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Global non-existence and decay of solutions for nonlinear wave equation

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Thanks

All gratitude and thanks to Allah our creator who gave me the strength to finish this modest work.

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DEDICATIONS

I dedicate this modest work:

- To those who gave me life, a symbol of beauty, pride, wisdom and patience.
- To those who are the source of my inspiration and courage, to whom I owe love and gratitude.
- To my husband, my daughter Miral and my son Anes.
- To my father, may Allah have mercy on him.
- To my mother, may God prolong her life and keep her safe.
- To my brothers Hamza Foued and Ramzi and my sisters Nani and Lamia.
- To my nephews Rafif, Douâa, and Sidra.
- To my aunts Assia & Nabila.
- To my cousins Feriel, Abir and Aymen.
- To Ikram & Amira.
- And to everyone who knows me and counts on me...

ملخص

ندرس في هذه المذكرة:

قمنا بتقديم الحالة الأكثر عمومية لمعادلة تفاضلية خطية من الدرجة الثانية ذات متغير منفصلين وتصنيف إلى ثلاثة أنواع أساسية زائدية، قطعية زائدية، قطعية ناقصية وتحويلها إلى صيغة قياسية، ثم قمنا بعرض النظريات والتعاريف المهمة عن فضاء الأس المتغير ليبسيغ $L^{p(.)}(\Omega)$ وفضاء صوبوليف $W^{p(.)}(\Omega)$ كما أشارنا إلى نتيجة الوجود. ثم نثبت عدم وجود الحل العالمي لفئة من المعادلات الموجية ذات التخميد غير الخطي وشروط المصدر، بعدها ذكرنا نظرية الوجود للمشكلة ونتيجة الاضمحلال للحلول معادلة موجبة غير خطية مخمدة ذات متغير غير خطي.

الكلمات المفتاحية:

- **المعادلات التفاضلية.**
- **معادلات الموجة.**
- **شروط المصدر.**
- **المتغيرات الأسية.**

Absract

In this thesis we examine:

We present the most general case of a second-order linear differential equation with two discrete variables and classify it into three basic types: hyperbolic, parabolic, and elliptic, and convert it to a standard form, We then presented important theories and definitions about the Lebesgue variable exponent space $L^{p(.)}(\Omega)$ and Sobolev space $W^{p(.)}(\Omega)$ We also pointed out the result of existence. Then we prove that there is no universal solution to a class of wave equations with nonlinear damping and source conditions. After that, we mention the theory of the existence of the problem and the decay result of the solutions of a positive nonlinear damped equation with a nonlinear variable.

Keys words :

- Differential equations.
- Wave equations.
- Source terms.
- Variable exponents.

Résumé

Dans cette thèse, nous examinons :

Nous présentons le cas le plus général d'une équation différentielle linéaire du second ordre avec deux variables discrètes et la classifions en trois types de base hyperbolique, paraboliques et elliptique et la convertissons en une forme standard, Ensuite, nous avons exposé des théories et des définitions cruciales sur l'espace d'exposant variable de Lebesgue $L^{p(.)}(\Omega)$ et l'espace de Sobolev $W^{p(.)}(\Omega)$, nous avons aussi mis en évidence le résultat de l'existence. Ensuite, nous prouvons qu'il n'y a pas de solution universelle à une classe d'équations ondulatoires avec un amortissement non linéaire et des conditions de source, après quoi nous avons mentionné la théorie d'existence du problème et le résultat de décroissance des solutions d'une équation positive non linéaire amortie avec une variable non linéaire.

Mots Clés :

- **Equation différentiel.**
- **Equation d'onde.**
- **Source terme.**
- **Exposants variables.**

CONTENTS

Introduction

Equation 4.1 appears in the models of non-linear viscoelasticity (see References [1]-[3]). It also can be considered s a system governing the longitudinal motion of a viscoelastic configuration obeying a non-linear Voight model (see reference [3],[4]).

In the absence of viscosity and strong damping, Equation 4.1 becomes

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{\alpha-2}\nabla u\right) + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, x \in \Omega, t > 0.$$

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitary initial data (see Reference [5],[6]). Then, for $a = 0$ the source term causes finite time blow up of solutions with negative initial energy if $p > \alpha$ (see Reference [7],[8]).

The interaction between the damping and the source terms was first considered by Levine [9],[10] in the linear damping case ($\alpha = m = 2$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [11] extended Levine's result to the non-linear damping case ($m > 2$). In their work, the authors considered 4.2 with ($\alpha = 2$) and introduced a method than the one know as the concavity method. They determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negativ energy continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [12] and Levine et al.[13]. In these work, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several non-continuation theorems. This generalization allowed them also to apply their result to quasilinear situation ($\alpha > 2$), of which the problem in Reference [11] is a particular case. Vitillaro [14] combined the arguments in References [11],[12] to extend these results to situations where the damping is non-linear and the solution has positive initial energy. Similar results have also been established by Todorova [15],[16] for different Cauchy problems.

In Reference [3] Yang studies 4.1 and proved a blow up result under the condition $p > \max\{\alpha, m\}, \alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) Theorem 2.1 of Reference [3]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$ (see Remark 2 of Reference [3]). We should note here that 4.1 corresponds to Equation (5) of [3] but the same conclusions hold for Equation 4.1 of the same work, under suitable conditions, stated in Theorem 2.3

of [3].

In this work we show that any weak solution of 4.1, with negative initial energy, cannot exists for all time if $p > \max\{\alpha, m\}, \alpha > \beta$, Therefore, our result improves the one of [3]. Our technique of proof follows closely the argument of [17] with the modifications needed for our problem.

Constant exponent

For $r = 2$ it is well-known that the damping term $u_t |u_t|^{m-2}$ assures global existence for arbitrary initial data (see [6]). Chen et al. [18] considered the nonlinear wave equation with the p-Laplacian operator.

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \Delta u_t + g(x, u) = f(x).$$

in a bounded domain $\Omega \in \mathbb{R}^n$, where $2 \leq p < n$ and f, g are given functions. Under suitable conditions on the initial data and the functions f, g global existence, uniqueness, and the asymptotic behavior of the solution are established. In [19], Benaissa et al. looked into the following equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \sigma(t) \operatorname{div}\left(|\nabla u_t|^{m-2} \nabla u_t\right) = 0.$$

where σ is a positive function, $p, m \geq 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. They gave an energy decay estimate for the solutions and extended the results obtained by Yang [20] and Messaoudi [21] by giving more precise decay rates. In 2015, Mokeddem and Mansour [22] considered the equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \sigma(t)\left(u_t - \operatorname{div}\left(|\nabla u_t|^{m-2} \nabla u_t\right)\right) = 0.$$

and gave the same decay result. Ibrahim and Lyaghfouri [23] considered the following p-wave equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |u|^{p_*-2} u.$$

in $\mathbb{R}^n$, where $n \geq 3, 2 < p < n$ and $p_* = \frac{pn}{n-p}$ is the critical Sobolev exponent. Under suitable conditions on the initial data, they proved the finite-time blow-up of solutions and, hence, they extended a result by Galaktionov and Pohozaev [24]. Wu and Xue [25] studied the following quasilinear hyperbolic:

$$u_{tt} - \psi(t) \operatorname{div}\left(|\nabla u|^{p-2} \nabla u_t\right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i\left(u_{x_i}\right) + \mu |u_t|^{\alpha} u_t = 0.$$

where $\mu, p \geq 0$, and $p \geq 2$ are constants, $\sigma_i (i = 1, ..., n)$ and ψ are given nonlinear functions, and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. By using the multiplier methods, they investigated the stability of weak solutions to the problem and obtained explicit decay rate estimation depending on strain-caused stress term $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i (u_{x_i})$ and the p-Laplacian damping terms. In [26], Messaoudi and Houari considered the nonlinear wave equation:

$$u_{tt} - \Delta u_t - \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{\beta-2} \nabla u_t\right) + a\,|u_t|^{m-2} u_t = b\,|u|^{p-2} u.$$

where $a, b > 0, \alpha, \beta, m, p > 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. They proved, under suitable conditions on $\alpha, \beta, m, p > 2$ and for negative initial energy, a global non-existence theorem. Ye [27] investigated the blow-up of solutions for a quasilinear hyperbolic equations system. He proved that certain solutions with positive initial energy blow up in finite time under suitable conditions. He also gave the estimates of the lifespan of solutions. In [28], Kafini and Messaoudi considered a nonlinear wave equation with damping delay terms and showed that, under suitable conditions on the initial data, the energy of solutions blows up in a finite time. For more results, we refer the reader to [29]-[33].

Variable exponent nonlinearity

In recent years, much attention has been paid to the study of mathematical nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents of nonlinearity. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on this subject can be found in [34] and [35]. We note here that most of the results regarding hyperbolic equations with variable exponents of nonlinearity deal with blow up and global nonexistence. Let us mention some of these works. For instance, Antontsev [36] considered the equation:

$$u_{tt} - \operatorname{div}\left(a\,(x,t)\,|\nabla u|^{p(x,t)-2} \nabla u\right) - \alpha \nabla u_t = b(x,t)\,|u|^{\sigma(x,t)-2} u.$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $\alpha > 0$ is a constant and a, b, p, σ are given functions. Under appropriate conditions on the initial data and the functions a, b, p, σ, he proved some blow up results, for certain solutions with non positive initial energy. He also discussed the case when $\alpha = 0$ and established a blow up result. Subsequently, Antontsev

[37] discussed the same equation and proved the local and the global existence of a weak solution under suitable conditions on a, b, p, σ.

