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جامعة 20 أوت 1955 -سكيكدة
كلية العــلوم
قسم الرياضيات
المرجع:..........

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Mathematics***

Numerical approaches by Finite Volumes of Fractional Klein-Gordon Equation

Presented by:

Naaima Latioui

Publicly discussed: 05/12/2023

In front of the Jury:

1. Khemis Rabah	**MCA**	**University 20 August 1955, Skikda**	**President**
2. Bouzettouta Lamine	**MCA**	**University 20 August 1955, Skikda**	**Examiner**
3. Ellagoune Fateh	**Professor**	**University 8 May 1955, Guelma**	**Examiner**
4. Chaoui Abd elrezak	**Professor**	**University 8 May 1955, Guelma**	**Examiner**
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Faculté des Sciences
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جامعة 20 أوت 1955 -سكيكدة
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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option: ***Mathématiques***

Approches Numériques par des Volumes Finis d'Equations de Klein-Gordon Fractionnaire

Présentée par:

Latioui Naaima

Soutenue publiquement: 05/12/2023

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Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
Option: *Mathematics*

Presented by:

Naaima LATIOUI

Topic:

Numerical approaches by Finite Volumes of Fractional Klein-Gordon Equations

Supervisor: Professor. A. Guesmia

University of 20 Août 1955-Skikda, 2022/2023.

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Abstract

The objective of this thesis, was to study some of the non-linear hyperbolic PDEs of the Klein-Gordon type by studying the existence, uniqueness and stability of the solution. First, we study the existence of the weak solution of the klein-Gordon equation using the Faedo-Galerkin method, and for the uniqueness of the solution we used the classical technique, and we also proved that the solution is stable over time using the appropriate Lyapunov function. Secondly, we used the Potential well method to prove the existence and uniqueness of the solution to another type of klein-Gordon equation. In the third part of our thesis, we studied the one dimension fractional Klein-Gordon equation and proved that there is a solution to it using the Galerkin method, and we also proved that the solution is unique using the classical technique finally we approximated the klein-Gordon fractional problem numerically using the finite volume method.

Keywords:

Klein-Gordon equation; Galerkin method, Potential well method, Faedo-Galerkin method; Finite volume method; Lyapunov function.

ملخص

الهدف من هذه الاطروحة هو دراسة بعض من المعادلات التفاضلية الجزئية الزائدية غير الخطية من نوع كلاين جوردون من خلال دراسة وجود الحل ووحدانيته واستقراره. اولا، درسنا وجود الحل الضعيف لمعادلة كلاين جوردون باستخدام طريقة فيدو جاليركين، اما بالنسبة لوحدانية الحل فقد استخدمنا الاسلوب الكلاسيكي، كما اننا اثبتنا ان الحل مستقر بمرور الوقت باستخدام دالة ليابونوف المناسبة. ثانيا، استخدمنا طريقة بئر الكمون لاثبات وجود الحل ووحدانيته لنوع اخر من معادلات كلاين جوردن.
ثالثا، درسنا معادلة كلاين جوردن الكسرية احادية البعد واثبتنا ان هناك حلا لها باستخدام طريقة جاليركين، كما اثبتناان الحل وحيد باسخدام الطريقة الكلاسيكية واخيرا قمنا بتقريب مشكلة كلاين جوردن الكسرية عدديا باستخدام طريقة الحجوم المتهية .

الكلمات المفتاحية: معادلة كلاين جوردن؛ طريقة جاليركين؛ طريقة بئر الكمون؛ طريقة فيدو جاليركين؛ طريقة الحجوم المتهية؛ وظيفة ليابونوف.

Résumé

L'objectif de cette thèse était d'étudier certaines des EDPs hyperbolique non-linéaires de type Klein-Gordon en étudiant l'existence, l'unicité et la stabilité de la solution. Premièrement, nous avons étudié l'existence de la solution faible de l'équation de Klein-Gordon en utilisant la méthode de Faedo- Galerkin, et pour l'unicité de la solution nous avons utilisé la méthode classique, et nous avons également prouve que la solution est stable dans le temps en utilisant la fonction de Lyapunov appropriée. Deuxiement, nous avons utilisé la méthode du puits potentiel pour prouver l'existence et l'unicité de la solution a un type d'équation de Klein-Gordon. Troisiement, nous avons étudié l'équation unidimensionnelle fractionnaire de Klein-Gordon et prouve qu'il existe une solution en utilisant la méthode de Galerkin, et nous avons également prouve que la solution est unique en utilisant la méthode classique et finalement, nous avons approché numériquement le problème fractionnaire de Klein-Gordon en utilisant la méthode des volumes finis.

Mots clés:

Equation de Klein-Gordon; Méthode de Galerkin; Méthode du puits potentiel; Méthode de Faedo-Galerkin; Méthode des volumes Finis; Fonction de Lyapunov.

Notations

- ∇ stands for the gradient operator.
- div is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- Δ is the Laplace operator.
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.
- $< .,. >$ or $(.,.)$ denotes the scalar product.
- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.
- $C^\infty(\Omega) = \underset{m\in\mathbb{N}}{\cap} C^m(\Omega)$.
- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.
- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.
- $L^2(\Omega)$ Hilbert space with norm $\|\cdot\|_2$.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W^{m,p}_{loc}(\Omega)$ the local Sobolev space.

- $W^{m,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{m,p}(\Omega)$.

- $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}_0(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $W^{s,p}(\Omega)$ fractional Sobolev space with norm $\|\cdot\|_{s,p}$.

- $W^{s,p}_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ in the norm $\|.\|_{W^{s,p}_0(\Omega)}$.

- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W^{s,2}_0(\mathbb{R}^n) = H^s_0(\mathbb{R}^n)$.

- $B(0, R)$ open ball centered at the point 0 with the radius R.

- Γ is the usual Gamma function.

Contents

Introduction

Partial differential equations, whether linear or non-linear, are concerned with the study of phenomena that occur in nature, each equation contains all the necessary informations to accuratly depict and read the event. Differential equations have become more important in the fields of mathematics, physics and chemistry, which enable us to understand an event such us knowing the time of its occurrence or the factors thar affect it and finding mathematicals solutions using several methods, including the method of separating variables..etc, or approximate solutions, when it is difficult to find approximate solutions, we resort to numerical methods, one of the most prominent of these methods is the finite volume method.
There are several types of partial differential equations, which interest ones in this thesis are the hyperbolic partial differential equations of the Klein-Gordon type. klein-Gordon's equation also sometimes called Klein-Gordon Fock's equations is a relativistic version of shrödinger's equation describing massive particles of zero spin without or with electric charge, established independently in **1926** by physicists **Oskar Klein** and **Walter Gordon**.
The Schrodinger equation, which is also called (the wave equation) is basically a differential equation that describes the energy and position of an electron in space and time and expresses it mathematically. it is based on three important foundations: the plane wave equations, Broglie's hypothesis about matter wave, and the principle of conservation of energy. This equation also looks at the shape of the waves that determine the movement of small particles, and clearly explains the influence of external factors on those waves, and provides information about the behavior of the electron associated with the nucleus based on the principle of energy conservation. By applying this equation to the hydrogen atom, Schrodinger was able to prove its validity, and he determined multiple propertises of the hydrogen atom, and this equation was widly used in atomic and nuclear phisics. Schrodinger's equations have an imprtant role in modeling phisical phenomenas as they represent an important mathematical object, and it is a wide field of research, including those that focus on understanding the connections between Schrodinge-type equations and some physical phenomina such as vortex dynamics in fluids and superfluids , as well as the propagation of signals in quantum wires.
Medeiros and Milla Miranda [23] considered the nonlinear system

