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Thesis

A view to obtaining the diploma of

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Option: ***Mathematics***

Numerical Study of the Thixotropic model

Presented by:

Amira Rahai

Publicly discussed: 15 /07/2021

In front of the Jury:

1. Khemis Rabah	**MCA**	**University 20 August 1955, Skikda**	**President**
2. Bouzettouta Lamine	**MCA**	**University 20 August 1955, Skikda**	**Examiner**
3. Ellagoune Fateh	**Professor**	**University 8 May 1945, Guelma**	**Examiner**
4. Chaoui Abd elrezak	**Professor**	**University 8 May1945, Guelma**	**Examiner**
5. Guesmia Amar	**Professor**	**University 20 August 1955, Skikda**	**Supervisor**
6. Messaoud Maouni	**Professor**	**University 20 August 1955, Skikda**	**Co-supervisor**

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قسم الرياضيات
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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option : *Mathématique*

Etude Numérique d'un Modèle de Thixotrope

Présentée par :

Amira Rahai

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Devant les Jury :

1. Khemis Rabah	**MCA, Université 20 Août 1955, Skikda**	**Président**
2. Bouzettouta Lamine	**MCA, Université 20 Août 1955, Skikda**	**Examinateur**
3. Ellagoune Fateh	**Professeur, Université 8 Mai 1955, Guelma**	**Examinateur**
4. Chaoui Abd elrezak	**Professeur, Université 8 Mai 1955, Guelma**	**Examinateur**
5 Guesmia Amar	**Professeur, Université 20 Août 1955, Skikda,**	**Directeur de thèse**
6. Messaoud Maouni	**Professeur, Université 20 Août 1955, Skikda,**	**Co-directeur de thèse**

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Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
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AMIRA RAHAI

Topic:

Numerical Study of the Thixotropic model

Supervisor: Professor. A. GUESMIA

Co-supervisor: Professor. M. MAOUNI

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Abstract

The Thixotropic model is described by a system of nonlinear PDEs. The system under consideration represents a convection-diffusion equation for the speed of the fluid thixotropic coupled with a reaction-diffusion equation for the temperature of the fluid.

The aim of this thesis is the study of the existence and uniqueness of solution for both Thixotropic model and its space fractional model by using certain assumptions. The proof is based on the Lax-Miligram Theorem, Galerkin's method , the Principle of the Maximum and the fixed point theory. And also the study of numerical solution for the thixotropic model and its space fractional model.

The finite volume method is utilized under certain hypotheses to prove the existence and uniqueness of an approximate positive solution. The study also demonstrates the stability and the convergence of finite volume method. Finally, we end this thesis with some numerical simulation carried out by the Matlab software.

Keywords:

Thixotropic model; Fractional differential equation; Global solution; Galerkin method; Maximum principle ; Fixed point theorem; Method of Line (MOL); Finite volume method.

Résumé

Le modèle de Thixotrope est décrit par un système d'EDP non linéaire. Le système considéré représente une équation de convection diffusion pour la vitesse du fluide thixotrope couplée d'une équation de réaction-diffusion pour la température du fluide.

L'objectif de cette thèse est l'étude de l'existence et de l'unicité de la solution pour les deux modèle thixotrope et son modèle fractionnaire spatial en utilisant certaines hypothèses. La démonstration est basé sur le théorème de Lax-Miligram, la méthode de Galerkin, le principe du maximum et la théorie du point fixe. Et aussi l'étude de solution numérique pour le modèle thixotrope et son modèle fractionnaire spatial.

La méthode des volumes finis est utilisée sous certaines hypothèses pour prouver l'existence et l'unicité d'une solution positive approchée. L'étude démontre également la stabilité et la convergence de la méthode des volumes finis. Enfin, nous terminons cette thèse par quelques simulation numérique réalisée par le logiciel Matlab.

Mots clés:

Modèle de Thixotrope; Equation différentielle fractionnaire; Solution Global; Méthode de Galerkin; Principe du Maximum; Théorème du point fixe; Méthode des lignes; Méthode des volumes finis.

Notations

- $\rightarrow$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- $\hookrightarrow$ indicates the continuous embedding.
- ∇ stands for the gradient operator.
- *div* is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- $\dfrac{\partial}{\partial n}$ outward normal derivative.
- Δ_p is the p-Laplace operator.
- Δ_p^{-1} is the p-Laplace inverse operator.
- Δ^α is the fractional Laplace operator of order α.
- *sp* denotes the spectrum of an operator.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.

- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.

- Ω^c the complement of Ω.

- $< .,. >$ denotes the scalar product.

- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.

- $C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega)$.

- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

- $L^p_{loc}(\Omega)$ the space of local p-integrable functions on Ω.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

- $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $W^{s,p}(\Omega)$ fractional Sobolev space with norm $\|\cdot\|_{s,p}$.

- $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$.

- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W_0^{s,2}(\mathbb{R}^n) = H_0^s(\mathbb{R}^n)$.

  with the norm, that we will denote by $\|.\|_U$

$$\|(u,v)\|^2_{\tilde{U}} = \|u\|^2_{D^{s,2}(\Omega)} + \|v\|^2_{D^{s,2}(\Omega)}$$

.

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Introduction

The phenomenon of thixotropy has recently attracted a great deal of attention. The term was first applied [23] to an "isothermal reversible sol-gel transformation". As the gel state is often merely one of high viscosity, the definition has been made more general, and the term is then applied [25] to any " isothermal reversible decrease of viscosity with increase of rate of shear". Colloidal solutions provide the more common examples of thixotropy and may be divided into three important classes :

- Solutions in Newtonian liquids of lyophilic substances whose molecules are of great length, e.g., gelatine, starch and many synthetic polymers.
- Suspensions of solid particles such as pigments in oils, or clays in water.
- Concentrated emulsions ([60],[61]) of oil droplets in water; foams of gas bubbles in water (with, of course, stabilising agents).

Thixotropic fluids are used widely in civil engineering, food, cosmetic as well as pharmaceutical industries, and impact every aspect of our lives. As emulsions, suspensions, or polymeric gels, they are very different from each other compositionally, but most of them have one thing in common, i.e., the existence of microstructures. The microstructures are changeable and may comprise a network of flocculated colloidal particles, tangles of polymers, or a spatial arrangement of suspended particles or drops[4].

Examples:

Ketchup

mustard

Yagurt

Painting

drilling muds

Gyps paste

A history of thixotropy

Origins

In 1923, Schalek and Szegvari found that aqueous iron oxide "gels have the remarkable property of becoming completely liquid through gentle shaking alone, to such an extent that the liquified gel is hardly distinguishable from the original sol. These sols were liquified by shaking, solidified again after a period of time ... the change of state process could be repeated a number of times without any visible change in the system" [58]. The term thixotropy was then coined by Peterfi in 1927 [50], in the first paper that properly described the phenomenon. The work combines the Greek words thixis (stirring or shaking) and trepo (turning or changing).
Although no mention of the phenomenon appeared in the seminal rheology text of the day 'The Viscosity of Liquids', by Emil Hatschek [30], (especially the chapter on colloidal solutions), by 1935 Freundlich had published a book called 'Thixotropie' [23] devoted to the subject, having been the first to introduce it into the title of a paper when he described the flow properties of aluminium hydroxide gels. Freundlich and co-workers soon found thixotropic effects manifested by a whole variety of systems including vanadium pentoxide sols, starch pastes, gelatin gels, pectin gels and many more.
Thixotropy originally therefore referred to the reversible changes from a flowable fluid to a solid-like elastic gel. Previously these kinds of physical changes had only been known to occur by changing the temperature, when such gels would melt on heating and then re-solidify on cooling. It was believed that a new kind of phase change had been found.

Progress

Early work in this area in the USA is exemplified by a series of three papers by McMillen in 1932 [41], reporting the results of his doctoral investigations into the thixotropy of a large number of flocculated paints. He showed that the fluidity (the inverse of viscosity) as a function of rest time decreased in some cases by four orders of magnitude, showing almost a quadratic dependence on rest time. Writing in the UK in 1942, Scott-Blair [59] stated that 'the whole subject [of thixotropy] is so very new'. But then went on to list over 80 papers on the subject (see pp. 61-64). (In the second edition of this book published in 1949, nearly 120 papers on thixotropy are cited.) Among the examples of thixotropic materials he gives are clays and soil suspensions, creams, drilling muds, flour doughs, flour suspensions, fibre greases, jellies,

paints, carbon black suspensions and starch pastes. He also lists a number of papers on so-called thixotrometers, instruments specially devised to characterise the phenomenon. In this respect he raised some interesting points, among them whether thixotropy ought to be studied at constant rate of shear or at constant stress? This is still a most controversial question.
Scott-Blair quotes Hamaker's explanation of thixotropy as being due to the secondary minimum so that 'particles can form a loose association which is easily destroyed by shaking but re-establishes itself on standing'. This explanation still stands. With our present knowledge of microstructural changes, it is probably safe to say that all materials that are shear thinning are thixotropic, in that they will always take a finite time to bring about the rearrangements needed in the microstructural elements that result in shear thinning. As Scott-Blair concluded all those years ago "If this recovery is very rapid, the phenomenon is observed as structural viscosity [shear thinning]; if slow, it is observed as thixotropy". However even Scott-Blair sometimes confused thixotropy with shear thinning, as in his example of the importance of thixotropy for drilling muds that must be runny [sic] when lubricating the drill, but "of a high enough consistency at rest to avoid settling of suspended matter".
An important point he made concerned a suggestion that certain results of flow in capillary tubes of suspensions–that we now believe showed migration of particles away from the wall and thus, have an easier flow in small rather than large tubes–was due to thixotropy. He refuted this by showing that doubling the tube length halved the flow rate for a given driving pressure.
Pryce-Jones [52] (the first well-known Welsh rheologist) studied about 250 paints all in a state of light flocculation, using his own thixotrometer [53]. He noted that "It is a well-established fact that thixotropy is more pronounced in systems containing non-spherical particles", this is obviously so because they have to find themselves in the best 3D structure by rotation as well as movement, and progress from a solid gel to a freely flowing liquid due to complete microstructural breakdown, see Fig. 1.

Thixotropy is one of the few original technical terms used in pre-war, European rheology circles that has survived, unlike 'structural viscosity' (Strukturviskositaet, which we now understand as shear thinning) and 'false body' (now understood as extreme shear thinning with thixotropy) which have fallen by the wayside .
However, as late as 1953, Roscoe [57] still referred to 'false body' as different from thixotropy. The 'false body' had an apparent yield stress [stress at low shear rate following shearing at a high shear rate] that recovered quickly, while the thixotropic material takes some times before relatively fast recovery takes place. Today we understand that they are both manifestations of thixotropy. False bodies were taking a long time to die.
Jobling and Roberts in 1957 [34] commented that "thixotropy now has an even less distinct connotation. Electronic methods of measurement have shown that the time-lag required before the original structure is regained may be very short indeed and it then becomes difficult to distinguish between a thixotropic material with a very short recovery time and a material whose viscosity falls with increasing rate of shear and depends for all practical purposes only on the instantaneous rate of shear. The latter effect is frequently called 'structural viscosity'". They went on to say "We endorse Pryce-Jones's plea that in the absence of authoritative definitions, terms such as ... thixotropy should not be used unless the intended meaning is made clear". In the Discussion section of this paper, Marcus Reiner notes that 'structural viscosity' and

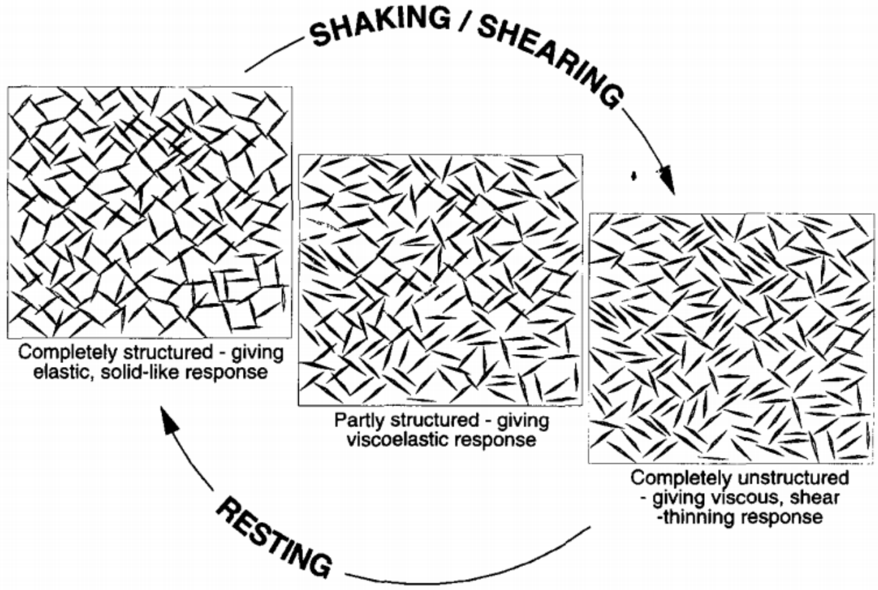

Fig. 1. Breakdown of a 3D thixotropic structure.

'thixotropy' are seen as the same thing by some, with structural viscosity seen as a material with "nearly zero time of recovery".

The full extent of thixotropy was maintained by Bauer and Collins in their 1967 review [5]: "When a reduction in magnitude of rheological properties of a system, such as elastic modulus, yield stress, and viscosity, for example, occurs reversibly and isothermally with a distinct time dependence on application of shear strain, the system is described as thixotropic". They went on to say that thixotropy was "usually conceived as an unusual property of very special materials, sol-gel systems such as aqueous iron oxide dispersions, thixotropy in the sense described above has been found to be exhibited by a great many and a large variety of systems. Along with the breakdown in structure, other non-rheological features change, such as conductivity and dielectric constant". Lastly they noted that "The terms used by Freundlich are now seen to be archaic, viz liquefaction, re-solidification, sol. These had some obvious meaning for the qualitative changes brought about in low concentration dispersions of highly insoluble oxides of needle-like crystals such as iron oxide and vanadium pentoxide in low-viscosity aqueous media". Nowadays thixotropy is sometimes used to include all time effects in a movement to non-linear behaviour, see for instance Cheng [15], but especially Lapasin and Pricl [38], who illustrate thixotropic behaviour by the transient response of viscosity and normal force of polymer solutions. They then noted that the stress overshoot on start-up increases with increasing rest time. This is an interesting point–build up in polymer solutions is usually considered to be rapid, and rest times are rarely considered necessary. Weakly cross-linked gels would give the same thixotropic effect as a flocculated system.

In this work, the general form of the thixotropic problem is given by

$$\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x \in \Omega \end{cases} \tag{1}$$

Where u is a function denotes the speed of fluid, $\lambda > 0$ is the viscosity of the fluid, v denotes the relative temperature of the fluid.

Research objectives

In this research work, we are interested in studying the following problems which are formulated by two subsystems, the first is the thixotropic problem coupled with the heat equation at the second subsystem:

$$(P)\begin{cases} (P_1)\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x \in \Omega \end{cases} \\ (P_2)\begin{cases} -\Delta v + \tau v = 0 & x \in \Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

and

$$(FP)\begin{cases} (P_1')\begin{cases} u_t + D^\alpha u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x \in \Omega \end{cases} \\ (P_2)\begin{cases} -\Delta v + \tau v = 0 & x \in \Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

Where Ω is bounded open convex domain in $\mathbb{R}^d$, ($d \leqslant 3$ for the problem (P) and $d = 1$ for the problem (FP)), with smooth boundary $\partial\Omega$, $u_0 \in H^1(\Omega), g \in H^{1/2}(\partial\Omega)$ and τ is a positive constant. Here we define $D^\alpha u(x) = F^{-1}(F(D^\alpha u))(x) = F^{-1}(|\xi|^\alpha Fu(\xi))(x)$, where F denotes the Fourier Transform and F^{-1} its inverse.
The work aims to archieve the following objectives:

1. To study the existence and uniqueness of the solutions for the problems (P) and (FP) via Lax Milgram theorem and Galerkin method.

2. To develop numerical solutions of the problems (P) and (FP) using FVM .

3. To solve numerically the problems (P) and (FP) using the FVM and investigate its convergence and stability.

Thesis outline

This thesis is composed of two parts. The first one focus on the existence and uniqueness of solutions for thixotropic and fractional derivative thixotropic models using the Galerkin method, Lax Milgram theorem and Maximum Principle while the second one focus on the study of these numerical problems using some numerical methods such as finite volume method.
The followings are a brief description of each chapter:

chapter1

In this chapter, we discuss about some definitions and important results in the $L^p(\Omega)$ spaces, Sobolev spaces and different theorems that has an essential role in the subsequent chapters. We also identify the numerical methods which is using to appraoch the solution of the problems.

chapter2

In this chapter, we study the elliptic-parabolic problem (P). The purpose of this work is to prove the uniqueness, existence and positivity of solutions for problem (P) on bounded convex domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ using Lax-Milgran 's Theorem, Maximum principle and Galerkin method. This study is the subject of a publication under the title :"**Global Existence and Uniqueness of the Weak Solution in Thixotropic Model** ". In: International Journal of Analysis and Applications, vol 19 (2021), 193-204.

chapter3

In this chapter, We study the existence and uniqueness of solutions for a following fractional Thixotropic problem (FP). The standard technique is used to employ a proof based on the Galerkin method, energy estimates and fixed point theorem. (**this latter is a subject of an article submitted**).

chapter4

In this chapter, we introduce an explicit finite volume method for the numerical solution of the one- dimensional space and an explicit finite volume method for the numerical solution of the multi- dimensional space for the Thixotropic system (P). And, We end with some numerical simulations, carried out under the Matlab software. (**this latter is a subject of an article submitted**).

chapter5

In this chapter, we present an explicit finite volume method for the numerical solution of the one- dimensional space for the fractional Thixotropic system (FP). Our method is based on the method of lines to get the finite volume scheme to this problem. Finally, We finish with some numerical simulations, carried out under the Matlab software. (**this latter is a subject of an article submitted**).

Chapter 1

Preliminaries

1.1 The Spaces of Continous Function

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$, we note*

$$C(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ continuous } \},$$

and

$$C^m(\Omega) : \text{ The space of functions } m \text{ times continuously differentiable on } \Omega,$$

where

$$C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega),$$

$$C_c(\Omega) = \{f \in C(\Omega); f(x) = 0 \quad \forall x \in \Omega \backslash K, \text{ where } K \text{ is compact}\},$$

and $D(\Omega)$ the space of functions C^∞ on Ω with compact support in Ω (also called the space of test functions).

1.2 The L^p Spaces

Definition 1.2.1. *[12] Let $p \in \mathbb{R}$ with $1 \leqslant p < \infty$ and Ω be an open subset in $\mathbb{R}^n$. We set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \,\mathrm{d}x < \infty \right\},$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

We shall check later on that $\| \, . \, \|_p$ is a the norm.

Definition 1.2.2. *[12] If $p = \infty$, we define*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leqslant C \text{ a.e. on } \Omega \right\},$$

with the norm

$$\|f\|_\infty = Inf\Big\{C; |f(x)| \leqslant C \;\; a.e. \;\; on \; \Omega\Big\}.$$

Proposition 1.2.1. *1. For $1 \leqslant p \leqslant \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space.*

2. For $1 \leqslant p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a separable space.

3. For $1 < p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.

1.3 The Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $\partial\Omega$ denote its boundary.

Definition 1.3.1. *[37] Let m be a positive integer and let $1 \leqslant p \leqslant \infty$. The Sobolev Space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \;\; for \; all \; |\alpha| \leqslant m\}.$$

The space $W^{m,p}(\Omega)$ is a vector space contained in $L^p(\Omega)$ and we endow it with the norm $\| \,.\, \|_{m,p,\Omega}$ defined as follows.

$$\| \; u \; \|_{m,p,\Omega} = \left(\sum_{|\alpha| \leqslant m} \| \; D^\alpha u \; \|^p_{L^p(\Omega)} \right)^{1\backslash p}$$

if $1 \leqslant p \leqslant \infty$ and

$$\| \; u \; \|_{m,\infty,\Omega} = max_{|\alpha| \leqslant m} \| \; D^\alpha u \; \|_{L^\infty(\Omega)} \; .$$

Notation:[**37**]

- When $p = 2$, we will write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The corresponding norm $\| \,.\, \|_{m,2,\Omega}$ will be written as $\| \,.\, \|_{m,\Omega}$ and it is generated by the inner-product

$$(u, v)_{m,\Omega} = \sum_{|\alpha| \leqslant m} \int_\Omega D^\alpha u D^\alpha v dx.$$

- We define a semi-norm om $W^{m,p}(\Omega)$ by

$$| \; u \; |_{m,p,\Omega} = \left(\sum_{|\alpha| = m} \| \; D^\alpha u \; \|^p_{L^p(\Omega)} \right)^{1\backslash p}$$

  When $1 \leqslant p \leqslant \infty$ and

$$| \; u \; |_{m,\infty,\Omega} = max_{|\alpha| = m} \| \; D^\alpha u \; \|_{L^\infty(\Omega)} \; .$$

Proposition 1.3.1. *Let Ω be an open subset of $\mathbb{R}^n$;*

1. *For* $1 \leqslant p \leqslant \infty$, $W^{m,p}(\Omega)$ *is a Banach space.*
2. *For* $1 \leqslant p < \infty$, $W^{m,p}(\Omega)$ *is a separable space.*
3. *For* $1 < p < \infty$, $W^{m,p}(\Omega)$ *is a reflexive space.*

Theorem 1. *The space* $W^{1,p}(\Omega)$ *is complete. It is reflexive if* $1 < p < \infty$ *and separable if* $1 \leqslant p \leqslant \infty$. *In particular,* $H^1(\Omega)$ *is a separable Hilbert space*

1.3.1 Some useful inequalities

Theorem 2 (**Young's inequality**[12]). *Let* $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant q \leqslant \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *, then*

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{q}b^q, \quad \forall a \geqslant 0, b \geqslant 0.$$

Lemma 1.3.1 (**Poincaré inequality**). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Then there is a positive constant* C_Ω *such that*

$$\|f\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega).$$

Theorem 3 (**Gagliardo-Nirenberg's inequality**). *Let* $1 \leqslant p < n$. *There exists* $C = c(n,p)$ *such that*

$$\left(\int_{\mathbb{R}^n} | u |^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant C \left(\int_{\mathbb{R}^n} | Du |^p dx\right)^{\frac{1}{p}}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$, *where* $p^* = \dfrac{np}{n-p}$ *is called the sobolev conjugate of* p.

1.4 Gronwall's Lemma

Lemma 1.4.1. *[40] Let* $T > 0$ *and* $c \geqslant 0$. *Let* $\varphi : v : [0,T] \to \mathbb{R}$ *be continuous and nonnegative functions. If*

$$\varphi(t) \leqslant c + \int_0^t v(\sigma)\varphi(\sigma)d\sigma, \text{ for all } t \in [0,T],$$

then

$$\varphi(t) \leqslant c \; exp\left(\int_0^t v(\sigma)d\sigma\right), \text{ for all } t \in [0,T].$$

1.5 Trace theorem

The trace operator can be defined for functions in the Sobolev spaces $W^{1,p}(\Omega)$ with $1 \leqslant p < \infty$

Theorem 4. *Let* $\Omega \subset \mathbb{R}^n$ *for* $n \in \mathbb{N}$ *be a bounded domain with Lipschitz boundary. Then there exists a bounded linear trace operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega).$$

such that

i) $Tu = u\backslash_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,

ii) $\| Tu \|_{L^p(\partial\Omega)} \leqslant C \| u \|_{W^{1,p}(\Omega)}$,

for each $u \in W^{1,p}(\Omega)$, *with the constant* C *depending only on* p *and* Ω.

Definition 1.5.1. *we call* Tu *the trace of* u *on* $\partial\Omega$.