He also established some blow up results for certain solutions with non positive initial energy. Guo and Gao [38] considered the same problem of [36] and established several blow-up results for certain solutions with positive initial energy. Precisely, they took $\sigma(x,t) = r > 2$, a constant, and proved a finite-time blow-up result. For the case $\sigma(x,t) = r(x)$, they claimed the same blow up result but no proof has been given. That work was considered to be an improvement for that of [36]. In [39], Sun et al. looked into the following equation:

$$u_{tt} - \operatorname{div}\left(a\left(x,t\right)\nabla u\right) + c\left(x,t\right)u_t\left|u_t\right|^{q(x,t)-1} = b(x,t)u\left|u\right|^{p(x,t)-1}.$$

in a bounded domain, with Dirichlet-boundary conditions, and established a blow-up result for certain solutions with positive initial energy. They also gave lower and upper bounds for the blow-up time and provided a numerical example to illustrate their result. Recently, Messaoudi and Talahmeh [40] studied the equation:

$$u_{tt} - \operatorname{div}\left(\left|\nabla u\right|^{m(x)-2}\nabla u\right) + \mu u_t = \left|u\right|^{p(x)-2}u.$$

with Dirichlet-boundary conditions and for $\mu \geq 0$. They established a blow up result for certain solutions with arbitrary positive initial energy. This result generalized that of Korpusov [41] established for 1.3, with m and p constants. Y. Gao and W. Gao [42] studied a nonlinear viscoelastic equation with variable exponents and proved the existence of weak solutions by using the Faedo–Galerkin method. Autuori et al. [43] looked into a nonlinear Kirchhoff system in the presence of the $\overrightarrow{p}(x,t)$ p-Laplace operator, a nonlinear force $f(t,x,u)$ and a nonlinear damping term $Q = Q(t,x,u,ut)$. They established a global non-existence result under suitable conditions on f, Q, p. For more results concerning the blow up of hyperbolic problems, we refer the reader to Antontsev and Ferreira [44] and the book by Antontsev and Shmarev [45].

Our aim in this work is to find sufficient conditions on m, r and the initial data for which the decay estimate for the solution takes place. This work consists of five sections in addition to the introduction. In Section one we begin with the classification of differential equations into second-order differential equations and the canonical forms, and in section two we presented important theories and definitions, In section three we recall the definitions of the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ and the Sobolev space $W^{1,p(.)}(\Omega)$ as well as some of their properties. In Section four and five we state and prove our decay result for solutions by using a lemma by Komornik [46].

CHAPTER 1

Classification of partial differential equation and canonical forms

1.1 Second-order partial differential equation

The most general case of second-order linear partial differential equation (PDE) in two independent variables is given by:

$$A\frac{\partial^{2u}}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1.1}$$

Where the coefficients A, B and C are functions of x and y and do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order.

Further the coefficients D, E and F are also assumed to be functions of x and y. We shall assume that the function $u(x, y)$ and the coefficients are twice continuously differentialble in some domain Ω. The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of notation, we combine the lower ordre terms and rewrite the above equation in the folowing form:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{1.2}$$

or using the short-hand notations for partial derivatives:

$$A(x,y)u_{x\,x} + B(x,y)u_{xy} + C(x,y)u_{yy} = \phi(x, y, u, u_x, u_y). \tag{1.3}$$

As we shall see, there are fondanmentally three types of PDE_{s-}hyperbolic, and elliptic $PDE_{S.}$From the physical point of view, these PDE_S respectively represents the wave propagation, the time-dipendent diffusion processes and the steady state or equilibrium processes. Thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems descorde evolutionary phenomena that lead to a steady state described by an elleptic equation. and elleptic equations are associated to a special state of a system, in principle corresponding to the minimum of the invergy. Mathematically, these classification of second-order PDE_S is based up on the possibility of redesing equation 1.3 by coordinate transformation to canonical or standard from at a point. It may be noted that, for the purposes of classification, it is not necessary to restrict considuration to linear equation. It is applicable to quasilinear second-order PDE as well. Aquasilinear second-ordre PDE is linear in the second derivatives only. The type of second-ordre PDE 1.2 at a point $(x_{0,}y_0)$ depents on the sign of the discriminant defined as:

$$\Delta(x_{0,}y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B\left(x_{0,}y_0\right)^2 \_4A\left(x_0, y_0\right)C\left(x_{0,}y_0\right). \tag{1.4}$$

The classification of second-order linear PDE_s is given by the following .If $\Delta\left(x_0 y_0\right) \succ 0$, the equation is hyperbolic, $\Delta\left(x_0, y_0\right) = 0$ the equation is parabolic, and $\Delta\left(x_0, y_0\right) \prec 0$ the equation is elliptic. In order to illustrate the significance of the discriminant Δ and thus the classification of the PDE 1.3. We try to reduce the given equation 1.3 to a canonical form.To do this, we transform the independent variables x and y to the new independent variables ζ and η through the change of variables:

$$\zeta = \zeta\left(x, y\right), \qquad \eta = \eta\left(x, y\right). \tag{1.5}$$

Where both ζ and η are twice continuously differentiable and that the Jacobian:

$$J = \frac{\partial\left(\zeta, \eta\right)}{\partial\left(x, y\right)} = \begin{vmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0. \tag{1.6}$$

in the region under consideration. The nonvanishing of the Jacobian of the transformation ensure that a one to one transformation exists between the new and old variables. This simply means that the new independent variables can serve as new coordinate variables without any apply the chain rule to compute the terms of the equation 1.3 in terms of ζ and η as follows:

$$\begin{aligned} u_x &= \omega_\zeta \zeta_x + \omega_\eta \eta_x. \\ u_y &= \omega_\zeta \zeta_y + \omega_\eta \eta_y. \\ u_{xx} &= \omega_{\zeta\zeta}\zeta_x^2 + 2\omega_{\zeta\eta}\zeta_x\eta_x + \omega_{\eta\eta}\eta_x^2 + \omega_\zeta\zeta_{xx} + \omega_\eta\eta_{xx}. \\ u_{yy} &= \omega_{\zeta\zeta}\zeta_y^2 + 2\omega_{\zeta\eta}\zeta_y\eta_y + \omega_{yy}\eta_y^2 + \omega_\zeta\zeta_{yy} + \omega\eta_{yy}. \\ u_{xy} &= \omega_{\zeta\zeta}\zeta_x\zeta_y + \omega_{\zeta\eta}\left(\zeta_x\eta_y + \zeta_y\eta_x\right) + \omega_{\eta\eta}\eta_x\eta_y + \omega_\zeta\zeta_{xy} + \omega_\eta\eta_{xy}. \end{aligned} \tag{1.7}$$

substituting these expressions into equation 1.3 we obtain the transformed PDE as:

$$a\omega_{\zeta\zeta} + b\omega_{\zeta\eta} + c\omega_{\eta\eta} = \phi\left(\zeta, \eta, \omega, \omega_\zeta, \omega_\eta\right). \tag{1.8}$$

Where Φ becomes ϕ and the new coefficients of the higher order terms a, b and c are expressed via the original coefficients and the change of variables formulas as follows :

$$\begin{aligned} a &= A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2, \\ b &= 2A\zeta_x\eta_y + B\left(\zeta_x\eta_y + \zeta_y\eta_x\right) + 2C\zeta_y\eta_y, \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned} \tag{1.9}$$

As this stage the form of the PDE 1.8 is no simpler than that of the original PDE 1.2, but this is to be expected because so far the choice of the new vaeiable ζ and has been that equation 1.9 can be written in matrix form as:

$$\begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} + \begin{vmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{vmatrix} J^2.$$

where J is Jacobian of the change of variables given by 1.6.Expanding the determinant and multiplying by the factor (-4) to obtain:

$$b - 4ac = J^2\left(B^2 - 4AC\right) \Rightarrow \delta = J^2\Delta. \tag{1.10}$$

Where $\delta = b^2 - 4ac$ is the discriminant of the equation 1.8 . This shows that the discriminant of 1.2 has the same sign as the discriminant of the transformed equation 1.8 and therfore it is clear that any real nonsingular $(J \neq 0)$ transformation does not change the type of PDE. Note that the discriminant involoves only the coefficients of second-order derivatives of the corresponding PDE.

1.2 Canonical forms

Let use now try to construct transformation, which will make one, or possibly two of the coefficients of the leading second-order terms of equation 1.8 vanish, thus reducing the equation to a simpler form called canonical form. For convenience, we reproduce below the original PDE :

$$A(x;y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = \Phi(x,y,u,u_x,u_y)\,. \tag{1.11}$$

and the corresponding transformrd PDE :

$$a(\zeta,\eta)\,\omega_{\zeta\zeta} + b(\zeta,\eta)\,\omega_{\eta\eta} + c(\zeta,\eta)\,\omega_{\eta\eta} = \phi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.12}$$

We again mention here that for the 1.2 or 1.8 to remain a second-order PDE, the coefficients A, B and C (or a, b and c) do not vanish simultaneously. By definition, a PDE is hyperbolic if the discriminant $\Delta = b^2 - 4ac \succ 0$. Since the sign of dicriminant is invariant under the change of coordinater (see equation 1.10), it follows that for a hyperboulic PDE, we should have $b^2 - 4ac \succ 0$. The simplest case of satisfying this condition is $a = c = 0$. So, if we try to choose the new variables ζ and η such that the cofficients a and c vanish, we get the following canonical form of hyperbolic equation :

$$\omega_{\zeta\eta} = \psi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.13}$$

Where $\psi = \frac{\phi}{b}$, this form is called the first canonical form of the hyperbolic equation, we also have another simple case for which $b^2 - 4ac \succ 0$ condition is satified. This the case when $b = 0$ and $c = -a$. Is this case 1.10 reduces to:

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi(\alpha,\beta,\omega,\omega_\alpha,\omega_\beta)\,. \tag{1.14}$$

Which is the second canonical form of the hyperbolic equation. By definition, a PDE is parabolic if the discriminant $\Delta = b^2 - 4ac = 0$. It follows that for a parabolic PDE, we should have $b^2 - 4ac = 0$. The simplest case of satisfying this condition is a (or $c = 0$). In this case another necessary requirement $b = 0$ will follow automatically (since $b^2 - 4ac$).