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + u - |v|^\rho |u|^\rho u = f_1, \\ \\ \frac{\partial^2 v}{\partial t^2} - \Delta v + v - |u|^\rho |v|^\rho v = f_2, \end{cases} \tag{1}$$

in $\Omega \times [0,T]$ where ρ is a real number satisfying certain condition and Ω is any domain of $\mathbb{R}^n$, they prove the existence and uniqueness of global weak solutions.
Doherty Andrade and Angela Mognon [4], considered the nonlinear system with memory term

$$\begin{cases} u_{tt} - \Delta u + f(u,v) + k * \Delta u = 0, \\ v_{tt} - \Delta v + g(u,v) + l * \Delta v = 0, \end{cases} \tag{2}$$

for x $\in \Omega$ and $t > 0$ where

$$f(u,v) = |u|^{\rho-2}u|v|^{\rho}, \qquad \text{and} \qquad g(u,v) = |v|^{\rho-2}v|u|^{\rho},$$

with

$$\rho > 0 \quad \text{if} \quad n = 1,2 \quad \text{and} \quad 1 < \rho \leqslant \frac{n-1}{n-2} \quad \text{if} \quad n \geqslant 3,$$

in $\Omega \times (0,T)$ with initial condition and boundary conditions, they prove the existence of weak and strong solution using argument from Komornik and Zuazua [20].
Aldo. T Louredo and M. Milla Miranda [9], considered the non-linear system

$$\begin{cases} u'' - \Delta u + \alpha v^2 u = 0, \\ v'' - \Delta v + \alpha u^2 v = 0, \end{cases} \tag{3}$$

with the non-linear boundary conditions,

$$\begin{aligned} \frac{\partial u}{\partial \nu} + h_1(.,u') = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \\ \frac{\partial v}{\partial \nu} + h_2(.,v') = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \end{aligned}$$

and boundary conditions $u = v = 0$ on $(\Gamma/\Gamma_1) \times (0,\infty)$ where Ω is a bounded open set of $\mathbb{R}^n$ $(n \leqslant 3)$, $\alpha > 0$ a real number, Γ_1 a subset of the boundary Γ of Ω and h_i a real function defined on $\Gamma_1 \times (0,\infty)$. They prove the existence of global solutions by using the Galerkin method.
Khaled Zennir and Amar Guesmia [6], considered the non-linear κth-order with nonlinear sources and memory terms

$$\begin{cases} u_1'' + (-1)^{\kappa}\Delta^{\kappa}u_1 + m_1^2 u_1 + \alpha_1(t)\int_0^t g_1(t-s)\Delta^{\kappa}u_1(x,s)ds + |u_1'|^{r-2}|u_1'| \\ = |u_1|^{p-2}u_1|u_2|^p, \\ u_2'' + (-1)^{\kappa}\Delta^{\kappa}u_2 m_2^2 u_2 + \alpha_2(t)\int_0^t g_2(t-s)\Delta^{\kappa}u_2(x,s)ds + |u_2'|^{r-2}|u_2'| \\ = |u_2|^{p-2}u_2|u_1|^p. \end{cases} \tag{4}$$

In a bounded domain Ω of $\mathbb{R}^n$, $m_i = 1,2$ are non-negative constants, $r,p \geqslant 2, \kappa \geqslant 1$, they prove the existence of global solution by using the potntial well method.
C. L. Frota and A. Vicente [15], studied the non-linear system of Klien-Gordon with acoustic boundary conditions

$$\begin{cases} u'' - \Delta u + |v|^{\rho+2}|u^{\rho}|u = f_1 \quad \text{in} \quad \Omega \times (0,T), \\ v'' - \Delta v + |u|^{\rho+2}|v^{\rho}|v = f_1 \quad \text{in} \quad \Omega \times (0,T), \end{cases} \tag{5}$$

they prove the existence of global and weak solution and the uniqueness of global solution.
Yaojun Ye [20], studied the nonlinear system with damping and source terms

$$\begin{cases} u_{tt} - div(|\nabla u|^{m-2}\nabla u) + a|u_t|^{p-2}u_t - \Delta u_t = f_1(u,v) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ v_{tt} - div(|\nabla v|^{m-2}\nabla v) + a|v_t|^{p-2}v_t - \Delta v_t = f_2(u,v) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{6}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. $a > 0$ and $m, p \geqslant 2$ are real numbers, they prove non existence theorem with positive initial energy blow in finite time under some conditions , the main tool of the proof is a technique introduced by paper [33].
M M Cavalcanti, V. N Domingos Cavalcanti, J. S. Prates Filho and J. A. Soriano [20], studied following system with boundary damping

$$\begin{cases} K_1(x,t)\frac{\partial^2 u}{\partial t^2} + \alpha_1(x,t)\frac{\partial u}{\partial t} - \Delta u + F(u,v) = f \quad \text{in} \quad Q = \Omega\times]0,\infty[, \\ \\ K_2(x,t)\frac{\partial^2 v}{\partial t^2} + \alpha_2(x,t)\frac{\partial v}{\partial t} - \Delta v + G(u,v) = g \quad \text{in} \quad Q = \Omega\times]0,\infty[, \\ \\ u = v = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1\times]0,\infty[, \\ \\ \frac{\partial u}{\partial \nu} + \beta_1(x)\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \nu} + \beta_2(x)\frac{\partial v}{\partial t} = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0\times]0,\infty[, \\ \\ u(0) = u^0, \frac{\partial u}{\partial t}(0) = u^1; v(0) = v^0, \frac{\partial v}{\partial t}(0) = v^1 \quad \text{in} \quad \Omega, \end{cases} \tag{7}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \leqslant 4$, with C^2 boundary Γ, which has a partition (Γ_0, Γ_1) where both parts has positive measure and let ν be the unit normal vector pointing toward Ω. They used Galerkin method to prove the existence of solutions and they used the perturbed energy method developed in Komornik and Zuazua [31].
Alfredo T. Cousin, Cícero L. Frota, and Nickolai A. Larkin [22], considered the nonlinear system

$$\begin{cases} u_1'' - \Delta u_1 + \alpha_1 u_1 + a_{12}u_1u_2^2 + a_{13}u_1u_3^2 + \cdots + a_{1k}u_1u_k^2 = f_1, \\ u_2'' - \Delta u_2 + \alpha_2 u_2 + a_{21}u_2u_1^2 + a_{23}u_2u_3^2 + \cdots + a_{2k}u_2u_k^2 = f_2, \\ \vdots \\ u_k'' - \Delta u_k + \alpha_k u_k + a_{k1}u_ku_1^2 + a_{k2}u_ku_2^2 + \cdots + a_{k(k-1)}u_ku_{k-1}^2 = f_k, \end{cases} \tag{8}$$

in $\Omega \times (0,\infty)$ with acoustic boundary conditions, they prove the existence, uniqueness and stabilization of global solutions.
Mathieu Colin and Tatsuya Watanabe [23], considered the non linear system with maxwell equation

$$\begin{cases} \psi_{tt} - \Delta\psi = & -2ie\phi\psi_t - ie\phi_t\psi + e^2|\phi|^2\psi - 2ie\nabla\psi.A \\ & -e^2|A|^2\psi - ie\psi divA - m^2\psi + W'(\psi), \\ A_{tt} - \Delta A = & eIm(\bar{\psi}\nabla\psi) - e^2|\psi|^2A - \nabla\phi_t - \nabla divA \\ & -\Delta\phi = eIm(\psi\bar{\psi_t}) - e^2|\psi|^2\phi + divA_t, \end{cases} \tag{9}$$

where ψ: $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, A: $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, $m > 0$, $e \in \mathbb{R}$ and i denotes the unit complex number, that is $i^2 = -1$. Moreover W(s) is a regular real valued function which is extended to the complex plane by setting $W'(\psi) = W'(|\psi|)\frac{\psi}{|\psi|}$ for $\psi \in \mathbb{C}$. Used the energy method, they obtained a local existence result for the Cauchy problem.
Salim A. Messaoudi, Belkacem Said Houari [20],considered the non linear viscolastic system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1}u_t = f_1(u,v), \quad x \in \Omega, t > 0, \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1}v_t = f_2(u,v), \quad x \in \Omega, t > 0, \end{cases} \tag{10}$$

for x $\in \Omega$ and $t > 0$

$$f_1(u,v) = \left[a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)}\right],$$
$$f_2(u,v) = \left[a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{(\rho)}v\right], \quad a,b > 0,$$

they prove a global non existence result of certain solution with positive initial energy, the main tool of the proof is a method used in [33].

