - d_\omega \omega(x,t) \end{cases}$$

In model (2.4), $U(x,t)$, $V(x,t)$, $M(x,t)$, and $\omega(x,t)$ represent the concentration of uninfected cells, infected cells, HMGB1, and virus at time t and location x , respectively. Uninfected cell production rate $\xi(x)$, infection rate of virus $\beta(x)$, measure of virus interference during infection $a(x)$, the speed of the infection rate approaches its saturation value $b(x)$, the loss rate of infected cells to produce HMGB1 $\alpha_1(x)$, the production rate of HMGB1 protein $\alpha_2(x)$, virus production rate $k(x)$, and $\gamma(x)$ (production of free virus is dependent on the presence of HMGB1) are spatially dependent. For mathematical considerations, these functions are positive and continuous on $\bar{\Omega}$, assuming

$$\begin{aligned} \bar{\xi} = \max_{x \in \bar{\Omega}} \xi(x) > 0, \quad \bar{\beta} = \max_{x \in \bar{\Omega}} \beta(x) > 0, \quad \bar{k} = \max_{x \in \bar{\Omega}} k(x) > 0 \\ \bar{\alpha}_2 = \max_{x \in \bar{\Omega}} \alpha_2(x) > 0, \quad \underline{\alpha}_1 = \min_{x \in \bar{\Omega}} \alpha_1(x) > 0, \quad \underline{\alpha} = \min_{x \in \bar{\Omega}} \alpha(x) > 0 \end{aligned}$$

where $\bar{\Omega}$ is the closure of Ω . The death rate of $U(x,t)$, $V(x,t)$, $M(x,t)$, and $\omega(x,t)$ are d_U , d_V , d_M and d_ω , respective. In model (2.4), uninfected cells, infected cells and HMGB1 can move according to the Fickian diffusion with the diffusion rate D_0 , D_0 and D_1 , respectively

$$\vec{J}_U = -D_0 \nabla U, \quad \vec{J}_V = -D_0 \nabla V, \quad \vec{J}_M = -D_1 \nabla M$$

2.3.7 A reaction diffusion model of cancer invasion

Gatenby-Gawlinski [20], they present a mathematical model of the tumor-host interface based on our acid mediation hypothesis of tumor invasion. It is important to point out that we are neither modeling early tumor formation (oncogenesis) nor large-scale tumor morphological structures such as central necrosis. Our model describes the interaction between a growing tumor and surrounding normal tissue only in the immediate vicinity of the tumor-host interface.

The model is a system of three coupled reaction diffusion equations which determine the spatial distribution and temporal evolution of three fields: $U(x,t)$, denoting the

density of healthy cells (a function of the spatial variable x in units of centimeters and time t , measured in seconds), $V(x, t)$, the density of tumour cells, and $W(x, t)$, the concentration of extracellular lactic acid in excess of normal tissue acid concentrations. The Gatenby-Gawlinski equations are:

$$(2.5) \quad \begin{aligned} \frac{\partial U}{\partial t} &= \underbrace{\rho_1 U \left(1 - \frac{U}{\kappa_1}\right)}_{\text{logistic growth}} - \underbrace{\delta_1 U W}_{\text{acid mediated healthy cell death}} \\ \frac{\partial V}{\partial t} &= \underbrace{\rho_2 V \left(1 - \frac{V}{\kappa_2}\right)}_{\text{logistic growth}} + \underbrace{\nabla \left[\eta_2 \left(1 - \frac{U}{\kappa_1}\right) \nabla V \right]}_{\text{density-limited tumor cell diffusion}} \\ \frac{\partial W}{\partial t} &= \underbrace{\rho_3 V - \delta_3 W}_{\text{acid production by tumor cells and clearance by vasculature}} + \underbrace{\eta_3 \Delta W}_{\text{chemical diffusion of excess acid}} \end{aligned}$$

Here the first equation represents parameter association with healthy cells, the second equation with tumor cells, and the third equation with excess extracellular lactic acid. The constants κ_1, κ_2 represent the tissue carrying capacities, δ_1 the rate of acid-mediated healthy cell death, δ_3 the rate of clearance of excess acid by combined buffering and vascular evacuation, ρ_1, ρ_2 the cell proliferation rates, ρ_3 the rate of production of excess acid by tumour cells, η_2 the free-space diffusion coefficient of tumour cells, and η_3 the chemical diffusion coefficient of excess acid. A key feature of this model is the density-limited tumour diffusion term in the second equation, the idea being that a healthy tissue operating at full carrying capacity will spatially constrain a tumour unless diminished. In [20], numerical solution of Eqs. (2.5) captured two types of behaviour: invasion by a heterogeneous tumour consisting of both healthy and malignant cells, and invasion by a homogeneous tumour killing all healthy cells behind its leading edge.

2.3.8 Reaction diffusion systems in protein networks

Assume that the domain Ω is an open, bounded and connected subset of \mathbb{R}^n with the boundary $\partial\Omega$ and outer normal vector v . [26] they considered the following reaction diffusion system to model the spatio-temporal dynamics of the biochemical concentrations (or densities) in a protein network: for $i = 1, \dots, N$

$$(2.6) \quad \begin{aligned} \frac{\partial u_i}{\partial t} - d_i \Delta u_i &= r_i(u, k) && \text{in } \Omega \times (0, T) \\ \frac{\partial u_i}{\partial v} &= 0 && \text{on } \partial\Omega \times (0, T) \\ u_i(x, 0) &= u_i^0(x) && \text{in } \Omega \times \{t = 0\} \end{aligned}$$

where $u := u(x, t) = (u_1(x, t), \dots, u_N(x, t))$ are the concentration levels of N proteins, $d = (d_1, \dots, d_N) \in (0, +\infty)^N$ are the mass diffusivities, and $k = (k_1, \dots, k_M) \in (0, +\infty)^M$ are the rate constants. Assume that the initial value u^0 is in $L^\infty(\Omega)^N$ and $u^0(x) > 0$ for all $x \in \Omega$. We call r_i the reaction function of the i th protein. The structure of the reaction function is determined by two factors: the reaction kinetics model and the protein network topology. The structure of r has drawn great interest because it affects the blow up property of (2.6).