1.6 Fixed point theorem

Theorem 5. *[12][**Banach fixed point theorem-the contraction mapping priciple**] Let* X *be a nonempty complete metric space and let* $S : X \longrightarrow X$ *be a strict contraction, i.e,*

$$d(Sv_1, Sv_2) \leqslant kd(v_1, v_2), \forall v_1, v_2 \in X$$

with $k < 1$.
Then S *has a unique fixed point,* $u = Su$.

1.7 Lax Milgram theorem

Definition 1.7.1. *[12] A bilinear form* $a : H \times H \longrightarrow \mathbb{R}$ *is said to be*

i. *Continuous if there is a constant* C *such that*

$$| a(u, v) \leqslant C | u || v |, \forall u, v \in H$$

ii. *Coersive if there is a constant* α *such that*

$$| a(v, v) \leqslant \alpha | v |^2, \forall v \in H$$

Theorem 6 (**Lax-Milgram**)**.** *Let* L *be a continuous linear form on Hilbert space* H *and* ***a*** *is a continuous and coercive bilinear form, then there is one and only one function* $u \in H$ *such that:*

$$a(u, v) = L(v), \quad \forall v \in H.$$

Moreover, if the bilinear form a *is symmetric, then* u *is the only element of* H *which minimizes the functional* $J : H \to \mathbb{R}$ *defined by*

$$J(v) = \frac{1}{2}a(v, v) - L(v), \quad \forall v \in H,$$

i.e.

$$J(u) = \min_{v \in H} J(v) \text{ and } J(u) < J(v) \text{ if } u \neq v.$$

1.8 Ascoli theorem

Next we examine what the concept of relative compactness means in the space $C_b(X)$. The following notions will be essential for this purpose.

Definition 1.8.1. *Let (X, d) be a metric space and $x_0 \in X$. Let F be a set of functions $f : X \to \mathbb{R}$. We say that F is* ***equicontinuous****[12] at x_0 if*

$$\forall \varepsilon > 0, \exists \varrho > 0 \text{ s.t. } \forall x \in X, \forall f \in F,$$

$$d(x, x_0) < \varrho \Rightarrow | f(x) - f(x_0) | < \varepsilon.$$

We say that F is ***equicontinuous*** *if it is equicontinuous at every point of X.*
This definition is quite analogous to continuity, but now ϱ must not depend on the choice of f in F. For comparison, the statement 'all functions in F are continuous at x_0 means that

$$\forall f \in F, \forall \varepsilon > 0, \exists \varrho > 0 \text{ s.t. } \forall x \in X,$$

$$d(x, x_0) < \varrho \Rightarrow | f(x) - f(x_0) | < \varepsilon.$$

Theorem 7 (**Arzelà-Ascoli**). *Let (X, d) be a compact metric space. A set of F of functions in $C_b(X)$ is relatively compact if and only if it is bounded and equicontinuous.*

1.9 Method of Galerkin

In mathematics, in the area of numerical analysis, Galerkin methods, named after the Russian mathematician Boris Galerkin, convert a continuous operator problem, such as a differential equation, commonly in a weak formulation, to a discrete problem by applying linear constraints determined by finite sets of basis functions.
For example, the wavelet-Galerkin method [35] has emerged as an accurate and efficient means of approximating the solution of partial differential equations (PDE's) [16]. Wavelets are well localized, oscillatory functions which provide a basis of $L^2(\mathbb{R})$ and can be modified to a basis of $L^2([a, b])$ where [a,b] is a bounded domain [39]. These localized characteristics of discrete, orthogonal wavelets allow sparse representation of piecewise signals, including transients and singularities, making them useful functions for use in the Galerkin approach when non-smooth or non-periodic solutions are predicted [63]. Discrete, orthogonal wavelets have been used by a number of investigators in a Galerkin approach to solving differential equations. Williams and Amaratunga provide a review of orthogonal wavelet use in engineering [66] and specifically to solutions of linear boundary value problems [2]. Beylkin and Keiser [7] attempt to efficiently capture shock-like responses in nonlinear equations described by the semigroup approach. Restrepo and Leaf [55] look specifically at periodic solutions using orthogonal wavelets, and Pernot and Lamarque [49] investigate transient vibrations and stability analysis. Beylkin [6], Chen et al. [14], and Romine and Peyton [56] all investigate the computation of inner products and other operators of orthogonal wavelets on bounded domains; the exact solution to these operators was paramount in the development of the discrete, orthogonal wavelet-Galerkin method.

1.10 Finite volume method[22]

The finite volume method is a discretization method which is well suited for the numerical simulation of various types (elliptic, parabolic or hyperbolic, for instance) of conservation laws; it has been extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer or petroleum engineering. Some of the important features of the finite volume method are similar to those of the finite element method, see Oden [48]: it may be used on arbitrary geometries, using structured or unstructured meshes, and it leads to robust schemes. An additional feature is the local conservativity of the numerical fluxes, that is the numerical flux is conserved from one discretization cell to its neighbour. This last feature makes the finite volume method quite attractive when modelling problems for which the flux is of importance, such as in fluid mechanics, semi-conductor device simulation, heat and mass transfer. . . The finite volume method is locally conservative because it is based on a " balance" approach: a local balance is written on each discretization cell which is often called "control volume"; by the divergence formula, an integral formulation of the fluxes over the boundary of the control volume is then obtained. The fluxes on the boundary are discretized with respect to the discrete unknowns.

1.11 Method of lines (MOL)[29]

The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDE with algebric approximation with only one remaing independent variable, we have a system of ODEs that approximate the oroginal PDE. The challenge, then, is to formulate the approximation system of ODEs. Once this is done, we can apply any integration algorithm for initial value ODEs to compute an approximate numerical solution to the PDE. thus, one of the salient feature of the MOL is the use of existing, and genersally well established, numerical methods for ODEs.

Chapter 2

Existence and Uniqueness of weak solution for Thixotropic problem

In this chapter, we study global existence, uniqueness and boundedness of the weak solution for the elliptic-parabolic system (P) which is formulated by two subsystems (P_1) and (P_2). Our model is defined as follows:

$$(P)\begin{cases} (P_1)\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x)\in\mathbb{R}^+\times\Omega \\ u = 0 & x\in\partial\Omega \\ u(0,x) = u_0 & x\in\Omega \end{cases} \\ \qquad (P_2)\begin{cases} -\Delta v + \tau v = 0 & x\in\Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

Where $u(t,x)$ is a function denotes the speed of fluid in the position $x \in \Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, Ω is a bounded convex domain with smooth boundary $\partial\Omega \in H^{\frac{3}{2}}(\partial\Omega)$, $\lambda > 0$ is the viscosity of the fluid, $\beta > 0$ is a parameter constant, v denotes the relative temperature of the fluid. The parameter τ is a time constant and it is expressed the thermal diffusivity of v. g is the initial relative temperature distribution.

To simplify the solution of the system (P), a decomposition of (P) into two subsystem (P_1) and (P_2) are adopted. Galerkin's method is very important to help us to demonstrate the existence and uniqueness of a weak solution for system (P_1). To prove the existence and uniqueness of a weak solution for system (P_2), we use Lax-Milgram's theorem and maximum principle. However this theorem can not be applied directly because it is nonhomogenous system. For this reason an adoptation of Trace theorem it used to simplify the system(P_2). Therefore we have the existence and uniqueness of a weak solution for system (P). Moreover we show that the solution is positive.

The following initial-boundary conditions on u_0 and g assumptions are used to prove the proposed solution of (P)

- H_1: $g \in L^{\frac{1}{2}}(\partial\Omega)$.
- H_2: $g \in L^{\frac{3}{2}}(\partial\Omega)$.

- $H_3:\ u_0 \in L^2(\Omega)$.
- $H_4:\ u_0 \geqslant 0$ and $g \geqslant 0$.

If the hypothesis H_1 is satisfies and using the theorem of trace, one can find a lifting of this trace which we denote $R(g) \in H_0^1(\Omega)$. Thus by definition it verefies $\gamma_0(R(g)) = g$. Now we looking for v having the form $v = \widetilde{v} + R(g)$ reduves the problem (P_2) to $\widetilde{v}$.

$$\left(\widetilde{P_2}\right)\begin{cases} -\Delta\widetilde{v} + \tau\widetilde{v} - \Delta R(g) + \tau R(g) = 0 & x \in \Omega \\ \widetilde{v} = 0 & \text{on } \partial\Omega \end{cases}$$

Definition 2.0.1. *We say $(u,\widetilde{v}) \in L^2(0,T,H_0^1(\Omega)) \times H_0^1(\Omega)$ with $u_t \in L^2(0,T,H^{-1}(\Omega))$ is a weak solution of the problem (P) if and only if*

$$\langle u_t, w\rangle + B(u,w,t) = (u_0, w) \tag{2.1}$$

$$a(\widetilde{v},q,t) = l(q) \tag{2.2}$$

where

$$\begin{cases} B(u,w,t) = -\int_\Omega \nabla u\nabla w dx + \int_\Omega (\delta\tau v + 1)\, uw dx - \delta\int_\Omega u\nabla u_0\nabla w dx \\ a(\widetilde{v},q,t) = \int_\Omega (\nabla\widetilde{v}\nabla q + \tau\widetilde{v}q)\, dx \\ l(q) = -\int_\Omega (\nabla R(g)\nabla q + \tau R(g)\, q)\, dx \end{cases}$$

for all $(w,q) \in (H_0^1(\Omega))^2$, $0 \leqslant t \leqslant T$,

$$u(0,x) = u_0 \in L^2(\Omega) \tag{2.3}$$

and

$$\delta = \frac{\lambda}{\sqrt{\beta + |\nabla(v-u_0)|^2}}. \tag{2.4}$$

Remark 1. *Note that $u \in C([0,T], L^2(\Omega))$ as $u \in L^2(0,T,H_0^1(\Omega))$ and $u_t \in L^2(0,T,H^{-1}(\Omega))$. Then equation 2.3 makes sense.*

2.1 Existence of weak solution of the problem (P)

In this section, to demonstrate the existence and uniqueness of weak solution for the system (P), firstly, we have to demonstrate the existence and uniqueness of weak solution for the system (P_2), which its variational formulat is given by equation 2.2 using the Lax-Milgram Theorem. Then, we will prove the same result for the system (P_1), which its variational formulat is given by equation 2.1 using the Galerkin's method.

2.1.1 Existence of weak solution of the problem (P_2)

Existence and uniqueness result:

Theorem 8. *(Existence and uniqueness of weak solution) If the hypothesis H_1 holds. Then the problem (P_2) has only one solution $v \in H^1(\Omega)$ for any $q \in H^1(\Omega)$.*

By applying the theorem of Lax-Milgram, the solution $\widetilde{v}$ of the problem 2.2 exists and it is unique. So (P_2) has unique solution.

Remark 2. *Elliptic regularity theorem remains valid provided that the boundary condition g is in the space $L^{\frac{3}{2}}(\partial\Omega)$ which is the image $H^2(\Omega)$ by the operator trace γ because $\gamma : H^2(\Omega) \hookrightarrow L^{\frac{3}{2}}(\Omega)$.*

Remark 3. *[20] If $v \in H^2(\Omega)$ and (v is a solution of problem (P_2)) this implies that $v \in W^{1,q}(\Omega)$ ($H^2(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for $1 \leqslant q \leqslant 2$).*

Positivity of a weak solution: Using the Maximum Principle one can show that the solution of the problem (P_2) is positive as follows. Multiplying the first equation of (P_2) by $q \in H_0^1(\Omega)$, we obtain other variational formulat for problem (P_2)

$$\left(\widetilde{P_3}\right) \int_\Omega (\nabla v \nabla q + \tau v q)\, dx = 0.$$

Proposition 2.1.1. *[1] If $g \in L^{\frac{3}{2}}(\partial\Omega)$ and $v \in H^1(\Omega) \cap C\left(\overline{\Omega}\right)$ then the problem $\left(\widetilde{P_3}\right)$ have a positive solution v.*

Proof. As $\partial\Omega$ is smooth enough and $g \in L^{\frac{3}{2}}(\partial\Omega)$ then $v \in H^2(\Omega)$. And as $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, by embedding of Sobolev spaces ($H^2\left(\overline{\Omega}\right) \hookrightarrow C\left(\overline{\Omega}\right)$) this implies that $v \in C\left(\overline{\Omega}\right)$. If $v = g \geqslant 0$ on $\partial\Omega$, then $v^- = \min(v, 0) \in H_0^1(\Omega)$. So, we have

$$\int_\Omega v v^- dx = \int_\Omega \left(v^-\right)^2 dx$$

$$\int_\Omega \nabla v \nabla v^- dx = \int_\Omega \left(\nabla v^-\right)^2 dx,$$

Since the support of functions v^- and $v^+ = \max(v, 0)$ is set $A(x) = \{x/u(x) = 0\}$.
This implies that $\nabla u = 0$ on $A(x)$. As $v = v^+ + v^-$, thus we have

$$0 = \int \left(\left(\nabla v^-\right)^2 + \tau \left(v^-\right)^2\right) dx \geqslant \min(1, \tau) \left\|v^-\right\|^2_{H_0^1(\Omega)}$$

Finally, we find $v^- = 0$. □

2.1.2 Existence of weak solution of the problem (P_1)

Before proving the existence and uniqueness of a weak solution of the problem (P_1), we need the following Lemma:

Lemma 2.1.1. *i) For all $w \in H_0^1(\Omega)$ then $B(.,.,t)$ is continuous in $H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a constant positive M such that*

$$|B(u,w,t)| \leqslant M \|u\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \tag{2.5}$$

ii) For any $u \in H_0^1(\Omega)$ and H_2 is hold. Then exists a constant positive α such that

$$B(u,u,t) \geqslant \alpha \|u\|^2_{H_0^1(\Omega)}. \tag{2.6}$$

Proof. i) We use the Cauchy-Shwartz inequality and $v \in H^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in \left[1, \frac{2n}{n-2}\right[$ with $n = 2$ or $n = 3$, we obtain i) as follows

$$\begin{aligned} |B(u,w,t)| &\leqslant \|\nabla u\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} + \left[|\delta\tau| \|v\|_{L^2(\Omega)} + 1\right] \|u\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\ &+ |\delta| \|u\|_{L^2(\Omega)} \|u_0\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\leqslant M \|u\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}. \end{aligned}$$

ii) Making use of $-\Delta v + \tau v = 0$ the expression of $B(u,u,t)$ becomes

$$\begin{aligned} B(u,u,t) &= -\int (\nabla u)^2 dx + \int (\delta\tau v + 1) u^2 dx - \delta \int u \nabla u \nabla u_0 dx \\ &= -\int (\nabla u)^2 dx + \int (\delta\tau v + 1) u^2 dx - \tfrac{\delta}{2} \int (\nabla u)^2 \nabla u_0 dx \\ &= \int \left(-1 - \tfrac{\delta}{2} \nabla u_0\right) (\nabla u)^2 dx + \int (\delta\tau v + 1) u^2 dx \\ &\geqslant \|\nabla u\|^2_{L^2(\Omega)}. \end{aligned}$$

Finally, by Poincarre inequality yields,

$$B(u,u,t) \geqslant \alpha \|u\|^2_{H_0^1(\Omega)}$$

□

Galerkin approximations:

To demonstrate the existence of weak solution of the problem (P_1) via the method of Galerkin, we assume $e_k = e_k(x)$ are smooth function verifying

$$\{e_k\}_{k=1}^{\infty} \quad \textit{is an arthogonal basis of} \quad H_0^1(\Omega) \tag{2.7}$$

and

$$\{e_k\}_{k=1}^{\infty} \quad \textit{is an arthonormal basis of} \quad L^2(\Omega). \tag{2.8}$$

Consider a positive integer m. We will look for a function $u_m\colon [0,T] \to H_0^1(\Omega)$ of the form

$$u_m: = \sum_{k=1}^{m} d_m^k(t) e_k \tag{2.9}$$

which satisfies

$$d_m^k(0) = (u_0, e_k) \tag{2.10}$$

and

$$\left\langle u_m^{'}, e_k \right\rangle + B(u_m, e_k, t) = (u_0, e_k), \quad 0 \leqslant t \leqslant T \quad and \quad k = 1, ..., m \tag{2.11}$$

where $u^{'} = u_t$ and here $(.,.)$ denotes the scalar product in $L^2(\Omega)$.

Theorem 9. *(construction of the approximate solution) For each integer m, there exists a unique function u_m of the form equation 2.9 satisfying equation 2.10 and equation 2.11 .*

Proof. Assuming u_m has the structure equation 2.9. Substituting equation 2.9 into equation 2.11 and using equation 2.8 we obtained

$$d_m^{\prime k}(t) + \sum_{l=1}^{m} d_m^l B(e_l, e_k, t) = d_m^k(0), \quad 0 \leqslant t \leqslant T \quad and \quad k = 1, ..., m \tag{2.12}$$

According to standard existence theory for ordinary differential equations, there exists a unique absolutely continuous functions $d_m(t) = (d_m^1, d_m^2, ..., d_m^m,)$ satisfying equation 2.10 and equation 2.12. So u_m of the form equation 2.9 satisfies equation 2.10 and equation 2.11 for all $t \in [0, T]$. □

Energy estimates:

We propose now to send m to infinity and show a subsequence of our solutions u_m of the approximation problems equation 2.10 and equation 2.11 converges to a weak solution of (P_1). For this we will need some uniform estimates.

Theorem 10. *(Energy estimates) [20]. There exists a constant C, depending only on Ω, T and w, such that*

$$\max_{0\leqslant t\leqslant T} \|u_m\|_{L^2(\Omega)} + \|u_m\|_{L^2\left(0,T,H_0^1(\Omega)\right)} + \|u_m'\|_{L^2(0,T,H^{-1}(\Omega))} \leqslant C\|u_0\|_{L^2(\Omega)} \quad for \quad m = 1, 2, ... \tag{2.13}$$

Proof. **Step1:** Multiplying equation 2.11 by $d_m^k(t)$, summing for $k = 1, ..., m$, and then recalling equation 2.9 we find

$$(u_m', u_m) + B(u_m, u_m, t) = (u_0, u_m) \tag{2.14}$$

for all $0 \leqslant t \leqslant T$. From Lemme 2.1.1, there exists constant $\alpha > 0$ such that

$$\alpha \|u_m\|_{H_0^1(\Omega)}^2 \leqslant B(u_m, u_m, t) \tag{2.15}$$

for all $0 \leqslant t \leqslant T$, $m = 1, ...$ Furthermore $|(u_0, u_m)| \leqslant \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_m\|_{L^2(\Omega)}^2$, and $(u_m', u_m) = \frac{d}{dt}\left(\|u_m\|_{L^2(\Omega)}^2\right)$ for a.e. $0 \leqslant t \leqslant T$. Consequently equation 2.14 yields the inequality

$$\frac{d}{dt}\left(\|u_m\|^2_{L^2(\Omega)}\right) + 2\alpha \|u_m\|^2_{H^1_0(\Omega)} \leqslant C_1 \|u_m\|^2_{L^2(\Omega)} + C_2 \|u_0\|^2_{L^2(\Omega)} \tag{2.16}$$

for all $0 \leqslant t \leqslant T$ and appropriate constants C_1 and C_2.

Step2: Now write

$$\varphi(t) := \|u_m\|^2_{L^2(\Omega)} \tag{2.17}$$

and

$$\zeta(t) := \|u_0\|^2_{L^2(\Omega)}. \tag{2.18}$$

Then equation 2.16 implies

$$\varphi'(t) \leqslant C_1\varphi(t) + C_2\zeta(t) \tag{2.19}$$

for a.e. $0 \leqslant t \leqslant T$. Thus the differential form of Gronwall's inequality yields the estimate

$$\varphi(t) \leqslant e^{C_1 t}\left(\varphi(0) + C_2\int_0^T \zeta(s)\,ds\right) (0 \leqslant t \leqslant T). \tag{2.20}$$

Since $\varphi(0) = \|u_m(0)\|^2_{L^2(\Omega)} \leqslant \|u_0\|^2_{L^2(\Omega)}$ by equation 2.10, we obtain from equations 2.17 - 2.20 the estimate

$$\max_{0\leqslant t\leqslant T} \|u_m\|_{L^2(\Omega)} \leqslant C \|u_0\|_{L^2(\Omega)}. \tag{2.21}$$

Step3: Integrate inequality equation 2.16 from 0 to T and we employ the inequality equation 2.21 to find

$$\|u_m\|^2_{L^2\left(0,T,H^1_0(\Omega)\right)} = \int_0^T \|u_m\|^2_{H^1_0(\Omega)}\,dt \leqslant C \|u_0\|^2_{L^2(\Omega)}.$$

Step4: Fix any $w \in H^1_0(\Omega)$, with $\|w\|^2_{H^1_0(\Omega)} \leqslant 1$, and write $w = w^1 + w^2$, where $w^1 \in span\,(e_k)_{k=1}^{k=m}$, and $(w^2, e_k) = 0\,(k = 1, ..., m)$. We use equation 2.11, we deduce for all $0 \leqslant t \leqslant T$ that

$$\left(u'_m, w^1\right) + B\left(u_m, w^1, t\right) = \left(u_0, w^1\right)$$

then equation 2.9 implies

$$\langle u'_m, w\rangle = (u'_m, w) = \left(u'_m, w^1\right) = \left(u_0, w^1\right) - B\left(u_m, w^1, t\right),$$

consequently

$$|\langle u'_m, w\rangle| \leqslant C\left(\|u_0\|^2_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}\right).$$

Simce $\|w^1\|^2_{H^1_0(\Omega)} \leqslant \|w\|^2_{H^1_0(\Omega)} \leqslant 1$. Thus

$$\|u'_m\|_{H^{-1}(\Omega)} \leqslant C\left(\|u_0\|^2_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}\right),$$

and therefore

$$\|u'_m\|^2_{L^2(0,T,H^{-1}(\Omega))} = \int_0^T \|u'_m\|^2_{H^{-1}(\Omega)}\,dt \leqslant C\int_0^T \left(\|u_0\|^2_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}\right)dt \leqslant C\,\|u_0\|_{L^2(\Omega)}.$$

□

Existence and uniqueness:

Next we pass to limit as $m \to \infty$, to build a weak solution of our initial boundary-value problem (P_1).

Theorem 11. *(Existence of weak solution). Under hypothesis H_2 and H_3, there exists a weak solution of (P_1).*

Proof.

Step1: According to the energy estimates equation 2.13, we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T,H^1_0(\Omega))$ and $\{u'_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T,H^{-1}(\Omega))$. Consequently there exists a subsequence which is also noted by $\{u_m\}_{m=1}^{\infty}$ and a function $u \in L^2(0,T,H^1_0(\Omega))$, with $u' \in L^2(0,T,H^{-1}(\Omega))$, such that

$$\begin{array}{llll} u_m \rightharpoonup u & weakly & in & L^2(0,T,H^1_0(\Omega)) \\ u'_m \rightharpoonup u' & weakly & in & L^2(0,T,H^{-1}(\Omega)). \end{array} \tag{2.22}$$

Step2: Next fix an integer N and choose a function $h \in C^1(0,T,H^1_0(\Omega))$ having the form

$$h(t) = \sum_{k=1}^{N} d^k(t)\,e_k \tag{2.23}$$

where $\{d^k\}_{k=1}^{N}$ are given smooth functions. We choose $m \geqslant N$, multiply equation 2.11 by $d^k(t)$, sum for $k = 1, ..., N$, and then integrate with respect to t to find

$$\int_0^T \langle u'_m, h\rangle + B(u_m, h, t)\,dt = \int_0^T (u_0, h)\,dt. \tag{2.24}$$

we recall equation 2.22 to find upon passing to weak limits that

$$\textstyle\int_0^T \langle u', h\rangle + B(u, h, t)\,dt = \int_0^T (u_0, h)\,dt \quad , \forall h \in L^2\left(0, T, H_0^1(\Omega)\right). \tag{2.25}$$

As functions of the from equation 2.23 are dense in $L^2\left(0, T, H_0^1(\Omega)\right)$. Hence in particular

$$\langle u', h\rangle + B(u, h, t) = (u_0, h) \quad , \forall h \in H_0^1(\Omega) \quad \text{and } \forall t \in [0, T], \tag{2.26}$$

and from remark 1 we have $u \in C\left(0, T, L^2(\Omega)\right)$.