Presentation of the thesis

This thesis has five chapters organized as follows

- The first chapter is mainly devoted to presenting important definitions and theories in research. These reminders are largely based on papers [1], [4], [23], [25], [31],..etc. We also remeber some important spaces, such as Sobolev spaces, as well as L^p spaces.

- The second chapter: in this chapter we proved the existence of the solution by applying the fadeo galerkin method, and we proved the uniqueness of the solution using classical technique . As for the stability we used the Lyapunov function (**this chapter is a published article**).

- The third chaper: in this chapter, we studied another type of non-linear Klein-Gordon equations coupled with the source terms. We proved the existence and uniqueness of the solution by using the potential well method.

- Chapter four: in this chapter we have studied a one-dimensinal fractional Klein-Gordon equation, we have proved the existence of the solution using the Galerkin method, as for the uniqueness of the solution we have used classical technique.

- Chapter five: in this chapter we searched for approximate solutions to the Klein-Gordon Fractional equation that we studied in the fourth chapter, the adopted method is the finite volume method.

Publications

- Published articles

  1. Well-Posedness and Stability for a System of Klein-Gordon Equations, International Journal of Analysis and Applications. VOL 20(2022), . DOI 10.28924/2291-8639-202022-10. [21].

- Communications

  1. Première conference nationale des mathematique pures et appliques-Universite Larbi Tebessi-Tebessa "CNMPA 2021", 15/04/2021, Oral communication entitled: **About the existence and uniqueness of a nonlinear hyperbolic problem** .
  2. 1st International conference on pure and applied mathematics, " IC-PAM21",may 26-27,2021, Ourgla, ALGERIA, Oral communication entitled : **Existence and Uniqueness for a system of klein-Gordon equations**.
  3. In the Nationale conference on applied mathematics and didactics "NCAMD2021", help online on Jun 26,2021, Oral communication entitled: **About the stabilization of a nonlinear hyperbolic system**.

Chapter 1

Preliminaries

In this chapter, we mentioned a set of important definitions and theories, as well as a set of spaces such as L^p spaces and Sobolev spaces.

Lemma 1.0.1. *[25] (**Young's Inequality**)*
Let a, $b \geqslant 0$ and $\frac{1}{q} + \frac{1}{p} = 1$ for $1 < p, q < +\infty$, then one has the inequality $ab \leqslant \delta a^q + c(\delta) b^p$, where $\delta > 0$ is an arbitrary constant, and $c(\delta)$ is a positive constant depending on δ.

Lemma 1.0.2. *[1] **(Sobolev-Poincaré inequality)***
Let s be a number with $2 \leqslant s < +\infty$ if $n \leqslant 2$ and $2 \leqslant s \leqslant \frac{2n}{n-2}$ if $n > 2$. Then there is a constant C depending on Ω and s such that

$$\|u\|_s \leqslant C \|\nabla u\|_2, \quad u \in H_0^1.$$

Lemma 1.0.3. *[6] (**Lemme de Gronwall**)*
Let $f \in L^\infty(0,T)$, $g \in L^1(0,T)$ and $f(t) \geqslant 0$, $g(t) \geqslant 0$.

$$\textit{If} \quad f(t) \leqslant c + \int_0^T f(s) g(s) ds; \qquad \textit{then} \quad f(t) \leqslant cexp(\int_0^T g(s) ds).$$

Lemma 1.0.4. *[33]*
Let B be a banch space, and $B = (B*)'$ with $B*$ being another Bnach space. Suppose that for $1 < p \leqslant \infty$,

$$\begin{cases} u_n \to u & \textit{weakly star in} \quad L^p([0,T], B), \\ u_n' \to u' & \textit{weakly star in} \quad L^p([0,T], B). \end{cases} \tag{1.1}$$

Then

$$u_n(0) \to u(0) \qquad \textit{weakly star in } B. \tag{1.2}$$

Lemma 1.0.5. *[32]*
Suppose that Ω is a bounded or unbounded domain in $\mathbb{R}^n$. Let $u_n(x)$, $u(x)$ be real functions in $L^p(\Omega)$, $(1 \leqslant p \leqslant \infty)$ such that u_n strongly converges to u in $L^p(\Omega)$. Then if $1 \leqslant p <$, u_n has a subsequence almost everywhere converging to u; if $p = \infty$, then u_n itself almost everywhere converges to u.

Lemma 1.0.6. *[9]*
Let B_0, B_1, B_2 be three Banach space where B_0, B_1 are reflexive. Suppose that B_0 is continuously imbedded into B, which is also continuously imbedded into B_1, and imbedding from B_0 into B is compact. For any given p_0, p_1 with $1 < p_0, p_1 < \infty$, let

$$W = \{v | v \in L^{p_0}([0,T], B_0), \quad v_t \in L^{p_1}([0,T], B_1)\}.$$

Then the imbedding from W into $L^{p_0}([0,T], B)$ is compact.

1.1 The L^p and the Bochner Spaces

Definition 1.1.1. ***(Lebesgue (Bochner) spaces)***
The Lebesgue spaces $L^p(Q;X)$ are spaces of (Bochner) measurable functions v ranging in Banach space X such that

$$\|v\|^p_{L^p(Q;X)} = \int_Q \|v\|^p_X dy \quad \textit{is finite}, \quad 1 \leqslant p < \infty. \tag{1.3}$$

Similarly, $v \in L^\infty(Q;X)$ if v is (Bochner) measurable and

$$\|v\|_{L^\infty(Q;X)} = ess \sup_{y \in Q} \|v(y)\|_X < \infty. \tag{1.4}$$

Definition 1.1.2. *[8] Let Ω be an open set of $\mathbb{R}^{\mathbb{N}}$ and $1 \leqslant p < \infty$, we define $L^p(\Omega)$ a Lebesgue space by*

$$L^p(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{R}^{\mathbb{N}} / f \textit{ is measurable and } \quad \int_\Omega |f(x)|^p dx \leqslant \infty \Big\}.$$

We define the norm of f in $L^p(\Omega)$ by

$$\|f\|_{L^p} = \|f\|_p = \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{R} / f \textit{ is measurable and,} \quad \exists C \geqslant 0; \quad |f(x)| \leqslant C \quad \textit{a.e. on} \quad \Omega \Big\}.$$

We notice

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \Big\{ C \geqslant 0 : |f(x)| \leqslant C, a.e. \textit{ on } \quad \Omega \Big\}.$$

Definition 1.1.3. ***Espaces*** *$L^2(\Omega)$ [8]*
For $p = 2$ we note $L^2(\Omega)$ the set of summable square functions, i.e.

$$L^2(\Omega) = \Big\{ f : \Omega \longrightarrow \mathbb{R} / \int_\Omega |f|^2 dx \leqslant \infty \Big\}.$$

The norm in $L^2(\Omega)$ is denoted by

$$\|f\|_{L^2} = \|f\|_2 = \left(\int_\Omega |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Remark 1.1.1. *The space $L^2(\Omega)$ is a Hilbert space for the inner product*

$$< f, g >= \int_\Omega f(x)g(x)dx.$$

Proposition [8]

1. For $1 \leqslant p \leqslant \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space.
2. For $1 \leqslant p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a separable space.
3. For $1 < p < \infty$, $(L^p(\Omega), \|\cdot\|_p)$ is a reflexive space.