So if we try to choose the new variables ζ and η such that the coefficients a and b vanish, we get the following canonical form of parabolic equation :

$$\omega_{\eta\eta} = \psi(\zeta,\eta,\omega,\omega_\zeta,\omega_\eta)\,. \tag{1.15}$$

where $\psi = \frac{\phi}{c}$.

By definition, a PDE is elliptic if the discriminant $\Delta = b^2 - 4ac \prec 0$. It follows that for a elliptic PDE, we should have $b^2 - 4ac \prec 0$. The simpler case of satisfying this condition is $b = 0$ and $c = a$. So if we tey to choose the new variables ζ and η such that b vanishes and $c = a$, we get the following canonical form of elliptic equation :

$$\omega_{\zeta\zeta} + \omega_{\eta\eta} = \psi\left(\zeta, \eta, \omega, \omega_\zeta, \omega_\eta\right). \tag{1.16}$$

where $\psi = \frac{\phi}{a}$

In summary equation 1.8 can be reduced to a canonical form if the coordinate transformation $\zeta = \zeta\left(x, y\right)$ and $\eta = \eta\left(x, y\right)$ can be selected such that:

$.a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by 1.15

$.b = 0$, $c = -a$ corresponds to the second canonical form of hyperbolic PDE given by 1.14

$.a = b = 0$ corresponds to the canonical form of parabolic PDE given by 1.15

$.b = 0$, $c = a$ corresponds to the canonical form of elliptic PDE given by 1.16

1.2.1 Hyperbolic equations

For a hyperbolic PDE the discriminant $(\Delta = b^2 - 4ac) \succ 0$.In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variablrs ζ and η such that the coefficients a and c vanish in 1.8.Thus form 1.9 we have:

$$a = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0, \tag{1.17}$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0. \tag{1.18}$$

Divising equation 1.7 and 1.8 throughout by ζ_y^2 and η_y^2 respectively to obtain:

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0, \tag{1.19}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \tag{1.20}$$

Equation 1.10 is a quadratic equation for $\left(\frac{\zeta_x}{\zeta_y}\right)$ whose roots are given by:

$$\mu_1\left(x, y\right) = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

$$\mu_2\left(x, y\right) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

The roots of the equation 1.20 can also be found in an identical manner, so as only two distinct roots are possible between the two equation 1.19 and 1.20. Here we may consider μ_1 as the root of 1.19 and μ_2 as that of 1.20 That is :

$$\mu_1(x,y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \tag{1.21}$$

$$\mu_2(x,y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{1.22}$$

The above equations lead to the following two first-order diferential equation:

$$\zeta_x - \mu_1(x,y)\,\zeta_Y = 0, \tag{1.23}$$

$$\eta_x - \mu_2(x,y)\,\eta_y = 0. \tag{1.24}$$

These are the equations that define the new coordinate variables ζ and η that are necessary to make $a = c = 0$ in 1.8.

As the total derivative of ζ along the coordinate $\zeta(x,y) = constant$, $d\zeta = 0$. It follows that:

$$d\zeta = \zeta_x dx = \zeta_y dy = 0.$$

and hence the slope of such curves is given by:

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$$

We also have a similar result along coordinate line $\eta(x,y) = constant$, $i.e$,

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

Using these results equation 1.19 can be written as:

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.25}$$

This is called the characteristic polynomial of the PDE 1.2 and its roots are given by:

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1(x,y), \tag{1.26}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2(x,y). \tag{1.27}$$

The required variables ζ and η are determined by the respective solutions of the two ordinary differential equations 1.26 and 1.27, known as the characteristic equation of the PDE 1.2. They are ordinary differential equation for families of curves in the xy-plane along wich $\zeta = constant$ and $\eta = constant$. Clearly these families of curves depend on the coefficients A, B and C in the original PDE 1.2.

Integation of equation 1.26 leads to the family of curvilinear coordinates $\zeta(x, y) = c_1$ while the integration of 1.27 gives another family of curvilinear coordinates $\eta(x, y) = c_2$, where c_1 and c_2 are arbitrary constants of integration. These two families of curvilinear

coordinates $\zeta(x, y) = c_1$ and $\eta(x, y) = c_2$ are called characteristic curves of the hyperbolic equation 1.3 or simply the characteristics of the equation. Hence second-order hyperbolic equation have two families of characteristic curves. The fact that $\Delta \succ 0$ means that the characteristic are real curves in xy-plane.

If the coefficients A, B and C are constants it is easy to integrate equations 1.26 and 1.27 to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical form. Thus integration of 1.26 produces:

$$y = \frac{B + \sqrt{B^2 - 4AC}}{2A}x + c_1 \qquad and \qquad y = \frac{B - \sqrt{B^2 - 4AC}}{2A}x + c_2. \tag{1.28}$$

Or

$$y = \frac{B + \sqrt{B^2 - 4AC}}{2A}x = c_1 \qquad and \qquad y = \frac{B - \sqrt{B^2 - 4AC}}{2A}x = c_2. \tag{1.29}$$

Thus when the coefficients A, B and C are two constants the two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines. Since the families of curves $\zeta = constant$ and $\eta = constant$ are the characteristic curves , the change of variables are given by the following equations: :

$$\zeta = y - \frac{B - \sqrt{B^2 - 4AC}}{2A}x = y - \lambda_1 x, \tag{1.30}$$

$$\eta = y - \frac{B + \sqrt{B^2 - 4AC}}{2A}x = y - \lambda_2 x. \tag{1.31}$$

The first canonical form of the hyperbolic is :

$$\omega_{\zeta\eta} = \psi(\zeta, \eta, \omega, \omega_\zeta, \omega_\eta). \tag{1.32}$$

where $\psi = \frac{\phi}{b}$ and b is calculated from 1.9

$$\begin{aligned} b &= 2A\zeta_x\eta_x + B\left(\zeta_x\eta_x + \zeta_y\eta_y\right) + 2C\zeta_y\eta_y \\ &= 2A\left(\frac{B^2-(B^2-4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C \\ &= 4C - \frac{B^2}{A} = \frac{\Delta}{A}. \end{aligned} \tag{1.33}$$

Each of the familiies $\zeta(x,y) = constant$ and $\eta(x,y) = constant$ forms an envelop of the domain of the xy-plane in which the PDE is hyperbolic.

The transformation $\zeta = \zeta(x,y)$ and $\eta = \eta(x,y)$ can be regarded as a mapping from the xy-plane to the $\zeta\eta$ xy-plane. and the curves along which ζ and η are constant in the xy-plane become coordinates lines in the $\zeta\eta - plane$. Since these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other worde, charactreristic curves of a hyperbolic PDE are those curves for the PDE. must be referred as coordinate curves in order that it take on canonical form. We now determine the Jacobian of transformation definded by 1.30 and 1.31. we have:

$$J = \begin{vmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{vmatrix} = \lambda_2 - \lambda_1.$$

We know that $\lambda_1 = \lambda_2$ only if $B^2 - 4AC = 0$. However, for an hyperbolic PDE , $B^2 - 4AC \neq 0$. Hence Jacobian is nonsingular for the given transformation. A consequence of $\lambda_1 \neq \lambda_2$ is that at no point can the particular curves from each family share a common trangent line. It is easy to show that the hyperbolic PDE has a second canonical form.

The following linear change of variables:

$$\alpha = \zeta + \eta \qquad\qquad \beta = \zeta - \eta.$$

converts 1.32 in to:

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \psi\left(\alpha, \beta, \omega, \omega_\alpha, \omega_\beta\right). \tag{1.34}$$

which is the second canonical form of the hyperbolic equations.

1.2.2 Parabolic equations

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ζ and η such that the coefficients a and b vanish in 1.8. Thus, from 1.9 we have:

$$a = A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0.$$

Dividing the above equation throughhout by ζ_y^2 to obtain:

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{1.35}$$

As the total derivative of ζ along the coordinate line $\zeta(x,y) = constant$, $d\zeta = 0$.It follows that:

$$d\zeta = \zeta_x dx + \zeta_y dy = 0.$$

and hence, the slope of such curves is given by:

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}.$$

Using this result, equation 1.35 can be written as:

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{1.36}$$

This is called the characteristic polynominal of the PDE 1.2. Since $B^2 - 4AC = 0$ in this case the characteristic 1.35 has only root, given by:

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda(x,y). \tag{1.37}$$

Hence we see that for a parabolic PDE there is only one family of real characteristic curves.

The required varibles ζ is determined by the ordinary differential equation 1.37, known as the characteristic equations of the PDE 1.2. This is an ordinary differential equation for families of curves in the xy-plane along which $\zeta = const.$ to determine the second transformation variable η, we set $b = 0$ in 1.9 so that:

$$\begin{aligned}
2A\xi_x\eta_x + B\left(\xi_x\eta_x + \xi_y\eta_y\right) + 2C\xi_y\eta_y &= 0,\\
2A\frac{\xi_x}{\xi_y}\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y &= 0,\\
2A\left(\frac{-B}{2A}\right)\eta_x + B\left[\left(\frac{-B}{2A}\right)\eta_y + \eta_x\right] + 2C\eta_y &= 0,\\
-B\eta_x + \frac{-B^2}{2A}\eta_y + B\eta_x + 2C\eta_y &= 0,\\
\left(B^2 - 4AC\right)\eta_y &= 0.
\end{aligned}$$

since $B^2 - 4AC = 0$ for a parabolic PDE, η_y could be an arbitrary function of (x,y) and consequently the transformation variable η can be chosen arbitrary, as long as the change of coordinates formulas define a non-degenerate transfoemation. If the coefficients

A, B and C are constants, it is easy to integrate equation 1.37 to obtain the expressions for change of variables formulas for reducing a parabolic PDE to the canonical form. thus integration of 1.37 produces:

$$y = \frac{B}{2A}x + c_1; \tag{1.38}$$

or

$$y - \frac{B}{2A}x = c_1. \tag{1.39}$$

since the families of curves $\zeta = const$ are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B}{2A}x, \tag{1.40}$$

$$\eta = x, \tag{1.41}$$

where we have set $\eta = x$. the Jacobian of this transformation is:

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -\frac{B}{2A} & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

Now, we have from 1.9:

$$\begin{aligned} b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\ &= 2A\left(-\frac{B}{2A}\right) + B + 0 = 0. \end{aligned}$$

in these new coordinate variables given by 1.39 and 1.40, equation 1.8 reduces to following canonical form:

$$\omega_{\eta\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_\eta). \tag{1.42}$$

where $\psi = \frac{\varphi}{\varsigma}$. As the choice of η is arbitrary the form taken by ψ will depend on the choice of η. we have from 1.9:

$$c = A\eta_x^2 + B\eta_\chi\eta_\gamma + C\eta_\gamma^2 = A. \tag{1.43}$$

Equation 1.8 may also assume the form:

$$\omega_{\xi\xi} = \psi(\xi, \eta, \omega_\xi, \omega_\eta), \tag{1.44}$$

if we choose $c = 0$ instead of $a = 0$.

