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Memory
Presented for the diploma of
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Homogenization of the Stokes problem in porous medium by unfolding operator method

Option : NAPDE

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Abstract

In this memory, we consider the Stockes system with boundary conditions depending a parameter γ on the boudary of the holes T, and boundary conditions wich are homogeneous (Dirchlet type condition) on the boundary of Ω, we consider the flow of an incompressible viscous fluid through a porous medium, under the action of an exterior electrical field. The aim is to give the asymptotic behavior of the velocity and of the pressure of the fluid as ε goes to zero. We use the periodic unfolding method in perforated domain. We give at the limit problems correspending to different values of γ.

Key words : stockes problem, periodic unfolding, perforated domain, homognisation.

Résumé

Nous considrons dans ce memoire le système de Stockes avec des conditions aux bords non homogéne dpendent de γ sur le bord des troux T, et des conditions homogénes (condition de type Dirchlet) au bord de Ω. Notre objectif est d'étudier le comportement asymptotique de la vitesse d'écoulement et de la pression du fluide quand ε tend vers zero par la méthode de l'éclatement périodique dans un domaine perforé. Nous obtenons plusieurs cas différents selon les valeurs de γ.

Mots clés : problème de Stockes, éclatement périodique, domaine perforé, homogénéisation.

ملخص

في هذا العمل نعتبر سائل لزج غير قابل للضغط يتدفق في وسط مسامي تحت تأثير حقل كهربائي و المتمثل في مشكلة ستوكس.
قمنا بدراسة مشكلة ستوكس مع شروط حدية غير متجانسة متعلقة ب γ على حافة الثقوب T و شروط حدية متجانسة من نوع ديركلي على حافة Ω ، الهدف من هذا العمل هو دراسة السلوك التقاربي لسرعة وضغط المائع عندما يؤول ε إلى الصفر و من اجل ذلك نستخدم طريقة الانفجار الدوري في مجالات مسامية . في النهاية نحصل على عدة حالات حسب قيم γ

كلمات مفتاحية: مسألة ستوكس، انفجار دوري ، وسط مسامي، التجانس.

Notations and definitions

Notations

Symbole	Signification
∇u	The gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right)$
Δu	$= div(\nabla u)$
$\mathcal{D}(\Omega)$	The set of continuous functions whose support is a compact set of $\mathbb{R}^N$ contained in Ω
$L^p(\Omega)$	The Lebesgue space, $\begin{cases} \{u : \Omega \longmapsto \mathbb{R} \text{ mesurable} ; \int_\Omega \lvert u\rvert^p < +\infty\} & \text{if } 1 \leq p \leq +\infty, \\ \{u : \Omega \longmapsto \mathbb{R},\ \text{u mesurable and such that there exists } C \in \mathbb{R};\ \text{with } \lvert u\rvert \leq C\} & \text{if } p = +\infty \end{cases}$
$L^p_{loc}(\Omega)$	$= \{u \in L^p(\omega),\ \text{for any open bounded set } \omega \text{ with } \overline{\omega} \subset \Omega\}$
$W^{1,p}(\Omega)$	The Sobolev spaces, $W^{1,p}(\Omega) = \{u \in L^p(\Omega); \lvert\nabla u\rvert \in L^p(\Omega)\}$
$H^1(\Omega)$	$\left\{u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \ldots, N\right\}$
$H^1_0(\Omega)$	The space of function in $H^1(\Omega)$ will vanish on the boundary in the sens of the trace.

Definitions

Holder Inequality :
Definition : if $1 < p < \infty$, and if q is combind exposure of p .We have , for f$\in L^p(m)$ and g $\in L^q(m)$, the inequality of Holder :

$$\int_s |fg|dm < (\int_s |f|^p dm)^{\frac{1}{q}} (\int_s |f|^q dm)^{\frac{1}{q}}$$

Cauchy shwarz Inequality :
Definition : for p=q=2, we have :

$$\int_s |fg|dm < (\int_s |f|^2 dm)^{\frac{1}{2}} (\int_s |f|^2 dm)^{\frac{1}{2}}$$

Young Inequality :
Definition :let a,b $\in \mathbb{R}^*$ and p,q $\in]1, +\infty[$ as: $\frac{1}{p} + \frac{1}{q} = 1$, so:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Inequality of Poincaré:
proposition: there exists a constant C such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \forall u \in H_0^1(\Omega)$$

Lax-Milgram :

theorem: Let a be a continuous bilinear form on a Hilbert space H and $F \in H^{'}$. Assume that a is H-elliptic with constant α_0. Then the variational equation has a unique solution u$\in H$

CONTENTS

INTRODUCTION

Various physical phenomens can be described in terms of fluid flow in porous medium. It occurs in the study of filtration in sandy soils or blood circulation in capillaries, see Bear [2] and Hormung [10] for more examples and motivation. In the study of such processes, one would like to find some averaged characteristics of the flow, e.g permeability, velocity, and pressure.

To obtain such quantities, ther exist several mathematical approches collectively referred to as homogenization theory. In the late 1960's and early 1970's, this theory became in its own right a new part of the branch of mathematics concerning partial differential equations and their numerical approximations. It had its origin in serval works from physics and mechanics, where equivalent (spatially) homogenous macroscopic structures were obtained for microscopic heterogeneous media (see for example Bruggeman [4] (1935), Hashin and Shtrikman [9] (1963)). Ther interest of applied mathematicians and numerical analysts started in earnest in the late 1960's. Seminal works appear in De Giorgi and Spagnolo [8] (1973), Babuska [1] (1975), Bensoussan, Lions and Papanicolaou [3] (1978), Sanchez-Palencia [12] (1980), among others.

This theory allows to replace problems with strongly oscillating coefficients by approximate problems with constant coefficients, this signifies mainly that the solution of a boundary value problem, depending on a small parameter, converge to the solution of a limit boundary value problem wich is explicity described, and therefore much easier to process numerically.

In this work, we study the stokes problem whith non homogeneous slip boundary condition depeding on a parameter $\gamma \in \mathbb{R}^N$ (whit still ε- size holes) was studied by Cioranescu,Donato and Ene in [3] by energy methods, and in 1980 Ene and Sanchez-Palencia study the stokes flow in a periodic porous medium whith Dirichlet condition on the boundary of the holes see [12].

In 2012 [13] Rachad Zaki study the stokes problem in a periodic porous medium with Robin condition on the boundary of the holes containing a terme $a\varepsilon^{\gamma}u_{\varepsilon}$ following the values of γ serval a priori estimates are obtained wich lead to different limits lows.

This memory is organized as folow:

In the first chapter, we present the multiple-scale for the Stokes system.

In the second chapter, we define the periodic infolding methode in perforated domains and some properties.

Finally, in chapter 03 we study the asymptotic behavior of the velocity and pressure of the fluid as $\varepsilon \to 0$ this leads to the main homogenization results, the first of them , for $\gamma < 0$, we get at the limit a Darcy law as enounced in theorem 3.1 , while, for $\gamma \geq 0$, it is a Brinkman law as enounced in theorem 3.2.

CHAPTER

1

THE MULTIPLE-SCALE METHODE FOR THE STOKES EQUATION

In this chapter we apply the multiple-scale method to the study of Stokes equation.

1.1 The multiple-scale methode

We consider a small positive parameter $\boldsymbol{\varepsilon}$; the fluid domain is defined by

$$\boldsymbol{\Omega}_{\varepsilon} = \{\boldsymbol{x}; \boldsymbol{x} \in \boldsymbol{Y}_{\varepsilon}\} \tag{1.1}$$

where $\boldsymbol{\Omega}$ is a given domain. In this chapter we only consider a formal expansion of the velocity and pressure out a neighbourhood of $\boldsymbol{\partial\Omega}$; consequently.

We consider the stokes problem

$$\begin{cases} -\boldsymbol{gradp_\varepsilon} + \boldsymbol{\Delta v_\varepsilon} + \boldsymbol{f} = \boldsymbol{0} & \text{in } \boldsymbol{\Omega_\varepsilon}, \\ \boldsymbol{divv_\varepsilon} = \boldsymbol{0} & \text{in } \boldsymbol{\Omega_\varepsilon}, \\ \boldsymbol{v_\varepsilon} = \boldsymbol{0} & \text{on } \boldsymbol{\partial\Omega_\varepsilon}. \end{cases} \tag{1.2}$$

Now, we postulate an asymptotic expansion

$$\begin{cases} \boldsymbol{v_\varepsilon(x) = \varepsilon^2 v_0(x,y) + \varepsilon^3 v_1(x,y) + \dots} \\ \boldsymbol{p_\varepsilon(x) = p_0 + \varepsilon p_1(x,y) + \dots} \end{cases} \tag{1.3}$$

with $\boldsymbol{y} = \frac{\boldsymbol{x}}{\boldsymbol{\varepsilon}}, \boldsymbol{v_i}, \boldsymbol{p_i}$ $\boldsymbol{Y_{per}}$ in $\boldsymbol{Y}$, for $\boldsymbol{x} \in \boldsymbol{\Omega}, \boldsymbol{y} \in \boldsymbol{Y}$. Note that we postulate that $\boldsymbol{p_0}$ does not depend on $\boldsymbol{Y}$, this is natural, because the very function in (1.2) is $\boldsymbol{gradp_\varepsilon}$, which depends on $\boldsymbol{y}$ from the first term because of the classical relation

$$\frac{\boldsymbol{d}}{\boldsymbol{dx_i}} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x_i}} + \frac{\boldsymbol{1}}{\boldsymbol{\varepsilon}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y_i}}. \tag{1.4}$$

The same relation (1.4) shows that the expansion of $\boldsymbol{\Delta}$ is

$$\boldsymbol{\Delta} = \frac{\boldsymbol{1}}{\boldsymbol{\varepsilon^2}}\boldsymbol{\Delta_{yy}} + \frac{\boldsymbol{1}}{\boldsymbol{\varepsilon}}\dots \tag{1.5}$$

where $\boldsymbol{\Delta_{yy}}$ denotes the laplacian with respect to the variables $\boldsymbol{y_i}$ ($\boldsymbol{x_i}$ begin parameters). The form of the relation (1.5) shows that the first significative term in (1.3). As usual in homogenization problems, if we postulate an expansion beginning by $\boldsymbol{\varepsilon_0}$ terms, the two first terms will be find to be zero.