Step3: In order to prove $u(0) = u_0$, we first note from equation 2.25 that

$$\int_0^T -\langle u, h'\rangle + B(u, h, t)\,dt = \int_0^T (u_0, h)\,dt + (u(0), h(0)) \tag{2.27}$$

for each $h \in C^1\left(0, T, H_0^1(\Omega)\right)$ with $h(t) = 0$. Similary, from equation 2.24 we deduce

$$\int_0^T -\langle u_m, h'\rangle + B(u_m, h, t)\,dt = \int_0^T (u_0, h)\,dt + (u_m(0), h(0)). \tag{2.28}$$

we use again equation 2.22, we obtain

$$\int_0^T -\langle u, h'\rangle + B(u, h, t)\,dt = \int_0^T (u_0, h)\,dt + (u_0, h(0)), \tag{2.29}$$

since $u_m(0) \to u_0$ in $L^2(\Omega)$. Comparing equation 2.27 and equation 2.29, we conclude $u(0) = u_0$.

□

Theorem 12. *(Uniqueness of a weak solutions) A weak solution of (P_1) is unique.*

Proof. We suppose there exists two weak solutions u_1 and u_2. We put

$$U = u_2 - u_1$$

then U is also a solution of (P_1) with $U_0 = (u_2 - u_1)(0) \equiv 0$. Setting $h = U$ in identity equation 2.23 we have

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2_{L^2(U)} + B(U, U, t)\right) = 0.$$

From Lemma 2.1.1 we have $B(U, U, t) \geqslant \alpha \|U\|^2_{H_0^1(U)} \geqslant 0$, so $\frac{d}{dt}(\frac{1}{2}\|U\|^2_{L^2(U)}) \leqslant 0$, then integrate with respect to t to find

$$\|U\|_{L^2(U)}^2 \leqslant \|U_0\|_{L^2(U)}^2 = 0,$$

thus $U \equiv 0$. □

2.1.3 Global solution of problem (P)

Our main results in this chapter are stated as follows.

Theorem 13. *if $v > v_0 > 0$ and $B(u,u,t) \geqslant \int_\Omega u_0 u dx$. then the solution (u,v) of problem (P) is global. Furthermore there exists $\sigma > 0$ such that $\|u\|_{L^2(\Omega)} \leqslant e^{\sigma t} \|u_0\|_{L^2(\Omega)}$.*

Proof. we put

$$Z(t) = \frac{1}{2} \int_\Omega u^2 dx \tag{2.30}$$

we derivate the equation 2.30 and we use first equations of (P_1) and (P_2) to find
i) we have

$$\frac{dZ}{dt} = \int_\Omega u_0 u dx - B(u,u,t) \leqslant 0$$

therefore

$$Z(t) \leqslant Z(0).$$

ii) we have

$$\begin{aligned} \frac{dZ}{dt} = \int_\Omega u_0 u dx - B(u,u,t) &= \int_\Omega u_0 u dx + \int_\Omega \nabla u^2 dx - \int_\Omega (\delta \tau v + 1) u^2 dx + \delta \int_\Omega u \nabla u \nabla u_0 dx \\ &= \int_\Omega u_0 u dx + \int_\Omega \left(1 + \frac{\delta \nabla u_0}{2}\right) \nabla u^2 dx - \int_\Omega (\delta \tau v + 1) u^2 dx \\ &\leqslant |\delta \tau v_0 + 1| \, \|u\|_{L^2(\Omega)}^2 = \sigma Z(t). \end{aligned}$$

This implies that

$$Z(t) \leqslant Z(0) e^{\sigma t}.$$

□

Proposition 2.1.2. *[1] Let $u_0 \in L^2(\Omega)$ and $u \in C([0,T]; L^2(\Omega) \cap L^2([0,T]; H_0^1(\Omega)))$ is the unique weak positive solution of (P_1). If $u_0 \geqslant 0$ in Ω, then $u \geqslant 0$ in $]0,T[\times \Omega$.*

Proof. If $u_0 \geqslant 0$ on $\partial\Omega$. Therefore $u^- = \min(u,0) \in L^2([0,T]; H_0^1(\Omega))$. We obtain for all $0 \leqslant t \leqslant T$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left(u^-\right)^2 dx + \int_\Omega B\left(u^-, u^-, t\right) dx = \int_\Omega u_0 u^- dx$$

Using the Lemma 2.1.1 and integrating with respect to t from 0 to T, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left(u^-\right)^2 dx + \alpha \int_0^T \|u(s)\|_{H_0^1(\Omega)}^2 ds \leqslant \frac{1}{2} \frac{d}{dt} \int_\Omega \left(u^-(0)\right)^2 dx = 0.$$

Since $u^-(0) = (u_0)^- = 0$. So $u^- = 0$. □

Summary

This study deals with the global existence, and uniqueness and boundedness of the weak solution for the elliptic-parabolic system (P) defined as

$$(P)\begin{cases} (P_1)\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x \in \Omega \end{cases} \\ \qquad (P_2)\begin{cases} -\Delta v + \tau v = 0 & x \in \Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

The system (P) is under Dirichlet boundary condition in a convex bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega, g$ and u_0 are two given functions. Based on Galerkin's method, Lax-Milgran's theorem and Maximum Principle, a proof of the existence and uniqueness of a global solution for the system (P) is determined for some properties adquate on initial and boundary conditions. The expential decay of $L^2(\Omega)$ norm of u has olso been shown. Moreover, we show that the unique solution is positive if the initial and boundary conditions are positive.

Chapter 3

Existence and uniqueness of solution for the fractional Thixotropic problem

In this chapter, we are interested in the study of the existence and uniqueness solution of a following spatial fractional derivative Thixotropic model (FP) in one dimensional case. This model is presenting another model that studies the phenomenon of thixotropy discussed with a new addition is the fractional derivative.
Our model is defined as follows:

$$(FP)\begin{cases} (P_1)\begin{cases} u_t + D^{\alpha}u - \lambda\dfrac{\partial}{\partial x}\left[u\dfrac{\dfrac{\partial}{\partial x}(v-u_0)}{\sqrt{\left|\dfrac{\partial}{\partial x}(v-u_0)\right|^2}}\right] + u = u_0 & (t,x)\in\mathbb{R}^+\times\Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x\in\Omega \end{cases} \\ (P_2)\begin{cases} -\dfrac{\partial^2 v}{\partial x^2} + \tau v = 0, & x\in\Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

Where $u(t,x)$ is a function denotes the speed of fluid in the position $x \in \Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, Ω is a bounded convex domain with smooth boundary $\partial\Omega \in H^{\frac{3}{2}}(\partial\Omega)$, $\lambda > 0$ is the viscosity of the fluid, . v denotes the relative temperature of the fluid. The parameter τ is a time constant and it is expressed the thermal diffusivity of v. g is the initial relative temperature distribution.
and

$$D^{\alpha} \equiv (\partial^2/\partial x^2)^{\alpha/2}$$

3.1 Existence and uniqueness

In this section, we are interested firstly in the study existence and uniqueness solution in one-dimensional cases of a following spatial modified fractional derivative Thixotropic model (FP')

and secondly to prove that the problem (FP') coverge to the problem (FP) when $\beta \to 0$. Our modified model is defined as follows:

$$(FP') \left\{ \begin{array}{l} (P_1') \left\{ \begin{array}{ll} u_t + D^{\alpha} u - \lambda \frac{\partial}{\partial x} \left[u \frac{\frac{\partial (v-u_0)}{\partial x}}{\sqrt{\beta + \left| \frac{\partial (v-u_0)}{\partial x} \right|^2}} \right] + u = u_0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x \in \Omega \end{array} \right. \\ \qquad (P_2) \left\{ \begin{array}{ll} -\frac{\partial^2 v}{\partial x^2} + \tau v = 0 & x \in \Omega \\ v = g & \partial\Omega \end{array} \right. \end{array} \right.$$

Where Ω is a bounded open domain with the smooth boundary $\partial\Omega \in C^1, u_0 \in H^1(I), g \in H^{1/2}(\partial\Omega)$; λ, τ and β are a positive constants.
Here we define

$$D^{\alpha} u(x) = F^{-1}\left(F\left(D^{\alpha} u\right)\right)(x) = F^{-1}\left(|\xi|^{\alpha} Fu(\xi)\right)(x), \tag{3.1}$$

where F denotes the Fourier transrform and F^{-1} its inverse.
We see that the solution of the problem (P_2) is given by

$$v(x) = Ae^{\sqrt{\tau}x} + Be^{-\sqrt{\tau}x} \in C^{\infty}\left(\overline{\Omega}\right), \tag{3.2}$$

Where $\Omega = [a,b]$ and the boundary conditions are $v(a) = \eta$ and $v(b) = \theta$.
So A and B verified

$$A = \left(\eta e^{-\sqrt{\tau}b} - \theta e^{-\sqrt{\tau}a}\right)\left(\eta e^{-\sqrt{\tau}(a-b)} - \theta e^{-\sqrt{\tau}x(a-b)}\right) \tag{3.3}$$

$$B = \left(\theta e^{\sqrt{\tau}a} - \eta e^{-\sqrt{\tau}b}\right)\left(\eta e^{-\sqrt{\tau}(a-b)} - \theta e^{-\sqrt{\tau}x(a-b)}\right) \tag{3.4}$$

The problem (FP') consist of two problem (P_1') and (P_2). Since the solution of (P_2) exists and belongs to $C^{\infty}\left(\overline{\Omega}\right)$, we need only to study the problem (P_1') and look for its weak solutions with initial data $u(x,0) = u_0(x)$ the function u in V_1 such that

$$V_1 = L^{\infty}\left(]0,T[\,; L^2(\Omega)\right) \cap L^2\left(]0,T[\,; H^1(\Omega)\right)$$

satisfying the following identity:

$$\begin{aligned} &\int_{\Omega} u\phi dx - \int_0^t \int_I u\phi_s dxds + \int_0^t \int_{\Omega} D^{\frac{\alpha}{2}} u D^{\frac{\alpha}{2}} \phi dxds \\ &-\lambda \int_0^t \int_{\Omega} \left[\left(u \frac{(v-u_0)_x}{\sqrt{\left|(v-u_0)_x\right|^2}} \right)_x \phi_x + u\phi - u_0\phi \right] dxds \\ &= \int_{\Omega} u_0 \phi_0 dx, \end{aligned} \tag{3.5}$$

for $a.e. t \in]0,T[$ and $\phi(t,x) \in H^1(]0,T[\times \Omega)$.

For simplicity we denote $u(t,x)$ by u , $\phi(x,t)$ by ϕ and $\phi(x,0)$ by ϕ_0. In order to simplify our construction, we suppose $u(t) \in H_0^1(\Omega)$ for $t \in]0,T[$ instead of $u(t) \in H^{\alpha/2}(\Omega)$ for $t \in]0,T[$ which could be easly generalized from the definition of weak solution of a parabolic second equation [8].

3.1.1 Existence and uniqueness solution of the problem (P_1')

In this part, we study existence and uniqueness of modify fractional Thixotropic problem (FP') using the Galerkin method, some prior estimate and Gronwall's lemma. We use the same the thechnique as in ([8],[26]) which allows us to show that the disipative operator D^α can control the nonlinearity $\frac{\partial}{\partial x}\left[u\frac{\frac{\partial(v-u_0)}{\partial x}}{\sqrt{\beta+\left|\frac{\partial(v-u_0)}{\partial x}\right|^2}}\right]$.
suppose

$$\delta=\frac{\lambda}{\sqrt{\beta+\left|\frac{\partial}{\partial x}(v-u_0)\right|^2}}$$

Then, we can rewrite the problem (P_1') as follows

$$(P')\begin{cases}(P_1')\begin{cases}u_t+D^\alpha u-\frac{\partial}{\partial x}\left[\delta u\frac{\partial}{\partial x}(v-u_0)\right]+u=u_0 & (t,x)\in\mathbb{R}^+\times\Omega\\ u=0 & \partial\Omega\\ u(0,x)=u_0 & x\in\Omega\end{cases}\\ \qquad (P_2)\begin{cases}-\frac{\partial^2 v}{\partial x^2}+\tau v=0 & x\in\Omega\\ v=g & \partial\Omega\end{cases}\end{cases}$$

Where Ω is a bounded open domain with the smooth boundary $\partial\Omega\in C^1, u_0\in H^1(\Omega), g\in H^{1/2}(\partial\Omega)$; λ, τ and β are a positive constants.

Theorem 14. *[8] Let $0<\alpha\leqslant 2$, $T>0$ and $u_0(x)\in H^1(\Omega)$. Then, Cauchy's problem (P_1') has a unique weak solution $u_\beta\in V_1$. Moreover, u_β satisfies the following regularity properties:*

$$u_\beta\in L^\infty\left(]0,T[\,;H_0^1(\Omega)\right)\cap L^2\left(]0,T[\,;H^{1+\alpha/2}(\Omega)\right),$$

and

$$(u_\beta)_t\in L^\infty\left(]0,T[\,;L^2(\Omega)\right)\cap L^2\left(]0,T[\,;H^{\alpha/2}(\Omega)\right),$$

For $t\to\infty,\alpha>1$ and $g>0$ this solution decays so that

$$\lim_{t\to\infty}\mid D^{\frac{\alpha}{2}}u_\beta\mid_2=\lim_{t\to\infty}\mid u_\beta\mid_\infty=0.$$

Proof. Suppose u_β is a weak solution of $\left(P_1'\right)$ and let S_n be a transaction operator that $(u_\beta)_n=S_n u_\beta$, (we denote $(u_\beta)_n(x,t)$ by $(u_\beta)_n$), then we can consider the following approximation problem:

$$((u_\beta)_n)_t+D^\alpha(u_\beta)_n-\frac{\partial}{\partial x}\left[\delta(u_\beta)_n\frac{\partial}{\partial x}\left(v-((u_\beta)_n)_0\right)\right]+(u_\beta)_n=((u_\beta)_n)_0 \qquad (3.6)$$

with initial data $(u_\beta)_n|_{t=0} = S_0(u_\beta)_0$
we suppose that $w = u_\beta$, then the equation 3.6 can write as follows

$$(w_n)_t + D^\alpha w_n - \frac{\partial}{\partial x}\left[\delta w_n \frac{\partial}{\partial x}\left(v - (w_n)_0\right)\right] + w_n = (w_n)_0 \tag{3.7}$$

Let us multiply (3.7) by w_n, then

$$\frac{d}{dt}\int w_n^2 dx + \int \left(D^{\frac{\alpha}{2}} w_n\right)^2 dx = \int \frac{\partial}{\partial x}\left[\delta w_n \frac{\partial}{\partial x}\left(v - (w_n)_0\right)\right] w_n dx - \int w_n^2 dx + \int (w_n)_0 w_n dx$$

which implies

$$\frac{d}{dt}\int w_n^2 dx + \int \left(D^{\frac{\alpha}{2}} w_n\right)^2 dx = \int \delta \frac{\partial u_n}{\partial x}\frac{\partial\left(v-(w_n)_0\right)}{\partial x} w_n dx + \int \frac{\partial \delta}{\partial x}\frac{\partial^2\left(v-(w_n)_0\right)}{\partial x^2} w_n^2 dx - \int w_n^2 dx + \int (w_n)_0 w_n dx$$

one has

$$\frac{d}{dt}\int w_n^2 dx + \int \left(D^{\frac{\alpha}{2}} w_n\right)^2 dx = \int \delta w_n \frac{\partial w_n}{\partial x}\frac{\partial v}{\partial x} dx - \int \delta w_n \frac{\partial w_n}{\partial x}\frac{\partial (w_n)_0}{\partial x} dx + \int \frac{\partial \delta}{\partial x}\frac{\partial^2 v}{\partial x^2} w_n^2 dx - \int \frac{\partial \delta}{\partial x}\frac{\partial^2 (w_n)_0}{\partial x^2} w_n^2 dx - \int w_n^2 dx + \int (w_n)_0 w_n dx$$

we have

$$\int \frac{\partial w_n}{\partial x}\frac{\partial v}{\partial x} dx = -\tau \int w_n v dx$$

and

$$\int \frac{\partial^2 v}{\partial x^2} w_n dx = -\tau \int w_n v dx$$

So, we find that

$$\frac{d}{dt}\int w_n^2 dx + \int \left(D^{\frac{\alpha}{2}} w_n\right)^2 dx = -\tau \int \delta w_n^2 v dx - \int \delta w_n \frac{\partial w_n}{\partial x}\frac{\partial (w_n)_0}{\partial x} dx - \tau \int \frac{\partial \delta}{\partial x} w_n^2 v dx - \int \frac{\partial \delta}{\partial x}\frac{\partial^2 (w_n)_0}{\partial x^2} w_n^2 dx - \int w_n^2 dx + \int (w_n)_0 w_n dx$$

integrate $\int w_n^2 \frac{\partial^2 (w_n)_0}{\partial x} dx$ by part, we obtain

$$\int w_n^2 \frac{\partial^2 (w_n)_0}{\partial x} dx = -\frac{1}{2}\int \frac{\partial (w_n)}{\partial x}\frac{\partial (w_n)_0}{\partial x} dx$$

hence, we obtain

$$\frac{d}{dt}\int w_n^2 dx + \int \left(D^{\frac{\alpha}{2}} w_n\right)^2 dx = \int \left(-2\tau\delta v - \delta \frac{\partial^2 (w_n)_0}{\partial x^2} - 1\right) w_n^2 dx - \frac{1}{2}\int \frac{\partial \delta}{\partial x}\frac{\partial w_n^2}{\partial x}\frac{\partial (w_n)_0}{\partial x} dx + \int (w_n)_0 w_n dx$$

then, it holds that

$$\begin{aligned}\frac{d}{dt}|w_n|_2^2+\left|D^{\frac{\alpha}{2}}w_n\right|_2^2 \leqslant\ & \left(|2\tau|\,|\delta|_{L^\infty}|v|_{L^\infty}+|\delta|_{L^\infty}\left|\frac{\partial^2(w_n)_0}{\partial x^2}\right|_{L^\infty}+1\right)|w_n|_2^2+\frac{1}{2}\left|\frac{\partial\delta}{\partial x}\right|_{L^\infty}\left|\frac{\partial w_n^2}{\partial x}\right|_{L^\infty}\left|\frac{\partial(w_n)_0}{\partial x}\right|_{L^\infty}\\ &+|(w_n)_0|_{L^\infty}|w_n|_2\end{aligned}$$

which implies

$$\frac{d}{dt}|w_n|_2^2+\left|D^{\frac{\alpha}{2}}w_n\right|_2^2\leqslant C_1\|w_n\|_{H^1}^2+C_2|w_n|_2\,.$$

Likewise, upon differentiation in formula (3.7) according to x and multiplying by $(w_n)_x$, we obtain

$$\begin{aligned}\frac{d}{dt}\int w_n(w_n)_x\,dx+\int\left(D^{\frac{\alpha}{2}}w_n\right)(w_n)_x\,dx &=\int\frac{\partial}{\partial x}\left[\delta w_n\frac{\partial}{\partial x}(v-(w_n)_0)\right](w_n)_x\,dx\\ &-\int w_n(w_n)_x\,dx+\int(w_n)_0(w_n)_x\,dx\end{aligned}$$

we find that

$$\begin{aligned}\frac{d}{dt}\int w_n(w_n)_x\,dx+\int\left(D^{\frac{\alpha}{2}}w_n\right)(w_n)_x\,dx = &-2\tau\int\delta w_n(w_n)_x\,v dx-\int\delta(w_n)_x\frac{\partial w_n}{\partial x}\frac{\partial(w_n)_0}{\partial x}dx\\ &-\int\frac{\partial\delta}{\partial x}\frac{\partial^2(w_n)_0}{\partial x^2}w_n(w_n)_x\,dx-\int w_n(w_n)_x\,dx\\ &+\int(w_n)_0(w_n)_x\,dx\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(w_n^2)_x\,dx+2\int\left(D^{1+\frac{\alpha}{2}}w_n\right)^2dx = &\int\left(-\tau\delta v-\frac{1}{2}\frac{\partial\delta}{\partial x}\frac{\partial^2(w_n)_0}{\partial x^2}-\frac{1}{2}\right)\frac{\partial w_n^2}{\partial x}dx-\int\delta\left(\frac{\partial w_n}{\partial x}\right)^2\\ &\times\frac{\partial(w_n)_0}{\partial x}dx+\int(w_n)_0(w_n)_x\,dx\end{aligned}$$

so, we have

$$\begin{aligned}\frac{d}{dt}|(w_n)_x|_2^2\,dx+2\left|D^{1+\frac{\alpha}{2}}w_n\right|_2^2 &\leqslant\left(|2\tau|\,|\delta|_{L\infty}|v|_{L\infty}+\left|\frac{\partial\delta}{\partial x}\right|_{L\infty}\left|\frac{\partial^2(w_n)_0}{\partial x^2}\right|_{L\infty}+1\right)|(w_n)_x|_2^2\,dx\\ &+|(w_n)_0|_{L\infty}|(w_n)_x|_2\\ &\leqslant M_1|(w_n)_x|_2^2+M_2|(w_n)_x|_2\end{aligned}\tag{3.8}$$

Now, a part of right member of (3.8) can be approximated by

$$\begin{aligned}|(w_n)_x|_2^2 &\leqslant K\|(w_n)_x\|_{\alpha/2}^{1/\alpha}|(w_n)_x|_2^{2-1/\alpha}\\ &\leqslant\|(w_n)_x\|_{1+\alpha/2}^2+C|(w_n)_x|_2\end{aligned}$$

Hence, we obtain

$$\|w_n(t)\|_1^2+\int_0^t\|w_n(s)\|_{1+\alpha/2}^2\,ds\leqslant C\tag{3.9}$$

Now, approximate a derivative according to the time of a solution.