2.3.9 Ca^{2+} diffusion in the presynaptic terminal or near the plasma

Most of these include mobile Ca^{2+} buffers. Examples of endogenous mobile buffers are calbindin-D28k and calretinin. Examples of exogenous mobile buffers are the Ca^{2+} chelators EGTA and BAPTA, and the fluorescent dye fura-2. The Ca^{2+} -buffer reaction diffusion equations have the form [12]:

$$\frac{\partial Ca}{\partial t} = D_c \nabla^2 Ca + R \frac{\partial B}{\partial t} = D_b \nabla^2 B + R \frac{\partial BC}{\partial t} = D_{bc} \nabla^2 BC - R$$

where Ca is the intraterminal free Ca^{2+} concentration, B is the free buffer concentration, and BC is the concentration of Ca^{2+} bound to buffer. It is assumed here that there is one mobile buffer, but others can be included in a natural way. The diffusion coefficients of the three species are D_c, D_b , and D_{bc} . The Ca^{2+} binding reaction (R) is given by

$$R = -k^+ BCa + k^- BC$$

where k^+ and k^- are the binding and unbinding rates, respectively. Physiological values for the diffusion coefficients and kinetic rates can be found in Allbritton et al. [5].

2.3.10 Reaction diffusion model for image restoration

Image processing is a discipline of computer science and applied mathematics that deals with digital images and their transformations in order to improve their quality or to extract information. There are a large number of applications of image processing in diverse spectrum of activities. One of the most active topics in this area has been image restoration. A number of different techniques have been proposed for digital image restoration, utilizing a number of different models and assumptions. The restoration of degraded images is an important problem because it allow to recovery lost information from the observed degraded image data.

In this model, [41] proposed a edge detection and image restoration, from the case of a single equation to the case of nonlinear reaction diffusion system. They are interested in the existence of weak solutions for this system for which two main properties hold: the positivity of the solutions and the total mass of the components are preserved with time.

In this example, we assume the system of the form

$$(2.7) \quad \begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} (g(|\nabla G_\sigma * u|)\nabla u) = A(t, x, u, v) & \text{in } Q_T \\ \frac{\partial v}{\partial t} - \operatorname{div} (h(|\nabla G_\sigma * v|)\nabla v) = B(t, x, u, v) & \text{in } Q_T \\ u(0, x) = u_0, v(0, x) = v_0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma_T \end{cases}$$

Here $Q_T =]0, T[\times \Omega$, $\Sigma_T =]0, T[\times \partial\Omega$, $T > 0, \Omega =]0, 1[\times]0, 1[$ denotes picture domaine with boundary $\partial\Omega$, with Neumann boundary conditions, $u = u(t, x)$, $v = v(t, x)$, ν is an outward Normal to domain Ω . The diffusivities $g, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are smooth decreasing functions with $g(0) = h(0) = 1$, $\lim_{t \rightarrow +\infty} g(t) = \lim_{t \rightarrow +\infty} h(t) = 0$. Let $\sigma > 0$, we suppose that G_σ is a Gaussian filter

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{|x|^2}{4\sigma}\right); x \in \mathbb{R}^2, \quad |\nabla G_\sigma * \omega| = \left[\sum_{i=1}^2 \left(\frac{\partial G_\sigma}{\partial x_i} * \tilde{\omega} \right)^2 \right]^{\frac{1}{2}}$$

where $\tilde{\omega}$ is a linear continuous extension of ω to \mathbb{R}^2 . The nonlinearity A, B are regular functions whose nonlinear structure is such that two main properties occur:

- The nonnegativity of the solution (u, v) of (2.7) is preserved with time, which is ensured by $A(t, x, u, v) \geq 0, B(t, x, u, v) \geq 0$ for all $u, v \geq 0$ and for a.e. $(t, x) \in Q_T$.

- The total mass of the components u, v is controlled with time, which is ensured by the structure condition $A(t, x, u, v) \leq 0, A(t, x, u, v) + B(t, x, u, v) \leq 0$ for all $u, v \geq 0$ and for a.e. $(t, x) \in Q_T$.

2.3.11 Spiral waves with interfacial oscillatory chemical reactions emerge in a model of reaction diffusion systems

In this example we propose a model for describing oscillatory chemical reactions involving a variety of immiscible solvents. This model describes the interaction between chemical oscillations on both sides of the interface of different solvents and the effect of the interface on chemical oscillations. Based on this model, [19] they studied the spiral chemical wave in a chemical oscillation with interfaces of different solvents, and reported a new type of spiral wave. The interface does not hinder the propagation of chemical waves, which can continue to propagate through the interface. The frequency of the chemical wave is increased after the chemical wave passed through the interface.

The reaction diffusion model equations are as follows

$$\begin{aligned}\frac{\partial \alpha}{\partial t} &= k_1 \beta + k_2 + D_\alpha \Delta \alpha, \\ \frac{\partial \beta}{\partial t} &= k_3 \gamma + k_4 + D_\beta \Delta \beta, \\ \frac{\partial \gamma}{\partial t} &= a_1 \sin(\omega \alpha) + a_2 \cos(\omega \alpha) + b \beta + k_5 c \gamma + k_6 (1 - c) + D_\gamma \Delta \gamma\end{aligned}$$

where, α, β and γ denote the concentrations of reactants, D_α, D_β and D_γ denote the diffusion coefficient of α, β and γ , respectively, parameters $k_1, k_2, k_3, k_4, k_5, k_6, a_1, a_2, b$ and c are determined by reaction conditions. Various values of the parameter c in the interval $[-0.8, -0.2]$ were considered, while the other parameters were fixed. For simplicity, we assume that all reactants have the same diffusion coefficient, i.e., $D_\alpha = D_\beta = D_\gamma$. Parameter values are given in the following table,

Par.	Value	Par.	Value
k_1	1 s^{-1}	a_1	$0.198 \text{ mM} \cdot \text{s}^{-1}$
k_2	$-0.6 \text{ mM} \cdot \text{s}^{-1}$	a_2	$0.028 \text{ mM} \cdot \text{s}^{-1}$
k_3	1 s^{-1}	b	-1.0 s^{-1}
k_4	$-0.6 \text{ mM} \cdot \text{s}^{-1}$	D_α	$0.1 \times 10^{-2} \text{ mm}^2 \cdot \text{s}^{-1}$
k_5	1.0 s^{-1}	D_β	$0.1 \times 10^{-2} \text{ mm}^2 \cdot \text{s}^{-1}$
k_6	0.6 mM	D_γ	$0.1 \times 10^{-2} \text{ mm}^2 \cdot \text{s}^{-1}$
ω	5.0 mM^{-1}	c	$[-0.8, -0.2]$

2.3.12 Nickel-iron alloy electrodeposition on rotating disk electrode model

To better understand the nickel-iron electrodeposition process, we are interested in the one-dimensional model. This model addresses dissociation, diffusion, electromigration, convection and deposition of multiple ion species. [1] They studied the global existence of solutions that are here different ion concentrations in the mixture as well as the electric potential. They presented the news techniques to obtain global existence and positivity of classical solution for our model in the quadratic case, without no restriction of growth on the non linear terms.