To study the problem (1.2), we replace (1.3) into (1.2). Then, by taking the $\boldsymbol{\varepsilon_0}$ term for the first equation of (1.2), the $\boldsymbol{\varepsilon}$ term for the second equation of (1.2) and the $\boldsymbol{\varepsilon_2}$ term for the last equation of (1.2), we have

$$\begin{cases} -\frac{\partial p_1}{\partial y_i} + \Delta_y v_i^0 + (f_i - \frac{\partial p_0}{\partial x_i}) = 0 & \text{in } \boldsymbol{Y}, \\ div_y v_0 = 0 & \text{in } \boldsymbol{Y}, \\ v_0|_\Gamma = 0 & \text{in } \boldsymbol{\Gamma}. \end{cases} \tag{1.6}$$

With the supplementary condition $\boldsymbol{v_0}, \boldsymbol{p_1}$ are $\boldsymbol{Y_{per}}$ in $\boldsymbol{Y}$. This is the local problem, where $\boldsymbol{x}$ is a parameter and the term in parenthesis of the equation of (1.6) plays the role of given force, $\boldsymbol{v_0}$ and $\boldsymbol{p_1}$ are the unknowns. We shall see later that (1.6) leads to the Darcy's law. Befor studying this, for the second equation we have

$$\begin{aligned} divv^\varepsilon &= \varepsilon^2 \left(\frac{\partial v^0}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial v^0}{\partial y_i} \right) + \varepsilon^3 \left(\frac{\partial v^1}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial v^1}{\partial y_i} \right) \\ &= \varepsilon^2 \frac{\partial v^0}{\partial x_i} + \varepsilon^1 \frac{\partial v^0}{\partial y_i} + \varepsilon^3 \frac{\partial v^1}{\partial x_i} + \varepsilon^2 \frac{\partial v^1}{\partial y_i} \\ &= 0 \end{aligned}$$

we consider the $\boldsymbol{\varepsilon^2}$ term for the second equation of (1.2)

$$div_x v_0 + div_y v_1 = 0 \tag{1.7}$$

and we apply the classical mean value operator

$$\tilde{\cdot} = \frac{1}{|\mathbf{Y}|} \int_Y \cdot \quad dy \tag{1.8}$$

not that $\boldsymbol{v_i}$ as functions of $\boldsymbol{y}$ are defined on $\boldsymbol{Y}$: it is natural to extend them to $\boldsymbol{Y}$ with value zero on $\boldsymbol{\partial Y}$. Then, we have

$$(div_y v_1)^\sim = \frac{1}{|\mathbf{Y}|} \int_Y \frac{\partial v_i^1}{\partial y_i} dy$$

we apply Green formule we get

$$\frac{1}{|\mathbf{Y}|} \int_{\partial Y} n_i v_i^1 d\sigma = 0. \tag{1.9}$$

To see that the surface integral of (1.9) is zero it suffices to see that $n_i v_i^1$ is zero on Γ and that the integral on the parts of ∂Y lying on ∂Y annihilate by periodicity. On the other hand, the operator $\partial/\partial x_i$ commutes with the mean value operator as usual. Then, by $\sim$ appliying to (1.7), we obtain

$$div_x \tilde{v}^0 = 0 \tag{1.10}$$

whiche is the macroscopic equation.

Now we study the local problem (1.6). We define an appropriate space of Y_{per} functions

$$V = \left\{ u; u \in H^1(Y); u|_\Gamma = 0, div_y u = 0; Y_{per} \right\} \tag{1.11}$$

$$(u, w)_V = \int_Y \frac{\partial u_i}{\partial y_k} \frac{\partial w_i}{\partial y_k} dy \tag{1.12}$$

which is Hilbert space; the associated norm is equivalent to the $H^1(Y)$ norm.

To obtain a variational formulation of (1.6), we take a test function $w \in V$ and we multiply the first equation of (1.6) by ω_i, by integrating over Y we have

$$- \int_Y \frac{\partial p_1}{\partial y_i} \omega_i dy = - \int_Y \frac{\partial}{\partial y_i} (p_1 \omega_i) \, dy = \int_{\partial Y} p_1 \omega_i \eta_i d\sigma = 0,$$

$$\int_Y \Delta_y v_i^0 \omega_i dy = \int_Y \left[\frac{\partial}{\partial y_k} \left(\frac{\partial v_i^0}{\partial y_k} \omega_i \right) - \frac{\partial v_i^0}{\partial y_k} \frac{\partial \omega_i}{\partial y_k} \right] dy,$$

$$\int_{\partial Y} n_k \frac{\partial v_i^0}{\partial y_k} \omega_i d\sigma - \int_Y \frac{\partial v_i^0}{\partial y_k} \frac{\partial \omega_i}{\partial y_k} dy = - \int_Y \frac{\partial v_i^0}{\partial y_k} \frac{\partial \omega_i}{\partial y_k} dy.$$

Then, by using (1.12), we get

$$(v_0, \omega)_V = \left(f_i - \frac{\partial p_0}{\partial x_i} \right) \int_Y \omega_i dy = 0 \quad \forall \omega \in V. \tag{1.13}$$

Conversely, if $v_0 \in V$ and satisfies (1.13) , by integrating by parts it satisfies

$$\int_Y \left[\Delta_y v_i^0 + \left(f_i - \frac{\partial p_0}{\partial x_i} \right) \right] \omega_i dy = 0 \quad \forall \omega \in V. \tag{1.14}$$

Now, we show that the function p_1 just found is Y-periodic. In fact, $grad p_1$ is periodic because p_1 satisfies the first equation of (1.14). We multiply this equation by ω_i and we integrate on $\boldsymbol{Y}$. By comparing with (1.14) we have

$$-\int_Y \frac{\partial p_1}{\partial y_i}\omega_i dy = 0$$

and because $div\omega = 0$, we have

$$\int_Y \frac{\partial}{\partial y_i}(p_1\omega_i)\, dy = \int_{\partial Y} p_1\omega_i n_i d\sigma = 0 \tag{1.15}$$

consequently, the local problem (1.14) is equivalent to the following variational problem

$$\begin{cases} \text{Find } v_0 \in V, \\ (v_0, \omega)_V = \left(f_i - \dfrac{\partial p_0}{\partial x_i}\right)\displaystyle\int_Y \omega_i dy = 0, \quad \forall \omega \in V. \end{cases} \tag{1.16}$$

Morever, by using the standard linearity property, we have

Proposition 1.1 *If we postulate an asymptotic expansion (1.3) the first term $v_0(x, y)$ is given by $f_i(x)$ and $\partial p_0/\partial x_i(x)$ as*

$$v_0 = (f_i - \frac{\partial p_0}{\partial x_i})v_i \tag{1.17}$$

where $v_i(y)(i = 1, 2, 3 \ldots N)$ are the solution of

$$\begin{cases} \text{Find } v_i \in V \text{ such that }, \\ (v_i, \omega)_V = \displaystyle\int_Y \omega_i dy \qquad \forall, \omega \in V. \end{cases} \tag{1.18}$$

The existence and uniqueness of solution of (1.16) or (1.18) are immediate consequences of the Lax-Milgram theorem, because the right hand sides of (1.16) , (1.18) are linear and bounded functionals on $\boldsymbol{V}$.

$$-\int_Y \frac{\partial v^0}{\partial y_k}\frac{\partial w_i}{\partial y_k}dy + \int_Y (f_i - \frac{\partial p^0}{\partial x_i})w_i dy = 0$$

$$-\int_Y \frac{\partial v^0}{\partial y_k}\frac{\partial w_i}{\partial y_k}dy = -\int_Y (f_i - \frac{\partial p^0}{\partial x_i})w_i dy$$
$$= \int_Y F w_i dy$$
$$F = f_i - \frac{\partial p^0}{\partial x_i}, \forall w_i \in H_0^1(Y).$$

Coercivity of $a(v^0, w)$:

$$a(v^0, v^0) = \int_Y \frac{\partial v^0}{\partial y_k}\frac{\partial v^0}{\partial y_k} = \|\nabla_y v^0\|^2_{L^2(Y)}$$
$$= \|v^0\|^2_{H_0^1(Y)}$$

so $a(v^0, w)$ is coercive.

Contunity of $a(v^0, w)$:

by Cauchy Shwarz inquality, we have

$$|a(v^0, w)| = \left|\int_Y \frac{\partial v^0}{\partial y_k}\frac{\partial w^i}{\partial y}dy\right| \leq \|\nabla v^0\|_{L^2(Y)}\|w\|_{L^2(Y)}$$
$$\leq \|v^0\|_{H_0^1(Y)}\|w\|_{H_0^1(Y)}$$

so $a(v^0, \omega)$ is continous.