1.2.3 Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC \prec 0$. In this case, we have seen that to reduce this PDE to canonical form we need to choose the new variables ξ and η to produce $b = 0$ and $a = 0$, or $b = 0$ and $a - c = 0$. then, from 1.9 we abtain the following equation :

$$A(\xi_\chi^2 - \eta_\chi^2) + B(\xi_\chi \xi_\gamma - \eta_\chi \eta_\gamma) + C(\xi_\gamma^2 6 - \eta_\gamma^2) = 0. \quad (1.45)$$

$$2A\xi_x \eta_x + B\left(\xi_x \eta_y + \xi_y \eta_x\right) + 2C\xi_y \eta_y = 0. \quad (1.46)$$

For hyperbolic and parabolic PDE_S, ξ and η are satisfied by equations that are not coupled each other (see 1.17 and 1.35). However, equation 1.46 are coupled since both unknowns ξ and η appear in both equations. In an attempt to separate them, we add the first of these equation to complex number i times the second to give:

$$A\left(\xi_x + i\eta_x^2\right) + B\left(\xi_x + i\eta_y\right)\left(\xi_x + i\eta_y\right) + C\left(\xi_y + i\eta_y\right)^2 = 0.$$

Dividing the above equation throughout by $\left(\xi_y + i\eta_y\right)^2$ to obtain:

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi^x + i\eta_y}{\xi^x + i\eta_y}\right) + C = 0. \quad (1.47)$$

This equation can be solved for two possible values of the ratio:

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \quad (1.48)$$

Clearly, these twe roots are complex conjugates and are given by:

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}; \quad (1.49)$$

$$\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}. \quad (1.50)$$

where $\beta(x, y)$ is the complex conjugate of $\alpha(x, y)$ .They are given by:

$$\alpha(x, y) = \xi(x, y) + i\eta(x, y); \quad (1.51)$$

$$\beta(x, y) = \xi(x, y) - i\eta(x, y). \quad (1.52)$$

We will now proceed in a purely formal fashion. As the total derivative of α along the coordinate line $\alpha(x, y) = constant$, $d\alpha = 0$, it follows that:

$$d\alpha = \alpha_x dx + \alpha_y dy = 0.$$

and hence, the slope of such curves is given by:

$$\frac{dy}{dx} = \frac{\alpha_x}{\alpha_y}.$$

we also have a similar result along coordinate line $\beta(x, y) = constant$,$i.e$,

$$\frac{dy}{dx} = -\frac{\beta_x}{\beta_y}.$$

From the forgoing discussion it follows that :

$$\frac{dy}{dx} = \lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A}; \tag{1.53}$$

$$\frac{dy}{dx} = \lambda_2 = \frac{B + i\sqrt{4AC - B^2}}{2A}. \tag{1.54}$$

Equations 1.53 and 1.54 are called the characteristic equation of the PDE 1.3.

Clearly, the solution of this differential equation are necessarily complex-valued and as consequence there are no real characteristic axist for an elliptic PDE. The complex variables α and β are determined by the respective solution of the two ordinary differential equations 1.53 and 1.54. Integration of equation 1.53 leads to the family of curvilinear coordinates $\alpha(x, y) = c_1$, where the integration of 1.54 gives another family of curvilinear coordinates $\beta(x, y) = c_1$ where $c_1$and c_2 are complex constants of integration.

Since α and β are complex function the characteristic curves of the elliptic equation 1.3 are not real.

Now the real and imaginary parts of α and β give the required transformation variables ξ and η . Thus, we have:

$$\xi = \frac{\alpha + \beta}{2}, \qquad \eta = \frac{\alpha - \beta}{2}. \tag{1.55}$$

With the choice of coordinate variables 1.55, equation 1.8 reduces to following canonical form.

$$\omega_{\xi\xi} + \omega_{\eta\eta} = \psi(\xi, \eta, \omega, \omega_\xi, \omega_\eta). \tag{1.56}$$

where $\psi = \frac{\phi}{\alpha}$

NOTE: It may be noted that the quasilinear second-order equation in two independent variables can also be classified in a similar way according to rule analogous to those developed above for semilinear equations. However, since A, B, and C are now functions of u_x, u_y and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

CHAPTER 2

Definitions and theorems importants

In this section, we state some definitions and Lemmas important for our problem.

Définition 2.1 The Faedo-Galerkin methode with de compctness argument yields a powerful method which allows us to deal with some nonlinear evolution equation.

Définition 2.2 We denote by $L^P([a,b];X)$, the space of L^P functions from $[a,b]$ into X. It is a Banach
space for the norm:

$$\|u\|_{L^P([a,b];X)} = \left(\int_a^b \|u(t)\|_X^p \; dt\right)^{\frac{1}{p}}.$$

Définition 2.3 $L^\infty([a,b];X)$ is the space of measurable functions from $[a,b]$ into X being essentially bounded. It is a banach space of the norm:

$$\|u\|_{L^\infty([a,b];X)} = \sup_{t\in[a,b]} \|u(t)\|_X .$$

let B be a banach space.

Définition 2.4 encounter the following three different concepts of convergence.
(i) **Strong convergence:** Let $u_n \in B, u \in B$ such that as $n \to +\infty$;

$$\|u_n - u\|_B \to 0.$$

Then u_n is said to strongly converge to u.
(ii) **Weak convergence:**

Let $u_n, u \in B$ such that for any $f \in B'$ as $n \to +\infty$;

$$f(u_n) \to f(u).$$

Then u_n is said to weakly converge to u
(iii) **Weakly star convergence:**
Let $u_n, u \in B$,and let B be the dual space of another banach space B^*, i.e , $B = (B^*)'$.
If for any $f \in B^*$, as $n \to +\infty$;

$$u_n(f) \to u(f).$$

Then u_n is said to weakly star.

Remarque 2.1 It is well known that strong convergence implies weak convergence, and weak convergence implies weakly star convergence. When B is a reflixive banach space, weak convergence is equivalent to weakly star convergence. Where B is a reflixive banach space, convergence is equivalent to weakly star convergence.

Lemme 2.1 *Let S be a number with $2 \leq S \leq M \prec +\infty$ then there is a constant C depending on Ω*
and S such that:

$$\|u\|_s \leq C \|\nabla u\|_m, \qquad u \in H_0^1(\Omega).$$

Lemme 2.2 *Any bounded set in reflexive banach space is weakly compact i.e any sequence in a bounded set has a weakly converging subsequence.*

Exemple 2.1 Since for $1 \prec P \prec \infty, L^\infty(\Omega)$ is a reflixive banach space and $L^\infty(\Omega) = (L^1(\Omega))$, any bounded set in $L^\infty(\Omega)$ with $1 \prec P \leq \infty$ is weakly star compact. In particular, any bounded set in $L^\infty(\Omega)$ is weakly star compact .

Lemme 2.3 *Let B_0, B, B_1 be three banach space.Suppose that B_0 is continuously imbedded into B, wich is also continuously imbedded into B_1, and imbedding from B_0 into B is compact. Then for any $\eta \succ 0$, there is a positive constant C_η depending only on η such that for any $v \in B_0$,the following holds.*

$$\|v\|_B \leq \eta \|v\|_{B_0} + C_\eta \|v\|_{B_1}.$$

Théorème 2.1 ***Local existence:*** *Suppose that $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, then there exists $T \succ 0$ such that problem $(1.1) - (1.3)$ has unique solution u stisfiying:*

$$u \in L^{\infty}\left([0,T],H_1^0(\Omega)\right) \cap L^2\left([0,T],L^2(\Omega)\right) \;\; u' \in L^{\infty}\left([0,T],L^2(\Omega)\right).$$

Remarque 2.2 Since for $1 \leq p \prec \infty, L^P(\Omega)$ is a reflexive banach space and $L^{\infty}(\Omega) = \left(L^1(\Omega)\right)'$, any bounded set in $L^P(\Omega)$ with $1 \prec p \leq \infty$ is weakly star compact . In particular, any bounded set in $L^{\infty}(\Omega)$ is weakly star compact.

Théorème 2.2 *Let B_0, B, B_1 be three banach space B_0, B_1 are reflexive .Suppose that $B_0$is continuously imbedded into B, which is also continuously imbedded into B_1, and imbedding from B_0 into B is compact. For any given P_0, P_1 with $1 \prec p_0, p_1 \prec \infty$, let:*

$$W = \left\{v/v \in L^{P_0}([O,T],B_0), v_t \in L^{p_1}[0,T],B_1\right\}.$$

Then the imbedding from W into $L^{P_0}([0,T],B)$ is compact.

Remarque 2.3 It can be seen from the proof that if the assumption of reflexivity of $B_0, B_1$is replaced by the assumption that $B_0, B_1$are the dual space of reflexive banach spaces B_0^*, B_1^*,respectively, then the conclusion of Theorem [2.1] still holds.