Theorem 1.1.1. *(**Hölder Inequality** [2]) Let $1 < p < \infty$ and let p' denote the conjugate exponent defined by*

$$p' = \frac{p}{p-1}, \quad \text{that is} \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

which also satisfies $1 < p < \infty$. If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$, and

$$\int_\Omega |u(x)v(x)|dx \leqslant \|u\|_p \|v\|_{p'}.$$

Equality holds if and only if $|u(x)|^p$ and $|v(x)|^{p'}$ are proportional a.e. in Ω.

Theorem 1.1.2. *(**Cauchy-Schwartz Inequality** [7])*
Let $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$, then $f.g \in L^2(\Omega)$ and

$$\int_\Omega \Big|f(x)g(x)\Big|dx \leqslant \Big(\int_\Omega |f(x)|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |g(x)|^2 dx\Big)^{\frac{1}{2}}.$$

1.2 The space of continuous function

Definition 1.2.1. *Let Ω be an open subset of $\mathbb{R}^n, n \geqslant 1$, we note*

$$C(\Omega) = \{f : \Omega \to \mathbb{R}; f \quad \text{continuous}\},$$

and

$C^m(\Omega)$: *The space of functions m times continuously differentiable on Ω where*

$$C^\infty(\Omega) = \cup_{m\in\mathbb{N}} C^m(\Omega).$$

1.3 The test functions space $C_0^\infty = D(\Omega)$

Definition 1.3.1. *Let Ω a non-empty set of $\mathbb{R}^{\mathbb{N}}$ $(N \geqslant 1)$, we call space of functions test and we denote $D(\Omega)$ the set*

$$C_0^\infty(\Omega) = D(\Omega) = \Big\{\varphi \in C^\infty(\Omega), \quad supp\varphi \subset \Omega\Big\},$$

$$supp\varphi = \overline{\{x \in \Omega : \varphi \neq 0\}}.$$

1.4 The Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leqslant p \leqslant \infty$.

1.4.1 The $W^{1,p}(\Omega)$ space

Definition 1.4.1. *[8] The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega \,\middle|\, \begin{array}{c} \exists g_1, g_2, .., g_N \in L^p(\Omega) \quad \text{such that} \\ \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, 2, .., N \end{array} \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ we define $\frac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = gradu = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right). \tag{1.5}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p.$$

Or sometimes with the equivalent norm $(\|u\|_p^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p^p)^{\frac{1}{p}}$ (if $1 \leqslant p < \infty$).
The space $H^1(\Omega)$ is equipped with scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2} = \int_\Omega u.v + \sum_{i=1}^N \frac{\partial u}{\partial x_i} . \frac{\partial v}{\partial x_i}.$$

The associted norm

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}},$$

is equivalent to the $W^{1,2}$ norm.

Theorem 1.4.1. *[8] There exists a constant C (depending only on $|I| \leqslant \infty$) such that*

$$\|u\|_{L^\infty(I)} \leqslant C \|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I), \forall 1 \leqslant p \leqslant \infty. \tag{1.6}$$

In orther words, $W^{1,p}(I) \subset L^\infty(I)$ with continious injection for all $1 \leqslant p \leqslant \infty$.
Further, if I is bounded then

$$\text{the injection} \quad W^{1,p}(I) \subset C(\bar{I}) \quad \text{is compact for all} \quad 1 < p \leqslant \infty, \tag{1.7}$$

$$\text{the injection} \quad W^{1,1}(I) \subset L^q(I) \quad \text{is compact for all} \quad 1 \leqslant p < \infty. \tag{1.8}$$

1.4.2 The Space $W_0^{1,p}(\Omega)$

Definition 1.4.2. *[8] Let* $1 \leqslant p < \infty$; $W_0^{1,p}(\Omega)$ *designates the closure of* $C_0^c(\Omega)$ *in* $W^{1,p}(\Omega)$. *We note*

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$, *equipped with the* $W^{1,p}$ *norm, is a separable Banach space, it is reflixive if* $1 < p < \infty$. H_0^1, *equipped with the* H^1 *scalar product, is a Hilbert space.*

Definition 1.4.3. ***(weak convergence and Weak star convergence)***
Let E *be a Banach space .*

1. ***Weak convergence***
   Let $(u_n)_{n\in\mathbb{N}} \subset E$ *and* $u \in E$. *We say that* $u_n \to u$ *weakly in* E *when* $n \to \infty$ *if* $T(u_n) \to T(u)$ *for all* $T \in E'$.

2. ***Weak star convergence***
   Let $(T_n)_{n\in\mathbb{N}} \subset E'$ *and* $x \in E$. *We say that* $T_n \to T$ *weak star in* E' *if* $T_n(u) \to T(u)$ *for all* $x \in E$.

1.4.3 Finite volume method [14]

Simulation of different conservation laws, such as elliptic, parabolic or hyperbolic laws, is widely utilized in engineering domains like fluid mechanics, heat and mass transfer, and petroleum engineering. According to **Oden** [24], the finite volume technique shres certain crucial characteristics with the finite element method, suchas the ability to be applied to arbitrary deometries with either structured or unstructured meshes and the ability to produce robust schemes, how they remain the same from one discretization cell to its neighbor, is another property. When modeling issues where the flux is significant, such as in fluid mechanics, semiconductor, the finite volume methode is highly appealing due to this final characteristic. The initial volume method is locally conservative because it relies on a "balance" approach: a local balance is written on each discretization cell, also known as a "control volume", and then by using the divergence formula over the control volume's boundary is obtained. With regard th the discrete unknowns, the fluxes on the border are discretized.

Chapter 2

Well-Posedness and Stability for a System of Klein-Gordon Equations

In this chapter, we study the existence of weak solution for a non-linear hyperbolic coupled system of Klein-Gordon equations with memory and source terms using the Faedo-Galerkin method techniques and compactness results, we have demonstrated the uniqueness of the solution by using the classical technique. In addition, we show that the solution remains stable over time. The reaction of the proper Lyapunov function is the primary tool of the proof.

2.1 Introduction

We consider a non-linear hyperbolic system of Klein-Gordon equations, defined as the following

$$\begin{cases} u_{tt} - \Delta u - \alpha u_t + k * \Delta u - div(|v|^2 \nabla u) + u|\nabla v|^2 = 0 \quad \text{in} \quad \Omega \times (0,T), \\ v_{tt} - \Delta v - \beta v_t + l * \Delta v - div(|u|^2 \nabla v) + v|\nabla u|^2 = 0 \quad \text{in} \quad \Omega \times (0,T), \end{cases} \tag{2.1}$$

with boundary conditions

$$u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \Gamma \times (0,T), \tag{2.2}$$

$$u_t(x,t) = v_t(x,t) = 0 \qquad \text{on} \quad \Gamma \times (0,T), \tag{2.3}$$

and Initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \qquad \text{on} \quad \Omega, \tag{2.4}$$

$$u_t(x,0) = u_1(x), \quad v_t(x,0) = v_1(x) \qquad \text{on} \quad \Omega. \tag{2.5}$$

Where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary Γ and let $T > 0$, α and β are non-positive constants, and

$$(n * w)(t) = \int_0^t n(t-s)w(s)ds. \tag{2.6}$$

Our objective is to prove that the problem (2.1)-(2.5) has a weak and unique solution such that the kernel terms k, l have some hypothesis as well as using some ideas from articles ([4]) and ([25]).