Contunity of $L(w)$:

again we use Cauchy Shwarz inquality, we have

$$|L(w)| = \left|\int_Y F w dy\right| \leq \|F\|_{L^2(Y)}\|w\|_{L^2(Y)} \leq c\|w\|_{L^2(Y)}$$
$$\leq c\|w\|_{H_0^1(Y)}$$

so L is continous.

Lax Milgram conditions are verified, so the problem have a weak solution in $H_0^1(Y)$.

Now, if we apply the mean operator $\sim$ (definded by (1.8) to (1.17)), we have

$$\tilde{v}_j^0 = k_{ij}\left(f_i - \frac{\partial p_0}{\partial x_i}\right); k_{ij} = \tilde{v}_j^i \qquad (1.19)$$

(as, usual, the indexes i denote the components of the vectors).

Relation (1.19) is the Darcy's law. The mean value of the velocity of the fluid is equal to

$\boldsymbol{f} - \boldsymbol{gradp_0}$ multiplied by a constant tensor with components $\boldsymbol{k_{ij}}$ which only depend on the geometry of the period $\boldsymbol{Y}$. It is noticeable that (1.19) was obtained from the first equation of (1.2) ,i.e, for the viscosity coefficient $\boldsymbol{\nu}$ equal to one. If we consider

$$-\boldsymbol{gradp_\varepsilon} + \boldsymbol{\nu\Delta v_\varepsilon} + \boldsymbol{f} = \boldsymbol{0} \tag{1.20}$$

instead of (1.2), we obtain

$$\tilde{\boldsymbol{v}}_j^0 = \frac{\boldsymbol{k_{ij}}}{\boldsymbol{\nu}}\left(\boldsymbol{f_i} - \frac{\boldsymbol{\partial p_0}}{\boldsymbol{\partial x_i}}\right); \boldsymbol{k_{ij}} = \tilde{\boldsymbol{v}}_j^i \tag{1.21}$$

instead of (1.19) . It is also necessaryto introduce the coefficient $\boldsymbol{\nu^{-1}}$ at the right hand side of (1.17).

Proposition 1.2 *The matrix* $\boldsymbol{k_{ij}}$*, defined by* (1.19) *will be called permeability tensor. It is a symmetric, positivee definite matrix.*

Proposition 1.3 *The macroscopic equation* (1.10) *may be writen in the form*

$$\boldsymbol{k_{ij}}\frac{\boldsymbol{\partial^2 p_0}}{\boldsymbol{\partial x_i \partial x_j}} = \boldsymbol{k_{ij}}\frac{\boldsymbol{\partial f_i}}{\boldsymbol{\partial x_j}}$$

which is an elliptic equation for the unknown $\boldsymbol{p_0(x)}$*. If* $\boldsymbol{p_0(x)}$ *is obtained, the velocity field* $\boldsymbol{v_0(x,y)}$ *is given by* (1.17) *and the mean value of the velocity satidfies the Darcy's law* (1.19).

CHAPTER

2

THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains $\mathcal{T}_\varepsilon$ for the classical homogenization.

2.1 The periodic unfolding operator $\mathcal{T}_\varepsilon$

In this section, we introduce the periodic unfolding operator in the case of perforated domains introduced by Cioranescu et all [5] and [6].
In the following we denote:

- $\boldsymbol{\Omega}$ an open set in $\mathbb{R}^{\boldsymbol{N}}$
- $\boldsymbol{Y} = [\mathbf{0}; \mathbf{1}\,[^{\boldsymbol{N}}$ the reference cell for, or more generally a set having the paving property to a basis $(\boldsymbol{b_1}, \ldots, \boldsymbol{b_N})$ defining the periods,

- $\boldsymbol{T}$ an open set included in $\boldsymbol{Y}$ such that $\boldsymbol{\partial T}$ does not contain the summits of $\boldsymbol{Y}$. We can be, sometimes, transported to this situation by a simple change of period,

we define

$$\boldsymbol{T_\varepsilon} = \bigcup_{\xi\in\mathbb{Z}^N} \boldsymbol{\varepsilon(\xi + T)} \quad \text{and} \quad \boldsymbol{\Omega_\varepsilon = \Omega \backslash T_\varepsilon}.$$

We assume in the following that $\boldsymbol{\Omega_\varepsilon}$ is a connected set and we take the regularity hypothesis

$$|\boldsymbol{\partial\Omega}| = \boldsymbol{0}.$$

and we define

$$\Xi_\varepsilon = \left\{\boldsymbol{\xi \in Z^N, \varepsilon(\xi + Y) \subset \Omega}\right\}$$

To do so, let us first define the following domain:

$$\widetilde{\boldsymbol{\Omega_\varepsilon}} = \boldsymbol{int}\left(\bigcup_{\xi\in\boldsymbol{\Lambda_\varepsilon}} \boldsymbol{(\xi + Y)}\right)$$

where

$$\boldsymbol{\Lambda_\varepsilon} = \left\{\boldsymbol{\xi} \in \mathbb{Z}^N; \boldsymbol{\varepsilon(\xi + \overline{Y}) \cap \Omega \neq \emptyset}\right\}$$

The set $\widetilde{\boldsymbol{\Omega_\varepsilon}}$ is the smallest finite union of $\boldsymbol{\varepsilon Y}$ cells containing $\boldsymbol{\Omega}$.

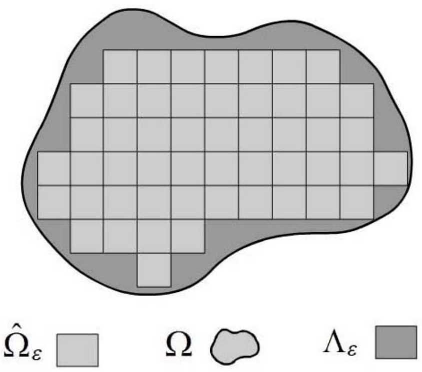

Fig. 1. The sets $\widehat{\boldsymbol{\Omega_\varepsilon}}, \boldsymbol{\Omega}$ and $\boldsymbol{\Lambda_\varepsilon}$

In the sequel, we will use the following notation:

- $\tilde{\varphi}$ for the extension by $\mathbf{0}$ outside $\boldsymbol{\Omega}_\varepsilon$ for any function $\varphi \in \boldsymbol{L}^p(\boldsymbol{\Omega}_\varepsilon)$,
- χ_ε for the characteristic function of $\boldsymbol{\Omega}_\varepsilon$,
- $\boldsymbol{\rho}(\boldsymbol{Y})$ for the diameter of the cell $\boldsymbol{Y}$,
- $\boldsymbol{T}_\varepsilon^{int}$ for the set of holes that do not intersect the boundary $\boldsymbol{\partial\Omega}$.

By analogy to the $\mathbf{1}\boldsymbol{D}$ notation, for $\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{N}}$, $[\boldsymbol{z}]_{\boldsymbol{Y}}$ denotes the unique integer combination $\sum_{j=1}^{j=N} \boldsymbol{k}_j \boldsymbol{b}_j$, such that $\boldsymbol{z})\ [\boldsymbol{z}]_{\boldsymbol{Y}}$ belongs to $\boldsymbol{Y}$. Set $\{\boldsymbol{z}\}_{\boldsymbol{Y}} = \boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}}$. Then, for almost every $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{N}}$, there exists a unique element in $\mathbb{R}^{\boldsymbol{N}}$, denoted by $\left[\frac{\boldsymbol{x}}{\boldsymbol{y}}\right]_{\boldsymbol{Y}}$, such that

$$\boldsymbol{x} - \varepsilon \left[\frac{\boldsymbol{x}}{\varepsilon}\right]_{\boldsymbol{Y}} = \varepsilon \left\{\frac{\boldsymbol{x}}{\varepsilon}\right\}_{\boldsymbol{Y}} \tag{2.1}$$

where

$$\left\{\frac{\boldsymbol{x}}{\varepsilon}\right\}_{\boldsymbol{Y}} \in \boldsymbol{Y}.$$

This decomposition (2.1) is essential in the definition of the unfollding operator in the next.

definition 2.1 *[5] Let $\varphi \in \boldsymbol{L}^p(\boldsymbol{\Omega}_\varepsilon)$, $\boldsymbol{p} \in [\mathbf{1}, +\infty]$. We define the function*

$$\mathcal{T}_\varepsilon(\varphi)(\boldsymbol{x}, \boldsymbol{y}) = \tilde{\varphi}\left(\varepsilon \left[\frac{\boldsymbol{x}}{\varepsilon}\right]_{\boldsymbol{Y}} + \varepsilon \boldsymbol{y}\right) \tag{2.2}$$

for every $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{N}}$ and $\boldsymbol{y} \in \boldsymbol{Y}$.

Remark 2.1 *Notice that the oscillations due to perforations are shifted into the second variable $\boldsymbol{y}$ which belongs to the fixed domain $\boldsymbol{Y}$, while the first variable $\boldsymbol{x}$ belongs to $\mathbb{R}^{\boldsymbol{N}}$. One see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on $\boldsymbol{\Omega}_\varepsilon$ one needs first, while using standard methods, to extend it to a fixed domain. With $\mathcal{T}_\varepsilon$, such extensions are more necessary.*

The main properties given in [5] for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are listed in the proposition below. To do so, let us first define the following domain:

$$\widetilde{\boldsymbol{\Omega}_\varepsilon} = \boldsymbol{int}\left(\bigcup_{\xi \in \boldsymbol{\Lambda}_\varepsilon} (\boldsymbol{\xi} + \boldsymbol{Y})\right),$$

where

$$\Lambda_\varepsilon \left\{\xi \in \mathbb{Z}^N; \varepsilon(\xi + \bar{Y}) \cap \Omega \neq \emptyset\right\}.$$

The set $\widetilde{\Omega_\varepsilon}$ is the smallest finite union of εY cells containing Ω.