Lemme 2.4 *Suppose that Ω is a bounded domain in R^n. Let $u_n(x), u(x)$ be real function in $L^P(\Omega), (1 \leq p \prec \infty)$ such that u_n strongly converges to u in $L^p(\Omega)$. Then if $1 \leq p \prec \infty$, has a subsequence allmost everywhere converging to u ; if $p = \infty$, then u_n it self allmost everywhere converges to u.*

Lemme 2.5 *Suppose that Ω is a bounded domain in R^n. Let $u_n(x)$ be bounded sequence in $L^p(\Omega)$, and u_n weakly converges in $L^p(\Omega)$ to u.*

Remarque 2.4 When $p = \infty$, then the conclusion becomes that u_n weakly star converges to u.

Lemme 2.6 *Let B be a banach space , and $B = (B^*)'$ with B^* being another banach space. Suppose that for $1 \prec p \leq \infty$;*

$$\begin{aligned} u_n &\rightarrow u \;\; \textit{weakly star in } L^p([0,T],B); \\ u_n' &\rightarrow u' \;\; \textit{weakly star in } L^p([0,T],B). \end{aligned}$$

CHAPTER 3

Preliminaries

In this section we present some preliminary facts about Lebesgue and Sobolev spaces with variable exponents (see [47],[48]). Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We define the Lebesgue space with a variable exponent $p(.)$ by:

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R}; \text{measurable in } \Omega : \varrho_{p(.)}(\lambda u) < \infty, \text{ for some } \lambda > 0 \right\}.$$

where:

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx.$$

Equipped with the following Luxembourg-type norm:

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

$L^{p(.)}(\Omega)$ is Banach space (see [48]).

We, next, define the variable exponent Lebesgue Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$. Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent under 5.3 (see [48]). The dual of $W_0^{1,p(.)}(\Omega)$ is defined as $W^{-1,p'(.)}(\Omega)$, in the same ways the classical Sobolev spaces, where:

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1.$$

Lemme 3.1 *(poincaré's Inequality [48]). Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(.)$ satisfies 5.3, then:*

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)}, \text{ for all } u \in W_0^{1,p(.)}(\Omega).$$

where the positive constant C depends on p_1, p_2 and Ω only. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by:

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}.$$

Lemme 3.2 *([48]). let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p : \Omega \to (1,\infty)$ is a measurable function such that:*

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \text{ for a.e } x \in \Omega.$$

If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ in $\overline{\Omega}$ with $p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, \text{ if } p_2<n \\ +\infty, \text{ if } p_2 \geq n \end{cases}$.
then the embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemme 3.3 *([48]).if $p : \Omega \to (1,\infty)$ is a measurable function with $p_2 < +\infty$, then $C_0^{\infty}(\Omega)$ is dense in $L^{p(.)}(\Omega)$.*

Lemme 3.4 *(Hölder's Inequality [48]). Let $p, q, s \geq 1$ be measurable functions defined on Ω such that:*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.$$

if $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$ and:

$$\|fg\|_{s(.)} \leq 2 \|f\|_{p(.)} \|g\|_{q(.)}.$$

Lemme 3.5 *(Unit Ball Property [48]). Let p be a measurable function on Ω then:*

$$\|f\|_{p(.)} \leq 1 \text{ if and only if } \varrho_{p(.)}(f) \leq 1.$$

Lemme 3.6 *([48]. if $1 < p_1 \leq p(x) \leq p_2 < \infty$ holds, then:*

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\}.$$

for any $u \in L^{p(.)}(\Omega)$.

CHAPTER 4

Global non-existence of solutions of a class of wave equations with non-linear damping and source terms

In this chapter we are concerned with the following initial boundary value problem:

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{\beta-2} \nabla u_t\right) \\ \quad +a\,|u_t|^{m-2} u_t = b\,|u|^{p-2} u, \qquad x \in \Omega, \qquad t > 0 \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), \qquad x \in \Omega \\ u(x,t) = 0 \qquad x \in \partial\Omega, \qquad t > 0 \end{cases}. \tag{4.1}$$

where $a, b > 0, \alpha, \beta, m, p > 2$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$.

BLOW UP In order to state and prove our result, we introduce the following function space:

$$Z = L^\infty\left([0,T); W_0^{1,\alpha}(\Omega)\right) \cap W^{1,\infty}\left([0,T); L^2(\Omega)\right) \cap W^{1,\beta}\left([0,T); W_0^{1,\beta}(\Omega)\right) \cap W^{1,m}\left([0,T); L^m(\Omega)\right).$$

For $T > 0$ and the energy functional:

$$E(t) = \frac{1}{2}\int_\Omega u_1^2 dx + \frac{1}{\alpha}\int_\Omega |\nabla u|^\alpha dx - \frac{b}{p}\int_\Omega |u|^p dx. \tag{4.2}$$

Assume the $\alpha, \beta, m, p \geq 2$ such that $\beta < \alpha$, and $max\{m, x\} < p < r_\alpha$, where r_α is the Sobolev critical exponent of $W_0^{1,\alpha}(\Omega)$. Assume further that:

$$E(0) < 0. \tag{4.3}$$

then the solution $u \in Z$, of 4.1, cannot exist for all time.

We remind that $r_\alpha = n\alpha/(n-\alpha)$, if $n > \alpha, r_\alpha > \alpha$ if $n = \alpha$, and $r_\alpha = \infty$ if $n < \alpha$ if the solution u is smooth enough then it blows up in finite time.

We suppose that the solution exists for all time and we reach to a contradiction. For this purpose we multiply Equation 4.1 by u_t and integrate over Ω to obtain:

$$E'(t) = -\int_\Omega |\nabla u_t|^2 dx - \int_\Omega |\nabla u_t|^\beta dx - a\int_\Omega |u_t|^m dx \le 0. \tag{4.4}$$

For any regular solution. this remains valid for $u \in Z$, by density argument. Hence

$$E(t) \le E(0), \qquad \forall t \ge 0.$$

By setting $H(t) = -E(t)$, we get:

$$0 < H(0) \le H(t) \le \frac{b}{p}\int_\Omega |u|^p dx, \qquad \forall t \ge 0. \tag{4.5}$$

When then define:

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t dx. \tag{4.6}$$

For ε small to be choosen later and:

$$0 < \sigma \le \min\left(\frac{\alpha-2}{p}, \frac{\alpha-\beta}{p(\beta-1)}, \frac{p-m}{p(m-1)}, \frac{\alpha-2}{2\alpha}\right) \tag{4.7}$$

Our goal is to show that $L(t)$ satisfies a differential inequality of the form:

$$L'(t) \ge \xi L^q(t), \qquad q > 1$$

This, of course, will lead to blow up in finite time. By taking a derivative of 4.5 we obtain:

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2 dx + \varepsilon\int_\Omega uu_{tt}dx. \tag{4.8}$$

By using Equation 4.1, the estimate 4.7 gives:

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2 dx - \varepsilon\int_\Omega \nabla u\nabla u_t dx \\ &\quad -\varepsilon\int_\Omega |\nabla u|^\alpha dx - \varepsilon\int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t\nabla u dx \\ &\quad -a\varepsilon\int_\Omega |u_t|^{m-2}u_t u dx + b\varepsilon\int_\Omega |u|^p dx. \end{aligned} \tag{4.9}$$

We then exploit Young's inequality to get:

$$\int_{\Omega} |u_t|^{m-2} u_t u dx \leq \frac{\delta^m}{m} \int_{\Omega} |u|^m dx + \frac{m-1}{m} \delta^{-m/(m-1)} \int_{\Omega} |u_t|^m dx. \tag{4.10}$$

$$\int_{\Omega} \nabla u \nabla u_t dx \leq \frac{1}{4\mu} \int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} |\nabla u_t|^2 dx. \tag{4.11}$$

$$\int_{\Omega} |\nabla u_t|^{\beta-1} \nabla u dx \leq \frac{\lambda^\beta}{\beta-1} \int_{\Omega} |\nabla u|^\beta dx + \frac{\beta-1}{\beta} \lambda^{-\beta(\beta-1)} \int_{\Omega} |\nabla u_t|^\beta dx. \tag{4.12}$$

A substitution of 4.9 - 4.11 in 4.8 yields:

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_{\Omega} u_t^2 dx - \frac{\varepsilon}{4\mu} \int_{\Omega} |\nabla u|^2 dx - \mu\varepsilon \int_{\Omega} |\nabla u_t|^2 dx \\ & -\varepsilon \int_{\Omega} |\nabla u|^\alpha dx - \varepsilon \frac{\lambda^\beta}{\beta} \int_{\Omega} |\nabla u|^\beta dx - \varepsilon \frac{\beta-1}{\beta} \lambda^{-\beta(\beta-1)} \int_{\Omega} |\nabla u_t|^\beta dx \\ & + b\varepsilon \int_{\Omega} |u|^p dx - a\varepsilon \frac{\delta^m}{m} \int_{\Omega} |u|^m dx - a\varepsilon \frac{m-1}{m} \delta^{-m/(m-1)} \int_{\Omega} |u_t|^m dx. \end{aligned} \tag{4.13}$$

therefore by choosing δ, μ, λ so that:

$$\begin{cases} \delta^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \mu = M_2 H^{-\sigma}(t) \\ \lambda^{-\beta/(\beta-1)} = M_3 H^{-\sigma}(t) \end{cases}.$$

For $M_1, M_2,$ and M_3 to be specified later, and using 4.12 we arrive at:

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_{\Omega} u_t^2 dx - \frac{\varepsilon}{4M_2} H^\sigma(t) \int_{\Omega} |\nabla u|^2 dx - \varepsilon \int_{\Omega} |\nabla u|^\alpha dx \\ & -\varepsilon \frac{M_3^{-(\beta-1)}}{\beta} H^{\sigma(\beta-1)}(t) \int_{\Omega} |\nabla u|^\beta dx - \frac{a\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t) \int_{\Omega} |u|^m dx + b\varepsilon \int_{\Omega} |u|^p dx \\ & -\varepsilon \left[M_2 \int_{\Omega} |\nabla u_t|^2 dx + \frac{\beta-1}{\beta} M_3 \int_{\Omega} |\nabla u_t|^\beta dx + a \frac{m-1}{m} M_1 \int_{\Omega} |u_t|^m dx \right] H^{-\sigma}(t) \end{aligned} \tag{4.14}$$