The reaction diffusion model equations are as follows

$$(2.8) \quad \left\{ \begin{array}{ll} \frac{\partial w_i}{\partial t} - d_i \frac{\partial^2 w_i}{\partial x^2} + b(x) \frac{\partial w_i}{\partial x} - m_i \frac{\partial}{\partial x} \left(w_i \frac{\partial \Phi}{\partial x} \right) = S_i(w) & \text{on } Q_T \\ -\frac{\partial^2 \Phi}{\partial x^2} = \frac{4\pi F}{\varepsilon_0} \sum_{i=1}^5 z_i w_i & \text{on } Q_T \\ -d_i \frac{\partial w_i}{\partial x}(t, 0) - m_i w_i(t, 0) \frac{\partial \Phi}{\partial x}(t, 0) = -\gamma_i(t) w_i(t, 0) & \text{for } t \in]0, T[\\ -d_i \frac{\partial w_i}{\partial x}(t, 3\delta) - m_i w_i(t, 3\delta) \frac{\partial \Phi}{\partial x}(t, 3\delta) = -b(3\delta) w_i(t, 3\delta) & \text{for } t \in]0, T[\\ \Phi(t, 0) = V(t), \Phi(t, 3\delta) = 0 & \text{for } t \in]0, T[\\ w_i(0, x) = w_{i,0}(x) & \text{for } x \in]0, 3\delta[\end{array} \right.$$

such system involve one-dimensional steady-state transport of the various species to a rotating disk electrode in the electrodeposition of $NiFe$. Where $Q_T =]0, T[\times]0, L[$, w_i is the concentration of species A_i in the reactor, S_i denotes the production rate of A_i due to all the homogeneous reactions, d_i is the diffusion coefficient of species A_i , v is the fluid velocity vector, Φ is the electric potential and u_i is the electrical mobility of species A_i . The mobility and diffusion coefficient are related through the Einstein equation

$$m_i = \frac{d_i z_i F}{RT}$$

where $z_i F$ is the charge carried by a mole of species A_i , R is the universal gas constant and T is the local temperature. The permittivity of the solvent is denoted by ε_0 and $\gamma_i(t) = \beta_i \exp\left(-\frac{\alpha_i z_i F}{RT} (V_m(t) - V(t))\right)$, $\gamma_i(t) = 0$ for $i = 4, 5$ The initial concentration is given by $w_i(0, x) = w_{i,0}(x)$, where $w_{i,0}$ represents the total amount of component A_i added to the bulk solution. The homogeneous reaction term in (2.8) takes the form

$$\left\{ \begin{array}{ll} S_i = 0 & \text{for } 1 \leq i \leq 2 \\ S_3 = S_4 = -S_5 = -S(w_3, w_4, w_5) \\ S(r, p, q) = k_1 r p - k_{-1} q \end{array} \right.$$

The aim of this work is to proof global existence and positivity of classical solution for the model (2.8).

2.3.13 On the dynamics of a non-linear duopoly game model

In order to take into account the territory in which the outputs are in the market and the time-depending firms' strategies, the discrete Cournot duopoly game (with adaptive expectations, modeled by Kopel) is generalized through a non autonomous reaction diffusion binary system of PDEs, with self and cross diffusion terms.

duopoly is the case where the market is controlled by two firms X, Y , producing similar products. As basic continuous model for the evolution of the outputs of the firms, we consider the continuous time-scale, in [55], it has been introduced the continuous autonomous model

$$(2.9) \quad \begin{cases} \partial_t u = -\alpha_1 u + \alpha_1 \mu_1 v(1-v) + \gamma_{11} \Delta u + \gamma_{12} \Delta v \\ \partial_t v = -\alpha_2 v + \alpha_2 \mu_2 u(1-u) + \gamma_{21} \Delta u + \gamma_{22} \Delta v \end{cases}$$

where:

u, v = outputs of the two firms X and Y respectively,

$\mu_1 = \text{const.} > 0$, measure of the intensity of the effect of the action of Y on X ,

$\mu_2 = \text{const.} > 0$, measure of the intensity of the effect of the action of X on Y ,

$0 < \alpha_i \leq 1 (i = 1, 2)$, adjustment coefficients,

with $\Omega \subset \mathbb{R}^3$, bounded domain, in which, the outputs are in the market,

$$\phi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \rightarrow \phi(\mathbf{x}, t) \in \mathbb{R}, \quad \phi \in \{u, v\}$$

$\gamma_{ii} = \text{const.} \geq 0 (i = 1, 2)$, self diffusion coefficients and γ_{12}, γ_{21} constant cross diffusion coefficients. In order to take into account that, according to the market evolution, the strategies of the firms may evolve, we let the adjustment coefficients be time-depending. Precisely, we assume that

$$\alpha_i : t \in [0, \infty[\rightarrow \alpha_i(t) \in]0, 1]$$

with $\alpha_i \in C^1(\mathbb{R})$.