Proposition 2.1 *[5] The unfolding operator $\mathcal{T}_\varepsilon$ has the following properties:*

1. *$\mathcal{T}_\varepsilon$ is a linear operator.*
2. $\mathcal{T}_\varepsilon(\varphi)\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right) = \varphi(x) \quad \forall \varphi \in L^p(\Omega_\varepsilon)$ *and* $x \in \mathbb{R}^N$.
3. $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi), \quad \forall \varphi, \psi \in L^p(\Omega_\varepsilon)$.
4. *Let φ in $L^p(Y)$ or $L^p(Y)$ be a Y-periodic function. Set $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$. Then,* $\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x, y) = \varphi(y), \quad$ *a.e.in* $\widetilde{\Omega_\varepsilon}$.
5. *One has the integration formula*
$$\int_{\Omega_\varepsilon} \varphi dx = \frac{1}{|Y|}\int_{\widetilde{\Omega_\varepsilon}\times Y} \mathcal{T}_\varepsilon(\varphi) dx dy \quad \forall \varphi \in L^1(\Omega_\varepsilon).$$
6. *For every $\varphi \in L^2(\Omega_\varepsilon)$, $\mathcal{T}_\varepsilon(\varphi)$ belongs to $L^2\left(\mathbb{R}^N \times Y\right)$. It also belongs to $L^2\left(\widetilde{\Omega_\varepsilon}\times Y\right)$.*
7. *For every $\varphi \in L^2(\Omega_\varepsilon)$, one has*
$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N\times Y)} = \sqrt{|Y|}\|\varphi\|_{L^2(\Omega_\varepsilon)}$$
.
8. $\nabla_y \mathcal{T}_\varepsilon(\varphi)(x,y) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(x,y) \quad$ *for every* $(x,y) \in \Omega \times Y$.
9. *If $\varphi \in H^1\left(\widetilde{\Omega_\varepsilon}\right)$, then $\mathcal{T}_\varepsilon(\varphi)$ is in $L^2\left(\mathbb{R}^N, H^1(Y)\right)$.*
10. *One has the estimate*
$$\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega\times Y))^N} = \varepsilon\sqrt{|Y|}\,\|\nabla_x \varphi\|_{(L^2(\Omega_\varepsilon))^N}$$

Proof.

1. Let $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in L^2(\Omega_\epsilon)$

$$\begin{aligned}\mathcal{T}_\varepsilon(\alpha\varphi + \beta\psi) &= \alpha\varphi + \beta\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \alpha\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right) + \beta\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \alpha\mathcal{T}_\varepsilon(\varphi) + \beta\mathcal{T}_\varepsilon(\psi).\end{aligned}$$

Then $\mathcal{T}_\varepsilon$ is a linear opperator

3. Let $\varphi, \psi \in L^2(\Omega_\varepsilon)$ we have by definition (2.1)

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi\psi) &= \varphi\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi).\end{aligned}$$

4. and by definition for all $\varphi_\epsilon = \varphi(\dfrac{x}{\varepsilon}), \varphi \in L^2(Y)$, we have

$$\begin{aligned}\mathcal{T}_\varepsilon\left(\varphi_\epsilon\right)(x,y) &= \varphi\left(\varepsilon\left[\frac{\left\{\frac{x}{\varepsilon}\right\}}{\varepsilon}\right] + \varepsilon\left\{\frac{\left\{\frac{x}{\varepsilon}\right\}}{\varepsilon}\right\}\right)\\ &= \varphi\{\frac{x}{\varepsilon}\} = \varphi(y).\end{aligned}$$

5. according to the definition (2.1) we have :

$$\begin{aligned}\frac{1}{|\bar{Y}|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy &= \frac{1}{|Y|}\int_{\widetilde{\Omega_\varepsilon}\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy\\ &= \frac{1}{|Y|}\sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy\end{aligned}$$

on each set $(\varepsilon\xi + \varepsilon Y) \times Y$ whith $\xi \in \Xi_\varepsilon$,the function $\mathcal{T}_\varepsilon(\varphi)(x,y) = \varphi(\varepsilon\xi + \varepsilon y)$ is constant in x, it is consquence of (2.1). So,for each integral in the sum of the right member, we have :

$$\begin{aligned}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy &= |\varepsilon\xi+\varepsilon Y|\int_Y \varphi(\varepsilon\xi+\varepsilon y)dy \\ &= \varepsilon^N|Y|\int_Y \varphi(\varepsilon\xi+\varepsilon y)dy = |Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x)dx\end{aligned} \tag{2.3}$$

we sppose $x=\varepsilon\xi+\varepsilon y$,this implies $dx=\varepsilon^N dy \implies dy=\frac{1}{\epsilon^N}dx$

by suming in Ξ ,the right member be $\int_{\widetilde{\Omega}_\varepsilon}\varphi(x)dx$

so we get:$\sum_{\xi\in\Xi}\varepsilon^N|Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x)\frac{1}{\varepsilon^N}dx=\int_{\widetilde{\Omega}}\varphi(x)dx.$

Now we remplace this expretion in (2.3), we get the result.

7. for all $\varphi\in L^2(\Omega_\varepsilon)$

$$\begin{aligned}\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N\times Y)} &= \left(\int |\mathcal{T}_\varepsilon(\varphi)|^2 dxdy\right)^{\frac{1}{2}} \\ &= \left(1\int_{\Omega\times y}|\varphi|^2 dxdy\right)^{\frac{1}{2}} = \left(\int 1dy\right)^{\frac{1}{2}}\left(\int|\varphi|^2dx\right)^{\frac{1}{2}} \\ &= \sqrt{|y|}\|\varphi\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

8. for all $\varphi\in L^2(\Omega_\varepsilon)$

$$\begin{aligned}\nabla_y\mathcal{T}_\varepsilon(\varphi)(x,y) &= \nabla_y\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y+\varepsilon y\right) \\ &= \varepsilon\nabla_x\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y+\varepsilon y\right) \\ &= \varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)(x,y).\end{aligned}$$

10. we applay proposition 2.1, we obtain

$$\begin{aligned}\|\nabla_y\mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega\times Y))^N} &= \|\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)\|_{(L^2(\Omega\times Y))^N} \\ &= \varepsilon\sqrt{|Y|}\|\nabla_x\varphi\|_{(L^2(\Omega_\varepsilon))^N}\end{aligned}$$

Proposition 2.2 *[5] Let $\varphi\in L^2(\Omega)$. Then,*

1. $\mathcal{T}_\varepsilon(\varphi)\longrightarrow\widetilde{\varphi}$ *strongly in* $L^2\left(\mathbb{R}^N\times Y\right)$.
2. $\varphi\chi_\varepsilon\to\varphi$ *weakly in* $L^2(\Omega)$.

3. Let (φ_ε) *be in* $L^2(\Omega)$ *such that*

$$\varphi_\varepsilon \longrightarrow \varphi \quad \textit{strongly in } L^2(\Omega).$$

Then

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \widetilde{\varphi} \quad \textit{strongly in } L^2\left(\mathbb{R}^N \times Y\right).$$

Proposition 2.3 *[5] Let* φ_ε *in* $L^2(\Omega_\varepsilon)$ *and* $\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C$ *for every* ε*, such that*

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi} \quad \textit{weakly in } L^2(\Omega \times Y).$$

Then,

$$\widetilde{\varphi_\varepsilon} \rightharpoonup \frac{1}{|Y|}\int_Y \widehat{\varphi}(.,y)dy \quad \textit{weakly in } L^2(\Omega).$$

Proposition 2.4 *[5] Let* φ_ε *be in* $L^2\left(\widetilde{\Omega}_\varepsilon\right)$ *for every* ε*, with*

$$\|\varphi_\varepsilon\|_{L^2\left(\widetilde{\Omega_\varepsilon}\right)} \leq C,$$

$$\varepsilon\,\|\nabla_x\varphi_\varepsilon\|_{\left(L^2\left(\widetilde{\Omega_\varepsilon}\right)\right)^N} \leq C.$$

Then, there exists $\hat{\varphi}$ *in* $L^2\left(\Omega; H^1(Y)\right)$ *such that, up to subsequences*

1. $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi}$ *weakly in* $L^2\left(\Omega; H^1(Y)\right)$.
2. $\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi_\varepsilon) \to \nabla_y\widehat{\varphi}$ *weakly in* $L^2(\Omega \times Y)$,

   where

$$y \longmapsto \widehat{\varphi} \in L^2\left(\Omega; H^1_{per}(Y)\right).$$

2.1.1 Macro-Micro Decomposition

Following [5], we decompose any function φ in the from

$$\varphi = \mathcal{Q}_\varepsilon(\varphi) + \mathcal{R}_\varepsilon(\varphi),$$

where $\mathcal{R}_\varepsilon$ is designed in order to capture the oscillations.

As in the case of fixed domains, we start by defining $\mathcal{Q}_\varepsilon$ on the nodes $\varepsilon\xi_k$ of the εY-lattice. Here, it is no longer possible to take the average on the entire cell Y as in [5], but it will be taken on a small ball B_ε centered on $\varepsilon\xi_k$ and not touching the holes. This is possible using the fact that ∂T does not contain the summits of Y. However, B_ε must be entirely contained in Ω_ε.