2.2 Preliminaries and Hypotheses

Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary Γ, let $T > 0$.

$$2E(t) = \|u_t\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(\Omega)} + (k \circ u)(t) + (l \circ v)(t) \tag{2.7}$$
$$+ \left(1 - \int_0^t k(s)ds\right) \|\nabla u\|_2^2 + \left(1 - \int_0^t l(s)ds\right) \|\nabla v\|_2^2 + \|v\nabla u\|_2^2 + \|u\nabla v\|_2^2.$$

Assumption
We assume that k, l: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions satisfying :

$$l_1 = \left(1 - \int_0^t l(s)ds\right) > 0 \quad \text{and} \quad k_1 = \left(1 - \int_0^t k(s)ds\right) > 0, \tag{2.8}$$

and

$$k'(t) \leqslant -k(t), \quad l'(t) \leqslant -l(t). \tag{2.9}$$

If $w = w(t,x)$ is a function in $L^2(0.T; H_0^1(\Omega))$ and k is continuous we put:

$$(k \circ w)(t) = \int_0^t k(t-s)\|\nabla w(t) - \nabla w(s)\|_2^2 ds.$$

Lemma 2.2.1. *[4] $w \in C^1((0,T); H_0^1(\Omega))$, $k \in C^1(0,\infty)$*

$$\int_0^t k(t-s)\left\langle \nabla w(s) \nabla w'(t)\right\rangle ds = -\frac{1}{2}\frac{d}{dt}(k \circ w)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t k(s)ds\right)\|\nabla w(t)\|_2^2 \tag{2.10}$$
$$+(k' \circ w)(t) - k(t)\|\nabla w(t)\|_2^2.$$

Theorem 2.2.1. *Let u_0, $v_0 \in L^2(\Omega)$ and u_1, $v_1 \in L^1(\Omega)$. Then, under assumptions on two functions k and l, the problem* (2.1)-(2.5) *has a local solution $(u(x,t), v(x,t))$ such that*

$$u, v \in L^\infty(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \tag{2.11}$$
$$u_t, v_t \in L^\infty(0,T; L^2(\Omega)). \tag{2.12}$$

Theorem 2.2.2. *Let $u, v :\rightarrow L^2(\Omega)$ be functions in the class* (2.11) *and* (2.12) *satisfying from* (2.1) *to* (2.5) *. Then the solution (u,v) obtained in Theorem* (2.2.1) *is unique.*

2.3 Global Existence

Step 1: Approximate solution

Using the Faedo-Galerkin process, we will determine the existence of a local solution to the problem (2.1)-(2.5) in this section. Let $\{w_i\}$ be a basis for both $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$ for each positive integer m we put

$$V = span\{w_1, w_2, \dots, w_m\}.$$

We look for an approximate solution in the form

$$u^m(t) = \sum_{i=1}^{m} u_i^m w_i \quad \text{and} \quad v^m(t) = \sum_{i=1}^{m} v_i^m w_i,$$

satisfying the approximate problem

$$\begin{aligned} &\int_\Omega \{u_{tt}^m - \Delta u^m - \alpha u_t^m\} w_j dx - \int_0^t k(t-s) \langle \nabla u^m(s), \nabla w_j \rangle ds \\ &+ \int_\Omega |v^m|^2 \nabla u^m \nabla w_j dx + \int_\Omega u^m |\nabla v^m| w_j dx = 0, \quad \forall j = 1, .., m \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\int_\Omega \{v_{tt}^m - \Delta v^m - \beta v_t^m\} w_j dx - \int_0^t l(t-s) \langle \nabla v^m(s), \nabla w_j \rangle ds \\ &+ \int_\Omega v^m |\nabla u^m| w_j dx + \int_\Omega |u^m|^2 \nabla v^m \nabla w_j dx = 0, \end{aligned} \tag{2.14}$$

with initial conditions satisfying

$$\begin{cases} u^m(0) = u_0^m, \sum_{i=1}^m a_{im} w_i = u_0^m \to u_0, & v^m(0) = v_0^m, \sum_{i=1}^m b_{im} w_i = v_0^m \to v_0 \quad \text{in } L^2(\Omega), \\ u_t^m(0) = u_1^m, \sum_{i=1}^m a_{im}^1 w_i = u_1^m \to u_1, & v_t^m(0) = v_1^m, \sum_{i=1}^m b_{im}^1 w_i = v_1^m \to v_1 \quad \text{in } L^1(\Omega). \end{cases} \tag{2.15}$$

Since the vectors $\{w_i\}$ are linearly independent, this means $det(w_i, w_j) \neq 0$, the latter ensuring that the problem admits a local solution $(u^m(t), v^m(t))$ in the interval $[0, T_m]$.

Step 2: A priori estimate

Our system's energy functional $E(t)$ is given by

$$\begin{aligned} 2E^m(t) &= \|u_t^m\|_2 + \|v_t^m\|_2 + (k \circ u^m)(t) + (l \circ v^m)(t) \\ &+ \left(1 - \int_0^t k(s) ds\right) \|\nabla u^m\|_2^2 + \left(1 - \int_0^t l(s) ds\right) \|\nabla v^m\|_2^2 + \|v^m \nabla u^m\|_2^2 + \|u^m \nabla v^m\|_2^2. \end{aligned} \tag{2.16}$$

After that, we multiply (2.13) by u_t, (2.14) by v_t, and use identity (2.10) to get

$$\frac{d}{dt} E^m(t) = (k' \circ u^m)(t) + (l' \circ v^m)(t) - k(t) \|\nabla u^m\|_2^2 - l(t) \|\nabla v^m\|_2^2 + \alpha \|u_t^m\|_2^2 + \beta \|v_t^m\|_2^2 \leqslant 0. \tag{2.17}$$

We found that $\frac{d}{dt} E^m(t)$ is a non-positive function, this last indicates that $E^m(t)$ is a non-increasing function, meaning there exists a positive constant C_1, independent of t and m such that

$$\|u_t^m\|_2^2 + \|v_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2 + \|u^m \nabla v^m\|_2^2 + \|v^m \nabla u^m\|_2^2 \leqslant C_1. \tag{2.18}$$

From this estimation, deduce that $T_m = T$. In addition, we get

$$\begin{cases} u^m, v^m & \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)),\\ u^m, v^m & \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),\\ u_t^m, v_t^m & \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.19}$$

By the Holder inequality, the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and (2.19), we obtain

$$\|u^m|\nabla v^m|^2\|_2^2 \leqslant \|u^m\|_{L^6(\Omega)}^2 \|\nabla v^m\|_{L^6(\Omega)}^4 \leqslant C_1, \tag{2.20}$$

$$\||v^m|^2\nabla u^m\|_2^2 \leqslant \|v^m\|_{L^6(\Omega)}^4 \|\nabla u^m\|_{L^6(\Omega)}^2 \leqslant C_2, \quad \forall (u^m, v^m) \quad \text{in} \quad H^2(\Omega). \tag{2.21}$$

Therefore

$$(u^m|\nabla v^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.22}$$

$$(|v^m|^2\nabla u^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.23}$$

Analogously

$$(v^m|\nabla u^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.24}$$

$$(|u^m|^2\nabla v^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.25}$$

Step 3: passage to the limit

From (2.19), (2.22), (2.23), (2.24) and (2.25) there exists a sub-sequence of (u^m) and a subsequence of (v^m), denoted by same symbols, such that

$$\begin{cases} u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T;H_0^1(\Omega)),\\ u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)),\\ u_t^m \to u_t \quad \text{and} \quad v_t^m \to v_t \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)),\\ u^m|\nabla v^m|^2 \to \chi_1 \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)),\\ v^m|\nabla u^m|^2 \to \chi_2 \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)),\\ |v^m|^2\nabla u^m \to \chi_3 \quad \text{weak star in} \; L^\infty(0,T;L^2(\Omega)),\\ |u^m|^2\nabla v^m \to \chi_4 \quad \text{weak star in} \; L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.26}$$