To (2.9) we append the initial data

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad v(\mathbf{x}, 0) = v_0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega$$

and the following boundary conditions: Dirichlet boundary conditions

$$u = \bar{u} \quad \text{and} \quad v = \bar{v} \text{ on } \partial\Omega \times \mathbb{R}^+ \text{ where } \bar{u}, \bar{v}$$

will be chosen among the constant steady-states solutions of (2.9), or Robin boundary data

$$\begin{cases} \beta u + (1 - \beta)\nabla u \cdot \mathbf{n} = \bar{u}\beta \\ \beta v + (1 - \beta)\nabla v \cdot \mathbf{n} = \bar{v}\beta, \end{cases} \quad \text{on } \partial\Omega \times \mathbb{R}^+ \text{ where } 0 < \beta < 1.$$

ON THE EXISTENCE OF POSITIVE SOLUTIONS OF A CLASS OF PARABOLIC REACTION DIFFUSION SYSTEMS

In this chapter, we show the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions using techniques of functional analysis and potential analysis. The work constituting this chapter is the subject of an article published in an international journal specialized in Mathematics (Studia Universitatis Babeş-Bolyai Mathematica, 0252-1938 (Print-ISSN) 2065-961X (E-ISSN)), in collaboration with S. Mesbahi.

Keywords : parabolic system ; reaction diffusion system ; positive solutions ; Green function.

2010 Mathematics Subject Classification : 34B27, 35B09, 35K41, 35K57.

3.1 Introduction

The modeling and the mathematical analysis of parabolic systems, in particular, reaction diffusion systems, has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena, not only in biology and chemistry, but also in engineering, economics and ecology, such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame

spread and others.

For the mathematical analysis of this type of problem, various methods and elaborate techniques have been proposed, see for example Mesbahi et al. [4], [2], [41], [40], Gontara [24], Lions [30], Maâgli et al. [34], [33], Pierre [50] and Zhang [62], [63]. We refer the reader to Arakelian and Gauthier [6], Armitage and Gardiner [7] and Port [51] for more details on the potential arguments of the theory that interest us mainly in this work.

The subject of this paper is in this context, we will take care to study the existence of positive solutions of the following nonlinear parabolic reaction diffusion system

$$(3.1) \quad \begin{cases} -\frac{\partial u}{\partial t} + \Delta u = \lambda p(x, t)f(v) \\ -\frac{\partial v}{\partial t} + \Delta v = \mu q(x, t)g(w) \\ -\frac{\partial w}{\partial t} + \Delta w = \eta r(x, t)h(z) \\ -\frac{\partial z}{\partial t} + \Delta z = \rho e(x, t)k(u) \end{cases}$$

with $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and the initial conditions

$$(3.2) \quad \begin{cases} u(x, 0) = \varphi(x) & , & v(x, 0) = \psi(x) \\ w(x, 0) = \gamma(x) & , & z(x, 0) = \zeta(x) \end{cases} , \forall x \in \mathbb{R}^n$$

where $n \geq 3$, φ, ψ, γ and $\zeta : \mathbb{R}^n \rightarrow [0, \infty)$ are continuous, the constants λ, μ, η and ρ are nonnegative, f, g, h and $k : (0, \infty) \rightarrow [0, \infty)$ are nondecreasing and continuous. p, q, r and $e : \mathbb{R}^n \times (0, \infty) \rightarrow [0, \infty)$ are measurable functions and satisfy an appropriate hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$ introduced in Zhang [63].

Before stating the main result of this work, it is worth mentioning that several mathematicians have dealt with this type of problem using various analytical and numerical techniques and methods, under different hypotheses as appropriate, see for example, Bachar et al. [8], Maâgli et al. [22], [21], [34]-[35], Ghergu and Radulescu [23], Gontara [24], Ma [32], Zhang [62], [63] and Zhao [64].

Concerning the problem (3.1) – (3.2) in the case of a single equation of the form

$$(3.3) \quad \begin{cases} \Delta u - \frac{\partial u}{\partial t} = q(x, t)u^{p+1} & , & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x) & , & x \in \mathbb{R}^n, n \geq 3 \end{cases}$$

Zhang in [62] discussed the existence and the asymptotic behavior of solutions to this problem, he proved the following result :

Theorem 3.1. *Suppose $p > 0$, $q \in P^\infty(\mathbb{R}^n)$. For any $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n)$ satisfying $\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq b_0$, there exists a positive and continuous solution u of (3.3) such that*

$$M^{-1} \int_{\mathbb{R}^n} G(x, t, y, 0) u_0(y) dy \leq u(x, t) \leq M \int_{\mathbb{R}^n} G(x, t, y, 0) u_0(y) dy$$

for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$.

G denotes the fundamental solution of the heat equation $\Delta u - \frac{\partial u}{\partial t} = 0$ in $\mathbb{R}^n \times (0, \infty)$ given for $t > s$ and $x, y \in \mathbb{R}^n$ by

$$G(x, t, y, s) = \frac{1}{[4\pi(t-s)]^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)$$

In [33], the authors considered the problem (3.3) with boundary condition u_0 , not necessarily bounded function. The nonlinearity $u\varphi(\cdot, u)$ is required to satisfy some conditions related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$. They gave existence results and similar estimates on the solutions as in [62].

In [34], a similar problem as (3.3) has been treated in the half space \mathbb{R}_+^n . The elliptical version of (3.3) was studied in [22]. In [24], the authors examined the problem (3.1)–(3.2) in the case of a system with two equations.

3.2 Statement of the main result

3.2.1 Assumptions

To study problem (3.1)–(3.2), we consider the following definition and hypotheses :

Definition 3.1. We say that a nonnegative superharmonic function ω satisfies condition (H_0) if ω is locally bounded in \mathbb{R}^n ($n \geq 3$) and the map $(x, t) \mapsto P\omega(x, t)$ is continuous in $\mathbb{R}^n \times (0, \infty)$, where P is defined below.

$(P_t)_{t>0}$ on \mathbb{R}^n denotes the Gauss semigroup defined for each nonnegative measurable function Φ on \mathbb{R}^n by

$$P_t\Phi(x) = P\Phi(x, t) = \int_{\mathbb{R}^n} G(x, t, y, 0)\Phi(y)dy, \quad t > 0, \quad x \in \mathbb{R}^n$$

The family $(P_t)_{t>0}$ is a Markovian semigroup. Moreover, a nonnegative superharmonic function ω on \mathbb{R}^n satisfies for every $t > 0$, $P_t\omega \leq \omega$, and consequently the mapping $t \mapsto P_t\omega$ is nonincreasing. We remark that for each nonnegative measurable function Φ on \mathbb{R}^n , the map $(x, t) \mapsto P_t\Phi(x)$ is lower semicontinuous on $\mathbb{R}^n \times (0, \infty)$ and becomes continuous if Φ is further bounded.