To guarantee that, we are let to define $\mathcal{Q}_\varepsilon(\varphi)$ on a subdomain of Ω_ε only. To do so, for every $\delta > 0$, let us set

$$\Omega_\delta^\varepsilon = \{x \in \Omega; d(x, \partial\Omega) > \delta\} \quad \text{and} \quad \widehat{\Omega_\delta^\varepsilon} = int\left(\bigcup_{\xi \in \Pi_\varepsilon^\delta} \varepsilon(\xi + \bar{Y})\right),$$

where

$$\Pi_\delta^\varepsilon = \left\{\xi \in \mathbb{Z}^N; \varepsilon(\xi + \bar{Y}) \subset \Omega_\delta^\varepsilon\right\}.$$

- For every node $\varepsilon\xi_k$ in $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ we define

$$\mathcal{Q}_\varepsilon(\varphi)\,(\varepsilon\xi_k) = \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} \varphi\,(\varepsilon\xi_k + \varepsilon z)\, dz.$$

  Observe that by definition, any ball B_ε centered in a node of $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$ s entirely contained in Ω_ε, since actually they all belong to $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$.

- We define $\mathcal{Q}_\varepsilon(\varphi)$ on the whole $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, by taking a Q_1-interpolate, as in the finite element method, of the discrete function to $\mathcal{Q}_\varepsilon(\varphi)\,(\varepsilon\xi_k)$.

- On $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$, $\mathcal{R}_\varepsilon$ will be defined as the remainder : $\mathcal{R}_\varepsilon(\varphi) = \varphi - \mathcal{Q}_\varepsilon(\varphi)$.

Proposition 2.5 *[5] For φ belonging to $H^1(\Omega_\varepsilon)$, one has the following properties*

1. $\|\mathcal{Q}_\varepsilon(\varphi)\|_{H^1(\hat{\Omega}_{2\varepsilon\rho(Y)})} \leq \|\varphi\|_{H^1(\hat{\Omega}_{2\varepsilon\rho(Y)})}$
2. $\|\mathcal{R}_\varepsilon(\varphi)\|_{L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)})} \leq C\varepsilon\,\|\nabla_x\varphi\|_{(L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)}))^N}$
3. $\|\nabla_x\mathcal{R}_\varepsilon(\varphi)\|_{(L^2(\widehat{\Omega}_{2\varepsilon\rho(Y)}))^N} \leq C\,\|\nabla_x\varphi\|_{(L^2(\hat{\Omega}_{2\varepsilon\rho(Y)}))^N}$ *We can now state the main result of this section.*

Theoreme 2.1 *[5] Let φ_ε be in $H^1(\Omega_\varepsilon)$ for every ε, with $\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ bounded. There exists φ in $H^1(\Omega)$ and $\widehat{\varphi}$ in $L^2\left(\Omega; H^1_{per}(Y)\right)$ such that, up to subsequences*

1. $\mathcal{Q}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ *weakly in* $H^1_{loc}(\Omega)$,
2. $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ *weakly in* $L^2_{loc}\left(\Omega; H^1(Y)\right)$,
3. $\dfrac{1}{\varepsilon}\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon(\varphi_\varepsilon)) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2_{loc}\left(\Omega; H^1(Y)\right)$,
4. $\mathcal{T}_\varepsilon(\nabla_x(\varphi_\varepsilon)) \rightharpoonup \nabla_x\varphi + \nabla_y\widehat{\varphi}$ *weakly in* $L^2_{loc}\left(\Omega; L^2(Y)\right)$.

2.1.2 The Averaging Operator $\mathcal{U}_\varepsilon$

definition 2.2 *[6] For $\varphi \in L^2(\Omega \times Y)$, we set*

$$\mathcal{U}_\varepsilon(\varphi)(x) = \frac{1}{|Y|}\int_Y \tilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz, \quad \text{for every } x \in \Omega.$$

Remark 2.2 *For $v \in L^1(\Omega \times Y)$, the function $x \longmapsto v\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right)$ is generally not measurable (for example, we refer to [18]). Hence, it cannot be used as a test function. We replace it by the function $\mathcal{U}_\varepsilon(v)$.*

The next result extends the corresponding one given in [5].

Proposition 2.6 *[6] One has the following properties*

1. *The operator $\mathcal{U}_\varepsilon$ is linear and continous from $L^2(\Omega \times Y)$ into $L^2(\Omega)$, and one has for every $\varphi \in L^2(\Omega \times Y)$*

$$\|\mathcal{U}_\varepsilon\|_{L^2(\Omega)} \leq \|\varphi\|_{L(\Omega\times Y)},$$

2. *$\mathcal{U}_\varepsilon$ is the left inverse of $\mathcal{T}_\varepsilon$ on Ω_ε, which means that $\mathcal{U}_\varepsilon \circ \mathcal{T}_\varepsilon = Id$ on Ω_ε,*
3. $\mathcal{T}_\varepsilon(\chi_\varepsilon \mathcal{U}_\varepsilon(\varphi))(x, y) = \dfrac{1}{|Y|}\displaystyle\int_Y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right) dz, \quad \forall \varphi \in L^2(\Omega \times Y),$
4. *$\mathcal{U}_\varepsilon$ is the formal adjoint of $\mathcal{T}_\varepsilon$.*

Theoreme 2.2 *[6] Let φ_ε be in $L^2(\Omega_\varepsilon)$ for every ε, and let $\varphi \in L^2\left(\mathbb{R}^N \times Y\right)$ Then,*

1. $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ *strongly in* $L^2\left(\mathbb{R}^N \times Y\right) \iff \widetilde{\varphi_\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \longrightarrow 0$ *strongly in* $L^2\left(\mathbb{R}^N\right)$

2. $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi$ *strongly in* $L^2_{loc}\left(\mathbb{R}^N; L^2(Y)\right) \iff \widetilde{\varphi_\varepsilon} - \mathcal{U}_\varepsilon(\varphi) \longrightarrow 0$ *strongly in* $L^2_{loc}\left(\mathbb{R}^N\right)$.

This result is essential for proving corrector results when studying homogenization problems.

Let us first state the following proposition.

Proposition 2.7 *[6] For every $\varphi \in H^1(\Omega_\varepsilon)$ one has*

$$\|\mathcal{R}_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} = \|\varphi - \mathcal{Q}_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla\varphi\|_{(L^2(\Omega_\varepsilon))^N}.$$

Theoreme 2.3 *[6] Let v_ε be in $H^1(\Omega_\varepsilon)$ for every ε and $v \in H^1(\Omega)$ such that*

1. $\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ *is bounded.*

2. $\tilde{v}_\varepsilon \to v$ *weakly in* $L^2(\Omega)$.

Then,

$$\mathcal{T}_\varepsilon(v_\varepsilon) \longrightarrow v \quad \text{strongly in } L^2_{loc}\left(\Omega, L^2(Y)\right).$$

2.1.3 The Boundary Unfolding Operator

We define here the unfolding operator on the boundary of the holes ∂T_ε, which is specific to the case of perforated domain. To do that, we need to suppose that T has Lipschitz boundary.

definition 2.3 *[6] Suppose that T has a Lipschitz boundary, and let $\varphi \in L^p(\partial T_\varepsilon)$, $p \in [1, +\infty]$.*

We define the function $\mathcal{T}^b_\varepsilon(\varphi) \in L^p\left(\mathbb{R}^N \times \partial T\right)$ by setting

$$\mathcal{T}^b_\varepsilon(\varphi)(x, y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right),$$

for every $x \in \mathbb{R}^N$ and $y \in \partial T$.

Observe that if $\varphi \in W^{1,p}(\Omega_\varepsilon)$ and $\varphi = 0$ on $\partial\Omega_\varepsilon \backslash \partial T_\varepsilon^{int}$ is the set holes that do not intersect the boundary $\partial\Omega$, one has $\mathcal{T}_\varepsilon^b(\varphi) = \mathcal{T}_\varepsilon(\varphi)$ on ∂T. The next assertion reformulate those presented in proposition 2.1, when functions are defined on the boundary ∂T_ε.

Proposition 2.8 *[6] The boundary unfolding operator $\mathcal{T}_\varepsilon^b$ has the following properties:*

1. *$\mathcal{T}_\varepsilon^b$ is a linear operator.*
2. *$\mathcal{T}_\varepsilon^b(\varphi)\left(x, \left\{\dfrac{x}{\varepsilon}\right\}_Y\right) = \varphi(x) \quad \forall\varphi \in L^p(\partial T_\varepsilon)$ and $x \in \mathbb{R}^N$.*
3. *$\mathcal{T}_\varepsilon^b(\varphi\psi) = \mathcal{T}_\varepsilon^b(\varphi)\mathcal{T}_\varepsilon^b(\psi), \quad \forall\varphi,\psi \in L^p(\partial T_\varepsilon)$.*
4. *Let φ in $L^p(\partial T)$ be a Y-periodic function. Set $\varphi_\varepsilon(x) = \varphi\left(\dfrac{x}{\varepsilon}\right)$. Then,*

$$\mathcal{T}_\varepsilon^b(\varphi_\varepsilon)(x,y) = \varphi(y)$$

5. *For every $\varphi \in L^1(\partial T_\varepsilon)$ we have the integration formula*

$$\begin{aligned}\int_{\partial T_\varepsilon} \varphi(x)d\sigma(x) &= \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}_\varepsilon^b(\varphi)dx\sigma(y) \\ &= \frac{1}{\varepsilon|Y|}\int_{\widetilde{\Omega_\varepsilon}\times\partial T} \mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y).\end{aligned}$$

6. *For every $\varphi \in L^2(\partial T_\varepsilon)$, $\mathcal{T}_\varepsilon^b(\varphi)$ belongs to $L^2\left(\mathbb{R}^N \times \partial T\right)$. It also belongs to $L^2\left(\widetilde{\Omega_\varepsilon} \times \partial T\right)$.*
7. *For every $\varphi \in L^2(\partial T_\varepsilon)$, one has*

$$\left\|\mathcal{T}_\varepsilon^b(\varphi)\right\|_{L^2(\mathbb{R}^N\times\partial T)} = \sqrt{\varepsilon|Y|}\|\varphi\|_{L^2(\partial T_\varepsilon)}.$$

Proposition 2.9 *[6] Let $g \in L^2(\partial T)$ and $\varphi \in H^1(\Omega)$ s.t. $\varphi = 0$ on $\partial\Omega_\varepsilon \backslash \partial T_\varepsilon^{int}$. One has the estimate*

$$\begin{aligned}\varepsilon|Y|\left|\int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x)\right| &= \left|\int_{\mathbb{R}\times\partial T} g(y)\mathcal{T}_\varepsilon^b(\varphi)(x,y)dxd\sigma(y)\right| \\ &\leq C\left(|\mathcal{M}_{\partial T}(g)| + \varepsilon\right)\|\nabla\varphi\|_{(L^2(\Omega_\varepsilon))^N},\end{aligned}$$

where $\mathcal{M}_{\partial T}(g) = \dfrac{1}{|\partial T|}\displaystyle\int_{\partial T} g(y)d\sigma(y)$.