Multiply (3.7) by $(w_n)_t$, we obtain

$$\begin{aligned}\frac{d}{dt}\int w_n(w_n)_t\,dx+\int\left(D^{\frac{\alpha}{2}}w_n\right)(w_n)_t\,dx &=\int\frac{\partial}{\partial x}\left[\delta w_n\frac{\partial}{\partial x}(v-(w_n)_0)\right](w_n)_t\,dx\\ &-\int w_n(w_n)_t\,dx+\int(w_n)_0(w_n)_t\,dx\end{aligned}$$

which implies

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\int (w_n^2)_t\,dx+\tfrac{1}{2}\int D^{\frac{\alpha}{2}}\,(w_n^2)_t\,dx = \;& -\tau\int\delta\,(w_n^2)_t\,v dx-\int\delta\,(w_n)_t\,\tfrac{\partial w_n}{\partial x}\tfrac{\partial (w_n)_0}{\partial x}dx\\ & -\tfrac{1}{2}\int\tfrac{\partial\delta}{\partial x}\tfrac{\partial^2 (w_n)_0}{\partial x^2}\,(w_n^2)_t\,dx-\tfrac{1}{2}\int (w_n^2)_t\,dx\\ & +\int (w_n)_0\,(w_n)_t\,dx\end{aligned}$$

After some calculations, we obtain

$$\begin{aligned}\tfrac{d}{dt}\left|(w_n)_t\right|_2^2+\left|D^{\frac{\alpha}{2}}\,(w_n)_t\right|_2^2\leqslant\;&\left(|\tau\delta|\,|v|_{L^\infty}+\left|\tfrac{\delta}{2}\right|\left|\tfrac{\partial^2 (w_n)_0}{\partial x^2}\right|_{L^\infty}+\tfrac{1}{2}\right)\left|(w_n)_t\right|_2^2+|\delta|\left|\tfrac{\partial (w_n)_0}{\partial x}\right|_{L^\infty}\\ &\times\left|(w_n)_t\right|_2\left|(w_n)_x\right|_2+\left|(w_n)_0\right|_2\left|(w_n)_t\right|_2\end{aligned}$$

so, we find that

$$\tfrac{d}{dt}\left|(w_n)_t\right|_2^2+\left|D^{\frac{\alpha}{2}}\,(w_n)_t\right|_2^2\leqslant\; C_1\left|(w_n)_t\right|_2^2+C_2\left|(w_n)_t\right|_2\left|(w_n)_x\right|_2+C_3\left|(w_n)_t\right|_2$$

Now, we approximate a right member $\left|(w_n)_t\right|_2\left|(w_n)_x\right|_2$ by

$$\begin{aligned}\left|(w_n)_t\right|_2\left|(w_n)_x\right|_2&\leqslant C\left\|(w_n)_t\right\|_{\alpha/2}^{1/\alpha}\left|(w_n)_t\right|_2^{2-1/\alpha}\left|(w_n)_x\right|_2\\ &\leqslant\left\|(w_n)_t\right\|_{1+\alpha/2}^2+C\left|(w_n)_t\right|_2\end{aligned}$$

Then,we have

$$\tfrac{d}{dt}\left|(w_n)_t\right|_2^2+\left|D^{\frac{\alpha}{2}}\,(w_n)_t\right|_2^2\leqslant\; C_1\left|(w_n)_t\right|_2^2+C\left|(w_n)_t\right|_2$$

A classical Gronwall inequality gives

$$\left|(w_n)_t\right|_2^2+\int_0^t\left\|w_n(s)\right\|_{\alpha/2}^2ds\leqslant C(T)\,. \tag{3.10}$$

It holds, from (3.9) and (3.10) , that a solution w_n is bounded. Then it is sufficient in order to apply approximation Galerkin's procedure. Hence, we can extract a subsequence which converge to a limit

$$w\in L^\infty\left(]0,T[\,;H_0^1(\Omega)\right)\cap L^2\left(]0,T[\,;H^{1+\alpha/2}(\Omega)\right),$$

and

$$w_t\in L^\infty\left(]0,T[\,;L^2(\Omega)\right)\cap L^2\left(]0,T[\,;H^{\alpha/2}(\Omega)\right),$$

To finish, it remains to know if w is a solution of problem? Since injection of $H^1(I)$ into $L^2(I)$ is compact, we can apply Ascoli theorem and conclude a strongly convergence of $(w_n)_{n\in\mathbb{N}}$ to w in $L^2(]0,T[\,;L^2(I))$.

In order to conclude, it is enough to prove that $(w_n)^2$ converge strongly to w^2 in $L^1(]0,T[\,;L^2(I))$. Remark that

$$\left\|(w_n)^2-w^2\right\|_{L^1(]0,T[;L^2(I))}\leqslant\left\|w_n-w\right\|_{L^1(]0,T[;L^4(I))}\left(\left\|w_n\right\|_{L^1(]0,T[;L^4(I))}+\left\|w\right\|_{L^1(]0,T[;L^4(I))}\right),$$

it is enough to prove that $w_n - w$ converge strongly in $L^1\left(]0,T[\,;L^4\left(I\right)\right)$. This last result holds by Gagliard-Nirenbrg's inequality ([3],[8])

$$\begin{aligned}\|w_n - w\|_{L^2(]0,T[;L^4(I))} &\leqslant C\|w_n - w\|^{1-\frac{1}{4}}_{L^2(]0,T[;L^4(I))}\|\nabla(w_n - w)\|^{\frac{1}{4}}_{L^2(]0,T[;L^4(I))}\\ &\leqslant C\|w_n - w\|^{1-\frac{1}{4}}_{L^2(]0,T[;L^4(I))}\end{aligned}$$

and to prove that $D^\alpha w_n$ converge strongly to $D^\alpha w$ in $L^1\left(]0,T[\,;L^2\left(I\right)\right)$.

In the same way, we remark that

$$\|D^\alpha w_n - D^\alpha w\|_{L^1(]0,T[;L^2(I))}\leqslant$$

$$\left\|\frac{\partial^2 w_n}{\partial x^2} - \frac{\partial^2 w}{\partial x^2}\right\|_{L^1\left(]0,T[;H^{1+\frac{\alpha}{2}}(I)\right)}\left(\left\|\frac{\partial^2 w_n}{\partial x^2}\right\|_{L^1\left(]0,T[;H^{1+\frac{\alpha}{2}}(I)\right)} + \left\|\frac{\partial^2 w}{\partial x^2}\right\|_{L^1\left(]0,T[;H^{1+\frac{\alpha}{2}}(I)\right)}\right)$$

and since the term $\frac{\partial^2}{\partial x^2}$ is linear, approach problem converge weakly to a limit point, then the existence holds.

Now, we prove uniqueness solution. Consider two weak solutions w_1 and w_2 of (P_1'), then the difference $z = w_1 - w_2$ satisfies

$$\frac{d}{dt}|z|_2^2 + \left|D^{\frac{\alpha}{2}}z\right|_2^2 = \delta\int\frac{\partial}{\partial x}\left[z\frac{\partial}{\partial x}(v - z_0)\right]zdx - \int z^2dx + \int(u_n)_0\,zdx$$

After some calculations, we have

$$\begin{aligned}\tfrac{d}{dt}|z|_2^2 + \left|D^{\frac{\alpha}{2}}z\right|_2^2 = &\int\left(-2\tau\delta v - \delta\tfrac{\partial^2(z)_0}{\partial x^2} - 1\right)z^2dx - \tfrac{\delta}{2}\int\tfrac{\partial z^2}{\partial x}\tfrac{\partial(u_n)_0}{\partial x}dx\\ &+\int(u_n)_0\,zdx\end{aligned}$$

$$\frac{d}{dt}|z|_2^2 + \left|D^{\frac{\alpha}{2}}z\right|_2^2 \leqslant C_1\|z\|_{H^1}^2 + C_2|z|_2$$

by Gronwall's lemma it holds that $z(t) \equiv 0$ on $[0,T]$.

Now, we will prove the asymptotic estimates

$$\lim_{t\to\infty}\left|D^{\alpha/2}u_\beta\right|_2 = \lim_{t\to\infty}|u_\beta|_\infty = 0.$$

The idea of this prove is to verify the hypothesis of the following Lemma:

Lemma 3.1.1. *Let f be the positive function such that f_t and $\int_0^\infty f(\tau)\,d\tau$ are bounded then* $\lim_{t\to\infty} f(t) = 0$.

Indeed, multiplying (3.7) by $D^\alpha w$

$$\frac{d}{dt}\int wD^\alpha wdx + \int D^\alpha wD^\alpha wdx = \delta\int\frac{\partial}{\partial x}\left[w\frac{\partial}{\partial x}(v - w_0)\right]D^\alpha wdx - \int wD^\alpha wdx + \int w_0D^\alpha wdx$$

we find that

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\int\left(D^{\alpha/2}w\right)^2dx + \int(D^\alpha w)^2dx = &\;\delta\int\tfrac{\partial w}{\partial x}\tfrac{\partial v}{\partial x}D^\alpha wdx - \delta\int\tfrac{\partial w}{\partial x}\tfrac{\partial w_0}{\partial x}D^\alpha wdx + \delta\int\tfrac{\partial^2 v}{\partial x^2}wD^\alpha wdx\\ &-\delta\int\tfrac{\partial^2 w_0}{\partial x^2}wD^\alpha wdx - \int wD^\alpha wdx + \int w_0D^\alpha wdx\end{aligned}$$

this implies

$$\begin{aligned}\tfrac{d}{dt}\left|D^{\alpha/2}w\right|_2^2 + 2|D^\alpha w|_2^2 = &\;2\int\left[\delta\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial w_0}{\partial x}\right)\tfrac{\partial w}{\partial x} + \left(\delta\tfrac{\partial^2 v}{\partial x^2} - \delta\tfrac{\partial^2 w_0}{\partial x^2} - 1\right)w\right]D^\alpha wdx\\ &+\int w_0D^\alpha wdx\end{aligned}$$

So, we obtain

$$\begin{aligned}\frac{d}{dt}\left|D^{\alpha/2}w\right|_2^2+2\left|D^{\alpha}w\right|_2^2 &\leqslant 4\max\left[\delta\left|\frac{\partial v}{\partial x}\right|_{L^\infty}+\delta\left|\frac{\partial w_0}{\partial x}\right|_{L^\infty},\delta\left|\frac{\partial^2 v}{\partial x^2}\right|_{L^\infty}+\delta\left|\frac{\partial^2 w_0}{\partial x^2}\right|_{L^\infty}+1\right]_2\\ &\times\|w\|_1\left|D^\alpha w\right|_2+\left|w_0\right|_{L^\infty}\left|D^\alpha w\right|_2\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\left|D^{\alpha/2}w\right|_2^2+2\left|D^{\alpha}w\right|_2^2 &\leqslant C_1\|w\|_1\left|D^\alpha w\right|_2+C_2\left|D^\alpha w\right|_2\\ &\leqslant C+\left|D^\alpha w\right|_2^2\end{aligned}$$

So

$$\frac{d}{dt}\left|D^{\alpha/2}w\right|_2^2\leqslant C$$

If $g>0$ and $v>w_0$ this implies that $v>0$, $v-w_0>0$ and from (3.7) we get

$$|w|_2^2+2\int_0^t\left|D^{\frac{\alpha}{2}}w\left(s\right)\right|_2^2ds-\lambda\int_0^t\int_\Omega\frac{\partial}{\partial x}\left[w^2\frac{\frac{\partial}{\partial x}(v-w_0)}{\sqrt{\beta+\left|\frac{\partial}{\partial x}(v-w_0)\right|}}\right]dx\quad\leqslant|w|_2^2+|w_0|_2\leqslant C\quad,0<\alpha<2.$$

for all t. Thus, $\lim_{t\to\infty}\left|D^{\frac{\alpha}{2}}w\right|_2=0$. The second asymptotic relation follows from the Sobolev embedding $H^{\alpha/2}\subset L^\infty$ which is valid for $\alpha>1$.

□

3.1.2 Convergence of the problem (FP') to the problem (FP)

In this part, we study the Convergence of the problem (FP') to the problem (FP); this study based on fixed point theorem.

Theorem 15. *We consider the application*

$$\begin{cases}T: & H_0^1(\Omega)\longrightarrow H_0^1(\Omega)\\ & u_\beta\longrightarrow T(u_\beta)=u_\beta\end{cases}$$

Where u_β is the solution of (P_1') and u is the solution of (P_1).
If $u_\beta\to u$. Then, the problem $(P_1')\to(P_1)$ which implies that the problem $(FP')\to(FP)$.

Proof. Suppose

$$f\left(u_\beta\right)=\left(u_\beta\right)_t+D^\alpha u_\beta-\delta\frac{\partial}{\partial x}\left[u_\beta\frac{\partial}{\partial x}(v-u_0)\right]+u_\beta-u_0. \tag{3.11}$$

If $f\left(u_\beta\right)=0$ then $u_\beta=T\left(u_\beta\right)$, i.e

$$f\left(u_\beta\right)=0\Rightarrow u_\beta=T\left(u_\beta\right) \tag{3.12}$$

from (3.11) and (3.12) ,we obtain

$$T\left(u_\beta\right)=-\left(u_\beta\right)_t-D^\alpha u_\beta+\delta\frac{\partial}{\partial x}\left[u_\beta\frac{\partial}{\partial x}(v-u_0)\right]+u_0$$

Now, we prove that T is contracting (so admits a fixed point) i.e.

$$\forall u_{\beta_1}, u_{\beta_2} \in V_1, \left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant Cste \left\|u_{\beta_1} - u_{\beta_2}\right\|?$$

We have

$$\begin{aligned}\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| &= \left\|-\left[\left(u_{\beta_1}\right)_t - \left(u_{\beta_2}\right)_t\right] - \left(D^\alpha u_{\beta_1} - D^\alpha u_{\beta_2}\right) + \delta\left(\frac{\partial}{\partial x}\left[u_{\beta_1}\frac{\partial}{\partial x}(v-u_0)\right]\right.\right. \\ &\left.\left. - \frac{\partial}{\partial x}\left[u_{\beta_2}\frac{\partial}{\partial x}(v-u_0)\right]\right) + u_0\right\|\end{aligned}$$

Which implies

$$\begin{aligned}\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| &\leqslant \left\|\left(u_{\beta_1}\right)_t - \left(u_{\beta_2}\right)_t\right\| + \left\|D^\alpha u_{\beta_1} - D^\alpha u_{\beta_2}\right\| \\ &+ |\delta|\left\|\frac{\partial}{\partial x}\left[u_{\beta_1}\frac{\partial}{\partial x}(v-u_0)\right] - \frac{\partial}{\partial x}\left[u_{\beta_2}\frac{\partial}{\partial x}(v-u_0)\right]\right\| + \|u_0\|\end{aligned}$$

Then

$$\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant 2|\delta|\left\|\frac{\partial}{\partial x}\left[\left(u_{\beta_1} - u_{\beta_2}\right)\frac{\partial}{\partial x}(v-u_0)\right]\right\|$$

We find that

$$\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant 2|\delta|\left[\left\|\frac{\partial\left(u_{\beta_1}-u_{\beta_2}\right)}{\partial x}\frac{\partial}{\partial x}(v-u_0)\right\| + \left\|\frac{\partial\left(\frac{\partial}{\partial x}(v-u_0)\right)}{\partial x}\left(u_{\beta_1} - u_{\beta_2}\right)\right\|\right]$$

Hence

$$\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant 2|\delta|\max\left[\left\|\frac{\partial}{\partial x}(v-u_0)\right\|;\left\|\frac{\partial\left(\frac{\partial}{\partial x}(v-u_0)\right)}{\partial x}\right\|\right]\left\|\left(u_{\beta_1} - u_{\beta_2}\right)\right\|_{H^1}$$

suppose

$$C(\beta) = 2\max\left[\left\|\frac{\partial}{\partial x}(v-u_0)\right\|;\left\|\frac{\partial\left(\frac{\partial}{\partial x}(v-u_0)\right)}{\partial x}\right\|\right]$$

So, we obtain;

$$\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant \delta C(\beta)\left\|\left(u_{\beta_1} - u_{\beta_2}\right)\right\|_{H^1}$$

suppose: $K = \delta C(\beta)$, so

$$\left\|T\left(u_{\beta_1}\right) - T\left(u_{\beta_2}\right)\right\| \leqslant K\left\|\left(u_{\beta_1} - u_{\beta_2}\right)\right\|_{H^1}$$

We chose λ like so that $K < 1$, which makes it strictly constant.
At the end, we conclude that: $u_\beta \to u$ and the problem $(P_1') \to (P_1)$, which implies that $(FP') \to (FP)$. □

Conclusion:
As a result, we conclude that the problem (P) has a unique solution $(u, v) \in V_1 \times C^\infty(\overline{\Omega})$.

Chapter 4

Numerical study for the Thixotropic problem

In this chapter, we study and approximate the thixotropic problem (P) by using the finite volume schemes in one dimensional case and multidimonsional case.

4.1 Numerical study in one dimensional case

In this section, we study and approximate the thixotropic problem (P) by using the finite volume schemes in one dimensional case which is based on Method of Lines (MOL).

$$(P)\begin{cases}(P_1)\begin{cases}u_t+\dfrac{\partial^2 u}{\partial x^2}-\lambda\dfrac{\partial}{\partial x}\left[u\dfrac{\dfrac{\partial}{\partial x}(v-u_0)}{\sqrt{\beta+\left|\dfrac{\partial}{\partial x}(v-u_0)\right|^2}}\right]+u=u_0, & (t,x)\in[0,T]\times\Omega\\ u=0, & \partial\Omega\\ u(0,x)=u_0, & x\in\Omega\end{cases}\\ (P_2)\begin{cases}-\dfrac{\partial^2 v}{\partial x^2}+\tau v=0, & x\in\Omega\\ v=g, & \partial\Omega\end{cases}\end{cases}$$

Where $\Omega=[0,1]\,,\beta,\ \tau$ and λ are positive constants.

4.1.1 Solution methods

In this part, we only study and approximate the problem (P_1) because the problem (P_2) with bounary condition $v(a)=\eta$ and $v(b)=\theta$ is easy given by the following equation:

$$v(x)=Ae^{\sqrt{\tau}x}+Be^{-\sqrt{\tau}x}\in C^{\infty}([a,b])\,,$$

where

$$A=\left(\eta e^{-\sqrt{\tau}b}-\theta e^{-\sqrt{\tau}a}\right)\setminus\left(e^{\sqrt{\tau}(a-b)}-e^{-\sqrt{\tau}(a-b)}\right),$$

and
$$B = \left(\theta e^{\sqrt{\tau}a} - \eta e^{\sqrt{\tau}b}\right) \setminus \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right).$$

Numerical Approximation by Method Of Lines (MOL) of finite volume for the problem (P_1)

Definition 4.1.1. *(Admisssible mesh) An admessible mesh of* (P_1)*, denoted by* T*, is given a familly* $(K_i)_{i=0,...,N}, N \in \mathbb{N}^*$*, such that* $K_i = [x_{i-1/2}, x_{i+1/2}]$*, and familly* $(x_i)_{i=0,...,N+1}$ *such that*

$$x_0 = x_{1/2} = 0 < x_1 < x_{3/2} < ... < x_{i-1/2} < x_i < x_{i+1/2}$$
$$< x_N < x_{N+1/2} = x_{N+1} = 1.$$

Ones set
$$h_i = m(K_i) = x_{i+1/2} - x_{i-1/2}, i = 1, ..., N$$
and therefore $\sum_{i=1}^{N} h_i = 1$,
$$h_i^- = x_i - x_{i-1/2}, i = 1, ..., N,$$
$$h_i^+ = x_{i+1/2} - x_i, i = 1, ..., N,$$
$$h_{i-1/2} = x_i - x_{i-1}, i = 0, ..., N,$$
$$h_{i+1/2} = x_{i+1} - x_i, i = 0, ..., N,$$
$$size(T) = h = max\{h_i, i = 1, ..., N\}.$$

A weak solution $u(x,t)$ is defined on $[0,T] \times [0,1]$, we introduce a mean value $u_i(t)$ of a solution, that one assumes exists in the following meaning

$$u_i(t) = \frac{1}{h_i}\int_{K_i} u(x,t)dx, i \in \mathbb{N},$$
$$\frac{\partial u_i(t)}{\partial t} = \frac{1}{h_i}\int_{K_i} \frac{\partial u(x,t)}{\partial t}dx, i \in \mathbb{N},$$
$$u_i(0) = \frac{1}{h_i}\int_{K_i} u(x,0)dx, i \in \mathbb{N},$$

Integrate the first equation of problem (P_1) on K_i, with

$$\delta(x) = \frac{\frac{\partial}{\partial x}(v - u_0)(x)}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v - u_0)(x)\right|^2}},$$

in order to obtain

$$\frac{d}{dt}\int_{K_i} u dx + \int_{K_i} \frac{\partial^2 u}{\partial x^2} dx - \lambda \int_{K_i} \frac{\partial}{\partial x}\left[u\delta\right] dx + \int_{K_i} u dx = \int_{K_i} u_0 dx$$

which implies

$$h_i \frac{\partial u_i}{\partial t} + \left[\frac{\partial u(x_{i+1/2},t)}{\partial x} - \frac{\partial u(x_{i-1/2},t)}{\partial x}\right] - \lambda\left[u(x_{i+1/2},t)\delta(x_{i+1/2}) - u(x_{i-1/2},t)\delta(x_{i-1/2})\right] + h_i u_i = h_i u_i^0.$$

we denote that

$$\begin{aligned} F(x_{i+1/2},t) &= \frac{\partial u(x_{i+1/2},t)}{\partial x} - \lambda u(x_{i+1/2},t)\delta(x_{i+1/2}) \\ &= \frac{u(x_{i+1},t) - u(x_i,t)}{h_{i+1/2}} - \frac{\lambda\delta(x_{i+1/2})}{2}\left[u(x_{i+1},t) + u(x_i,t)\right] \\ &= \left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta(x_{i+1/2})}{2}\right] u(x_{i+1},t) - \left[\frac{1}{h_{i+1/2}} + \frac{\lambda\delta(x_{i+1/2})}{2}\right] u(x_i,t). \end{aligned}$$

Where

$$\begin{aligned} \delta(x_{i+1/2}) &= \frac{\frac{\partial}{\partial x}\left(v - u_0\right)(x_{i+1/2})}{\sqrt{\beta + \left|\frac{\partial}{\partial x}\left(v - u_0\right)(x_{i+1/2})\right|^2}} \\ &= \frac{\frac{\partial v}{\partial x}(x_{i+1/2}) - \frac{\partial u_0}{\partial x}(x_{i+1/2})}{\sqrt{\beta + \left|\frac{\partial v}{\partial x}(x_{i+1/2}) - \frac{\partial u_0}{\partial x}(x_{i+1/2})\right|^2}} \end{aligned}$$

and

$$\begin{aligned} F(x_{i-1/2},t) &= \frac{\partial u(x_{i-1/2},t)}{\partial x} - \lambda u(x_{i-1/2},t)\delta(x_{i-1/2}) \\ &= \frac{u(x_i,t) - u(x_{i-1},t)}{h_{i-1/2}} - \frac{\lambda\delta(x_{i-1/2})}{2}\left[u(x_i,t) + u(x_{i-1},t)\right] \\ &= \left[\frac{1}{h_{i-1/2}} - \frac{\lambda\delta(x_{i-1/2})}{2}\right] u(x_i,t) - \left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta(x_{i-1/2})}{2}\right] u(x_{i-1},t). \end{aligned}$$

Where

$$\delta(x_{i-1/2}) = \frac{\frac{\partial}{\partial x}(v-u_0)(x_{i-1/2})}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v-u_0)(x_{i-1/2})\right|^2}}$$
$$= \frac{\frac{\partial v}{\partial x}(x_{i-1/2}) - \frac{\partial u_0}{\partial x}(x_{i-1/2})}{\sqrt{\beta + \left|\frac{\partial v}{\partial x}(x_{i-1/2}) - \frac{\partial u_0}{\partial x}(x_{i-1/2})\right|^2}}$$

So, we have that

$$h_i \frac{\partial u_i(t)}{\partial t} + F(x_{i+1/2}, t) - F(x_{i-1/2}, t) + h_i u_i = h_i u_i^0.$$

As x_i is the midpoint of K_i, one has

$$|u_i(t) - u(x_i, t)| \leqslant ch^2.$$

Approach the flux of an exact solution at the vertex $x_{i+1/2}$ by numerical flux which depends on mean values $\Psi(u_{i+1}, u_i)$

$$\Psi(u_{i+1}, u_i) = \left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta_{i+1/2}}{2}\right] u_{i+1} - \left[\frac{1}{h_{i+1/2}} + \frac{\lambda\delta_{i+1/2}}{2}\right] u_i.$$

Where

$$\delta_{i+1/2} = \frac{\frac{\partial v_{i+1/2}}{\partial x} - \frac{\partial (u_0)_{i+1/2}}{\partial x}}{\sqrt{\beta + \left|\frac{\partial v_{i+1/2}}{\partial x} - \frac{\partial (u_0)_{i+1/2}}{\partial x}\right|^2}}$$

and at the vertex $x_{i-1/2}$ by the numerical flux which depends on mean values $\Psi(u_i, u_{i-1})$

$$\Psi(u_i, u_{i-1}) = \left[\frac{1}{h_{i-1/2}} - \frac{\lambda\delta_{i-1/2}}{2}\right] u_i - \left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta_{i-1/2}}{2}\right] u_{i-1}.$$

Where

$$\delta_{i-1/2} = \frac{\frac{\partial v_{i-1/2}}{\partial x} - \frac{\partial (u_0)_{i-1/2}}{\partial x}}{\sqrt{\beta + \left|\frac{\partial v_{i-1/2}}{\partial x} - \frac{\partial (u_0)_{i-1/2}}{\partial x}\right|^2}}$$

to obtain an ordinary defferential equation

$$\frac{\partial u_i(t)}{\partial t} + \frac{1}{h_i}\Psi(u_{i+1}, u_i) - \frac{1}{h_i}\Psi(u_i, u_{i-1}) + u_i = u_i^0$$

which implies

$$\frac{\partial u_i(t)}{\partial t} = -\frac{1}{h_i}\Psi(u_{i+1}, u_i) + \frac{1}{h_i}\Psi(u_i, u_{i-1}) - u_i + u_i^0$$

For time-discretisation, we apply one of the various basic schemes to solve a general ordinary differential equation

$$\begin{cases} \dfrac{du}{dt} = F(t, u) \\ u(0) = u_0 \end{cases}$$

where

$$F_i(t, u) = -\frac{1}{h_i}\left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta_{i+1/2}}{2}\right]u_{i+1} + \left[\frac{1}{h_i}\left(\frac{1}{h_{i+1/2}} + \frac{1}{h_{i-1/2}}\right) + \frac{\lambda}{2}\left(\delta_{i+1/2} - \delta_{i-1/2}\right)\right]u_i$$
$$- \left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta_{i-1/2}}{2}\right]u_{i-1} + u_i^0$$

and

$$F(t, u) = (F_1(t, u), F_2(t, u), ..., F_N(t, u)).$$

Stability for Explicit Schemes

Theorem 16. *Let the assumption* $(H1)$ *and* $(H2)$ *holds:*

(H1): $u_0 \in L^\infty([0, T])$;

(H2): *a condition C-F-L (Courant-Friedrichs-Lewy)*

$$\Delta t \leqslant \frac{inf_{i\in\mathbb{N}} h_i}{Lm_1}$$

where Lm_1 *is a Lpschitz constant, takes place, then a solution* u_i^n *defined by*

$$u_i^{n+1} = (1 - \Delta t)u_i^n - \frac{\Delta t}{h_i}\Psi(u_{i+1}^n, u_i^n) + \frac{\Delta t}{h_i}\Psi(u_i^n, u_{i-1}^n) + \Delta t u_i^0,$$

and

$$u_i^0 = \frac{1}{h_i}\int_{K_i} u_0(x)dx, i \in \mathbb{N},$$

verifies

$$A \leqslant u_i^n \leqslant B, \text{ for all } i \in \mathbb{N},$$

and

$$\| u_i^n \|_\infty \leqslant C \| u_0 \|_\infty \leqslant B.$$

Proof. According to assumption $A \leqslant u_0 \leqslant B$, a.e, and a definition of u_i^0, we see that $\overline{A} \leqslant u_i^0 \leqslant \overline{B}$ for all $i \in \mathbb{N}$,

Let us show that this property is still true in the rank $n+1$.