From (2.26) and Aubin-Lions compactness Lemma in ([9]), we obtain

$$u^m \to u, \quad v^m \to v \quad \text{strongly in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.27}$$

since ∇u^m and ∇v^m are bounded, then we have

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 & \text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 & \text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\ |u^m|^2\nabla v^m \to |u|^2\nabla v & \text{strongly in} \quad L^2(0,T;L^2(\Omega)),\\ |v^m|^2\nabla u^m \to |v|^2\nabla u & \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \end{cases} \tag{2.28}$$

Then, there exists a subsequences of u^m and v^m, which we will denote by u^m, v^m respectively, such that

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 & \text{almost everywhere in} \quad (0,T)\times\Omega, \\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 & \text{almost everywhere in} \quad (0,T)\times\Omega, \\ |u^m|^2\nabla v^m \to |u|^2\nabla v & \text{almost everywhere in} \quad (0,T)\times\Omega, \\ |v^m|^2\nabla u^m \to |v|^2\nabla u & \text{almost everywhere in} \quad (0,T)\times\Omega. \end{cases} \tag{2.29}$$

From Lemma (1.0.5) in ([32]) and (2.29) we deduce

$$\begin{cases} u^m|\nabla v^m|^2 \to u|\nabla v|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ v^m|\nabla u^m|^2 \to v|\nabla u|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ |u^m|^2\nabla v^m \to |u|^2\nabla v & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)), \\ |v^m|^2\nabla u^m \to |v|^2\nabla u & \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.30}$$

By the last formula (2.30) and (2.26) we get

$$\begin{aligned} \chi_1 &= u|\nabla v|^2, \\ \chi_2 &= v|\nabla u|^2, \\ \chi_3 &= |v|^2\nabla u, \\ \chi_4 &= |u|^2\nabla v. \end{aligned} \tag{2.31}$$

Taking $w_i = 1$ in (2.13) become

$$(u_{tt}^m, 1) - \alpha(u_t^m, 1) + (u^m|\nabla v^m|^2, 1) = 0 \tag{2.32}$$
$$|(u_{tt}^m, 1)| = \left|\alpha(u_t^m, 1) - (u|\nabla v^m|^2, 1)\right|.$$

Using the Cauchy Schwartz inequality, we have

$$\|u_{tt}^m\|_{L^1(\Omega)} \leqslant |\alpha|\|u_t^m\|_2 m^{\frac{1}{2}}(\Omega) + \|u^m|\nabla v^m|^2\|_2 m^{\frac{1}{2}}(\Omega),$$

such that, $m(\Omega)$ is a measure of Ω.
Since, the measure of Ω is finite, and (2.26), we obtain

$$\|u_{tt}^m\|_{L^1(\Omega)} \leqslant C_1. \tag{2.33}$$

Analogously

$$\|v_{tt}^m\|_{L^1(\Omega)} \leqslant C_2. \tag{2.34}$$

Then

$$\begin{cases} u_{tt}^m & \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)), \\ v_{tt}^m & \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)). \end{cases} \tag{2.35}$$

Similarly we have

$$\begin{cases} u_{tt}^m \to u_{tt} & \text{weakly star in} \quad L^\infty(0,T;L^1(\Omega)), \\ v_{tt}^m \to v_{tt} & \text{weakly star in} \quad L^\infty(0,T;L^1(\Omega)). \end{cases} \tag{2.36}$$

From (2.26), (2.36) and Lemma (1.0.4) in ([33]) with $B = L^2(\Omega)$ and $B = L^1(\Omega)$ we get

$$\begin{cases} u_0^m \to u(0), \quad v_0^m \to v(0) \quad \text{weakly star in} \quad L^2(\Omega), \\ u_1^m \to u_1(0), \quad v_1^m \to v_1(0) \quad \text{weakly star in} \quad L^1(\Omega). \end{cases} \tag{2.37}$$

From (2.37) and (2.15), we get

$$u(0) = u_0, \quad v(0) = v_0, \tag{2.38}$$

$$u_1(0) = u_1, \quad v_1(0) = v_1. \tag{2.39}$$

Setting up $m \to \infty$ and passing to the limit in (2.8), (2.9), we obtained

$$\int_\Omega \{u_{tt} - \Delta u - \alpha u_t\} w_i dx - \int_0^t k(t-s) \langle \nabla u(s), \nabla w_i \rangle ds \tag{2.40}$$
$$+ \int_\Omega |v|^2 \nabla u \nabla w_i dx + \int_\Omega u|\nabla v| w_i dx = 0,$$

$$\int_\Omega \{v_{tt} - \Delta v - \beta v_t\} w_i dx - \int_0^t l(t-s) \langle \nabla v(s), \nabla w_i \rangle ds \tag{2.41}$$
$$+ \int_\Omega v|\nabla u| w_i dx + \int_\Omega |u|^2 \nabla v \nabla w_i = 0.$$

$i = 1, .., m$. Since $(wi)_{i=1}^\infty$ is a base of $H_0^1(\Omega)$, we deduce that (u, v) satisfies (2.1).
The proof is complete.

Lemma 2.3.1. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Assume that* (2.8) *and* (2.9) *are true. Then the problem's local solution* (2.1)-(2.5) *is global in time.*

proof. Using integration by part in domain Ω, multiply the first equation of system 2.1 by u_t, and second equation by v_t, and use identity (2.10) to get

$$\frac{d}{dt}E(t) = (k' \circ u)(t) + (l' \circ v)(t) - k(t)\|\nabla u\|_2^2 - l(t)\|\nabla v\|_2^2 + \alpha\|u_t\|_2^2 + \beta\|v_t\|_2^2 \leqslant 0. \tag{2.42}$$

Since the map $t \to E(t)$ is a non-increasing function, i.e there exists a positive constant C_1, independent of t and m such that

$$C_1 \geqslant 2E(t) = \|u_t\|_2 + \|v_t\|_2 + (k \circ u)(t) + (l \circ v)(t) \tag{2.43}$$
$$+ \left(1 - \int_0^t k(s)ds\right) \|\nabla u\|_2^2 + \left(1 - \int_0^t l(s)ds\right) \|\nabla v\|_2^2 + \|v\nabla u\|_2^2 + \|u \nabla v\|_2^2 > 0,$$

which give

$$C_1 \geqslant 2E(t) \geqslant \|u_t\|_2 + \|v_t\|_2 + \left(1 - \int_0^t k(s)ds\right) \|\nabla u\|_2^2 + \left(1 - \int_0^t l(s)ds\right) \|\nabla v\|_2^2 \tag{2.44}$$
$$+ \|v\nabla u\|_2^2 + \|u\nabla v\|_2^2 > 0,$$

consequently, $\forall t \in [0, T)$, we have $\|u_t\|_2 + \|v_t\|_2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|v\nabla u\|_2^2 + \|u\nabla v\|_2^2 \leqslant C_1$.
This deduces that the solution of (2.1)-(3.5) is global in time.