If $M = M_2 + (\beta-1) M_3/\beta + (m-1) M_1/m$ then 4.14 takes the form:

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma) - \varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon \int_{\Omega} u_t^2 dx - \frac{\varepsilon}{4M_2} H^\sigma(t) \int_{\Omega} |\nabla u|^2 dx \\ & -\varepsilon \int_{\Omega} |\nabla u|^\alpha dx - \varepsilon \frac{M_3^{-(\beta-1)}}{\beta} H^{\sigma(\beta-1)}(t) \int_{\Omega} |\nabla u|^\beta dx \\ & -\frac{a\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t) \int_{\Omega} |u|^m dx + b\varepsilon \int_{\Omega} |u|^p dx. \end{aligned} \tag{4.15}$$

We then use the embedding $L^p(\Omega) \to L^m(\Omega)$ and 4.5 to get:

$$H^{\sigma(m-1)}(t)\int_\Omega |u|^m\,dx \le \left(\frac{b}{p}\right)^{\sigma(m-1)}\left(\int_\Omega |u|^p\,dx\right)^{\frac{m+\sigma p(m-1)}{p}}. \tag{4.16}$$

We also exploit te inequality

$$\int_\Omega |\nabla u|^2\,dx \le C\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{2/\alpha}.$$

the embedding $W_0^{1,\alpha}(\Omega) \to L^p(\Omega)$, and 4.3 to obtain:

$$H^{\sigma}(t)\int_\Omega |\nabla u|^2\,dx \le C\left(\frac{b}{p}\right)^{\sigma}\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{p\sigma+2}{\alpha}}. \tag{4.17}$$

Since $\alpha > \beta$ we have:

$$\int_\Omega |\nabla u|^\beta\,dx \le C\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\beta/\alpha}.$$

consequently

$$H^{\sigma(\beta-1)}(t)\int_\Omega |\nabla u|^\beta\,dx \le C\left(\frac{b}{p}\right)^{\sigma(\beta-1)}\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}}. \tag{4.18}$$

Where C is a constant depending on Ω only. By using 4.7 and

$$z^v \le z+1 \le \left(1+\frac{1}{a}\right)(z+a), \qquad \forall z \ge 0, 0 < v \le 1, a \ge 0. \tag{4.19}$$

we have the following

$$\begin{aligned}\left(\int_\Omega |u|^p\,dx\right)^{\frac{m+\sigma p(m-1)}{p}} &\le \left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{m+\sigma p(m-1)}{\alpha}}\\ &\le d\left(\int_\Omega |\nabla u|^\alpha\,dx + H(0)\right)\\ &\le d\left(\int_\Omega |\nabla u|^\alpha\,dx + H(t)\right) \qquad \forall t\ge 0.\end{aligned} \tag{4.20}$$

$$\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{p\sigma+2}{\alpha}} \le d\left(\int_\Omega |\nabla u|^\alpha\,dx + H(t)\right), \qquad \forall t \ge 0. \tag{4.21}$$

$$\left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}} \le d\left(\int_\Omega |\nabla u|^\alpha\,dx + H(t)\right), \qquad \forall t \ge 0. \tag{4.22}$$

where $d = 1 + 1/H(0)$. Inserting the estimates 4.15 - 4.17and 4.19 - 4.21 into 4.15 we get

$$\begin{aligned} L'(t) &\geq ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t) + kH(t) + \left(\varepsilon + \frac{k}{2}\right) \int_\Omega u_t^2 dx \\ &-\frac{\varepsilon C_2}{M_2}\left(\int_\Omega |\nabla u|^\alpha dx + H(t)\right) - \varepsilon \int_\Omega |\nabla u|^\alpha dx - \frac{\varepsilon C_3}{M_3^{\beta-1}}\left(\int_\Omega |\nabla u|^\alpha dx + H(t)\right) \\ &-\frac{k}{\alpha}\int_\Omega |\nabla u|^\alpha dx - \frac{\varepsilon C_1}{M_1^{m-1}}\left(\int_\Omega |\nabla u|^\alpha dx + H(t)\right) - b\left(\varepsilon - \frac{k}{p}\right)\int_\Omega |u|^p dx. \end{aligned} \quad (4.23)$$

for some constant k and

$$C_1 = \frac{aCd}{m}\left(\frac{b}{p}\right)^{\sigma(m-1)}, C_2 = \frac{Cd}{4}\left(\frac{b}{p}\right)^{\sigma}, C_3 = \frac{Cd}{\beta}\left(\frac{b}{p}\right)^{\sigma(\beta-1)}.$$

Using $k = \varepsilon p$, we arrive at:

$$\begin{aligned} L'(t) &\geq ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon\left(\frac{p+2}{2}\right)\int_\Omega u_t^2 dx \\ &+\varepsilon\left(p - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{m-1}}\right) H(t) \\ &+\varepsilon\left(\frac{p}{a} - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{m-1}} - 1\right)\int_\Omega |\nabla u|^\alpha dx. \end{aligned} \quad (4.24)$$

At this point, we choose $M_1, M_{2,} M_3$ large enough so that:

$$L'(t) \geq ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t) + \gamma\varepsilon\left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^\alpha dx\right]. \quad (4.25)$$

where γ is a positive constant (this is possible since $p > \alpha$). By choosing $\varepsilon < (1-\sigma)/M$ so that:

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 dx > 0.$$

we obtain:

$$L(t) \geq L(0) > 0, \forall t \geq 0.$$

and

$$L'(t) \geq \gamma\varepsilon\left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^\alpha dx\right]. \quad (4.26)$$

Next it is clear that:

$$L^{\frac{1}{1-\sigma}}(t) \leq 2^{\frac{1}{1-\sigma}}\left\{H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left(\int_\Omega u_t u dx\right)^{\frac{1}{1-\sigma}}\right\}.$$

By the Cauchy-schwarz inequality and embedding of the $L^p(\Omega)$ spaces we have:

$$\left|\int_\Omega u_t u dx\right| \leq \left(\int_\Omega u^2 dx\right)^{1/2}\left(\int_\Omega u_t^2 dx\right)^{1/2} \leq C\left(\int_\Omega |u|^\alpha dx\right)^{1/\alpha}\left(\int_\Omega u_t^2 dx\right)^{1/2}.$$

which implies

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\int_\Omega |u|^\alpha dx\right)^{\frac{1}{(1-\sigma)\alpha}}\left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2(1-\sigma)}}.$$

Also Young's inquality gives

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_\Omega |u|^\alpha dx\right)^{\frac{\mu}{(1-\sigma)\alpha}}\left(\int_\Omega u_t^2 dx\right)^{\frac{\theta}{2(1-\sigma)}}\right].$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\sigma)$, (hence $\mu = 2(1-\sigma)/(1-2\sigma)$ to get:

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_\Omega |u|^\alpha dx\right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2 dx\right].$$

By Poincaré's inequality, we obtain:

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_\Omega |\nabla u|^\alpha dx\right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2 dx\right].$$

By Using 4.6 and 4.18 we deduce:

$$\left(\int_\Omega |\nabla u|^\alpha dx\right)^{\frac{2}{(1-2\sigma)\alpha}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^\alpha dx + H(t)\right).$$

therefore

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C\left[H(t) + \int_\Omega |\nabla u|^\alpha dx + \int_\Omega u_t^2 dx\right], \forall t \geq 0.$$

consequently

$$L^{\frac{1}{1-\sigma}}(t) \leq \Gamma\left[H(t) + \int_\Omega |\nabla u|^\alpha dx + \int_\Omega u_t^2 dx\right]. \tag{4.27}$$

where Γ is positive constant. A combination of 4.26 and 4.27, thus, yields

$$L'(t) \geq \xi L^{\frac{1}{1-\sigma}}(t), \forall t \geq 0. \tag{4.28}$$

Integration of 4.28 over $(0, t)$ gives:

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma}{(1-\sigma)}t}.$$

hence $L(t)$ blow up in time

$$T^* \leq \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(0)}. \tag{4.29}$$

The time estimate 4.29 shows that the larger $L(0)$ is the quicker the blow up takes places.

In 4.4 we only require that $H(0) > 0$, Unlie Yang [3], where it is required that $H(0) > A$, a constant depending on the size of Ω. See condition (ii), Theorem 2.1 of [3].

If we consider:

$$u_{tt} - \Delta u_t - \operatorname{div}(\sigma(\Delta u)\Delta u) - \operatorname{div}(\beta(\Delta u)\Delta u_t) + f(u_t) = g(u), \ x \in \Omega, t > 0.$$

with the initial and boundary conditions of 4.1 we can establish a similar blow up result under the growth conditions of Theorem 2.3 of [3] on f, g, σ and β.

CHAPTER 5

Decay for solutions of a nonlinear damped weve equation with variable exponent nonlinearities

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{r(.)-2}\nabla u\right) + |u_t|^{m(.)-2}u_t = 0, \text{ in } \Omega\times(0,T) \\ u(x,t) = 0, \text{ on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \text{ in } \Omega \end{cases}. \tag{5.1}$$

where the exponent $m(.)$ and $r(.)$ are given measurable function on Ω satisfying:

$$2 \le r_1 \le r(x) \le r_2 \le m_1 \le m(x) \le m_2 \le r_1^*. \tag{5.2}$$

with:

$$\begin{aligned} r_1 &: = ess_{x\in\Omega}\inf r(x), \quad r_2 := ess_{x\in\Omega}\sup r(x); \\ m_1 &: = ess_{x\in\Omega}\inf m(x), \quad m_2 := ess_{x\in\Omega}\sup r(x). \end{aligned}$$

and

$$r_1^* = \begin{cases} \frac{nr_1}{n-r_1} \text{ if } n > r_1 \\ +\infty \text{ if } n \le r_1 \end{cases}.$$

and the log-Hölder continuity condition:

$$|q(x) - q(y)| \le \frac{A}{\log|x-y|}, \text{for a.e } x, y \in \Omega, \text{with } |x-y| < \delta,$$

$$A > 0, 0 < \delta < 1. \tag{5.3}$$

The term div $\left(|\nabla u|^{r(.)-2}\nabla u\right)$ is called $r(.)-$ Laplacian and denoted by $\Delta_{r(.)}u$.