Proof. See [6]

Proposition 2.10 *[6] Let* $g \in L^2(\partial T)$ *a Y-periodic function, and set* $g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$. *On has the following convergence results as* $\varepsilon \to 0$

1. *If* $\mathcal{M}_{\partial T}(g) \neq 0$, *then*

$$\varepsilon \int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x) \longrightarrow \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \varphi(x)dx \quad \forall \varphi \in H^1(\Omega)$$

2. *If* $\mathcal{M}_{\partial T}(g) = 0$, *then*

$$\int_{\partial T_\varepsilon} g_\varepsilon(x)\varphi(x)d\sigma(x) \longrightarrow 0 \quad \forall \varphi \in H^1(\Omega).$$

The next result is the equivalent of proposition (2.2) (1) and (2.3), to the case of functions defined on the boundaries of the holes.

Proposition 2.11 *[6]*

1. *Let* $\varphi \in H^1(\Omega)$. *Then, as* $\varepsilon \to 0$, *one has the convergence*

$$\int_{\mathbb{R}^N \times \partial T} \mathcal{T}_\varepsilon^b(\varphi)(x,y)dxd\sigma(y) \to \int_{\mathbb{R}^N \times \partial T} \tilde{\varphi} dxd\sigma(y).$$

2. *Let* $\varphi \in H^1(\Omega)$. *Then*

$$\mathcal{T}_\varepsilon^b(\varphi) \to \tilde{\varphi} \text{ strongly in } L^2\left(\mathbb{R}^N \times \partial T\right).$$

3. *Let* φ_ε *be in* $L^2(\partial T_\varepsilon)$ *for every* ε, *such that*

$$\mathcal{T}_\varepsilon^b(\varphi_\varepsilon) \rightharpoonup \hat{\varphi} \text{ weakly in } L^2\left(\mathbb{R}^N \times \partial T\right).$$

Then,

$$\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \psi d\sigma(x) \to \frac{1}{|Y|}\int_{\mathbb{R}^N \times \partial T} \hat{\varphi}(x,y)\psi(x)dxd\sigma(y) \quad \forall \psi \in H^1(\Omega).$$

Proposition 2.12 *[6] There exists a constant* C *independent of* ε *such that*

$$\left|\int_{\partial T_\varepsilon} g_\varepsilon(x)\, v(x)\, d\sigma\right| \leq C \|g\|_{(L^2(\partial T))^N} \|\nabla v\|_{(L^2(\Omega_\varepsilon))^{N\times N}}, \forall v \in V.$$

Lemma 2.1 *[7] There exists a positive constant* C*, independent of* ε*, such that*

$$\|v\|_{(L^2(\Omega_\varepsilon))^N} \leq C\varepsilon \|\nabla v\|_{(L^2(\Omega_\varepsilon))^{N\times N}}, \forall v \in V.$$

CHAPTER

3

ASYMPTOTIC BEHAVIOR OF THE STOKES PROBLEM BY UNFOLDING METHOD

In this chapter we study the stokes problem in a domain $\boldsymbol{\Omega_\varepsilon}$ of $\mathbb{R}^{\boldsymbol{N}}$ $\boldsymbol{\varepsilon}$-periodically perforated by holes of size $\boldsymbol{\varepsilon}$. Our aim is to descrabe the asymptotic behavior of the velocity and the pressure of the fluid as $\boldsymbol{\varepsilon} \rightarrow \mathbf{0}$ and give, if possible, a limite ("homogenized") problem . To do so, we use the periodic unfloding method introduced by Cioranescu, Damlamian and Griso (C.R. Acad. Sci. Paris, Ser. I 335(2002)99-104; SIAM J. of math. A,al. 40(4)(2008)1585-1620).

3.1 Setting of the problem and variational formulation

Let Ω be an open set in $\mathbb{R}^N (N \geq 2)$ and $Y = [0,1[\times \cdots \times [0,1[$ be a reference cell. Let T be a strictly closed subset of $\bar{Y}$.

We define the following domain:

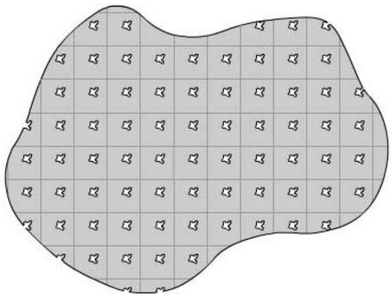

Fig. 2. The sets Ω_ε

Now, we consider the following stokes system :

$$\begin{cases} -\nu \Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ div u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \nu \dfrac{\partial u_\varepsilon}{\partial n} + \alpha \varepsilon^\gamma u_\varepsilon = g_\varepsilon & \text{on } \partial T_\varepsilon. \end{cases} \tag{3.1}$$

This problem modelize the flow of an incomprissible viscous fluid through a porous medium under the action of an exterior electric filed. Where, u_ε is the velocity field, p_ε is the pressure, f_ε is the field of exterior body forces,g_ε is the field of exterior surface forces. Recall that $\alpha \geq 0$ and γ are constants.We assume that the data f_ε and g_ε satisfing:

$$f_\varepsilon = \frac{f}{\varepsilon}, \qquad f_\varepsilon \in L^2(\Omega_\varepsilon) \tag{3.2}$$

$$g_\varepsilon = \varepsilon g, \qquad g_\varepsilon \in L^2(\partial T) \tag{3.3}$$

variational formulation

We multiply (3.1) by a test function φ where $\varphi \in \mathcal{D}(\Omega)$, we get

$$-\nu \Delta u_\varepsilon \varphi + \nabla p_\varepsilon \varphi = f_\varepsilon \varphi,$$

we integrate on Ω_ε,we get

$$-\nu \int_{\Omega_\varepsilon} \Delta u_\varepsilon \varphi dx + \int_{\Omega_\varepsilon} \nabla p_\varepsilon \varphi dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx,$$

by Green's formula

$$\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx - \nu \int_{\partial\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} \varphi d\sigma(x) - \int_{\Omega_\varepsilon} p_\varepsilon div \varphi dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx,$$

so

$$\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx - \nu \int_{\partial T_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} \varphi d\sigma(x) - \int_{\Omega_\varepsilon} p_\varepsilon div \varphi dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx,$$

on other hand, we have

$$\nu \frac{\partial u_\varepsilon}{\partial n} = -g_\varepsilon + \alpha \varepsilon^\gamma u_\varepsilon,$$

we obtain

$$\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx - \int_{\partial T_\varepsilon} g_\varepsilon \varphi d\sigma(x) + \alpha \varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x) - \int_{\Omega_\varepsilon} p_\varepsilon div \varphi dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx.$$

The variational formulation of system (3.1) is the following:

$$\begin{cases} \text{Find } u_\varepsilon \in V_\varepsilon, p_\varepsilon \in L^2(\Omega_\varepsilon), \text{ such that:} \\ \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx + \alpha \varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x) - \int_{\Omega_\varepsilon} p_\varepsilon div \varphi dx = \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx + \int_{\partial T_\varepsilon} g_\varepsilon \varphi d\sigma(x). \end{cases} \tag{3.4}$$

Finally, with the functional space

$$V_\varepsilon = \left\{ v | v \in H^1(\Omega_\varepsilon), v = 0 \text{ on } \partial\Omega_\varepsilon / \partial T_\varepsilon \right\}.$$

3.2 Apriori estimates for u_ε and p_ε

For the proof of proposition 3.1 we need the following two Lemmas due to Conca, they will be essential in the sequel. We use the first one in other to establish apriori estimates for the solution u_ε of problem (3.4) for different values of γ. The second Lemma is used to estimate the pressure p_ε.

Lemma 3.1 *([7] Lemma 6.1) There exists a positive constant C independent of ε such that*

$$\|\psi\|_{(L^2(\Omega_\varepsilon))^N} \leq C\left(\varepsilon \|\nabla\psi\|_{(L^2(\Omega_\varepsilon))^{N\times N}} + \varepsilon^{1/2}\|\psi\|_{(L^2(\partial T_\varepsilon))^N}\right),$$

for all $\psi \in V_\varepsilon$.

Lemma 3.2 *([7] Lemma 5.1) For every $\phi \in L^2(\Omega_\varepsilon)$,there exists $\varphi \in V_\varepsilon$such that*

$$\begin{cases} div\varphi = \phi, \\ \|\varphi\|_{V_\varepsilon} \leq C\|\phi\|_{L^2(\Omega_\varepsilon)}. \end{cases}$$

The next result gives the a priori estimates for u_ε and p_ε for the different values of γ.