2.4 Uniqueness

Let (u, v) and (u_1, v_1) two solutions of (2.1), we assume that $U = u - u_1$ and $V = v - v_1$ satisfy

$$U_{tt} - \Delta U - \alpha U_t + k * \Delta U - div(|v|^2\nabla u - |v_1|^2\nabla u_1) + (u|\nabla v|^2 - u_1|\nabla v_1|^2) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.45}$$

$$V_{tt} - \Delta V - \beta V_t + k * \Delta V - div(|u|^2\nabla v - |u_1|^2\nabla v_1) + (v|\nabla u|^2 - v_1|\nabla u_1|^2) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.46}$$

with

$$U(0) = V(0) = 0 \quad U_t(0) = V_t(0) = 0. \tag{2.47}$$

Multiplying (2.45) by $U(t)$ and (2.46) by $(V(t))$, we have

$$\int_\Omega \left(U_{tt} - \Delta U - \alpha U_t + k * \Delta U - div(|v|^2\nabla u - |v_1|^2\nabla u_1) + (u|\nabla v|^2 - u_1|\nabla v_1|^2)\right) U dx = 0, \tag{2.48}$$

$$\int_\Omega \left(V_{tt} - \Delta V - \beta V_t + k * \Delta V - div(|u|^2\nabla v - |u_1|^2\nabla v_1) + (v|\nabla u|^2 - v_1|\nabla u_1|^2)\right) V dx = 0, \tag{2.49}$$

by the Cauchy Schwartz inequality, Holder inequality, Young inequality and $H_0^1 \hookrightarrow C^0(\Omega)$, we have

$$-\alpha \frac{d}{dt}\|U\|_2^2 \leqslant c_1 + c_2\|U\|_2^2, \tag{2.50}$$

$$-\beta \frac{d}{dt}\|V\|_2^2 \leqslant c_1 + c_2\|V\|_2^2. \tag{2.51}$$

Then, by using Gronwall's Lemma (1.0.3) in ([6]) we get

$$\|U\|_2^2 = \|V\|_2^2 = 0. \tag{2.52}$$

This proves the uniqueness of the solution. ■

2.5 Stability

Theorem 2.5.1. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Assume that (2.8) and (2.9) hold. then there exists two positive constants μ_1 and μ_2 independent of t such that $0 < E(t) \leqslant \mu_1 e^{-\mu_2 t}, \forall t \geqslant 0$.*

Proof. We define the function of Laypunov, for $\epsilon > 0$ as follows

$$L(t) = E(t) + \epsilon \int_\Omega u_t u + v_t v dx. \tag{2.53}$$

We prove that $L(t)$ and $E(t)$ are equivalent, meaning that there exist two positive constants N and M depending on ϵ such that for $t \geqslant 0$

$$NE(t) \leqslant L(t) \leqslant ME(t). \tag{2.54}$$

From the Lemma (1.0.1), we have

$$L(t) \leqslant E(t) + \epsilon \left[\frac{1}{2\delta} \|u_t\|_2^2 + \delta \|u\|_2^2 \right] + \epsilon \left[\frac{1}{2\delta} \|v_t\|_2^2 + \delta \|v\|_2^2 \right].$$

By using the Poincaré inequality (1.0.2), we get

$$L(t) \leqslant E(t) + \epsilon \left[\frac{1}{2\delta} \|u_t\|_2^2 + \delta C_1 \|\nabla u\|_2^2 \right] + \epsilon \left[\frac{1}{2\delta} \|v_t\|_2^2 + \delta C_2 \|\nabla v\|_2^2 \right].$$

From (2.44) we have

$$\begin{aligned}
L(t) &\leqslant E(t) + \epsilon \left[\frac{1}{\delta} E(t) + 2\delta \frac{C_1}{k_1} E(t) \right] + \epsilon \left[\frac{1}{\delta} E(t) + 2\delta \frac{C_2}{l_1} E(t) \right] \\
L(t) &\leqslant E(t) + 2\epsilon \frac{1}{\delta} E(t) + 2\epsilon\delta \frac{C_1}{k_1} E(t) + 2\epsilon\delta \frac{C_2}{l_1} E(t) \\
L(t) &\leqslant ME(t) \quad \text{such that} \quad M = 1 + 2\epsilon \frac{1}{\delta} + 2\epsilon\delta \frac{C_1}{k_1} + 2\epsilon\delta \frac{C_2}{l_1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
L(t) &\geqslant E(t) - \epsilon \left[\frac{1}{2\delta} \|u_t\|_2^2 + \delta \|u\|_2^2 \right] - \epsilon \left[\frac{1}{2\delta} \|v_t\|_2^2 + \delta \|v\|_2^2 \right] \\
&\geqslant E(t) - \epsilon \left[\frac{1}{2\delta} \|u_t\|_2^2 + \delta C_1 \|\nabla u\|_2^2 \right] - \epsilon \left[\frac{1}{2\delta} |v_t|_2^2 + \delta C_2 \|\nabla v\|_2^2 \right] \\
&\geqslant E(t) - \epsilon \left[\frac{1}{\delta} E(t) + 2\delta \frac{C_1}{k_1} E(t) \right] - \epsilon \left[\frac{1}{\delta} E(t) + 2\delta \frac{C_2}{l_1} E(t) \right], \\
L(t) &\geqslant NE(t) \quad \text{such that} \quad N = 1 - 2\epsilon \frac{1}{\delta} - 2\epsilon\delta \frac{C_1}{k_1} - 2\epsilon\delta \frac{C_2}{l_1}.
\end{aligned}$$

Now we have

$$\frac{d}{dt} L(t) = \frac{d}{dt} E(t) + \epsilon \int_\Omega \left[u_t^2 + u_{tt} u + v_t^2 + v_{tt} v \right] dx \tag{2.55}$$

$$\begin{aligned}
\epsilon \int_\Omega u_{tt} u dx &= \epsilon \int_\Omega \left[u.\Delta u + \alpha u u_t - u.k * \Delta u + u.div(|v|^2 \nabla u) - u|\nabla v|^2.u \right] dx \\
&\leqslant \epsilon \left[-\|\nabla u\|_2^2 + \alpha \frac{1}{2\delta} \|u_t\|_2^2 + \alpha\delta \|u\|_2^2 - \|v \nabla u\|_2^2 - \|u \nabla v\|_2^2 + \int_\Omega \nabla u \int_0^t k(t-s) \nabla u(s) ds dx \right] \qquad (2.56) \\
&\leqslant \epsilon \left[-\|\nabla u\|_2^2 + \alpha \frac{1}{2\delta} \|u_t\|_2^2 + \alpha C_1 \delta \|\nabla u\|_2^2 - \|v \nabla u\|_2^2 - \|u \nabla v\|_2^2 + \int_\Omega \nabla u \int_0^t k(t-s) \nabla u(s) ds dx \right]
\end{aligned}$$

Analogous

$$\epsilon\int_\Omega v_{tt}vdx = \epsilon\int_\Omega \left[v.\Delta v + \beta v v_t - v.k*\Delta v + v.div(|u|^2\nabla v) - v|\nabla u|^2.v\right]dx \tag{2.57}$$
$$\leqslant \epsilon\left[-\|\nabla v\|_2^2 + \beta\frac{1}{2\delta}\|v_t\|_2^2 + \beta C_1\delta\|\nabla v\|_2^2 - \|u\nabla v\|_2^2 - \|v\nabla u\|_2^2 + \int_\Omega \nabla v\int_0^t l(t-s)\nabla v(s)dsdx\right].$$

So

$$\begin{aligned}\frac{d}{dt}L(t) \leqslant{}& \frac{d}{dt}E(t) + \epsilon\|u_t\|_2^2 + \epsilon\|v_t\|_2^2\\ &- \epsilon\|\nabla u\|_2^2 + \epsilon\alpha\frac{1}{2\delta}\|u_t\|_2^2 + \epsilon\alpha C_1\delta\|\nabla u\|_2^2 - \epsilon\|v\nabla u\|_2^2 - \epsilon\|u\nabla v\|_2^2 + \epsilon\int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\\ &- \epsilon\|\nabla v\|_2^2 + \epsilon\beta\frac{1}{2\delta}\|v_t\|_2^2 + \epsilon\beta C_1\delta\|\nabla v\|_2^2 - \epsilon\|u\nabla v\|_2^2 - \epsilon\|v\nabla u\|_2^2 + \epsilon\int_\Omega\nabla v\int_0^t l(t-s)\nabla v(s)ds.dx.\end{aligned} \tag{2.58}$$

The last term of relation (2.58) can be estimated as follow.