We have

$$u_i^{n+1} = (1-\Delta t)u_i^n - \frac{\Delta t}{h_i}\Psi(u_{i+1}^n, u_i^n) + \frac{\Delta t}{h_i}\Psi(u_i^n, u_{i-1}^n) + \Delta t u_i^0,$$

and

$$u_i^{n+1} = (1-\Delta t)\, u_i^n - \frac{\Delta t}{h_i}\left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta_{i+1/2}}{2}\right] u_{i+1}^n + \left[\frac{\Delta t}{h_i}\left(\frac{1}{h_{i+1/2}} + \frac{1}{h_{i-1/2}}\right) + \frac{\lambda}{2}\left(\delta_{i+1/2} - \delta_{i-1/2}\right)\right] u_i^n$$
$$- \frac{\Delta t}{h_i}\left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta_{i-1/2}}{2}\right] u_{i-1}^n + +\Delta t u_i^0.$$

Therefore

$$u_i^{n+1} = -\frac{\Delta t}{h_i}\left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta_{i+1/2}}{2}\right] u_{i+1}^n + \left[1 - \Delta t + \frac{\Delta t}{h_i}\left(\frac{1}{h_{i+1/2}} + \frac{1}{h_{i-1/2}}\right) + \frac{\lambda}{2}\left(\delta_{i+1/2} - \delta_{i-1/2}\right)\right] u_i^n$$
$$- \frac{\Delta t}{h_i}\left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta_{i-1/2}}{2}\right] u_{i-1}^n + +\Delta t u_i^0.$$

hence

$$u_i^{n+1} = \vartheta_{i+1}u_{i+1}^n + \vartheta_i u_i^n + \vartheta_{i-1}u_{i-1}^n + \Delta t u_i^0 \tag{4.1}$$

where

$$\vartheta_{i+1} = -\frac{\Delta t}{h_i}\left[\frac{1}{h_{i+1/2}} - \frac{\lambda\delta_{i+1/2}}{2}\right],$$

$$\vartheta_i = \left[1 - \Delta t + \frac{\Delta t}{h_i}\left(\frac{1}{h_{i+1/2}} + \frac{1}{h_{i-1/2}}\right) + \frac{\lambda}{2}\left(\delta_{i+1/2} - \delta_{i-1/2}\right)\right],$$

and

$$\vartheta_{i-1} = -\frac{\Delta t}{h_i}\left[\frac{1}{h_{i-1/2}} + \frac{\lambda\delta_{i-1/2}}{2}\right].$$

We have

$$\vartheta_{i+1} \leqslant Lm_1\frac{\Delta t}{h_i}, \text{ for } i \in \mathbb{N},$$

$$\vartheta_i \leqslant Lm_1\frac{\Delta t}{h_i}, \text{ for } i \in \mathbb{N},$$

$$\vartheta_{i-1} \leqslant Lm_1\frac{\Delta t}{h_i}, \text{ for } i \in \mathbb{N},$$

and

$$\mid \vartheta \mid \leqslant 1$$

$\Psi(p,q)$ is a Lipschitz function on $[A,B]^2$ with the same Lipschitz constant in p and q : Lm_1. Consequently, u_i^{n+1} is a convex combination of u_i^n, u_i^{n+1} and u_i^{n-1} on one part and linear combination on the other part, then until (5.1), for all $i = 1, ..., N$, we have

$$| u_i^{n+1} | \leqslant \vartheta_{i+1} | u_{i+1}^n | + \vartheta_i | u_i^n | + \vartheta_{i-1} | u_{i-1}^n | + | \Delta t u_i^0 |,$$

and

$$| u_i^{n+1} | \leqslant (\vartheta_{i+1} + \vartheta_i + \vartheta_{i-1}) \| u^n \|_\infty + | \Delta t | \| u_0 \|_\infty$$

which implies

$$\| u^{n+1} \|_\infty \leqslant C_1 \| u^n \|_\infty + C_2 \| u_0 \|_\infty$$

and a recurrence assumption implies

$$\| u_i^{n+1} \|_\infty \leqslant C_1 \| u_i^n \|_\infty + C_2 \| u_i^0 \|_\infty$$

and gives us

$$\| u^n \|_\infty \leqslant C^{Nmax} \| u_0 \|_\infty \leqslant B.$$

Convergence

Fix an initial condition $u^0 \in L^\infty([0,1])$, that we discretize on a mesh T_m of step $h_m > 0$:

$$(u_m)_i^0 = \frac{1}{h_i} \int_{K_i} u_m(x,0) dx, i \in \mathbb{N},$$

we use a step of time Δt_m and search a function u_m.
Supposed constant on every product of form

$$]ih_m, (i+1)h_m[\times]n\Delta t_m, (n+1)\Delta t_m[:$$

$$u_m(x,t) = (u_m)_i^n, (x,t) \in]ih_m, (i+1)h_m[\times]n\Delta t_m, (n+1)\Delta t_m[$$

Calculate $(u_m)_i^{n+1}$

$$\frac{1}{\Delta t_m} \left((u_m)_i^{n+1} - (u_m)_i^n \right) = -\frac{1}{h_m} \Psi(u_{i+1}^n, u_i^n) + \frac{1}{h_m} \Psi(u_i^n, u_{i-1}^n) - (u_m)_i^n + (u_m)_i^0.$$

When $\Delta t_m \to 0$ and $h_m \to 0$ for $m \to 0$, the function family $(u_m)_{m \in \mathbb{N}}$ can converge to weak solution of problem. □

4.2 Numerical study in multidimensional case

In this section, we study the finite volume scheme applied to the elliptic-parabolic model (P) in multidimensional case
The model (P) defined as:

$$(P)\begin{cases}(P_1)\begin{cases}u_t+\Delta u-\lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right]+u=u_0 & (t,x)\in\mathbb{R}^+\times\Omega\\ u=0 & \partial\Omega\\ u(0,x)=u_0 & x\in\Omega\end{cases}\\ \qquad (P_2)\begin{cases}-\Delta v+\tau v=0 & x\in\Omega\\ v=g & \partial\Omega\end{cases}\end{cases}$$

Where

Assumption 1

1. Ω is an open bounded polygonal subset of $\mathbb{R}^3$.
2. $\tau\geqslant 0,\beta\geqslant 0$ and $\lambda\geqslant 0$.
3. $g\in C(\partial\Omega,\mathbb{R})$.
4. $u_0\in C^2(\overline{\Omega},\mathbb{R})$.

4.2.1 Definition and Theorems

We introduce some definition and theorems needed in our work. First we state the existence and the uniqueness of solution.

Theorem 17. *If $g\in L^{\frac{3}{2}}(\Omega), u_0\in L^2(\Omega)$. Then, the problem (P) has a unique solution.*

Remark 4. *The third condition of the assumption assures existence and uniqueness as well as positive of variational solution to problem (P_2) via Lax-Milgran's Theorem and Maximum principle.*

To obtain a finite volume discretization of problem (P), we introduce some important notations and definitions:

Definition 4.2.1. *(admissible meshes) Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, 2 or 3. An admissible finite volume mesh of Ω, denoted by T, is given by a family of "control volume", which are open polygonal convex subset of Ω, a familly of subset of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$,denoted by Ξ (these are the edges (two-dimemsional measure) or sides (three-dimemsional) of the control volumes), with strictly positive $(d-1)$-dimemsional measure, and a familly of points of Ω denoted by PT satisfying the followimg properties:*

1. *The closure of the union of all the control volumes is $\overline{\Omega}$.*

2. *For any $K \in T$, there existe a subset Ξ_K of Ξ such that*

$$\partial K = \overline{K} \backslash K = \bigcup_{\sigma \in \Xi_K} \overline{\sigma}.$$

Furthermore, $\Xi = \cup_{K \in T} \Xi_K$.

3. *For any $(K, L) \in T^2$ with $K \neq L$, either the $(d-1)$-dimemsional Lebesgue measure of $\overline{K} \cap \overline{L} = \overline{\sigma}$, for some $\sigma \in \Xi$, which will then be denoted by $K \backslash L$.*

4. *The family $PT = (x_K)_{K \in T}$ is such that $x_K \in \overline{K}$ (for all $K \in T$) and, if $\sigma = K \backslash L$, it is assumed that $x_k \neq x_L$, and that the straight line $D_{K,L}$ going through x_k and x_L is orthogonal to $K \backslash L$.*

5. *For any $\sigma \in \Xi$ such that $\sigma \subset \partial\Omega$, let K be the control volume such that $\sigma \in \Xi_K$. If $x_K \notin \sigma$, let $D_{K,\sigma}$ be straight line going through x_K and orthogonal to σ, then the condition $D_{k,\sigma} \cap \sigma \neq \emptyset$ is assumed, let $y_\sigma = D_{K,\sigma} \cap \sigma$.*

In this study, we use the following notations

1. *size(T) = $supdiam(K), K \in T$.*

2. *For any $K \in T$ and $\sigma \in \Xi, m(K)$ is the d-dimemsional Lebesgue measure of K and $m(\sigma)$ is the $(d-1)$-dimemsional measure of σ.*

3. *Denonting the set of interior (resp. boundary) edges by Ξ_{int} (resp.by Ξ_{ext}).*

4. *Let $N(x)$ be the set of neighbours of K. If $\sigma = K \backslash L$, we denote d_σ or $d_{K,L}$ the Euclidean distance between x_K and x_L and $d_{K,\sigma}$ is the distance from X_K to σ. If $\sigma \in \Xi_K \cap \Xi_{ext}, d_\sigma$ or $d_{K,\sigma}$ denotes the Euclidean distance between x_K and y_σ.*

5. *The transmutability through σ is defined by $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$ and $n_{K,\sigma}$ resp (n_K) denotes the outhward normal unit vector to σ resp (∂K).*

we note that throughout this chapter, T is an admissible mesh in the sense of Definition 3.1 in [22].

Exemple:

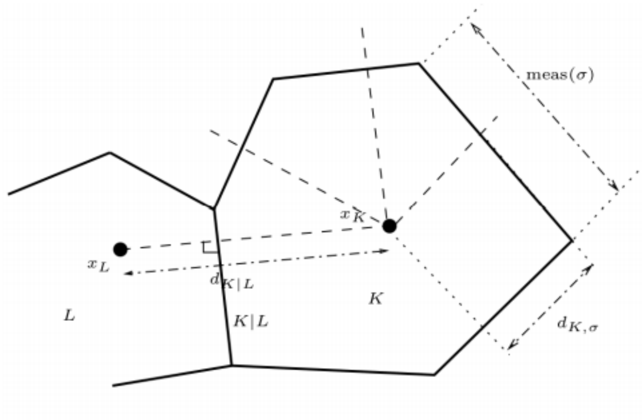

Figure 4.1: Two control volumes of an admissible mesh [22]

For the time discretization, we define a temporal partition $t^n = nk$, for $n \in 0, ...N_k + 1$ where $k \in (0, N_k)$ is the time step and $N_k = max\{n \in \aleph, nk < T\}$. The value (u^n_K, v^n_K) is an approximationof $(u(nk, x_K), v(x_K)$ at $(x_K, t^n) \in K \times (0, T)$.

Definition 4.2.2. *Let Ω be an open bounded polygonal subset of $\mathbb{R}, d = 3$ or 3, and T is an admissible mesh. Define $X(T)$ as the set of function from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.*

Definition 4.2.3. *(Discrete H^1_0 norme) Let Ω be an open bounded polygonal subset of $\mathbb{R}, d = 3$ or 3, and T an admessible finite volume mesh. For $v \in X(T)$, the discrete H^1_0 norm is defined as follows.*

$$\parallel v \parallel_{1,T} = \left(\sum_{\sigma \in \Xi} \tau_\sigma (D_\sigma v)^2 \right)^{\frac{1}{2}}, \tag{4.2}$$

where

$$D_\sigma v = \begin{cases} | v_L - v_K |, & \forall \sigma \in K \backslash L, \\ | v_K |, & if \sigma \in \Xi_K \cap \Xi_{ext}. \end{cases} \tag{4.3}$$

4.2.2 Finite volume method for the elliptic problem (P_2)

A finite volume scheme for the problem (P_2) is given by

$$\sum_{\sigma \in \Xi_K} F_{K,\sigma} + \tau m(K) v_K = 0, \forall K \in T, \tag{4.4}$$

$$F_{\sigma \in \Xi_K} = -\tau_{K \backslash L}(v_L - v_K), \forall \sigma \in \Xi_{int}, \text{ if } \sigma = K \backslash L, \tag{4.5}$$

$$F_{\sigma \in \Xi_K} = -\tau_{K \backslash L}(g(y_\sigma) - v_K), \forall \sigma \in \Xi_{ext}, \text{ where } v_\sigma = (g(y_\sigma), \forall \sigma \in \Xi_{ext}. \tag{4.6}$$

We recall some important result of the approximation solution of elliptic problem (P_2) (for proof we can see [22]).

Proposition 4.2.1. *[22]. Under Assumption 1. Let T be an admissible mesh. If $(g(y_\sigma) \geqslant 0$ for all $\sigma \in \Xi_{ext}$, then the solution $(v_K)_{K\in T}$ of (4.5)-(4.6) is positive for any $K \in T$.*

Lemma 4.2.1. *(Existence and uniqueness)[22]. Under Assumption 1, let T be an admissible mesh; there exists a unique solution $(v_K)_{K\in T}$ To the equations (4.5)-(4.6).*

Theorem 18. *(C^2 error estimation)[22]. Under Assumption 1, let T be an admissible mesh, $(v_K)_{K\in T}$ is the solution to (4.5)-(4.6). Assume that the unique variational solution v of problem (P_2) satisfies $v \in C^2(\overline{\Omega})$. Let $e_K \in X(T)$ be defined by $e_T = e_K = (v(x_K) - v_K)$ for a.e. $x \in K$ and for all $K \in T$. Then, three exists $C > 0$ depends only on v, τ and Ω such that*

$$\| e_T \|_{1,T} \leqslant Csize(T), \tag{4.7}$$

$$\| e_T \|_{L^2(\Omega)} \leqslant Csize(T), \tag{4.8}$$

and

$$\begin{gathered}\sum_{\sigma\in\Xi_{int}} m(\sigma)d_\sigma \left(\frac{(v_L - v_K)}{d_\sigma} - \frac{1}{m(\sigma)}\int_\sigma \nabla v(x).n_{K,\sigma} d\gamma(x)\right)^2 \\ \sigma = K\backslash L \\ \sum_{\sigma\in\Xi_{ext}} m(\sigma)d_\sigma \left(\frac{((g(y_\sigma) - v_K)}{d_\sigma} - \frac{1}{m(\sigma)}\int_\sigma \nabla v(x).n_{K,\sigma} d\gamma(x)\right)^2 \\ \sigma = \overline{K} \cap \partial\Omega \\ \leqslant C(size(T))^2\end{gathered}$$

4.2.3 Finite volume method for the parabolic problem (P_1)

Appplying the following implicit finite volume scheme for the discretization of parabolic problem (P_1) yield:

$$m(K)\frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma\in\Xi_K} F_{K,\sigma}^{n+1} + \sum_{\sigma\in\Xi_K} \delta V_{K,\sigma} u_{\sigma,+}^{n+1} + m(K)u_K^{n+1} = m(K)u_K^0, \tag{4.9}$$

for all $K \in T$, and for all $n \in \{0, ..., N_K\}$,

$$u_K^0 = u_0(x_K), \text{ for all } \ K \in T, \tag{4.10}$$

where

$$F_{K,\sigma}^n = -\tau_{K\backslash L}(u_K^n - u_L^n), \text{ for all } \sigma \in \Xi_{int} \text{ such that } \sigma = K\backslash L, \tag{4.11}$$

for all $n \in \{1, ..., N_K + 1\}$,

$$F_{K,\sigma}^n = -\tau_\sigma(u_L^n), \forall\sigma \in \Xi_{ext} \cap \partial\Omega, \text{ for } n \in \{0, ..., N_K + 1\}\,, \tag{4.12}$$

$$\begin{cases} u_{\sigma,+}^n = u_K^n, & \text{if } V_{k,\sigma} \geqslant 0 \\ u_{\sigma,+}^n = u_L^n, & \text{if } V_{,\sigma} < 0 \end{cases} \text{for all } \sigma \in \Xi_{int} \text{ such that } \sigma = K\backslash L, \tag{4.13}$$

$$\begin{cases} u^n_{\sigma,+} = u^n_K, & \text{if } V_{k,\sigma} \geqslant 0 \\ u^n_{\sigma,+} = 0, & \text{if } V_{,\sigma} < 0 \end{cases} \text{ for all } \sigma \in \Xi_K \text{ such that } \sigma \subset \partial\Omega, \tag{4.14}$$

$$V_{k,\sigma} = -\tau_\sigma \left[(v_L - v_K) - \left(u^0_L - u^0_K \right) \right], \text{ for all } \sigma \in \Xi_{int} \text{ such that } \sigma = K \backslash L, \tag{4.15}$$

$$V_{k,\sigma} = -\tau_\sigma \left[(g(y_\sigma) - v_K) - \left(u^0(y_\sigma) - u^0_K \right) \right], \text{ for all } \sigma \in \Xi_{ext} \text{ such that } \sigma \subset \partial\Omega, \tag{4.16}$$

with v_K satisfies the equation (4.4)-(4.6) for all $K \in T$,
and

$$\delta = \frac{\lambda}{\sqrt{\beta + \mid \nabla (v - u_0) \mid^2}}.$$

Error estimate

In this following theorem we give a L^∞ estimate as well as the estimate error

Theorem 19. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d, T > 0$. Let $u \in C^2\left(\mathbb{R} \times \overline{\Omega}, \mathbb{R}\right)$ be defined in (P_1) and $v \in C^2\left(\overline{\Omega}, \mathbb{R}\right)$ defined in (P_2). Let $u_0 \in C^2\left(\overline{\Omega}, \mathbb{R}\right)$ and $g \in C^0\left(\partial\Omega, \mathbb{R}_+\right)$.Let T be an admessible mesh and $k \in (0, T)$. Then there exists a unique vector $(u_K)_{K \in T}$ satisfying (4.9)-(4.16). In addition to that, there exists C depends only on u_0 such that*

$$Sup \left\{ \mid u^n_K \mid, K \in T, n \in \{1, ..., N_K + 1\} \leqslant C \right) \tag{4.17}$$

Furthermore, let $e^n_K = u(x_K, t_n) - u^n_K$, for $K \in T, n \in \{1, ..., N_K + 1\}$ and $h = size(T)$. Then there exists $C > 0$ depends only on u, v, Ω and T such that

$$\left(\sum_{K \in T} m(K)(e^n_K)^2 \right)^{1\backslash 2} \leqslant C(h + k), \forall K \in T, \forall n \in \{1, ..., N_K + 1\}. \tag{4.18}$$

Proof. **i) Existence and uniqueness and L^∞ estimate:** For the existence and uniqueness of the solution of limite volume scheme, we can follow the proof of Lemma 3.2 page 42 in [22].