Remark 3.1. We note that every bounded superharmonic function in \mathbb{R}^n satisfies (H_0) , see Gontara and Turki [24] and Mâagli et al. [33].

We fix four nonnegative superharmonic functions ω , θ , δ and ϕ satisfying condition (H_0) . Let us introduce the required hypotheses on the initial values φ , ψ , γ and ζ the nonlinear terms :

(H_1) There exist four constants $c_i > 1$, $1 \leq i \leq 4$, such that

$$\begin{aligned} \frac{1}{c_1}\omega(x) \leq \psi(x) \leq c_1\omega(x) & \quad , \quad \frac{1}{c_2}\theta(x) \leq \varphi(x) \leq c_2\theta(x) \\ \frac{1}{c_3}\delta(x) \leq \gamma(x) \leq c_3\delta(x) & \quad , \quad \frac{1}{c_4}\phi(x) \leq \zeta(x) \leq c_4\phi(x) \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0} P_t\psi(x) = \psi(x) & \quad , \quad \lim_{t \rightarrow 0} P_t\varphi(x) = \varphi(x) \\ \lim_{t \rightarrow 0} P_t\gamma(x) = \gamma(x) & \quad , \quad \lim_{t \rightarrow 0} P_t\zeta(x) = \zeta(x) \end{aligned}$$

(H_2) $f, g, h, k : (0, \infty) \rightarrow [0, \infty)$ are nondecreasing and continuous.

(H_3) The functions p, q, r and e are measurable nonnegative and for each $c > 0$, the functions

$$\tilde{p}_c = \frac{pf(cP\omega)}{P\theta} \quad , \quad \tilde{q}_c = \frac{qg(cP\delta)}{P\omega} \quad , \quad \tilde{r}_c = \frac{rh(cP\phi)}{P\delta} \quad , \quad \tilde{e}_c = \frac{ek(cP\theta)}{P\phi}$$

belong to the parabolic Kato class $P^\infty(\mathbb{R}^n)$.

To study (3.1) – (3.2), a basic assumptions on p, q, r and e requires to fix four superharmonic functions ω, θ, δ and ϕ on \mathbb{R}^n satisfying condition (H_0) .

3.2.2 The main result

Now, we can state the main result of this work :

Theorem 3.2. *Assume (H_1) – (H_3) . Then there exist four constants λ_0, μ_0, η_0 and ρ_0 such that for each $\lambda \in [0, \lambda_0), \mu \in [0, \mu_0), \eta \in [0, \eta_0)$ and $\rho \in [0, \rho_0)$, the problem (3.1)–(3.2) has a positive continuous solution (u, v, w, z) in $(\mathbb{R}^n \times (0, \infty))^4$ satisfying for each $t > 0$ and $x \in \mathbb{R}^n$*

$$(3.4) \quad \left\{ \begin{array}{l} (1 - \frac{\lambda}{\lambda_0})P\varphi(x, t) \leq u(x, t) \leq P\varphi(x, t) \\ (1 - \frac{\mu}{\mu_0})P\psi(x, t) \leq v(x, t) \leq P\psi(x, t) \\ (1 - \frac{\eta}{\eta_0})P\gamma(x, t) \leq w(x, t) \leq P\gamma(x, t) \\ (1 - \frac{\rho}{\rho_0})P\zeta(x, t) \leq z(x, t) \leq P\zeta(x, t) \end{array} \right.$$

This document is organized as follows : In the next section, we give some technical results and to recall some theoretical tools that are essential to prove our main result. The last section is devoted to the proof of the main result, Theorem 3.2. The difficulties in this section are similar to those in [8]-[24], [34]-[35] and [62]-[64], and the techniques are of the same spirit, but specific new difficulties due to the nature of the system must be handled.

3.3 Preliminary results

We give here some essential results proved in [33], we can also see [63], [64], which were retained for the proof of our result. Now, we recall the definition of the Kato class $P^\infty(\mathbb{R}^n)$.

Definition 3.2. A Borel measurable function q in \mathbb{R}^{n+1} belongs to the Kato class $P^\infty(\mathbb{R}^n)$ if for all $c > 0$,

$$\lim_{\epsilon \rightarrow 0} \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} \int_{t-\epsilon}^{t+\epsilon} \int_{B(x, \sqrt{\epsilon})} G_c(x, |t-s|, y, 0) |q(y, s)| dy ds = 0$$

and

$$\sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} G_c(x, |t-s|, y, 0) |q(y, s)| dy ds < \infty$$

where

$$G_c(x, t, y, s) = \frac{1}{(t-s)^{\frac{n}{2}}} \exp(-c \frac{|x-y|^2}{t-s}) \quad \text{for } t > s \text{ and } x, y \in \mathbb{R}^n$$

In the following, we give a class of functions belonging to $P^\infty(\mathbb{R}^n)$.

Proposition 3.1. (i) $L^\infty(\mathbb{R}^n) \otimes L^1(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$.

(ii) Let $1 \leq p < \infty$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\sigma > \frac{np}{2}$ and $\tau < \frac{2}{p} - \frac{n}{\sigma} < \nu$, we have

$$\frac{L^\sigma(\mathbb{R}^n)}{|\cdot|^\tau (1 + |\cdot|)^{\nu-\tau}} \otimes L^q(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$$

(iii) $P^\infty(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^{n+1})$.

We denote for any measurable function Φ on $\mathbb{R}^n \times (0, \infty)$, the potential

$$V\Phi(x, t) = \int_0^t \int_{\mathbb{R}^n} G(x, t, y, s) \Phi(y, s) dy ds = \int_0^t P_{t-s}(\Phi(\cdot, s))(x) ds$$

Proposition 3.2. Let q be a nonnegative function in $P^\infty(\mathbb{R}^n)$, then there exists a positive constant α_q such that for each superharmonic function v in \mathbb{R}^n , we have

$$V(qPv)(x, t) \leq \alpha_q Pv(x, t) \quad , \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty)$$

Proposition 3.3. Let v be a superharmonic function in \mathbb{R}^n satisfying (H_0) and q be a nonnegative function in $P^\infty(\mathbb{R}^n)$. Then the family of functions

$$\left\{ (x, t) \rightarrow Vf(x, t) = \int_0^t \int_{\mathbb{R}^n} G(x, t, y, s) f(y, s) dy ds, |f| \leq qPv \right\}$$

is equicontinuous in $\mathbb{R}^n \times [0, \infty)$.