A considerable effort has been devoted to the study of problem 5.1 in case of constant and variable exponent nonlinearities.

Decay estimates

In this section we state an existence theorem for our problem 5.1. We also state and prove our decay result. For this purpose we define the energy of the solution by:

$$E(t) := \frac{1}{2}\int_\Omega u_1^2 dx + \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} dx. \tag{5.4}$$

Simple calculations show:

$$E'(t) := -\int_\Omega |u_t|^{m(x)} dx \leq 0.$$

Proposition 5.1 *Let $u_0 \in W_0^{1,r(.)}(\Omega), u_1 \in L^2(\Omega)$ and the exponent m and r satisfy conditions 5.2 and 5.3. Then problem 5.1 has a unique weak solution such that:*

$$\begin{aligned} u &\in L^\infty\left((0,T), W_0^{1,r(.)}(\Omega)\right), \\ u_t &\in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}(\Omega\times(0,T)), \\ u_{tt} &\in L^\infty\left((0,T), W^{-1,r'(.)}(\Omega)\right). \end{aligned} \tag{5.5}$$

where:

$\frac{1}{r(.)} + \frac{1}{r'(.)} = 1.$

Remarque 5.1 This result can be established using the Galerkin method as in [49]. See also [50].

Lemme 5.1 *[46]. Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. Assume that there exist $\sigma > 0, \omega > 0$ such that:*

$$\int_s^\infty E^{1+\sigma}(t)\,dt \leq \frac{1}{\omega}E^\sigma(0)\,E(s) = cE(s), \forall s > 0.$$

Then, $\forall t \geq 0$.

$$E(t) \leq= \begin{cases} \frac{cE(0)}{(1+t)^{1/\sigma}}, & \text{if } \sigma > 0 \\ cE(0)\,e^{-\omega t} & \text{if } \sigma = 0 \end{cases}.$$

We will establish a decay result for problem 5.1, with the exponent r satisfying conditions 5.2 and 5.3. For this purpose, we assume the existence of a global solution with the required regularity. We need a technical lemma before we state and prove our main decay result.

Lemme 5.2 *Let u be a solution of 5.1 and $0 \leq E(t) \leq E(0)$. Then, for some $C > 0$.*

$$\varrho_{r(.)}(\nabla u) \geq C \|\nabla u\|_{r_1}^{r_2}. \tag{5.6}$$

Preuve.

$$\varrho_{r(.)}(\nabla u) = \int_{\Omega} |\nabla u|^{r(x)}\, dx = \int_{\Omega+} |\nabla u|^{r(x)}\, dx + \int_{\Omega-} |\nabla u|^{r(x)}\, dx.$$

■

where:

$$\Omega_{+} = \{x \in \Omega / |\nabla u(x,t)| \geq 1\} \text{ and } \Omega_{-} = \{x \in \Omega / |\nabla u(x,t)| < 1\}.$$

So we get

$$\varrho_{r(.)}(\nabla u) \geq \int_{\Omega+} |\nabla u|^{r_1} + \int_{\Omega-} |\nabla u|^{r_2} \geq \int_{\Omega+} |\nabla u|^{r_1} + c_1 \left(\int_{\Omega-} |\nabla u|^{r_1}\right)^{\frac{r_2}{r_1}}.$$

This implies that:

$$c_2 \left(\varrho_{r(.)}(\nabla u)\right)^{\frac{r_1}{r_2}} \geq \int_{\Omega-} |\nabla u|^{r_1} \text{ and } \varrho_{r(.)}(\nabla u) \geq \int_{\Omega+} |\nabla u|^{r_1}. \tag{5.7}$$

and, hence;

$$c_2 \left(\varrho_{r(.)}(\nabla u)\right)^{\frac{r_1}{r_2}} + \varrho_{r(.)}(\nabla u) \geq \|\nabla u\|_{r_1}^{r_1}.$$

Since:

$$\varrho_{r(.)}(\nabla u) \leq r_2 E(t) \leq r_2 E(0). \tag{5.8}$$

Then 5.7 leads to:

$$\left(\varrho_{r(.)}(\nabla u)\right)^{\frac{r_1}{r_2}} \left[c_2 + (r_2 E(0))^{1-\frac{r_1}{r_2}}\right] \geq \|\nabla u\|_{r_1}^{r_1}.$$

Thus, 5.4 follows.

Théorème 5.1 *Suppose conditions 5.2 and 5.3 hold. Then there exist two constants $c, \alpha > 0$ independent of t and may depend on $E(0)$ such that the energy $E(t)$ satisfies, $\forall t \geq 0$,*

$$E(t) \leq \begin{cases} \frac{c}{(1+t)^{2/(m_2-2)}}, & \text{if } m_2 > 2 \\ ce^{-\alpha t}, & \text{if } m(x) = 2 \end{cases}.$$

Preuve. Multiplying 5.1 by $uE^q(t)$, for $q > 0$ to be specified later, and integrating over $\Omega \times (s,T), T > s$,we obtain:

$$\int_s^T E^q(t) \int_{\Omega} \left(\mathrm{u}u_{tt} - u \operatorname{div}\left(|\nabla u|^{r(.)-2} \nabla u\right) + |u_t|^{m(x)-2} u_t u\right) = 0.$$

■

which implies that:

$$\int_s^T E^q(t)\int_\Omega \left[(u_tu)_t - u_t^2 + |\nabla u|^{r(x)} + |u_t|^{m(x)-2}u_tu\right] = 0.$$

It follows that:

$$\begin{aligned}\int_s^T E^q(t)\frac{d}{dt}\int_\Omega \mathrm{u}u_t dx + \int_s^T E^q(t)\int_\Omega \left(u_t^2 + |\nabla u|^{r(x)}\right)dx \\ -2\int_s^T E^q(t)\int_\Omega u_t^2 + \\ \int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_tu &= 0. \end{aligned} \tag{5.9}$$

By using the definition of $E(t)$ and the relation:

$$\frac{d}{dt}\left(E^q(t)\int_\Omega \mathrm{u}u_t dx\right) = qE^{q-1}(t)E'(t)\int_\Omega \mathrm{u}u_t dx + E^q(t)\frac{d}{dt}\int_\Omega \mathrm{u}u_t dx.$$

Eq.5.7 becomes:

$$\begin{aligned}2\int_s^T E^{q+1}(t)\,dt &\le -\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega \mathrm{u}u_t dx\right) + q\int_s^T E^{q-1}(t)E'(t)\int_\Omega \mathrm{u}u_t dx \\ &+2\int_s^T E^q(t)\int_\Omega u_t^2 - \int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_tu.\end{aligned} \tag{5.10}$$

Estimates

$$\begin{aligned}\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega \mathrm{u}u_t dx\right)\right| &= \left|E^q(s)\int_\Omega \mathrm{u}u_t(x,s)\,dx - E^q(t)\int_\Omega \mathrm{u}u_t(x,T)\,dx\right| \\ &\le E^q(s)\left[\frac{1}{2}\int_\Omega u^2(x,s)\,dx + \frac{1}{2}\int_\Omega u_t^2(x,s)\,dx\right] \\ &\quad +E^q(s)\left[\frac{1}{2}\int_\Omega u^2(x,T)\,dx + \frac{1}{2}\int_\Omega u_t^2(x,T)\,dx\right] \\ &\le E^q(s)\left[\frac{1}{2}c_*\int_\Omega |\nabla u(x,s)|^2 dx + E(s)\right] \\ &\quad +E^q(T)\left[\frac{1}{2}c_*\int_\Omega |\nabla u(x,T)|^2 dx + E(T)\right].\end{aligned}$$

where c_* is the embedding constant. So, we have:

$$\begin{aligned}\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega \mathrm{u}u_t dx\right)\right| &\le E^q(s)\left[c\,||\nabla u(s)||_{r_1}^2 dx + E(s)\right] \\ &\quad +E^q(T)\left[c\,||\nabla u(T)||_{r_1}^2 dx + E(T)\right] \\ &\le E^{q+1}(s) + cE^q(s)\left(||\nabla u(s)||_{r_1}^{r_2}\right)^{\frac{2}{r_2}} + E^{q+1}(T) \\ &\quad +cE^q(T)\left(||\nabla u(T)||_{r_1}^{r_2}\right)^{\frac{2}{r_2}}.\end{aligned} \tag{5.11}$$

where c is a generic positive constant which may change values from a line to another. We then use 5.6 and recall that $E(t)$ is nonincreasing to arrive at:

$$\begin{aligned}\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)\right| &\leq E^{q+1}(s)+cE^q(s)\left(\varrho_{r(x)}(\nabla u(s))\right)^{\frac{2}{r_2}}+E^{q+1}(T) \quad (5.12)\\ &+cE^q(T)\left(\varrho_{r(x)}(\nabla u(T))\right)^{\frac{2}{r_2}}\\ &\leq E^{q+1}(s)+c(E(s))^{q+\frac{2}{r_2}}+E^{q+1}(T)+c(E(T))^{q+\frac{2}{r_2}}\\ &\leq E^{q+1}(s)+c(E(s))^{q+\frac{2}{r_2}}.\end{aligned}$$

Notice that in the last estimate, we have applied the following Hölder inequality for $r_1 > 2$.

$$\int_\Omega |\nabla u|^2 dx \leq |\Omega|^{\frac{r_1-2}{r_1}}\left(\int_\Omega |\nabla u|^{r_1} dx\right)^{\frac{2}{r_1}}.$$