Proposition 3.1 *Let $(u_\varepsilon, p_\varepsilon)$ be the solution of problem (3.4) then the following a priori estimates hold true:*

$\forall\gamma \in \mathbb{R}$

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C1, \tag{3.5}$$

$$\left\|\varepsilon^{-1}u_\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \leq C2, \tag{3.6}$$

If $\gamma < 0$

$$\left\|\varepsilon^{-\gamma}p_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leq C3, \tag{3.7}$$

If $\gamma \geq 0$

$$\|\varepsilon p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C4. \tag{3.8}$$

Proposition 3.2 *[11] Let $u_\varepsilon \in V_\varepsilon$, $\varphi \in L^2(\Omega_\varepsilon)$ ther exists constant C independent of ε such that:*

$$\left| \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x) \right| \leq \left(C\|\varphi\|_{L^2(\partial T_\varepsilon)}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{-1}\|\varphi\|_{L^2(\partial T_\varepsilon)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right).$$

Let u_ε a test function in (3.4),

$$\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon^2 dx + \alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon^2 d\sigma(x) - \int_{\Omega_\varepsilon} p_\varepsilon div u_\varepsilon dx = \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx + \int_{\partial T_\varepsilon} g_\varepsilon u_\varepsilon d\sigma(x),$$

we have $div u_\varepsilon = 0$ in Ω_ε, so

$$\left|\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon^2 dx + \alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon^2 d\sigma(x)\right| = \left|\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx + \int_{\partial T_\varepsilon} g_\varepsilon u_\varepsilon d\sigma(x)\right|,$$

by triangle inequality

$$\left|\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon^2 dx\right| + \left|\alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon^2 d\sigma(x)\right| \leq \left|\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx\right| + \left|\int_{\partial T_\varepsilon} g_\varepsilon u_\varepsilon d\sigma(x)\right|,$$

$$\nu \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} |u_\varepsilon|^2 d\sigma(x) \leq \int_{\Omega_\varepsilon} |f_\varepsilon u_\varepsilon| dx + \int_{\partial T_\varepsilon} |g_\varepsilon u_\varepsilon| d\sigma(x),$$

we use the Cauchy Shwarz inequality

$$\nu \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \alpha\varepsilon^\gamma \|u_\varepsilon\|^2_{L^2(\partial T_\varepsilon)} \leq \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|g_\varepsilon\|_{L^2(\partial T_\varepsilon)} \|u_\varepsilon\|_{L^2(\partial T_\varepsilon)},$$

using proposition 2.12, we get

$$\nu \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \alpha\varepsilon^\gamma \|u_\varepsilon\|^2_{L^2(\partial T_\varepsilon)} \leq \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\|g_\varepsilon\|_{L^2(\partial T_\varepsilon)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)},$$

$$\nu \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \alpha\varepsilon^\gamma \|u_\varepsilon\|^2_{L^2(\partial T_\varepsilon)} \leq C\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)},$$

by Poincaré inequality

$$\begin{aligned} \nu \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \alpha\varepsilon^\gamma \|u_\varepsilon\|^2_{L^2(\partial T_\varepsilon)} &\leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\ &\leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

from

$$\nu \left\|\nabla u_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} \leq C \left\|\nabla u_\varepsilon\right\|_{L^2(\Omega_\varepsilon)},$$

we get

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C,$$

and

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C,$$

so

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C.$$

It remains to obtain the a priori estimates for p_ε to do so ,use $\phi \in L^2(\Omega_\varepsilon)$ a test function in variational formulation of system due to lemma 3.1 and lemma 3.2.

Using lemma 3.2 into the variational formulation we get

$$\int_{\Omega_\varepsilon} p_\varepsilon \phi dx = \int_{\Omega_\varepsilon} p_\varepsilon div\varphi dx = \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi dx + \alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x)$$

$$- \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx - \int_{\partial T_\varepsilon} g_\varepsilon \varphi d\sigma(x)$$

Then,

$$\left|\int_{\Omega_\varepsilon} p_\varepsilon \phi dx\right| \leq \nu\left|\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi dx\right| + \left|\alpha\varepsilon^\gamma \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x)\right| + \left|\int_{\Omega_\varepsilon} f_\varepsilon \varphi dx\right| + \left|\int_{\partial T_\varepsilon} g_\varepsilon \varphi d\sigma(x)\right| \tag{3.9}$$

One has

• $\left|\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi dx\right| \leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}$

Then,

$$\left|\nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi dx\right| \leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\varphi\|_{V_\varepsilon} \tag{3.10}$$

Using proposition 3.2, we have

• $\alpha\varepsilon^\gamma\left|\int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x)\right| \leq \alpha\varepsilon^\gamma\left(C\|\varphi\|_{L^2(\partial T_\varepsilon)}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{-1}\|\varphi\|_{L^2(\partial T_\varepsilon)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)$

by Trace theoreme, we get

$\alpha\varepsilon^\gamma\left|\int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x)\right| \leq \alpha\varepsilon^\gamma\left(C\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{-1}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)$

then

$$\alpha\varepsilon^\gamma\left|\int_{\partial T_\varepsilon} u_\varepsilon\varphi d\sigma(x)\right| \leq \left(C\alpha\varepsilon^\gamma\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^\gamma\varepsilon^{-1}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)\|\varphi\|_{V_\varepsilon} \tag{3.11}$$

• $\left|\int_{\Omega_\varepsilon} f_\varepsilon\varphi d\sigma(x)\right| \leq C\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\varphi\|_{L^2(\Omega_\varepsilon)}$

$$\left|\int_{\Omega_\varepsilon} f_\varepsilon\varphi dx\right| \leq C\|\varphi\|_{V_\varepsilon} \tag{3.12}$$

On the other hand and by proposition 3.2, we have

• $\left|\int_{\partial T_\varepsilon} g_\varepsilon\varphi d\sigma(x)\right| \leq \left(C\|\varphi\|_{L^2(\partial T_\varepsilon)}\|\nabla g_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-1}\|\varphi\|_{L^2(\partial T_\varepsilon)}\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)$

by Trace theoreme, we get

$\left|\int_{\partial T_\varepsilon} g_\varepsilon\varphi d\sigma(x)\right| \leq \left(C\|\varphi\|_{V_\varepsilon}\|\nabla g_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-1}\|\varphi\|_{V_\varepsilon}\|g\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)$ so that

$$\left|\int_{\partial T_\varepsilon} g_\varepsilon\varphi d\sigma(x)\right| \leq C\|\varphi\|_{V_\varepsilon}+C\varepsilon^{-1}\|\varphi\|_{V_\varepsilon} \tag{3.13}$$

Then, turning back to (3.14) and using (3.10),(3.11),(3.12),(3.13) one has

$$\left|\int_{\Omega_\varepsilon} p_\varepsilon\phi dx\right| \leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.\|\varphi\|_{V_\varepsilon}+\left(C\alpha\varepsilon^\gamma\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^\gamma\varepsilon^{-1}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)\|\varphi\|_{V_\varepsilon}$$

$$+C\|\varphi\|_{V_\varepsilon}+C\varepsilon^{-1}\|\varphi\|_{V_\varepsilon}+C\|\varphi\|_{V_\varepsilon}$$

then

$$\left|\int_{\Omega_\varepsilon} p_\varepsilon\phi dx\right| \leq \left[C(1+\varepsilon^\gamma)\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^\gamma\varepsilon^{-1}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-1}\right]\|\varphi\|_{V_\varepsilon}$$

Due to lemma 3.2 one has

$$\left|\int_{\Omega_\varepsilon} p_\varepsilon\phi dx\right| \leq C\left[(1+\varepsilon^\gamma)\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+\varepsilon^\gamma\varepsilon^{-1}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}+\varepsilon^{-1}\right]\|\phi\|_{L^2_{(\Omega_\varepsilon)}} \tag{3.14}$$

The a priori estimates for the pressure follow now from (3.14) and estimates the u_ε and ∇u_ε corresponding to the different values of γ.

we obtain:

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\left(1+\varepsilon^\gamma+\varepsilon^{-1}\right)$$

this imply two cases:

for $\gamma \geq 0$

$$\|\varepsilon p_\varepsilon\| \leq C \tag{3.15}$$

for $\gamma < 0$

$$\left\|\varepsilon^{-\gamma} p_\varepsilon\right\| \leq C \tag{3.16}$$

in the auther hand by Young incquality, we get :

$$\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{1}{\eta}\varepsilon^2 + \eta\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C$$

this imply :

$$\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c \Rightarrow \|\varepsilon\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \tag{3.17}$$

3.3 Statement of the homogenization results

Theoreme 3.1 (Darcy type law) *Let $(u_\varepsilon, p_\varepsilon)$ be the solusion of (3.4) for $\gamma < 0$, the sequence $\left\{\varepsilon^{-1}u_\varepsilon, \varepsilon^{-\gamma}p_\varepsilon\right\}_\varepsilon$ is uniformely bounded in $L^2(\Omega)^N \times L^2(\Omega)$.*
So,up to subsequence , $\forall\alpha > 0$ there exists $(u, p) \in L^2(\Omega) \times L^2(\Omega)$ such that:

$$\varepsilon^{-1}u_\varepsilon \to u \text{ in } L^2(\Omega)^N,$$

$$\varepsilon^{-\gamma}p_\varepsilon \to p \text{ in } L^2(\Omega),$$

$$u = \frac{-1}{\alpha|\partial T|}\nabla p.$$

Proof.