Let us now prove the estimate (4.17). We set $m_0 = min\ u(x, 0) = u_0(x), x \in \overline{\Omega}$. Let $n \in \{0, ..., N_K\}$. Then, we claim that

$$min\left\{ u^{n+1}_K, K \in T \right\} \geqslant min\left\{ min\left\{ u^n_K, K \in T \right\} + Tm_0, 0 \right\} \tag{4.19}$$

Indeed, if $min\left\{ u^n_K, K \in T \right\} < 0$, let $K_0 \in T$ such that

$$u^{n+1}_{K_0} = min\left\{ u^{n+1}_K, K \in T \right\}.$$

Since $u^{n+1}_K < 0$, and replacing in relation (4.9) K by K_0 to get

$$\begin{aligned} m(K)\frac{u^{n+1}_{K_0} - u^n_{K_0}}{k} = &- \sum_{\sigma \in \Xi_{K_0}} F^{n+1}_{K_0,\sigma} - \sum_{\sigma \in \Xi_{K_0}} \delta V_{K_0,\sigma}\left(u^{n+1}_{\sigma,+} - u^{n+1}_{K_0} \right) - \sum_{\sigma \in \Xi_{K_0}} \delta V_{K_0,\sigma} u^{n+1}_{K_0} \\ &- m(K)u^{n+1}_{K_0} + m(K)u^0_{K_0} \end{aligned} \tag{4.20}$$

with $K_0 \in T$ ana $n \in \{0, ..., N_K\}$.
From the definition of $F^{n+1}_{K_0,\sigma}$ and $u^{n+1}_{\sigma,+}$, we deduce that

$$\sum_{\sigma \in \Xi_{K_0}} F^{n+1}_{K_0,\sigma} \leqslant 0 \text{ and } \sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma}\left(u^{n+1}_{\sigma,+} - u^{n+1}_{K_0}\right) \leqslant 0. \tag{4.21}$$

Next, in view the proposition 4.2.1, the function v is positive verifying $-\Delta v + \tau v = 0$, then we find

$$\sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma} u^{n+1}_{K_0} = u^{n+1}_{K_0} \int_{\partial K_0} \nabla v . n_{K_0}(x)\gamma(x) = \tau C \int_{K_0} v dx \leqslant 0. \tag{4.22}$$

Thus, we have

$$u^{n+1}_{K_0} \geqslant u^n_{K_0} + u^0_{K_0} \geqslant min\left\{u^n_K, K \in T\right\} + K m_0. \tag{4.23}$$

This prove (4.19) and by induction, we deduce that

$$min\left\{u^n_K, K \in T\right\} \geqslant min\left\{min\left\{u^0_K, K \in T\right\}, 0\right\} + nkmin\left\{m_0, 0\right\}, n \in \{0, ..., N_K + 1\} \tag{4.24}$$

Similarly,

$$max\left\{u^n_K, K \in T\right\} \leqslant max\left\{max\left\{u^0_K, K \in T\right\}, 0\right\} + nkmax\left\{M_0, 0\right\}, n \in \{0, ..., N_K + 1\}, \tag{4.25}$$

with $M_0 = max u_0(x), x \in \overline{\Omega}$.
this proves (4.17) with C depends only by u_0.

ii) Error estimate: One uses the regularity of the data and the solution to write an equation for the error $e^n_K = u(x_K, t_n) - u^n_K$, defined for $K \in T$ and $n \in \{1, ..., N_K + 1\}$. Note that $e^0_K = 0$ for $K \in T$. Let $n \in \{1, ..., N_K\}$.
Integrating the first equation of problem (P_1) over each control volume K of T, at time $t = t_{n+1}$

$$\begin{aligned}\int_K u_t(x, t_{n+1})dx + \int_{\partial K} \left(\nabla u(x, t_{n+1}) - \delta \nabla (v - u_0) u(x, t_{n+1})\right) n_K d\gamma(x) \\ + \int_K u(x, t_{n+1})dx = m(K) u^K_0.\end{aligned} \tag{4.26}$$

Note that, for all $x \in K$ and all $K \in T$, a Taylor exeption yields, thanks to the regularity of u:

$$u_t(x, t) = \frac{1}{k}\left(u(x, t_{n+1}) - u(x_K, t_n)\right) + s^n_K(x)$$

with $\mid s^n_K(x) \mid \leqslant C_1(h + k)$
with some C_1 only depending on u and T. Therefore, defining $S^n_K = \int_K s^n_K(x)dx$, on has

$$\mid S^n_K(x) \mid \leqslant C_1 m(K)(h + k) \tag{4.27}$$

Substracting (4.9) from (4.26)

$$m(K)\frac{e_K^{n+1}-e_K^n}{k}+\sum_{\sigma\in\Xi_K}\left(G_{K,\sigma}^{n+1}+\delta W_{K,\sigma}^{n+1}\right)+m(K)e_K^{n+1}=m(K)\rho_K^n \tag{4.28}$$
$$-\sum_{\sigma\in\Xi_K}m(\sigma)\left(R_{K,\sigma}^n+\delta r_{K,\sigma}^n\right)-S_K^n,\forall K\in T$$

where

$$\begin{aligned}G_{K,\sigma}^{n+1}&=\tau_\sigma\left(e_L^{n+1}-e_K^{n+1}\right),\forall\sigma\in\Xi_{int}\cap\Xi_K,\sigma=K\backslash L,\\ G_{K,\sigma}^{n+1}&=-\tau_\sigma e_K^{n+1},\forall\sigma\in\Xi_{ext}\cap\Xi_K,\\ W_{K,\sigma}^{n+1}&=W_{K,\sigma,1}^{n+1}+W_{K,\sigma,2}^{n+1}\end{aligned}$$

with

$$\begin{aligned}e_K^n&=(v(x_K)-v_K)=\left(u_0(x_K)-u_K^0\right),\\ W_{K,\sigma,1}^{n+1}&=-\tau_\sigma\left(e_L-e_K\right)u_{\sigma,+}^{n+1}=-\tau_\sigma D_\sigma(e)u_{\sigma,+}^{n+1}\\ W_{K,\sigma,2}^{n+1}&=-\tau_\sigma\left[(v(x_L)-u_0(x_L))-(v(x_K)-u_0(x_K))\right]\left(u(x_{\sigma,+},t_{n+1})-u_{\sigma,+}^{n+1}\right)\\ &=-\tau_\sigma D_\sigma(v-u_0)e_{\sigma,+}^{n+1},\end{aligned}$$

where $x_{\sigma,+}=x_K$ (resp x_L) if $\sigma\in\Xi_{int},\sigma=K\backslash L$ and $\nabla(v-u_0)n_{K,\sigma}\geqslant 0$ (resp $\nabla(v-u_0)n_{K,\sigma}<0$) and $x_{\sigma,+}=x_K$ (resp y_σ) if $\sigma\in\Xi_{ext}\cap\Xi_K$ and $\nabla(v-u_0)n_{K,\sigma}\geqslant 0$ (resp $\nabla(v-u_0)n_{K,\sigma}<0$),

$$\rho_K^n=u(x_K,t_{n+1})-\frac{1}{m(K)}\int_K u(x,t_{n+1})dx,$$

$$-m(\sigma)R_{K,\sigma}^n=\begin{cases}-\tau_\sigma\left(u(x_K,t_{n+1})-u(x_L,t_{n+1})\right)-\int_\sigma\nabla u(x,t_{n+1})n_{K,\sigma}d\gamma(x), & if\sigma=K\backslash L\in\Xi_{int}\\ -\tau_\sigma\left(u(x_K,t_{n+1})\right)-\int_\sigma\nabla u(x,t_{n+1})n_{K,\sigma}d\gamma(x), & if\sigma\in\Xi_{ext}\cap\Xi_K,\end{cases}$$

$$m(\sigma)r_{K,\sigma}^n=\begin{cases}-\tau_\sigma\left[(v(x_K)-u_0(x_K))-(v(x_L)-u_0(x_L))\right]+\int_\sigma\nabla(v-u_0)(x)n_{K,\sigma}u(x,t_{n+1}),\\ if\sigma=K\backslash L\in\Xi_{int},\\ -\tau_\sigma\left[(v(x_K)-u_0(x_K))-(g(y_\sigma)-u_0(y_\sigma))\right]+\int_\sigma\nabla(v-u_0)(x)n_{K,\sigma}u(x,t_{n+1}),\\ if\sigma\in\Xi_{ext}\cap\Xi_K,\end{cases}$$

Thanks to the regularity of u, there exists C_2, only depending on u,v,u_0 and T such that

$$\mid R_{K,\sigma}^n\mid+\mid r_{K,\sigma}^n\mid\leqslant C_2h \tag{4.29}$$

and

$$\mid\rho_K^n\mid\leqslant C_2h \tag{4.30}$$

for any $K \in T$ and $\sigma \in \Xi_K$.
Multiplying (4.28) by e_K^{n+1}, summing for $K \in T$, and noting that

$$\sum_{K\in T}\sum_{\sigma\in\Xi_K} G_{K,\sigma}^{n+1}e_K^{n+1} = \sum_{\sigma\in\Xi_K} -\tau_\sigma(e_K^{n+1}) = \| e_T^{n+1} \|_{1,T}^2,$$
$$\sum_{K\in T}\sum_{\sigma\in\Xi_K} \delta W_{K,\sigma,2}^{n+1}e_K^{n+1} \geqslant 0,$$
$$\sum_{K\in T} m(K)(e_K^{n+1})^2 = \| e_T^{n+1} \|_{L^2(\Omega)}^2,$$

where $e_K^n \in X(T), e_T^n(x) = e_K^n$ for a.e. $x \in T$, we obtain

$$(1+k) \| e_K^{n+1} \|_{L^2(\Omega)}^2 + k \| e_T^{n+1} \|_{1,T}^2 \leqslant km(K)\rho_K^n e_K^{n+1} - k\sum_{K\in T}\sum_{\sigma\in\Xi_K} mes(\sigma)(R_{K,\sigma}^n + \delta r_{K,\sigma}^n)e_K^{n+1}$$
$$+ \sum_{K\in T} k \mid S_K^n \mid e_K^{n+1} - k\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_K^{n+1} + \sum_{K\in T} m(K)e_K^{n+1}e_K^n. \tag{4.31}$$

by young's inequality, the first term of the left hand side satisfies:

$$\mid \sum_{K\in T} m(K)\rho_K^n e_K^{n+1} \mid \leqslant \frac{1}{2} \| e_K^{n+1} \|_{L^2(\Omega)}^2 + \frac{1}{2}C_3^2(size(T))^2 m(\Omega). \tag{4.32}$$

Hence, (4.31) and (4.32) yield that there exists C_4 only depending on u and Ω such that

$$(1+k) \| e_K^{n+1} \|_{L^2(\Omega)}^2 + \frac{k}{2} \| e_T^{n+1} \|_{1,T}^2 \leqslant C_4kh^2 - k\sum_{K\in T}\sum_{\sigma\in\Xi_K} mes(\sigma)(R_{K,\sigma}^n + \delta r_{K,\sigma}^n)e_K^{n+1}$$
$$+ \sum_{K\in T} k \mid S_K^n \mid e_K^{n+1} - k\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_K^{n+1} + \sum_{K\in T} m(K)e_K^{n+1}e_K^n. \tag{4.33}$$

Using (4.27), (4.29) and (4.30) and Cauchy-Schwarz's inequality yields,

$$(1+k) \| e_K^{n+1} \|_{L^2(\Omega)}^2 + \frac{k}{2} \| e_T^{n+1} \|_{1,T}^2 \leqslant C_4kh^2 + C_1k(k+h)\sum_{K\in T} m(K) \mid e_K^{n+1} \mid$$
$$- k\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_K^{n+1} + \sum_{K\in T} m(K)e_K^{n+1}e_K^n. \tag{4.34}$$

Applying again the Cauchy-Schwarz's inequality and the the estimate ($ab \leqslant \frac{1}{2}(a^2+b^2)$) to estimate

$$k(k+h)\sum_{K\in T} m(K)e_K^{n+1} \leqslant k(k+h)\left(\sum_{K\in T} m(K)\right)^{\frac{1}{2}}\left(\sum_{K\in T} m(K)(e_K^{n+1})^2\right)^{\frac{1}{2}}$$
$$= k(k+h)(m(\Omega))^{\frac{1}{2}} \| e_K^{n+1} \|_{L^2(\Omega)}$$

and

$$\sum_{K\in T} m(K)e_K^{n+1}e_K^n \leqslant \frac{1}{2}m(K)(e_K^{n+1})^2 + \frac{1}{2}m(K)(e_K^n)^2$$
$$= \frac{1}{2} \parallel e_K^{n+1} \parallel_{L^2(\Omega)}^2 + \frac{1}{2} \parallel e_K^n \parallel_{L^2(\Omega)}^2$$

Using the estimate (4.7) of the Theorem 18 to find

$$\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_{\sigma,+}^{n+1} = \sum_{\sigma\in\Xi_K} -\tau_\sigma D_\sigma(e)u_{\sigma,+}^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,-}^{n+1})$$
$$\leqslant C \parallel e_T \parallel_{1,T} \parallel e_T^{n+1} \parallel_{1,T}$$
$$\leqslant C_5 h \parallel e_T^{n+1} \parallel_{1,T},$$

where for all $\sigma \in \Xi_K$,

$$e_{\sigma,+}^{n+1} = \begin{cases} e_K^n, & \text{if } V_{K,\sigma} \geqslant 0, \\ e_K^n, & \text{if } V_{K,\sigma} < 0 \end{cases}$$

$$e_{\sigma,-}^{n+1} = \begin{cases} e_K^n, & \text{if } V_{K,\sigma} \geqslant 0, \\ e_K^n, & \text{if } V_{K,\sigma} < 0 \end{cases}$$

and $V_{K,\sigma} = -\tau_\sigma D_\sigma(v - u_0)$.
We note that if $\sigma \in \Xi_{ext}$ then $e_{\sigma,+}^{n+1} = e_{\sigma,-}^{n+1} = 0$.
Then, we obtain

$$(\frac{1}{2} + k) \parallel e_K^{n+1} \parallel_{L^2(\Omega)}^2 + \frac{k}{2} \parallel e_T^{n+1} \parallel_{1,T} \leqslant C_4 kh^2 + C_1 k(k+h)(m(\Omega))^{\frac{1}{2}} \parallel e_K^{n+1} \parallel_{L^2(\Omega)}^2$$
$$+ C_6 kh \parallel e_T^{n+1} \parallel_{1,T} + \frac{1}{2} \parallel e_T^n \parallel_{L^2(\Omega)}^2 \quad (4.35)$$

So, applying again the estimate ($ab \leqslant \frac{1}{2}(a^2 + b^2)$) to estimate

$$kh \parallel e_T^{n+1} \parallel_{1,T} \leqslant \frac{k}{2}h^2 + \frac{k}{2} \parallel e_T^{n+1} \parallel_{1,T}^2 .$$

Then, we deduce

$$(1 + 2k) \parallel e_K^{n+1} \parallel_{L^2(\Omega)}^2 \leqslant C_7 kh^2 + 2C_1 k(k+h)(m(\Omega))^{\frac{1}{2}} \parallel e_K^{n+1} \parallel_{L^2(\Omega)} + \parallel e_T^n \parallel_{L^2(\Omega)}^2 . \quad (4.36)$$

Since, we obtain

$$(1 + 2k) \parallel e_K^{n+1} \parallel_{L^2(\Omega)}^2 \leqslant C_8 \left[kh^2 + k(k+h) \parallel e_K^{n+1} \parallel_{L^2(\Omega)}\right] + \parallel e_T^n \parallel_{L^2(\Omega)}^2 . \quad (4.37)$$

Where $C_8 \in \mathbb{R}$ only depends on u, v, u_0, Ω and T.
Remarking that for $\varepsilon > 0$, the following inequality holds

$$C_8 k(k+h) \parallel e_T^{n+1} \parallel_{L^2(\Omega)} \leqslant \varepsilon^2 \parallel e_T^{n+1} \parallel_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2} C_8^2 k^2 (k+h)^2$$

taking $\varepsilon^2 = 2k$, (4.37) yeils,

$$\| e_T^{n+1} \|^2_{L^2(\Omega)} \leqslant C_8 k h^2 + \frac{1}{2} C_8^2 k (k+h)^2 + \| e_T^n \|^2_{L^2(\Omega)} . \tag{4.38}$$

Then, if $\| e_T^n \|^2_{L^2(\Omega)} \leqslant C_n (k+h)^2$, with $C_n \in \mathbb{R}_+$, one deduces from (4.38), using $h \leqslant k+h$ and $k < T$, that

$$\| e_T^{n+1} \|^2_{L^2(\Omega)} \leqslant C_n (k+h)^2$$

with $C_{n+1} = 2kC_n + C_9$ and $C_9 = C_8(0+T) + C_8^2(0+T)^2$.
Note that C_9 only depends on u, v, u_0, Ω and T.
Choosing $C_0 = 0$ (since $\| e_T^0 \|^2_{L^2(\Omega)} = 0$), the relation between C_n and C_{n+1} yeilds $C_n \leqslant C_9 e^{2kn}$.
Estimate (4.18) follows with $C^2 \leqslant C_8 e^{4T}$.

Proposition 4.2.2. *Under assumption 1. Let T be an admissible mesh. If $u_0 \geqslant 0$ for all $K \in T$, then the solution $(u_K)_{K \in T}$ of (4.9)-(4.16) satisfied $u_K \geqslant 0$ for all $K \in T$.*

Proof. We have $u_K) \geqslant 0$ and from the relation (4.24), we deduce that

$$min \{u_K^n, K \in T\} \geqslant min \left\{ min \{u_K^0, K \in T\}, 0 \right\} + nk min \{m_0, 0\} = 0,$$

therefore $u_K^n \geqslant 0$. So this proves this Proposition. □

□

4.2.4 Finite volume method for the elliptic-parabolic problem (P)

A finite volume scheme of the problem (P) can be defined by the following set of equations:

$$m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \Xi_K} F_{K,\sigma}^{n+1} + \sum_{\sigma \in \Xi_K} \delta V_{K,\sigma} u_{\sigma,+}^{n+1} + m(K) u_K^{n+1} = m(K) u_K^0, \forall K \in T \tag{4.39}$$

$$\sum_{\sigma \in \Xi_K} F_{K,\sigma} + \tau m(K) v_K = 0, \forall K \in T, \tag{4.40}$$

$$u_K^0 = u_0(x_K), \forall K \in T, \tag{4.41}$$

$$F_{K,\sigma}^n = -\tau_{K \backslash L}(u_K^n - u_L^n), \forall \sigma \in \Xi_{int}, \text{ if } \sigma = K \backslash L, \tag{4.42}$$

$$F_{K,\sigma}^n = -\tau_\sigma(u_L^n), \forall \sigma \in \Xi_{ext} \cap \partial\Omega, \tag{4.43}$$

$$V_{k,\sigma} = -\tau_\sigma \left[(v_L - v_K) - \left(u_L^0 - u_K^0 \right) \right], \forall \sigma \in \Xi_{int} \text{ if } \sigma = K \backslash L, \tag{4.44}$$

$$V_{k,\sigma} = -\tau_\sigma \left[(g(y_\sigma) - v_K) - \left(u^0(y_\sigma) - u_K^0 \right) \right], \forall \sigma \in \Xi_{ext}, \text{ if } \sigma \subset \partial\Omega, \tag{4.45}$$

$$F_{\sigma \in \Xi_K} = -\tau_{K \backslash L}(v_L - v_K), \forall \sigma \in \Xi_{int}, \text{ if } \sigma = K \backslash L, \tag{4.46}$$

$$F_{\sigma \in \Xi_K} = -\tau_{K \backslash L}(g(y_\sigma) - v_K), \forall \sigma \in \Xi_{ext}, \forall \sigma \in \Xi_{ext}. \tag{4.47}$$

With $u_{\sigma,+}^{n+1} + m(K)u_K^{n+1}$ is defined by the relation (4.13)-(4.14). In this work, we assume that the unknowns u and v are constants over each control volume K of the mesh T.
Now, from section 4.2.2 and 4.2.3 we are able to conclude that the main result is represented by the following Theorem.

Theorem 20. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d, T > 0$. Let $(u, v) \in C^2\left(\mathbb{R} \times \overline{\Omega}, \mathbb{R}\right) \times C^2\left(\overline{\Omega}, \mathbb{R}\right)$ be defined in (P). Let $u_0 \in C^2\left(\overline{\Omega}, \mathbb{R}\right)$ and $g \in C^2\left(\partial\overline{\Omega}, \mathbb{R}_+\right)$. Let T be an admissible mesh and $k \in (0, T)$. Then there exists a unique vector $(u_K, v_K)_{K\in T}$ satisfying (4.39)-(4.47). Furthermore, let $e_K^n = u(x_K, t_n) - u_K^n$, for $K \in T, n \in \{1, ..., N_K + 1\}$ and $h = size(T)$. Then, there exists C depends only on u, v, u_0, Ω and T such that*

$$\left(\sum_{K\in T} m(K)(e_K^n)^2\right)^{1\backslash 2} + \left(\sum_{K\in T} m(K)(e_K)^2\right)^{1\backslash 2} \leqslant C(h + k), \forall K \in T, \forall n \in \{1, ..., N_K + 1\}.$$

Proof. The demonstration of this Theorem is a consequence of Lemma 4.2.1, Theorem 18 and 19. □

Proposition 4.2.3. *Under assumption 1. Let T be an admissible mesh. If $u_0 \geqslant 0$ for all $K \in T$, then the solution $(u_K, v_K)_{K\in T}$ of (4.39)-(4.47) is positive for all $K \in T$, ($u_K \geqslant 0$ and $v_K \geqslant 0$ for all $K \in T$).*

Proof. The demonstration of this Proposition follows from the Proposition 4.2.1 and 4.2.2 □

4.3 Numerical analysis

In this section, we simulate the numerical solutions of the following Thixotropic problem

$$u_t + \frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial}{\partial x}\left[u \frac{\frac{\partial}{\partial x}(v - u_0)}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v - u_0)\right|^2}} \right] + u = u_0, \ (t, x) \in [0, 1] \times \Omega$$

$$-\frac{\partial^2 v}{\partial x^2} + \tau v = 0, \ x \in \Omega$$

In our numerical method, we consider $\Omega = [0, 1]$ and the time (in second) $t = 1$, the spatial step size $h = 10^{-1}$ and the time step size $k = 10^{-4}s$, $\tau = 0.01$, $\lambda = 0.02$ and

$$u(x, 0) = x^2 + 1$$

$$v(x) = Ae^{\sqrt{\tau}x} + Be^{-\sqrt{\tau}x} \in C^\infty([a, b]),$$

where

$$A = \left(\eta e^{-\sqrt{\tau}b} - \theta e^{-\sqrt{\tau}a}\right) \backslash \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right),$$

and

$$B = \left(\theta e^{\sqrt{\tau}a} - \eta e^{\sqrt{\tau}b}\right) \setminus \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right).$$

The boundary conditions are:

$$u(a,t) = u(b,t) = 0,$$
$$v(a) = \eta,\ v(b) = \theta$$

Let us $a = 0$, $b = 1$, $\eta = 1$ and $\theta = 2$. So, we have the following numerical results which defined in Figures 4.1-4.3.

Discussion

Note that for all $0 < \beta < 1$, the flow speed of fluid increase with time, this reason leads to a decrease in viscosity of the fluid which explains the phenomeon of thixotripy.
We also note that for all $0 < \lambda < 1$ (initial parameter of viscosity) with increasing of the temperature, the viscosity decrease with time and this is what we observe with Ketchup that remains knit until its viscosity is decreasing with time and by high the temperature, while its flow speed of fluid is increasing.