Of course, for the case $r_1 = 2$, the estimate 5.12 is also true.

$$\cdot\left|q\int_s^T E^{q-1}(t)E'(t)\int_\Omega uu_t dx\right| \leq -c\int_s^T E^{q-1}(t)E'(t)\left[E(t)+cE^{\frac{2}{r_2}}(t)\right]dt$$
$$\leq -c\left[\int_s^T E^q(t)E'(t)\,dt+\int_s^T (E(s))^{q+\frac{2}{r_2}-1}E'(t)\,dt\right]$$
$$\leq c\left[E^{q+1}(s)+(E(s))^{q+\frac{2}{r_2}}\right].$$

For the third term of the right-hand side of 5.10, we set, as in [46],

$$\Omega_+ = \{x\in\Omega/\,|u_t(x,t)|\}\geq 1 \text{ and } \Omega_- = \{x\in\Omega/\,|u_t(x,t)|\}<1.$$

and exploit Hölder's and Young's inequalities and 5.2 as follows:

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &= \left|2\int_s^T E^q(t)\left[\int_{\Omega_-} u_t^2+\int_{\Omega_+} u_t^2\right]\right|\\ &\leq c\int_s^T E^q(t)\left[\left(\int_{\Omega_-}|u_t|^{m_2}\right)^{2/m_2}+\left(\int_{\Omega_+}|u_t|^{m_1}\right)^{2/m_1}\right]\\ &\leq c\int_s^T E^q(t)\left[\left(\int_{\Omega}|u_t|^{m(x)}\right)^{2/m_2}+\left(\int_{\Omega}|u_t|^{m(x)}\right)^{2/m_1}\right]\\ &\leq c\int_s^T E^q(t)\left(-E'(t)\right)^{2/m_2}dt+c\int_s^T E^q(t)\left(-E'(t)\right)^{2/m_1}dt\\ &\leq c\varepsilon\int_s^T E^{q+1}(t)\,dt+c\varepsilon\int_s^T\left(-E'(t)\right)^{2(q+1)/m_2}dt\\ &+c\varepsilon\int_s^T\left(E(t)\right)^{qm_1/(m_1-2)}dt+c_\varepsilon\int_s^T\left(-E'(t)\right)dt.\end{aligned}$$

where $c_{\varepsilon}=\frac{1}{q+1}\left(\frac{\varepsilon(q+1)}{q}\right)^{-q}$.

Choose q such that $q=\frac{m_2}{2}-1$ and notice that $\frac{qm_1}{m_1-2}=q+1+\frac{m_2-m_1}{m_1-2}$. In the rest of the proof, we consider two cases $m_1>2$ and $m_1=2$. If $m_1>2$, we arrive at:

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &\le c\varepsilon\int_s^T E^{q+1}(t)\,dt+c\varepsilon\,(E(0))^{(m_2-m_1)/(m_1-2)}\int_s^T E^{q+1}(t)\,dt+c_\varepsilon E(s) \\ &\le \widetilde{c}\varepsilon\int_s^T E^{q+1}(t)\,dt+c_\varepsilon E(s).\end{aligned} \tag{5.13}$$

where $\widetilde{c}$ is a positive constant independent of ε.

For the case $m_1=2$, we have:

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &= \left|2\int_s^T E^q(t)\left[\int_{\Omega_-}u_t^2+\int_{\Omega_+}u_t^2\right]\right| \\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega_-}|u_t|^{m_2}\right)^{2/m_2}+\int_{\Omega_+}|u_t|^{m(x)}\right] \\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega}|u_t|^{m_2}\right)^{2/m_2}+\int_{\Omega}|u_t|^{m(x)}\right] \\ &\le c\int_s^T E^q(t)\left(-E'(t)\right)^{2/m_2}dt+c\int_s^T E^q(t)\left(-E'(t)\right)dt \\ &\le c\varepsilon\int_s^T E^{q+1}(t)\,dt+c_\varepsilon\int_s^T\left(-E'(t)\right)^{2(q+1)/m_2}dt+cE^{q+1}(s).\end{aligned}$$

Choose q such that $q=\frac{m_2}{2}-1$. Hence, w arrive at:

$$\begin{aligned}\left|2\int_s^T E^q(t)\int_\Omega u_t^2\right| &\le c\varepsilon\int_s^T E^{q+1}(t)\,dt+c_\varepsilon E(s)+cE^{q+1}(s) \\ &\le c\varepsilon\int_s^T E^{q+1}(t)\,dt+\left(c_\varepsilon+cE^q(0)\right)E(s) \\ &\le c\varepsilon\int_s^T E^{q+1}(t)\,dt+\widetilde{c}_\varepsilon E(s).\end{aligned} \tag{5.14}$$

where: $\widetilde{c}_\varepsilon=c_\varepsilon+cE^q(0)$.

Since: $q=\frac{m_2}{2}-1$ and $m_2\ge r_2$, then $q+\frac{2}{r_2}-1\ge 0$, and, consequently, the estimates 5.11 and 5.12 become, respectively:

$$\begin{aligned}\left|-\int_s^T\frac{d}{dt}\left(E^q(t)\int_\Omega \mathrm{u}u_t dx\right)\right| &\le E^{q+1}(s)+c\,(E(s))^{q+\frac{2}{r_2}} \\ &\le \left[E^q(0)+c\,(E(0))^{q+\frac{2}{r_2}-1}\right]E(s)=\widetilde{c}_\varepsilon E(s).\end{aligned} \tag{5.15}$$

$$\begin{aligned}\left|q\int_s^T E^{q-1}(t)\,E'(t)\int_\Omega \mathrm{u}u_t dx\right| &\le c\left[E^{q+1}(s)+(E(s))^{q+\frac{2}{r_2}}\right] \\ &\le c\left[E^q(0)+(E(0))^{q+\frac{2}{r_2}-1}\right]E(s)=\widetilde{c}E(s).\end{aligned} \tag{5.16}$$

For the last term of 5.10, we use Young's inequality with $p(x) = \frac{m(x)}{m((x)-1)}, p'(x) = m(x)$, so for a.e. $x \in \Omega$. we have:

$$|u_t|^{m(x)-1}|u| \leq \varepsilon |u|^{m(x)} + c_\varepsilon(x)|u_t|^{m(x)};$$

where:

$$c_\varepsilon(x) = \varepsilon^{1-m(x)}(m(x))^{-m(x)}(m(x)-1)^{m(x)-1}.$$

Therefore.

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_t u\right| &\leq \varepsilon\int_s^T E^q(t)\int_\Omega |u|^{m(x)} + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx\\ &\leq \varepsilon\int_s^T E^q(t)\left[\int_\Omega |u|^{m_1}dx + \int_\Omega |u|^{m_2}dx\right]\\ &+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx.\end{aligned}$$

By using the embedding, we obtain:

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_t u\right| &\leq \varepsilon\int_s^T E^q(t)\left[c_1\|\nabla u\|_{r_1}^{m_1} + c_2\|\nabla u\|_{r_1}^{m_2}\right]\\ &+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx.\end{aligned}$$

where $c_1$and c_2 are positive constants independent of ε.

We recall 5.4 and 5.6 to get:

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega |u_t|^{m(x)-2}u_t u\right| &\leq \varepsilon\int_s^T E^q(t)\left[c_1\left(\varrho_{r(x)}(\nabla u)\right)^{\frac{m_1}{r_2}} + c_2\left(\varrho_{r(x)}(\nabla u)\right)^{\frac{m_2}{r_2}}\right] \quad (5.17)\\ &+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx\\ &\leq \varepsilon c_1'\int_s^T E^{q+1}(t)(E(t))^{\frac{m_1}{r_2}-1} + \varepsilon c_2'\int_s^T E^{q+1}(t)(E(t))^{\frac{m_2}{r_2}-1}\\ &+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx\\ &\leq c'\varepsilon\left((E(0))^{\frac{m_1}{r_2}-1} + (E(0))^{\frac{m_2}{r_2}-1}\right)\int_s^T E^{q+1}(t)\\ &\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx.\end{aligned}$$

where c_1', c_2', and c' are positive constants independent of ε.

Therefore, a combination of 5.10 - 5.17 leads to:

$$\begin{aligned}2\int_s^T E^{q+1}(t)\,dt &\leq c_\varepsilon\left(1 + (E(0))^{\frac{m_1}{r_2}-1} + (E(0))^{\frac{m_2}{r_2}-1}\right)\int_s^T E^{q+1}(t)\,dt \quad (5.18)\\ &+c_\varepsilon E(s) + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx.\end{aligned}$$

Choose $\varepsilon > 0$ so small so that:

$$c_{\varepsilon}\left(1+(E(0))^{\frac{m_1}{r_2}-1}+(E(0))^{\frac{m_2}{r_2}-1}\right)<1.$$

Once ε is fixed, then $c_{\varepsilon}(x) \leq M$, since $m(x)$ is bounded. Thus, we arrive at:

$$\begin{aligned}
\int_s^T E^{q+1}(t)\,dt &\leq cE(s)+M\int_s^T E^q(t)\int_{\Omega}|u_t|^{m(x)}\,dx \\
&\leq cE(s)-M\int_s^T E^q(t)\,E'(t)\,dt \\
&\leq cE(s)+\frac{M}{q+1}\left(E^{q+1}(s)-E^{q+1}(T)\right) \\
&\leq c\left(E(s)+E^{q+1}(s)\right) \\
&\leq c\left(1+E^q(0)\right)E(s)=\widetilde{c}E(s),\forall T>s>0.
\end{aligned}$$

By taking $T \to \infty$, we get:

$$\int_s^{\infty} E^{\frac{m_2}{2}}(t)\,dt \leq \widetilde{c}E(s).$$

Hence, Komornik's Lemma (with $\sigma = \frac{m_2}{2} - 1$) implies the desired result.

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