Moreover, for each $x \in \mathbb{R}^n$ we have $\lim_{t \rightarrow 0} Vf(x, t) = 0$, uniformly on f .

We therefore conclude the following result on the continuity needed to obtain the proof of Theorem 3.2.

Proposition 3.4. Assuming the hypothesis (H_1) . Then the functions $P\varphi$, $P\psi$, $P\gamma$ and $P\zeta$ are continuous in $\mathbb{R}^n \times (0, \infty)$.

Proof. We prove that $P\varphi$ is continuous in $\mathbb{R}^n \times (0, \infty)$.

Let c_2 be the constant given in (H_1) . We write for each $t > 0$ and $x \in \mathbb{R}^n$

$$c_2 P_t \theta(x) = P_t(c_2 \theta - \varphi)(x) + P_t \varphi(x)$$

So, from (H_0) we have $(x, t) \mapsto P\theta(x, t)$ is continuous in $\mathbb{R}^n \times (0, \infty)$ and from the fact that $(x, t) \mapsto P_t(c_2 \theta - \varphi)(x)$ and $(x, t) \mapsto P_t \varphi(x)$ are lower semi-continuous in $\mathbb{R}^n \times (0, \infty)$, we deduce that $(x, t) \mapsto P_t \varphi(x)$ is continuous in $\mathbb{R}^n \times (0, \infty)$.

Similarly, we can prove the continuity of $P\psi$, $P\gamma$ and $P\zeta$ in $\mathbb{R}^n \times (0, \infty)$. ■

3.4 Proof of the main result

Let

$$\begin{aligned} \lambda_0 &= \inf_{(x,t) \in \mathbb{R}^n \times (0, \infty)} \frac{P\varphi(x, t)}{V(pf(P\psi))(x, t)} \\ \mu_0 &= \inf_{(x,t) \in \mathbb{R}^n \times (0, \infty)} \frac{P\psi(x, t)}{V(qg(P\gamma))(x, t)} \\ \eta_0 &= \inf_{(x,t) \in \mathbb{R}^n \times (0, \infty)} \frac{P\gamma(x, t)}{V(rh(P\zeta))(x, t)} \\ \varrho_0 &= \inf_{(x,t) \in \mathbb{R}^n \times (0, \infty)} \frac{P\zeta(x, t)}{V(ek(P\varphi))(x, t)} \end{aligned}$$

Proposition 3.5. *Suppose that the hypotheses (H_1) – (H_3) are satisfied, then the constants λ_0 , μ_0 , η_0 and ϱ_0 are positive.*

Proof. The hypothesis (H_1) leads to $\psi \leq c_1 \omega$

From the fact that f is nondecreasing and p is nonnegative, we have

$$V(pf(P\psi)) \leq V(pf(c_1 P\omega))$$

Hence, by hypothesis (H_3) and Proposition 3.2, there exist $\tilde{p}_{c_1} \in P^\infty(\mathbb{R}^n)$ and a positive constant $\alpha_{\tilde{p}_{c_1}}$ such that for each $(x, t) \in \mathbb{R}^n \times (0, \infty)$, we have

$$V(pf(P\psi))(x, t) \leq V(\tilde{p}_{c_1} P\theta)(x, t) \leq \alpha_{\tilde{p}_{c_1}} P\theta(x, t)$$

So, using again (H_1) we find for each $(x, t) \in \mathbb{R}^n \times (0, \infty)$

$$\frac{P\varphi(x, t)}{V(pf(P\psi))(x, t)} \geq \frac{\frac{1}{c_1}P\theta(x, t)}{\alpha_{\tilde{p}_{c_1}}P\theta(x, t)} = \frac{1}{c_1\alpha_{\tilde{p}_{c_1}}} > 0$$

In the same way, we prove that

$$\frac{P\psi(x, t)}{V(qg(P\gamma))(x, t)} > 0, \quad \frac{P\gamma(x, t)}{V(rh(P\zeta))(x, t)} > 0, \quad \frac{P\zeta(x, t)}{V(ek(P\varphi))(x, t)} > 0$$

which implies that $\lambda_0 > 0$, $\mu_0 > 0$, $\eta_0 > 0$, $\rho_0 > 0$. ■

Proof. (of Theorem 3.2). Let $\lambda \in [0, \lambda_0)$, $\mu \in [0, \mu_0)$, $\eta \in [0, \eta_0)$ and $\rho \in [0, \rho_0)$. We define the sequences $(u_j)_{j \geq 0}$, $(v_j)_{j \geq 0}$, $(w_j)_{j \geq 0}$, and $(z_j)_{j \geq 0}$ by

$$\left\{ \begin{array}{l} v_0 = P\psi, \quad z_0 = P\zeta \\ u_j = P\varphi - \lambda V(pf(v_j)) \\ w_j = P\gamma - \eta V(rh(z_j)) \\ z_{j+1} = P\zeta - \rho V(ek(u_j)) \\ v_{j+1} = P\psi - \mu V(qg(w_j)) \end{array} \right.$$

We are determined to prove for all $j \in \mathbb{N}$,

$$(3.5) \quad 0 < \left(1 - \frac{\lambda}{\lambda_0}\right)P\varphi \leq u_j \leq u_{j+1} \leq P\varphi$$

$$(3.6) \quad 0 < \left(1 - \frac{\eta}{\eta_0}\right)P\gamma \leq w_j \leq w_{j+1} \leq P\gamma$$

$$(3.7) \quad 0 < \left(1 - \frac{\mu}{\mu_0}\right)P\psi \leq v_{j+1} \leq v_j \leq P\psi$$

$$(3.8) \quad 0 < \left(1 - \frac{\rho}{\rho_0}\right)P\zeta \leq z_{j+1} \leq z_j \leq P\zeta$$