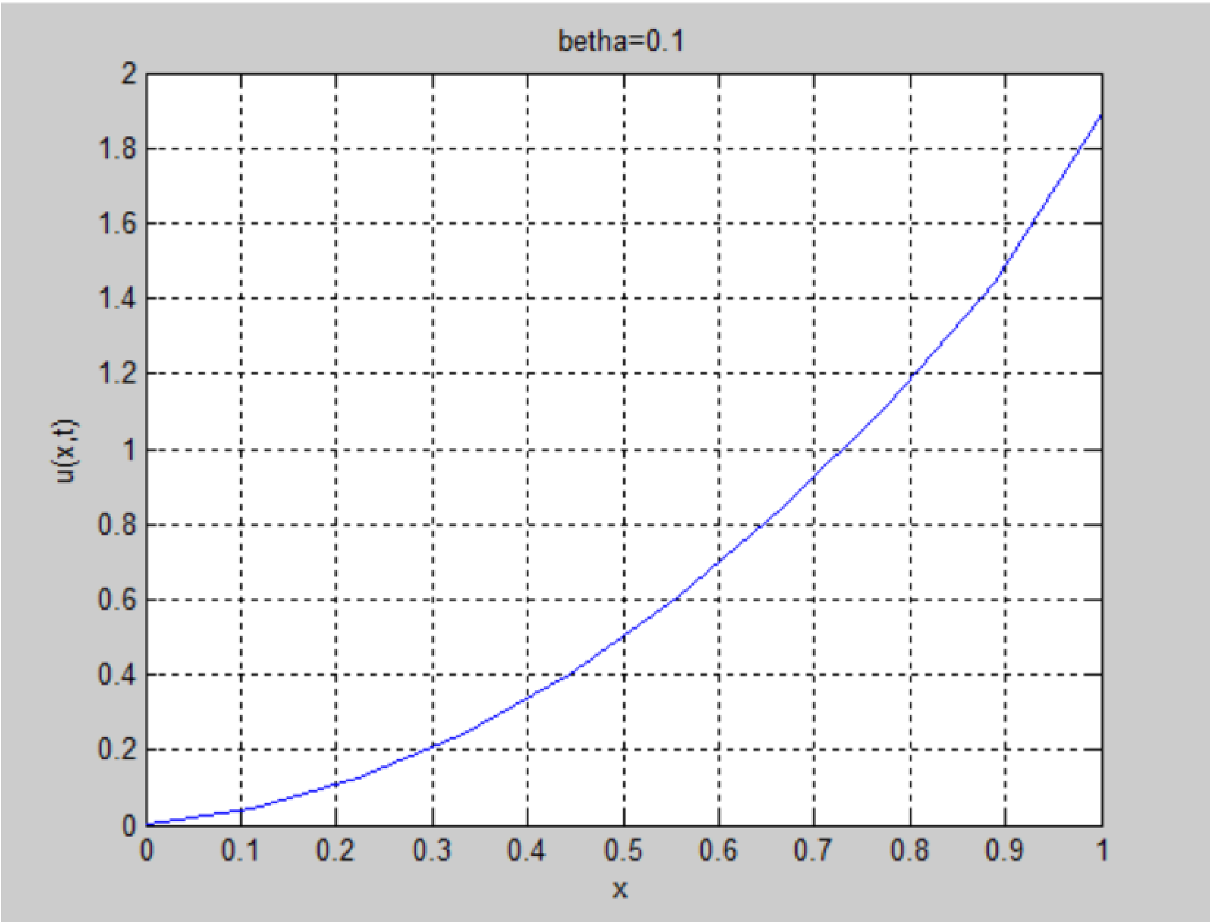

Figure 4.1: Numerical solution of the thixotropic problem for $\beta = 0.1$

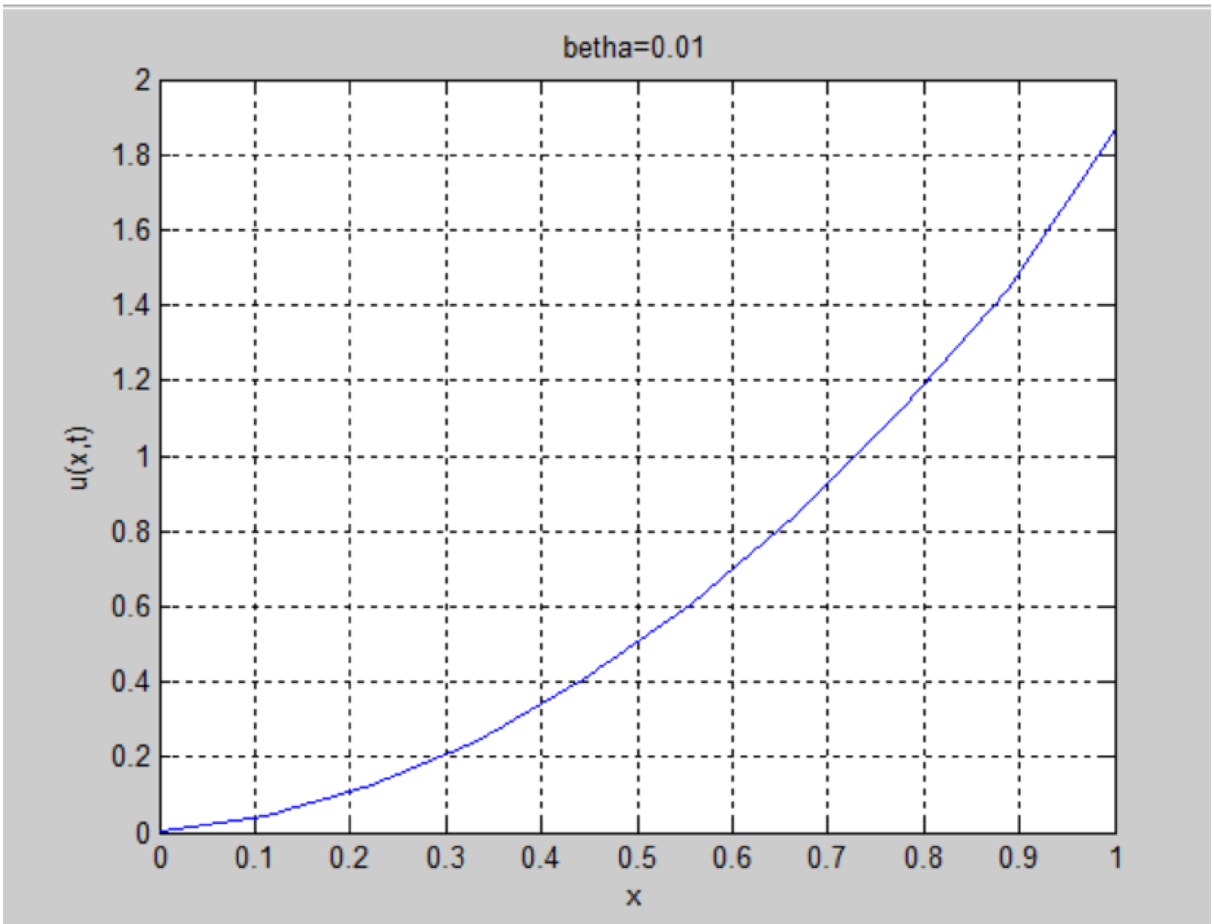

Figure 4.2: Numerical solution of the thixotropic problem for $\beta = 0.01$

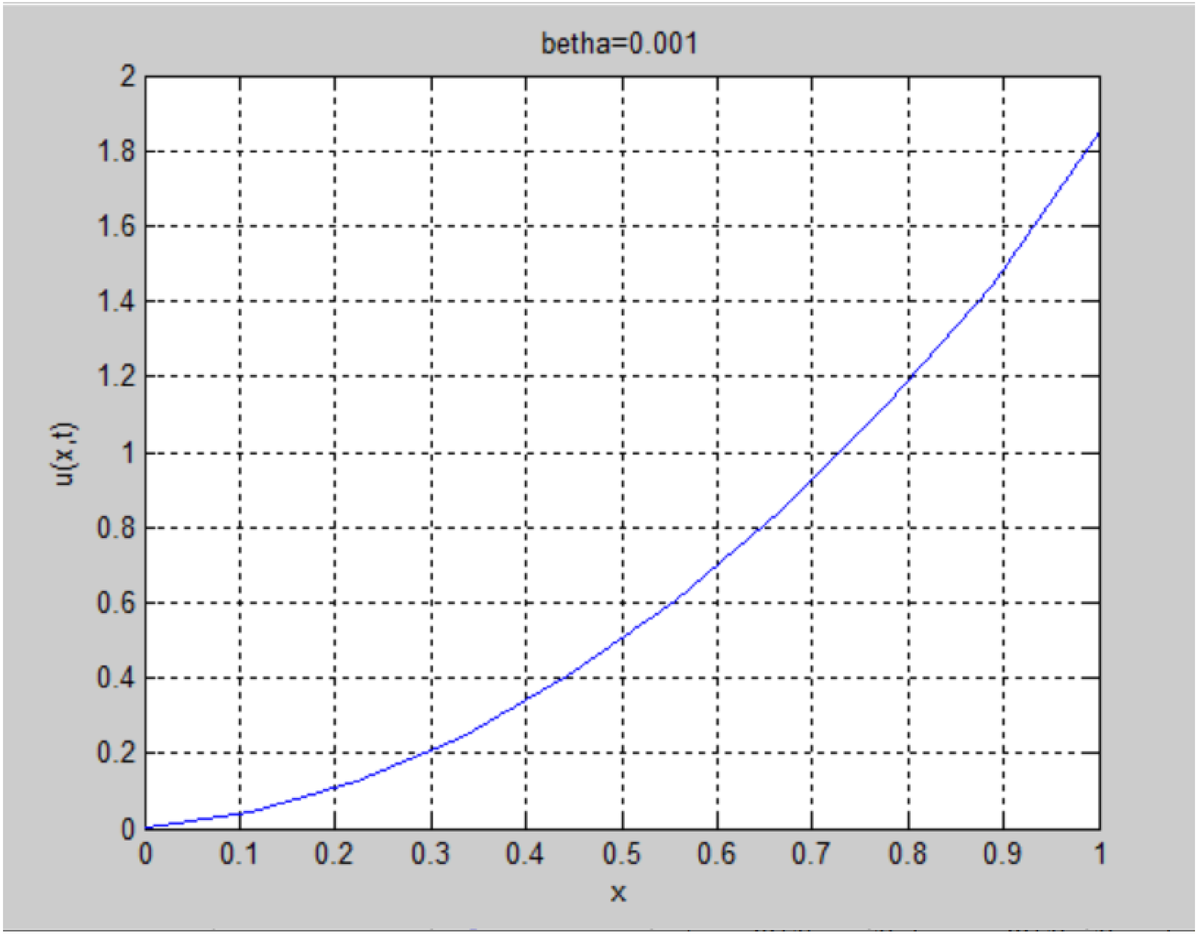

Figure 4.3: Numerical solution of the thixotropic problem for $\beta = 0.001$

Chapter 5

Numerical study in one dimensional case for the fractional Thixotropic problem

In this chapter, we study and approximate the fractional thixotropic problem (FP) by using the finite volume schemes in one dimensional case which is based on Method of Lines (MOL).

$$(FP)\begin{cases}(P_1)\begin{cases}u_t + D^{\alpha}u - \lambda\dfrac{\partial}{\partial x}\left[u\dfrac{\dfrac{\partial}{\partial x}(v-u_0)}{\sqrt{\beta+\left|\dfrac{\partial}{\partial x}(v-u_0)\right|^2}}\right]+u=u_0 & (t,x)\in\mathbb{R}^+\times\Omega\\ u=0 & \partial\Omega\\ u(0,x)=u_0 & x\in\Omega\end{cases}\\ \qquad (P_2)\begin{cases}-\dfrac{\partial^2 v}{\partial x^2}+\tau v=0, & x\in\Omega\\ v=g & \partial\Omega\end{cases}\end{cases}$$

Where $\Omega = [0,1]$, β, τ and λ are positive constants and $D^{\alpha} \equiv (-\partial^2 \backslash \partial x^2)^{\frac{\alpha}{2}}$ defined by

$$(D^{\alpha}C)(x) = -\frac{1}{\Gamma(\alpha)}\int_0^x (x-Z)^{\alpha-1}C(Z)dZ.$$

Definition 5.0.1. *(Riemann-Liouville fractional derivative on $[a,b]$)*

$$D^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} = \frac{\partial^{\alpha}}{\partial x^2}(I^{2-\alpha})u(x,t),\ \textit{for}\ \alpha\in\,]1,2[\,,$$

Where $I^{\alpha}(.)$ is called the Liouville integral and it is given by the following integral

$$I^{\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{u(\xi,t)}{(x-\xi)^{1-\alpha}}d\xi,$$

Where $\Gamma(.)$is the well known Gamma function.

5.1 Discretization method of the problem

In this section, we only discretize the first equation of the problem (P_1) because the first equation of the problem (P_2) with the boundary conditions $v(a) = \eta$ and $v(b) = \theta$ is easy given by the following equation:

$$v(x) = Ae^{\sqrt{\tau}x} + Be^{-\sqrt{\tau}x} \in C^{\infty}([a, b]),$$

where

$$A = \left(\eta e^{-\sqrt{\tau}b} - \theta e^{-\sqrt{\tau}a}\right) \setminus \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right),$$

and

$$B = \left(\theta e^{\sqrt{\tau}a} - \eta e^{\sqrt{\tau}b}\right) \setminus \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right).$$

Definition 5.1.1. *(admessible mesh) An admessible mesh of (P_1), denoted by T, is given a familly $(K_i)_{i=0,...,N}$, $N \in \mathbb{N}^*$, such that $K_i = [x_{i-1/2}, x_{i+1/2}]$, and familly $(x_i)_{i=0,...,N+1}$ such that*

$$x_0 = x_{1/2} = 0 < x_1 < x_{3/2} < ... < x_{i-1/2} < x_i < x_{i+1/2}$$

$$< x_N < x_{N+1/2} = x_{N+1} = 1.$$

Ones set

$$h_i = m(K_i) = x_{i+1/2} - x_{i-1/2}, i = 1, ..., N$$

and therefore $\sum_{i=1}^{N} h_i = 1$,

$$h_i^- = x_i - x_{i-1/2}, i = 1, ..., N,$$

$$h_i^+ = x_{i+1/2} - x_i, i = 1, ..., N,$$

$$h_{i-1/2} = x_i - x_{i-1}, i = 0, ..., N,$$

$$h_{i+1/2} = x_{i+1} - x_i, i = 0, ..., N,$$

$$size(T) = h = max\{h_i, i = 1, ..., N\}.$$

A weak solution $u(x, t)$ is defined on $[0, T] \times [0, 1]$, we introduce a mean value $u_i(t)$ of a solution, that one assumes exists in the following meaning

$$u_i(t) = \frac{1}{h_i}\int_{K_i} u(x, t)dx, i \in \mathbb{N},$$

$$\frac{\partial u_i(t)}{\partial t} = \frac{1}{h_i}\int_{K_i} \frac{\partial u(x, t)}{\partial t}dx, i \in \mathbb{N},$$

$$u_i(0) = \frac{1}{h_i}\int_{K_i} u(x, 0)dx, i \in \mathbb{N},$$

Now, we get to approach $(D^\alpha C)(x)$ on K_i (namely $x \in K_i$) by a quadrature formula. A new quadrature formula has been proposed which uses weight functions. This formula has the form given below:

$$(D^\alpha C)(x) = -\frac{1}{\Gamma(\alpha)} \int_0^x (x-Z)^{\alpha-1} C(Z) dZ = -\frac{1}{\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} (x-sh)^{\alpha-1} C(sh),$$

where sh are nodes of a quadrature formula and $w_{s,i}$ are weight functions with $\sum_{s=0}^{i} w_{s,i} = 1$. Integrate the first equation of problem (P_2) on K_i, with

$$\delta(x) = \frac{\frac{\partial}{\partial x}(v-u_0)(x)}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v-u_0)(x)\right|^2}},$$

in order to obtain

$$\begin{aligned} \frac{d}{dt}\int_{K_i} u dx + \int_{K_i} D^\alpha u dx - \lambda \int_{K_i} \frac{\partial}{\partial x}(u\delta)\, dx \\ + \int_{K_i} u dx = \int_{K_i} u_0 dx \end{aligned}$$

which implies

$$\begin{aligned} & h_i \frac{\partial u_i}{\partial t} - \frac{1}{\Gamma(\alpha)} \int_{K_i}\int_0^x (x-Z)^{\alpha-1} C(Z) dZ dx \\ & - \lambda \left[u(x_{i+1/2},t)\delta(x_{i+1/2}) - u(x_{i-1/2},t)\delta(x_{i-1/2})\right] \\ & + h_i u_i = h_i u_i^0. \end{aligned}$$

namely,

$$\begin{aligned} & h_i \frac{\partial u_i}{\partial t} - \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \int_{K_i} (x-x_s)^{\alpha-1} dx \\ & - \lambda \left[u(x_{i+1/2},t)\delta(x_{i+1/2}) - u(x_{i-1/2},t)\delta(x_{i-1/2})\right] \\ & + h_i u_i = h_i u_i^0. \end{aligned}$$

implies

$$\begin{aligned} & h_i \frac{\partial u_i}{\partial t} - \frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \left[(x_{i+1/2}-x_s)^\alpha - (x_{i-1/2}-x_s)^\alpha\right] \\ & - \lambda \left[u(x_{i+1/2},t)\delta(x_{i+1/2}) - u(x_{i-1/2},t)\delta(x_{i-1/2})\right] \\ & + h_i u_i = h_i u_i^0. \end{aligned}$$

implies

$$
\begin{aligned}
& h_i \frac{\partial u_i}{\partial t} - \left[\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)(x_{i+1/2} - x_s)^{\alpha} + \lambda u(x_{i+1/2},t)\delta(x_{i+1/2}) \right] \\
& + \left[\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)((x_{i-1/2} - x_s)^{\alpha} + \lambda u(x_{i-1/2},t)\delta(x_{i-1/2}) \right] \\
& + h_i u_i = h_i u_i^0.
\end{aligned}
$$

Denote the flux of exact solution at the vertex $x_{i+1/2}$ and $x_{i-1/2}$ by

$$
\begin{aligned}
F(x_{i+1/2},t) &= -\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)(x_{i+1/2} - x_s)^{\alpha} - \lambda u(x_{i+1/2},t)\delta(x_{i+1/2}) \\
&= -\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)(x_{i+1/2} - x_s)^{\alpha} - \frac{\lambda\delta(x_{i+1/2})}{2} \left[u(x_{i+1},t) + u(x_i,t) \right]
\end{aligned}
$$

Where

$$
\begin{aligned}
\delta(x_{i+1/2}) &= \frac{\frac{\partial}{\partial x}\left(v - u_0\right)\left(x_{i+1/2}\right)}{\sqrt{\beta + \left| \frac{\partial}{\partial x}\left(v - u_0\right)\left(x_{i+1/2}\right) \right|^2}} \\
&= \frac{\frac{\partial v}{\partial x}(x_{i+1/2}) - \frac{\partial u_0}{\partial x}(x_{i+1/2})}{\sqrt{\beta + \left| \frac{\partial v}{\partial x}(x_{i+1/2}) - \frac{\partial u_0}{\partial x}(x_{i+1/2}) \right|^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
F(x_{i-1/2},t) &= -\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)((x_{i-1/2} - x_s)^{\alpha} - \lambda u(x_{i-1/2},t)\delta(x_{i-1/2}) \\
&= -\frac{1}{\alpha\Gamma(\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t)((x_{i-1/2} - x_s)^{\alpha} - \frac{\lambda\delta(x_{i-1/2})}{2} \left[u(x_i,t) + u(x_{i-1},t) \right]
\end{aligned}
$$

Where

$$\delta(x_{i-1/2}) = \frac{\frac{\partial}{\partial x}(v-u_0)(x_{i-1/2})}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v-u_0)(x_{i-1/2})\right|^2}}$$
$$= \frac{\frac{\partial v}{\partial x}(x_{i-1/2}) - \frac{\partial u_0}{\partial x}(x_{i-1/2})}{\sqrt{\beta + \left|\frac{\partial v}{\partial x}(x_{i-1/2}) - \frac{\partial u_0}{\partial x}(x_{i-1/2})\right|^2}}$$

As x_i is the midpoint of K_i, one has

$$|u_i(t) - u(x_i, t)| \leqslant ch^2.$$

Approach the flux of an exact solution at the vertex $x_{i+1/2}$ by numerical flux which depends on mean values $\Xi(u_0, u_1, ..., u_i, u_{i+1})$

$$\begin{aligned}
&\Xi(u_0, u_1, ..., u_i, u_{i+1}) \\
&= -\frac{1}{\alpha\Gamma(\alpha)}\sum_{s=0}^{i} w_{s,i}u_s(t)((x_{i+1/2}-x_s)^\alpha - \frac{\lambda\delta(x_{i+1/2})}{2}\left[u_{i+1}(t)+u_i(t)\right] \\
&= -\frac{1}{\alpha\Gamma(\alpha)}\sum_{s=0}^{i} w_{s,i}u_s(t)((x_{i+1/2}-x_s)^\alpha - \Psi\left(u_{i+1}(t), u_i(t)\right)
\end{aligned}$$

where

$$\delta_{i+1/2} = \frac{\frac{\partial v_{i+1/2}}{\partial x} - \frac{\partial u^0_{i+1/2}}{\partial x}}{\sqrt{\beta + \left|\frac{\partial v_{i+1/2}}{\partial x} - \frac{\partial u^0_{i+1/2}}{\partial x}\right|^2}}$$

and at the vertex $x_{i-1/2}$ by numerical flux which depends on mean values $\Xi(u_0, u_1, ..., u_{i-1}, u_i)$

$$\begin{aligned}
&\Xi(u_0, u_1, ..., u_{i-1}, u_i) \\
&= -\frac{1}{\alpha\Gamma(\alpha)}\sum_{s=0}^{i} w_{s,i}u_s(t)((x_{i-1/2}-x_s)^\alpha + \frac{\lambda\delta(x_{i-1/2})}{2}\left[u_i(t)+u_{i-1}(t)\right] \\
&= -\frac{1}{\alpha\Gamma(\alpha)}\sum_{s=0}^{i} w_{s,i}u_s(t)((x_{i-1/2}-x_s)^\alpha - \Psi\left(u_i(t), u_{i-1}(t)\right)
\end{aligned}$$

where

$$\delta_{i-1/2} = \frac{\frac{\partial v_{i-1/2}}{\partial x} - \frac{\partial u^0_{i-1/2}}{\partial x}}{\sqrt{\beta + \left|\frac{\partial v_{i-1/2}}{\partial x} - \frac{\partial u^0_{i-1/2}}{\partial x}\right|^2}}$$

to obtain an ordinary defferential equation

$$\frac{\partial u_i(t)}{\partial t} = -\frac{1}{h_i}\Xi(u_0, u_1, ..., u_i, u_{i+1}) + \frac{1}{h_i}\Xi(u_0, u_1, ..., u_{i-1}, u_i) - u_i + u_i^0$$

For time-discretisation, we apply one of the various basic schemes to solve a general ordinary differential equation

$$\begin{cases} \dfrac{du}{dt} = F(t, u) \\ u(0) = u_0 \end{cases}$$

5.1.1 Stability for Explicit Schemes

Theorem 21. *Let the assumption* $(H1)$ *and* $(H2)$ *holds:*

(H1): $u_0 \in L^\infty([0, T])$*;*

(H2): *a condition C-F-L (Courant-Friedrichs-Lewy)*

$$\Delta t \leqslant \frac{inf_{i\in\mathbb{N}} h_i}{Lm_1}$$

where Lm_1 *is a Lpschitz constant, takes place, then a solution* u_i^n *defined by*

$$u_i^{n+1} = (1 - \Delta t)u_i^n - \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, ..., u_i^n, u_{i+1}^n) + \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, ..., u_{i-1}^n, u_i^n) + \Delta t u_i^0,$$

and

$$u_i^0 = \frac{1}{h_i}\int_{K_i} u_0(x)dx, i \in \mathbb{N},$$

verifies

$$A \leqslant u_i^n \leqslant B, \text{ for all } i \in \mathbb{N},$$

and

$$\| u_i^n \|_\infty \leqslant C \| u_0 \|_\infty \leqslant B.$$

Proof. According to assumption $A \leqslant u_0 \leqslant B$, a.e, and a definition of u_i^0, we see that $\overline{A} \leqslant u_i^0 \leqslant \overline{B}$ for all $i \in \mathbb{N}$,
Let us show that this property is still true in the rank $n + 1$.
We have

$$u_i^{n+1} = (1 - \Delta t)u_i^n - \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, ..., u_i^n, u_{i+1}^n) + \frac{\Delta t}{h_i}\Xi(u_0^n, u_1^n, ..., u_{i-1}^n, u_i^n) + \Delta t u_i^0,$$

and

$$u_i^{n+1} = (1 - \Delta t)\, u_i^n + \frac{\Delta t}{h_i}\frac{1}{\alpha\Gamma(\alpha)}\sum_{s=0}^{i} w_{s,i}u_s(t)\left[(x_{i+1/2} - x_s)^\alpha - (x_{i-1/2} - x_s)^\alpha\right]$$
$$- \frac{\lambda\Delta t}{2h_i}\left[\delta(x_{i+1/2})\,(u_{i+1}(t) + u_i(t)) - \delta(x_{i-1/2})\,(u_i(t) + u_{i-1}(t))\right] + \Delta t u_i^0.$$