we multiply (3.4) by $\varepsilon^{-\gamma}$, we get :

$$\nu\varepsilon^{-\gamma}\int_{\Omega_\varepsilon}\nabla\varphi dx+\alpha\int_{\partial T_\varepsilon}u_\varepsilon\varphi d\sigma(x)-\varepsilon^{-\gamma}\int_{\Omega_\varepsilon}p_\varepsilon div\varphi dx$$
$$=\varepsilon^{-\gamma}\int_{\Omega_\varepsilon}f_\varepsilon\varphi dx+\varepsilon^{-\gamma}\int_{\partial T_\varepsilon}g_\varepsilon\varphi d\sigma(x).$$

By unfloding operator one has :

$$\varepsilon^{-\gamma}\nu\int_{\Omega\times y}\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla\varphi)dxdy+\alpha\int_{\Omega\times\partial T}\mathcal{T}^b_\varepsilon(\varepsilon^{-1}u_\varepsilon).\mathcal{T}^b_\varepsilon(\varphi)dxd\sigma(y)$$
$$-\int_{\Omega_\varepsilon}\varepsilon^{-\gamma}p_\varepsilon div\varphi dx=\varepsilon^{-\gamma}\int_{\Omega\times y}\mathcal{T}_\varepsilon(f_\varepsilon)\mathcal{T}_\varepsilon(\varphi)dxdy+\frac{\varepsilon^{-\gamma}}{\varepsilon}\int_{\Omega\times\partial T}\mathcal{T}^b_\varepsilon(y_\varepsilon)\mathcal{T}^b_\varepsilon(\varphi)dxd\sigma(y). \tag{3.18}$$

Let now pars to the limit in the (3.18), using ∇u_ε is bonded in $L^2(\Omega_\varepsilon)^{N\times N}$, and $-\gamma>0$, we get at the limit, all the integrals vanish axept for the second, and third integrales this imply the following:

$$\alpha\int_{\Omega\times\partial T}u\varphi dxd\sigma y-\int_\Omega p div\varphi dx=0,$$

then,

$$\alpha\int_{\Omega\times\partial T}u\varphi dxd\sigma y+\int_\Omega\nabla p\varphi dx=0, \tag{3.19}$$

we get:

$$\alpha|\partial T|\int_\Omega u\varphi dx=-\int_\Omega\nabla p\varphi dx.$$

This leads the Darcy low to theorem 3.1.

Theoreme 3.2 (Brinkman type law) *Let* $(u_\varepsilon,p_\varepsilon)$ *be the solution of problem* (3.4). *If* $\gamma\geq 0$, *the sequence* $\{\varepsilon u_\varepsilon,p_\varepsilon\}$ *is uniformly bounded in* $L^2(\Omega)^N\times L^2(\Omega)$. *So, up to a subsequence, there exists* $(u,p)\in L^2(\Omega)^N\times L^2(\Omega)$ *such that*

$$\varepsilon u_\varepsilon\to u \textit{ weakly in } L^2(\Omega)^N.$$

$$\varepsilon p_\varepsilon\to p \textit{ weakly in } L^2(\Omega).$$

Morever, there exists $\hat{u} \in L^2(\Omega, H^1_{per}(Y))^N$ *such that* $(u, \hat{u}, p)$ *solves the equations :*

$$\nu \int_{\Omega\times Y} (\nabla u + \nabla \hat{u}) \nabla_y \Psi(x,y) dxdy = 0, \tag{3.20}$$

if $\gamma = 0$,

$$\nu \int_{\Omega\times Y} (\nabla_x u + \nabla_y \hat{u}) \nabla \varphi dxdy + \alpha \int_{\Omega\times \partial T} u\varphi dxd\sigma y + \int_{\Omega\times Y} \nabla p \varphi dxdy = \int_{\Omega\times Y} f\varphi dxdy, \tag{3.21}$$

if $\gamma > 0$,

$$\nu \int_{Y} (\nabla_x u + \nabla_y \hat{u}) \nabla \varphi dxdy + \int_{\Omega\times Y} \nabla p \varphi dxdy = \int_{\Omega\times Y} f\varphi dxdy. \tag{3.22}$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ a test function in (3.4) and we multiply by ε, we get

$$\nu \int_{\Omega_\varepsilon} \varepsilon \nabla u_\varepsilon \nabla \varphi dx + \alpha\varepsilon^\gamma \varepsilon \int_{\partial T_\varepsilon} u_\varepsilon \varphi d\sigma(x) - \varepsilon \int_{\Omega_\varepsilon} p_\varepsilon div \varphi dx = \varepsilon \int_{\Omega_\varepsilon} f_\varepsilon \varphi dx + \varepsilon \int_{\partial T_\varepsilon} g_\varepsilon \varphi d\sigma(x), \tag{3.23}$$

with reads as $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon = I_4^\varepsilon + I_5^\varepsilon$.
For to pass to the limit in each terme, we use the unfloding method.

We start by the first integral, we use the $\mathcal{T}_\varepsilon$ operater, we get

$$I_1^\varepsilon = \nu \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varepsilon \nabla u_\varepsilon) \mathcal{T}_\varepsilon(\nabla \varphi) dxdy,$$

from corresponding estimates (3.17) we can applay theorem 2.1, we get the convergence

$$\mathcal{T}_\varepsilon(\varepsilon \nabla u_\varepsilon) \longrightarrow \nabla_x u + \nabla_y \hat{u} \text{ in } L^2(\Omega \times Y)^{N\times N},$$

so, we obtain

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \nu \int_{\Omega\times Y} (\nabla_x u + \nabla_y \hat{u}) \nabla \varphi dxdy. \tag{3.24}$$

For the second integral I_2^ε we have two cases, the first cases if $\gamma = 0$, using the boundary operator $\mathcal{T}_\varepsilon^b$, we get :

$$I_2^\varepsilon = \frac{1}{\varepsilon}\varepsilon\alpha \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(u_\varepsilon)\mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y),$$
$$I_2^\varepsilon = \alpha \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(u_\varepsilon)\mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y).$$

Using estimate (3.5) and by theorem 2.1, we get

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \alpha \int_{\Omega\times\partial T} u\varphi dxd\sigma(y), \tag{3.25}$$

for $\gamma > 0$,

$$I_2^\varepsilon = \alpha\varepsilon^\gamma \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(u_\varepsilon)\mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y),$$
$$\lim_{\varepsilon\to 0} I_2^\varepsilon = 0. \tag{3.26}$$

For the third integral and corresponding estimates (3.8) it follows that $\varepsilon p_\varepsilon \longrightarrow p$ weakly in $L^2(\Omega)$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_{\Omega\times Y} pdiv\varphi dxdy = \int_{\Omega\times Y} \nabla p\varphi dxdy. \tag{3.27}$$

In the forth integral, we use the unfolding $\mathcal{T}_\varepsilon$, and assumption (3.2)

$$I_4^\varepsilon = \varepsilon\frac{1}{\varepsilon}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\varphi)dxdy,$$

$$\lim_{\varepsilon\to 0} I_4^\varepsilon = \int_{\Omega\times Y} f\varphi dxdy. \tag{3.28}$$

For the last integral, we use $\mathcal{T}_\varepsilon^b$ and assumption (3.3), we get

$$I_5^\varepsilon = \frac{1}{\varepsilon}\varepsilon \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(\varepsilon g)\mathcal{T}_\varepsilon^b(\varphi)dxd\sigma(y),$$

$$\lim_{\varepsilon\to 0} I_5^\varepsilon = 0. \tag{3.29}$$

We are now in a position to pass to the limit in (3.23), using (3.24),(3.25),(3.27),(3.28),and (3.29) we get (3.21).

For the equation (3.22), we also pass at the limit at (3.23) and we use (3.24),(3.26),(3.27),(3.28),and (3.29), we get the results.

Now, for the equation (3.20) we use v_ε a test function in (3.4), where $v_\varepsilon = \varepsilon\psi(x)\phi(\frac{x}{\varepsilon})$,

then, $\nabla v_\varepsilon = \nabla \phi(\frac{x}{\varepsilon})\psi(x) + \varepsilon \nabla \psi(x)\phi(\frac{x}{\varepsilon})$,
where $\varepsilon \to 0$, all the terms gos to zero except the second one ,unfolding it with $\mathcal{T}_\varepsilon$ yields

$$\nu \int_{\Omega_\varepsilon} \nabla u_{\varepsilon(x)} \psi(x) \nabla \varphi(\frac{x}{\varepsilon}) dx \simeq \nu \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \mathcal{T}_\varepsilon(\psi) \mathcal{T}_\varepsilon(\nabla \varphi(\frac{x}{\varepsilon})) dx dy,$$
$$= \nu \int_{\Omega \times Y} (\nabla u + \nabla \hat{u}) \varphi(x) \nabla_y \varphi(y) dx dy,$$

suppose that $\Psi = \psi(x)\varphi(y)$,then$\nabla_y \Psi = \psi(x) \nabla_y \varphi(y)$this imply,

$$\nu \int_{\Omega \times Y} (\nabla u + \nabla \hat{u}) \nabla_y \Psi(x, y) dx dy = 0.$$

CONCLUSION

In this memory , we studied the homogenization of the stokes problem in a perforat domain. This problem describe the flow of an incompressible viscous medium under the action of an external electrical field.
In the third chapter, we have treated asymptotic behavior of the stokes system whith boundary condition, to do so , we used the periodic unfolding method introduced by [5] we obtained at the limit, following the values of a real parametre γ , a Darcy or Brinkmann type laws.

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