We note that according to the definition of λ_0 , μ_0 , η_0 and ρ_0 that, for each $(x, t) \in \mathbb{R}^n \times (0, \infty)$

$$(3.9) \quad \lambda_0 V(pf(P\psi))(x, t) \leq P\varphi(x, t)$$

$$(3.10) \quad \mu_0 V(qg(P\gamma))(x, t) \leq P\psi(x, t)$$

$$(3.11) \quad \eta_0 V(rh(P\zeta))(x, t) \leq P\gamma(x, t)$$

$$(3.12) \quad \rho_0 V(ek(P\varphi))(x, t) \leq P\zeta(x, t)$$

From (3.9) and (3.11), we have

$$\begin{aligned} u_0 &= P\varphi - \lambda V(pf(P\psi)) \geq P\varphi - \frac{\lambda}{\lambda_0} P\varphi = (1 - \frac{\lambda}{\lambda_0}) P\varphi > 0 \\ w_0 &= P\gamma - \eta V(rh(P\zeta)) \geq P\gamma - \frac{\eta}{\eta_0} P\gamma = (1 - \frac{\eta}{\eta_0}) P\gamma > 0 \end{aligned}$$

Then

$$\begin{aligned} z_1 - z_0 &= -\rho V(ek(u_0)) \leq 0 \\ v_1 - v_0 &= -\mu V(qg(w_0)) \leq 0 \end{aligned}$$

Since f and h are nondecreasing, we obtain

$$\begin{aligned} u_1 - u_0 &= \lambda V(p(f(v_0) - f(v_1))) \geq 0 \\ w_1 - w_0 &= \eta V(r(h(z_0) - h(z_1))) \geq 0 \end{aligned}$$

Now, since v_0, z_0 are nonnegatives ($v_0 > 0 \Rightarrow u_0 \leq P\varphi, z_0 > 0 \Rightarrow w_0 \leq P\gamma$) and g, k are nondecreasing, we deduce from (3.10) and (3.12) that

$$\begin{aligned} z_1 &= P\zeta - \rho V(ek(u_0)) \geq (1 - \frac{\rho}{\rho_0}) P\zeta > 0 \\ v_1 &= P\psi - \mu V(qg(w_0)) \geq (1 - \frac{\mu}{\mu_0}) P\psi > 0 \end{aligned}$$

which gives us

$$u_1 \leq P\varphi \text{ and } w_1 \leq P\gamma$$

Finally, we find

$$\left\{ \begin{array}{l} 0 < (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_0 \leq u_1 \leq P\varphi \\ 0 < (1 - \frac{\eta}{\eta_0})P\gamma \leq w_0 \leq w_1 \leq P\gamma \\ 0 < (1 - \frac{\mu}{\mu_0})P\psi \leq v_1 \leq v_0 \leq P\psi \\ 0 < (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_1 \leq z_0 \leq P\zeta \end{array} \right.$$

By induction, we suppose that (3.5), (3.6), (3.7) and (3.8) hold for j . Since g, k are nondecreasing and $u_{j+1} \leq P\varphi, w_{j+1} \leq P\gamma$, we have

$$\begin{aligned} z_{j+2} - z_{j+1} &= \varrho V(e(k(u_j) - k(u_{j+1}))) \leq 0 \\ v_{j+2} - v_{j+1} &= \mu V(q(g(w_j) - g(w_{j+1}))) \leq 0 \end{aligned}$$

and

$$\begin{aligned} z_{k+2} &= P\zeta - \varrho V(ek(u_{k+1})) \geq P\zeta - \varrho V(ek(P\varphi)) \geq (1 - \frac{\varrho}{\varrho_0})P\zeta \\ v_{k+2} &= P\psi - \mu V(qg(w_{k+1})) \geq P\psi - \mu V(qg(P\gamma)) \geq (1 - \frac{\mu}{\mu_0})P\psi \end{aligned}$$

Using the two relations (3.10) and (3.12), we have

$$\begin{aligned} 0 &< (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_{j+2} \leq z_{j+1} \leq P\zeta \\ 0 &< (1 - \frac{\mu}{\mu_0})P\psi \leq v_{j+2} \leq v_{j+1} \leq P\psi \end{aligned}$$

Now, using that f, h are nondecreasing, we have

$$\begin{aligned} u_{j+2} - u_{j+1} &= \lambda V(p(f(v_{j+1}) - f(v_{j+2}))) \geq 0 \\ w_{j+2} - w_{j+1} &= \eta V(r(h(z_{j+1}) - h(z_{j+2}))) \geq 0 \end{aligned}$$

Since $z_{j+2} > 0, v_{j+2} > 0$, we obtain

$$\begin{aligned} 0 &< (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_{j+1} \leq u_{j+2} \leq P\varphi \\ 0 &< (1 - \frac{\eta}{\eta_0})P\gamma \leq w_{j+1} \leq w_{j+2} \leq P\gamma \end{aligned}$$

Therefore, the sequences $(u_j)_{j \geq 0}$, $(v_j)_{j \geq 0}$, $(w_j)_{j \geq 0}$ and $(z_j)_{j \geq 0}$ converge respectively to u , v , w and z satisfying (3.4). We claim that

$$(3.13) \quad u = P\varphi - \lambda V(pf(v))$$

$$(3.14) \quad w = P\gamma - \eta V(rh(z))$$

$$(3.15) \quad z = P\zeta - \rho V(ek(u))$$

$$(3.16) \quad v = P\psi - \mu V(qg(w))$$

Since $v_j \leq P\psi$ and $z_j \leq P\zeta$ for all $j \in \mathbb{N}$, using hypotheses $(H_1), (H_3)$ and the fact that f, h are nondecreasing, there exist $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^\infty(\mathbb{R}^n)$ such that

$$(3.17) \quad pf(v) \leq pf(c_1 P\omega) \leq \tilde{p}_{c_1} P\theta$$

$$(3.18) \quad rh(z) \leq rh(c_4 P\phi) \leq \tilde{r}_{c_4} P\delta$$

and so

$$p |f(v_j) - f(v)| \leq 2\tilde{p}_{c_1} P\theta, \text{ for all } j \in \mathbb{N}$$

$$r |h(z_j) - h(z)| \leq 2\tilde{r}_{c_4} P\delta, \text{ for all } j \in \mathbb{N}$$

Now, from Proposition 3.3 and by Lebesgue's theorem, we can deduce

$$\lim_{k \rightarrow \infty} V(pf(v_k)) = V(pf(v))$$

$$\lim_{k \rightarrow \infty} V(rh(z_k)) = V(rh(z))$$

So, letting $j \rightarrow \infty$ in equations

$$u_j = P\varphi - \lambda V(pf(v_j)) \quad , \quad w_j = P\gamma - \eta V(rh(z_j))$$

we have (3.13) and (3.14). Similarly, we obtain (3.15) and (3.16).