Let us explicit $\left[(x_{i+1/2}-x_s)^\alpha-(x_{i-1/2}-x_s)^\alpha\right]$. Then, we have

$$\begin{aligned}(x_{i+1/2}-x_s)^\alpha-(x_{i-1/2}-x_s)^\alpha &= \frac{(x_{i+1/2}-x_s)^\alpha-(x_{i-1/2}-x_s)^\alpha}{(x_{i+1/2}-x_{i-1/2})}(x_{i+1/2}-x_{i-1/2})\\ &= \alpha(x_{i+1/2}-x_{i-1/2})^{\alpha-1}(x_{i+1/2}-x_{i-1/2})\\ &= \alpha h_i^\alpha.\end{aligned}$$

Therefore

$$\begin{aligned}u_i^{n+1} = {} & \frac{\lambda\Delta t}{2h_i}\delta_{i+1/2}u_{i+1}^n+\left[1-\Delta t+\frac{\lambda\Delta t}{2h_i}\left(\delta_{i+1/2}-\delta_{i-1/2}\right)\right]u_i^n-\frac{\lambda\Delta t}{2h_i}\delta_{i-1/2}u_{i-1}^n\\ &+\sum_{s=0}^{i}\chi(\alpha,i,s)u_s^n+\Delta t u_i^0.\end{aligned}$$

hence

$$u_i^{n+1}=\vartheta_{i+1}u_{i+1}^n+\vartheta_i u_i^n+\vartheta_{i-1}u_{i-1}^n+\sum_{s=0}^{i}\chi(\alpha,i,s)u_s^n+\Delta t u_i^0 \tag{5.1}$$

where

$$\vartheta_{i+1}=\frac{\lambda\Delta t}{2h_i}\delta_{i+1/2},$$

$$\vartheta_i=\left[1-\Delta t+\frac{\lambda\Delta t}{2h_i}\left(\delta_{i+1/2}-\delta_{i-1/2}\right)\right],$$

$$\vartheta_{i-1}=-\frac{\lambda\Delta t}{2h_i}\delta_{i-1/2}$$

and

$$\chi(\alpha,i,s)=\frac{1}{\Gamma(\alpha)}w_{s,i}h_i^\alpha.$$

We have

$$\vartheta_{i+1}\leqslant Lm_1\frac{\Delta t}{h_i},\ \text{ for } i\in\mathbb{N},$$

$$\vartheta_i\leqslant Lm_1\frac{\Delta t}{h_i},\ \text{ for } i\in\mathbb{N},$$

$$\vartheta_{i-1}\leqslant Lm_1\frac{\Delta t}{h_i},\ \text{ for } i\in\mathbb{N},$$

and

$$\sum_{s=0}^{i}\chi(\alpha,i,s)\leqslant 1,$$

$$\mid\vartheta\mid\leqslant 1$$

$\Psi(p,q)$ is a Lipschitz function on $[A,B]^2$ with the same Lipschitz constant in p and q : Lm_1. Consequently, u_i^{n+1} is a convex combination of u_i^n, u_i^{n+1} and u_i^{n-1} on one part and linear combination on the other part, then until (5.1), for all $i = 1, ..., N$, we have

$$| u_i^{n+1} | \leqslant | \vartheta_{i+1} || u_{i+1}^n | + | \vartheta_i || u_i^n | + \vartheta_{i-1} | u_{i-1}^n | + | \sum_{s=0}^{i} \chi(\alpha, i, s) || u_s^n | + \Delta t | u_i^0 |,$$

and

$$| u_i^{n+1} | \leqslant | (\vartheta_{i+1} + \vartheta_i + \vartheta_{i-1} + \sum_{s=0}^{i} \chi(\alpha, i, s)) ||\| u^n \|_\infty + \Delta t \| u_0 \|_\infty$$

which implies

$$\| u^{n+1} \|_\infty \leqslant C_1 \| u^n \|_\infty + C_2 \| u_0 \|_\infty$$

and a recurrence assumption implies

$$\| u_i^{n+1} \|_\infty \leqslant C_3 \| u_i^n \|_\infty + C_2 \| u_0 \|_\infty$$

and gives us

$$\| u^n \|_\infty \leqslant C^{Nmax} \| u_0 \|_\infty \leqslant B.$$

□

5.1.2 Convergence

Fix an initial condition $u^0 \in L^\infty([0,1])$, that we discretize on a mesh T_m of step $h_m > 0$:

$$(u_m)_i^0 = \frac{1}{h_i} \int_{K_i} u_m(x,0) dx, i \in \mathbb{N},$$

we use a step of time Δt_m and search a function u_m.
Supposed constant on every product of form

$$]ih_m, (i+1)h_m[\times]n\Delta t_m, (n+1)\Delta t_m[:$$

$$u_m(x,t) = (u_m)_i^n, (x,t) \in]ih_m, (i+1)h_m[\times]n\Delta t_m, (n+1)\Delta t_m[$$

Calculate $(u_m)_i^{n+1}$

$$\frac{1}{\Delta t_m}\left((u_m)_i^{n+1} - (u_m)_i^n\right) = -\frac{1}{h_m} \Xi((u_m)_0, (u_m)_1, ..., (u_m)_i, (u_m)_{i+1}) + \frac{1}{h_m} \Xi(u_0, (u_m)_1, ..., (u_m)_{i-1}, (u_m)_i) \\ - (u_m)_i + (u_m)_i^0.$$

When $\Delta t_m \to 0$ and $h_m \to 0$ for $m \to 0$, the function family $(u_m)_{m \in \mathbb{N}}$ can converge to weak solution of problem.

Numerical analysis

In this section, we simulate the numerical solutions of the following Thixotropic problem

$$u_t + D^\alpha u - \lambda \frac{\partial}{\partial x}\left[u \frac{\frac{\partial}{\partial x}(v-u_0)}{\sqrt{\beta + \left|\frac{\partial}{\partial x}(v-u_0)\right|^2}} \right] + u = u_0,\ (t,x) \in [0,1]\times\Omega$$

$$-\frac{\partial^2 v}{\partial x^2} + \tau v = 0,\ x \in \Omega$$

In our numerical method, we consider $\Omega = [0,1]$ and the time (in second) $t = 1$, the spatial step size $h = 10^{-1}$ and the time step size $k = 10^{-4}s$, $\tau = 0.01$, $\lambda = 0.02$ and

$$u(x,0) = x^2 + 1$$

$$v(x) = Ae^{\sqrt{\tau}x} + Be^{-\sqrt{\tau}x} \in C^\infty([a,b]),$$

where

$$A = \left(\eta e^{-\sqrt{\tau}b} - \theta e^{-\sqrt{\tau}a}\right) \backslash \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right),$$

and

$$B = \left(\theta e^{\sqrt{\tau}a} - \eta e^{\sqrt{\tau}b}\right) \backslash \left(e^{\sqrt{\tau}(a-b)} - e^{-\sqrt{\tau}(a-b)}\right).$$

The boundary conditions are:

$$u(a,t) = u(b,t) = 0,$$
$$v(a) = \eta,\ v(b) = \theta$$

Let us $\alpha = 1\backslash 2$, $a = 0$, $b = 1$, $\eta = 1$ and $\theta = 2$. So, we have the following numerical results which defined in Figure 5.1-5.3.

Discussion

In the case $\alpha = 1\backslash 2$, it is clear that the previous result in the chapter4 is correct and it expound more the phenomenon of thixotropy, where we note also that for all $0 < \beta < 1$ the speed of fluid is increasing with time, this reason leads to a decrease in viscosity of the fluid .

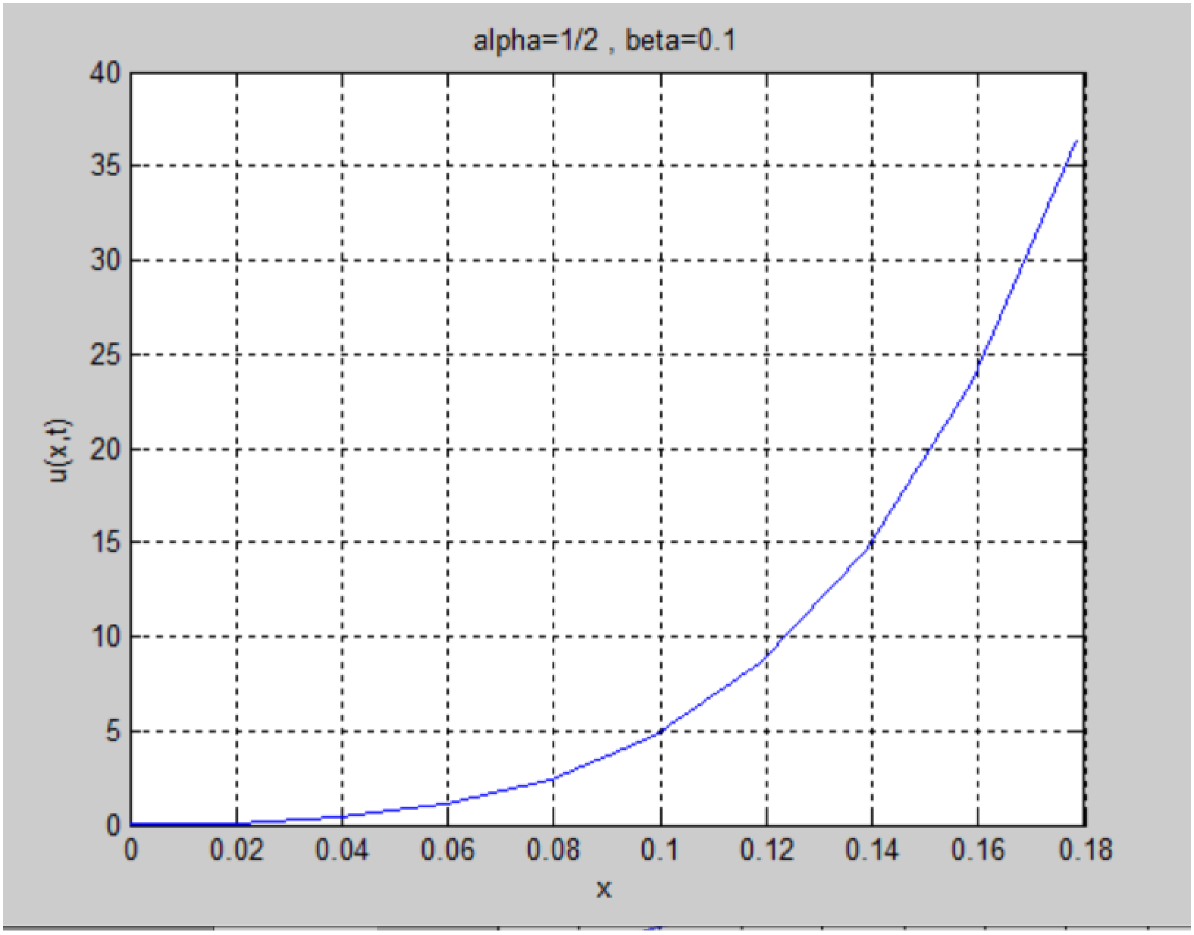

Figure 5.1: Numerical solution for the fractional thixotropic problem

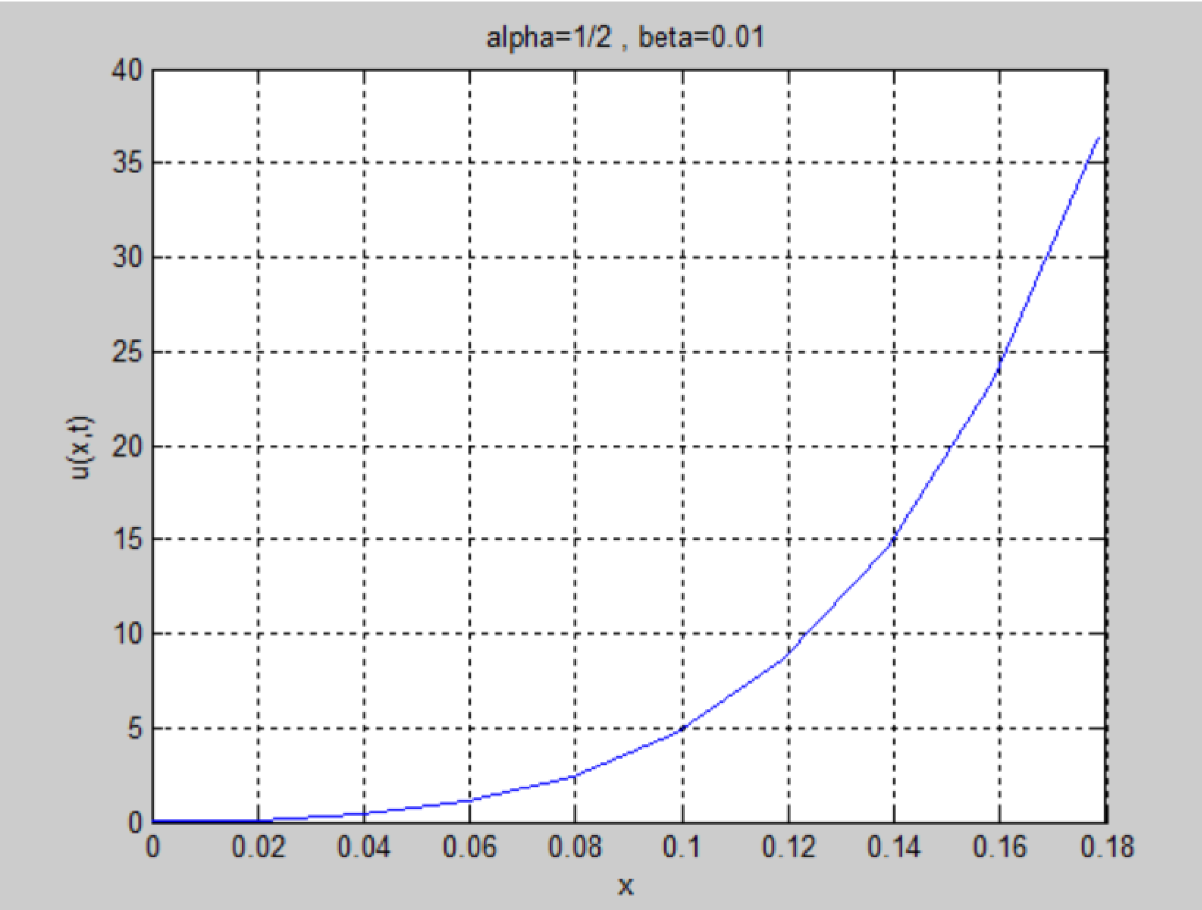

Figure 5.2: Numerical solution for the fractional thixotropic problem

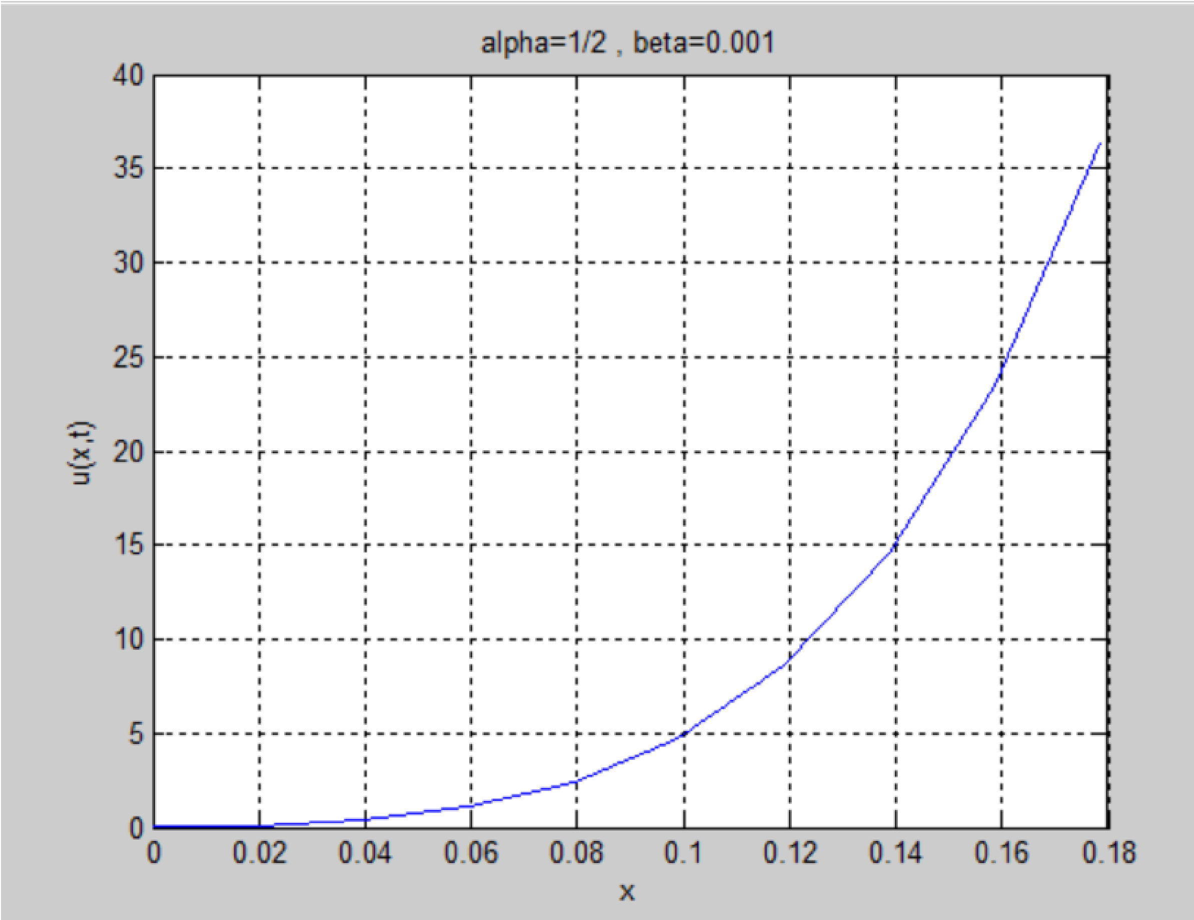

Figure 5.3: Numerical solution for the fractional thixotropic problem

Conclusion

Thixotropic systems are time dependent partial differential equation systems. It is composed of two equations; the first is a convection-diffusion equation, the second is a reaction–diffusion equation.
In this research work, we have studied the following Thixotropic problems

$$(P)\begin{cases} (P_1)\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x)\in\mathbb{R}^+\times\Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x\in\Omega \end{cases} \\ \qquad (P_2)\begin{cases} -\Delta v + \tau v = 0 & x\in\Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

and its space fractional problem (FP).

$$(FP)\begin{cases} (P_1')\begin{cases} u_t + D^\alpha u - \lambda div\left[u\frac{\nabla(v-u_0)}{\sqrt{\beta+|\nabla(v-u_0)|^2}}\right] + u = u_0 & (t,x)\in\mathbb{R}^+\times\Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x\in\Omega \end{cases} \\ \qquad (P_2)\begin{cases} -\Delta v + \tau v = 0 & x\in\Omega \\ v = g & \partial\Omega \end{cases} \end{cases}$$

For the Thixotropic Model(P), under some assumptions on initial and boundary data we have proved :

1. The existence and uniqueness results by using a Lax-Milgran Theorem, trace Theorem and Galerkin method.
2. the positivity of the solution using the Maximum Principle.
3. The decay exponential solution using the energy estimates.
4. The existence and uniqueness for a numerical solution obtained by FVM.
5. the stability and convergence for a numerical scheme obtained by FVM.

Besides, for the fractional Thixotropic Model(FP), under some assumptions on initial and boundary data we have proved :

1. The existence and uniqueness results by using the Fractional Gagliardo–Nirenberg inequality, Galerkin method, fixed point theorem and some Sobolev inequalities.

2. the positivity of the solution using the Maximum Principle.

3. the existence and uniqueness for a numerical solution obtained by FVM.

4. the stability and convergence for a numerical scheme obtained by FVM.

Contributions

The contribution of the presented research work can be viewed from different aspects, the important one can be identified as follows:

1. Analysis and proof determination for the Thixotropic model (P) solutions such as the existence, uniqueness and positivity.

2. Proving the existence, uniqueness and positivity of solutions for the fractional Thixotropic model (FP).

3. Determination of the approximate solution for the Thixotropic problems (P) using FVM.

4. Determination of the approximate solution for the fractional Thixotropic problem (FP) via FVM.

5. Study the stability and the convergence of approximation solutions.

Future work

In view of the importance of the Thixotropic problem in sciences and its various applications, many research areas still not be covered and can be suggested for future investigations. In particular, the following areas could lead to a fruitful research:

1. Investigate our numerical methods with other boundary conditions, and nonlinear source terms.

2. Develop high-efficiency finite volume methods for solving fractional differential equations.

3. Investigate our numerical methods in higher-dimensional problems.

4. Study the problem temporal fractional Thixotropic system , and the temporal and spatial fractional Thixotropic.

5. Investigate other numerical methods for the same problem conditions.

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Abstract

The Thixotropic model is described by a system of nonlinear PDEs. The system under consideration represents a convection-diffusion equation for the speed of the fluid thixotropic coupled with a reaction-diffusion equation for the temperature of the fluid.
The aim of this thesis is the study of the existence and uniqueness of solution for both Thixotropic model and its space fractional model by using certain assumptions. The proof is based on the Lax-Milgram Theorem, Galerkin's method, the Principle of the Maximum and the fixed point theory. And also the study of numerical solution for the thixotropic model and its space fractional model.
The finite volume method is utilized under certain hypotheses to prove the existence and uniqueness of an approximate positive solution. The study also demonstrates the stability and the convergence of finite volume method. Finally, we end this thesis with some numerical simulation carried out by the Matlab software.
Keywords:
Thixotropic model; Fractional differential equation; Global solution; Galerkin method; Maximum principle; Fixed point theorem; Method of Line (MOL); Finite volume method.

Résumé

Le modèle de Thixotrope est décrit par un système d'EDP non linéaire. Le système considéré représente une équation de convection diffusion pour la vitesse du fluide thixotrope couplée d'une équation de réaction-diffusion pour la température du fluide.
L'objectif de cette thèse est l'étude de l'existence et de l'unicité de la solution pour les deux modèle thixotrope et son modèle fractionnaire spatial en utilisant certaines hypothèses. La démonstration est basé sur le théorème de Lax-Milgram, la méthode de Galerkin, le principe du maximum et la théorie du point fixe. Et aussi l'étude de solution numérique pour le modèle thixotrope et son modèle fractionnaire spatial.
La méthode des volumes finis est utilisée sous certaines hypothèses pour prouver l'existence et l'unicité d'une solution positive approchée. L'étude démontre également la stabilité et la convergence de la méthode des volumes finis. Enfin, nous terminons cette thèse par quelques simulation numérique réalisée par le logiciel Matlab.
Mots clés:
Modèle de Thixotrope; Equation différentielle fractionnaire; Solution Global; Méthode de Galerkin; Principe du Maximum; Théorème du point fixe; Méthode des lignes; Méthode des volumes finis.

ملخص

في هذه الأطروحة قمنا بدراسة وجود و وحدانية الحلول الموجبة لمشكلتين: الأولى هي لنموذج المتغيرة الانسيابية وأما الثانية فهي نموذج المشتق الكسري للمتغيرة الانسيابية باستخدام نظرية لكس ميلغرام، المبدأ الأعظمي و نظرية النقطة الثابثة.
كما قمنا بدراسة التحليل العددية لهاتين المشكلتين باستعمال طريقة الحجم المحدود و بالاعتماد على طريقة الخطوط. و انهينا الأطروحة بمحاكاة عددية باستعمال برنامج الماطلاب.

الكلمات المفتاحية: نموذج المتغيرة الانسيابية ، المعادلة التفاضلية الكسرية، نظرية لكس ميلغرام، مبدأ الحد الأقصى المنفصل، نظرية النقطة الصامدة ، طريقة الحجم المحدود و طريقة الخطوط .