Next, we affirm that (u, v, w, z) satisfies

$$(3.19) \quad \begin{cases} \Delta u - \frac{\partial u}{\partial t} = \lambda p f(v) \\ \Delta v - \frac{\partial v}{\partial t} = \mu q g(w) \\ \Delta w - \frac{\partial w}{\partial t} = \eta r h(z) \\ \Delta z - \frac{\partial z}{\partial t} = \rho e k(u) \end{cases}$$

Since θ, δ satisfies (H_0) and $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^\infty(\mathbb{R}^n)$, using Proposition 3.1, we have

$$\tilde{p}_{c_1} P \theta, \tilde{r}_{c_4} P \delta \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Moreover (3.17), (3.18) and Proposition 3.3 imply that

$$p f(v), r h(z) \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

and

$$V(p f(v)), V(r h(z)) \in C(\mathbb{R}^n \times (0, \infty)) \subset L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Similarly

$$q g(w), V(q g(w)), e k(u), V(e k(u)) \in L^1_{loc}(\mathbb{R}^n \times (0, \infty))$$

Now, applying the heat operator $\Delta - \frac{\partial}{\partial t}$ in (3.13), (3.14), (3.15) and (3.16), (u, v, w, z) is clearly a positive solution (in the sense of distributions) of (3.19).

Furthermore since $V(p f(v)), V(q g(w)), V(r h(z))$ and $V(e k(u))$ are continuous in $\mathbb{R}^n \times (0, \infty)$ and using Proposition 3.4, we deduce from (3.13), (3.14), (3.15) and (3.16) that

$$(u, v, w, z) \in (C(\mathbb{R}^n \times (0, \infty)))^4$$

which implies according to hypothesis (H_1) and proposition 3.3 that

$$\begin{aligned} \lim_{t \rightarrow 0} u(x, t) &= \lim_{t \rightarrow 0} P \varphi(x, t) = \varphi(x), \quad x \in \mathbb{R}^n \\ \lim_{t \rightarrow 0} v(x, t) &= \lim_{t \rightarrow 0} P \psi(x, t) = \psi(x), \quad x \in \mathbb{R}^n \\ \lim_{t \rightarrow 0} w(x, t) &= \lim_{t \rightarrow 0} P \gamma(x, t) = \gamma(x), \quad x \in \mathbb{R}^n \\ \lim_{t \rightarrow 0} z(x, t) &= \lim_{t \rightarrow 0} P \zeta(x, t) = \zeta(x), \quad x \in \mathbb{R}^n \end{aligned}$$

This completes the proof of our theorem. ■



GENERAL CONCLUSION AND OUTLOOK

This study focuses on the analysis and mathematical modeling of reaction diffusion systems :

- **This work has mainly focused on the question of the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions using techniques of functional analysis and potential analysis.**
- **Many important results have been obtained with additional assumptions that can be applied to several models in biology, ecology, physics and others as appropriate.**
- **We have developed original methods to overcome certain difficulties, and despite the complexity of the model studied, we have succeeded in obtaining an existence result.**

In addition to this work, we can address the following interesting questions :

- (i) **The mathematical analysis of anisotropic system, which consists in adding diffusion coefficients to the studied system depending on (t, x) or more generally depending on $(t, x, u, \nabla u)$.**
- (ii) **The asymptotic behaviour of solutions.**
- (iii) **Numerical simulation.**
- (iv) **It is important to study the same model with other conditions.**

This list of loose themes corresponds to work in progress or prospective. Some are a continuation of the work already done, and some are new research projects.

This not only makes it possible to be closer to the reality and concerns of the current industrial world, but also goes beyond the theoretical framework by developing models and tools that can be used and transferred to various industries.

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العنوان : التحليل الرياضي لفئة من أنظمة التفاعل والانتشار.

ملخص : العمل الذي يشكل هذه الأطروحة هو مساهمة في النمذجة والتحليل الرياضي لأنظمة التفاعل والانتشار. نحن مهتمون بوجود حلول إيجابية مستمرة لفئة من أنظمة التفاعل والانتشار المكافئة غير الخطية مع شروط ابتدائية؛ باستخدام تقنيات تعتمد على التحليل الوظيفي والتحليل الكموني. يتكون هذا العمل من ثلاثة فصول مستقلة؛ مسبقة بمقدمة عامة تبرز فن هذا الموضوع والمشاكل التي تم تناولها.

كلمات مفتاحية : أنظمة التفاعل والانتشار، أنظمة مكافئة، حلول إيجابية، نمذجة رياضية.

Titre : Analyse mathématique d'une classe de systèmes de réaction diffusion.

Résumé : Le travail constituant cette thèse est une contribution à la modélisation et l'analyse mathématique de systèmes de réaction diffusion. Nous nous intéressons à l'existence de solutions continues positives d'une classe de systèmes de réaction diffusion paraboliques non linéaires avec des conditions initiales ; utilisant des techniques basées sur l'analyse fonctionnelle et l'analyse de potentiel. Ce travail est alors composé de trois chapitres indépendants, il est précédé par une introduction générale qui met en évidence l'art du sujet et les problèmes abordés.

Mots-clés : systèmes de réaction diffusion, systèmes paraboliques, solutions positives, modélisation mathématique.

Title : Mathematical analysis of a class of reaction diffusion systems.

Abstract : The work that constitutes this thesis is a contribution to the modeling and mathematical analysis of reaction diffusion systems. We are interested in the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions; using techniques based on functional analysis and potential analysis. This work is then composed of three independent chapters, it is preceded by a general introduction that highlights the art of the subject and the problems addressed.

Keywords : reaction diffusion systems, parabolic systems, positive solutions, mathematical modeling.