$$\begin{aligned}&\left|\int_\Omega\nabla u\int_0^t k(t-s)\nabla u(s)dsdx\right|\\ &\leqslant \int_\Omega\left(\int_0^t k(t-s)|\nabla u(s)-\nabla u(t)|ds\right)dx + \int_0^t k(s)ds\|\nabla u\|_2^2\\ &\leqslant (1+\eta)(1-k_1)\|\nabla u\|_2^2 + \frac{1}{4\eta}(k\circ\nabla u)(t) \quad \text{for} \quad \eta>0.\end{aligned} \tag{2.59}$$

Analogously

$$\begin{aligned}&\left|\int_\Omega\nabla v\int_0^t l(t-s)\nabla v(s)dsdx\right|\\ &\leqslant (1+\eta)(1-l_1)\|\nabla v\|_2^2 + \frac{1}{4\eta}(l\circ\nabla v)(t) \quad \text{for} \quad \eta>0.\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dt}L(t) \leqslant{}& (k'\circ u)(t) + (l'\circ v)(t) - k(t)\|\nabla u\|_2^2 - l(t)\|\nabla v\|_2^2 + \alpha\|u_t\|_2^2 + \beta\|v_t\|_2^2 + \epsilon\|u_t\|_2^2 + \epsilon\|v_t\|_2^2\\ &- \epsilon\|\nabla u\|_2^2 + \epsilon\alpha\frac{1}{2\delta}\|u_t\|_2^2 + \epsilon\alpha C_1\delta\|\nabla u\|_2^2 - \epsilon\|v\nabla u\|_2^2 - \epsilon\|u\nabla v\|_2^2 + \epsilon(1+\eta)(1-k_1)\|\nabla u\|_2^2\\ &+ \epsilon\frac{1}{4\eta}(k\circ\nabla u)(t) - \epsilon\|\nabla v\|_2^2 + \epsilon\beta\frac{1}{2\delta}\|v_t\|_2^2 + \epsilon\beta C_1\delta\|\nabla v\|_2^2 - \epsilon\|u\nabla v\|_2^2 - \epsilon\|u\nabla u\|_2^2\\ &+ \epsilon(1+\eta)(1-l_1)\|\nabla v\|_2^2 + \epsilon\frac{1}{4\eta}(l\circ\nabla v)(t),\end{aligned} \tag{2.60}$$

so

$$\frac{d}{dt}L(t) \leqslant \left(\alpha+\epsilon+\epsilon\alpha\frac{1}{2\delta}\right)\|u_t\|_2^2 + \left(\beta+\epsilon+\epsilon\beta\frac{1}{2\delta}\right)\|v_t\|_2^2$$
$$+ (-k(t)-\epsilon+\epsilon\alpha C_1\delta+\epsilon(1+\eta)(1-k_1))\|\nabla u\|_2^2$$
$$+ (-l(t)-\epsilon+\epsilon\beta C_1\delta+\epsilon(1+\eta)(1-l_1))\|\nabla v\|_2^2 + (-2\epsilon-1)\|v\nabla u\|_2^2 + (-2\epsilon-1)\|v\nabla u\|_2^2$$
$$-(k\circ u)(t)-(l\circ v)(t)+\|u\nabla v\|_2^2+\|v\nabla u\|_2^2+\epsilon\frac{1}{4\eta}(k\circ\nabla u)(t)+\epsilon\frac{1}{4\eta}(l\circ\nabla v)(t), \quad (2.61)$$

so

$$\frac{d}{dt}L(t) \leqslant \gamma E(t)+\lambda. \quad (2.62)$$

We choosing ϵ small enough, such that

$$\gamma = Min(\alpha+\epsilon+\epsilon\alpha\frac{1}{2\delta};\beta+\epsilon+\epsilon\beta\frac{1}{2\delta};-k(t)-\epsilon+\epsilon\alpha C_1\delta+\epsilon(1+\eta)(1-k_1) \quad (2.63)$$
$$-l(t)-\epsilon+\epsilon\beta C_1\delta+\epsilon(1+\eta)(1-l_1);(-2\epsilon-1);-1)<0,$$

and

$$\lambda = \|u\nabla v\|_2^2+\|v\nabla u\|_2^2+\epsilon\frac{1}{4\eta}(k\circ\nabla u)(t)+\epsilon\frac{1}{4\eta}(l\circ\nabla v)(t). \quad (2.64)$$

From (2.54), we have

$$\frac{d}{dt}L(t) \leqslant \frac{\gamma}{M}L(t)+\lambda, \quad (2.65)$$

like that

$$\frac{d}{dt}L(t)-\frac{\gamma}{M}L(t) \leqslant \lambda. \quad (2.66)$$

For obtain the estimate of the function L we solving the previous differential inequality (2.65) as following. Solve the homogeneous inequality of relation (2.66), we have

$$\int\frac{dL}{L} \leqslant \int\frac{\gamma}{M}dt+c,$$
$$ln(L) \leqslant \frac{\gamma}{M}t+c,$$
$$L(t) \leqslant e^{\frac{\gamma}{M}t}c_1,$$

to complete the solve we apply the method of the variation of the constant.
Let

$$L(t) \leqslant e^{\frac{\gamma}{M}t}c_1(t), \quad (2.67)$$

so

$$\frac{dL}{dt} \leqslant \frac{\gamma}{M}e^{\frac{\gamma}{M}t}c_1(t)+e^{\frac{\gamma}{M}t}c_1'(t) \leqslant \frac{\gamma}{M}e^{\frac{\gamma}{M}t}c_1(t)+\lambda,$$
$$e^{\frac{\gamma}{M}t}c_1'(t) \leqslant \lambda,$$
$$c_1'(t) \leqslant e^{\frac{-\gamma}{M}t}\lambda,$$
$$\int dc_1(t) \leqslant \int e^{\frac{-\gamma}{M}t}\lambda dt+c_2,$$

so

$$c_1(t) \leqslant \frac{-M\lambda}{\gamma} e^{\frac{-\gamma}{M}t} + c_2, \tag{2.68}$$

so, from relations (2.67) and (2.68), we get

$$L(t) \leqslant c_2 e^{\frac{\gamma}{M}t} - \frac{\lambda M}{\gamma}, \quad \forall t \geqslant 0, \tag{2.69}$$

by using (2.54), we conclude

$$E(t) \leqslant c_3 e^{\frac{\gamma}{M}t} - \frac{\lambda M}{\gamma N}, \quad \forall t \geqslant 0. \tag{2.70}$$

Chapter 3

Global existence and stabilization of solution for a non-linear hyperbolic system of Klein-Gordon Equations

In this chapter, we have studied the existence and uniqueness of solution, and we also prove that the solution is stable over time by applying Lyapunov function.

3.1 Introduction

We look at the Klein-Gordon non-linear problem in this work, which is defined as follows

$$\begin{cases} u_{tt} - \Delta u - div\,(\nabla u|\nabla u|^{p-2}|u|^{p}) + u|u|^{p-2}|\nabla u|^{p} - \alpha u_t = u|u|^{m-2}, \\ v_{tt} - \Delta v - div\,(\nabla v|\nabla v|^{p-2}|v|^{p}) + v|v|^{p-2}|\nabla v|^{p} - \beta v_t = v|v|^{m-2}, \end{cases} \tag{3.1}$$

with boundary conditions

$$u(x,t) = v(x,t) = 0 \quad \text{in} \quad \partial\Omega \times (0,t), \tag{3.2}$$

and initial conditions

$$u(x,o) = u_0(x) \qquad v(x,o) = v_0(x) \quad \text{on} \quad \Omega, \tag{3.3}$$

$$u_t(x,o) = u_1(x) \qquad v_t(x,o) = v_1(x) \quad \text{on} \quad \Omega. \tag{3.4}$$

Where Ω is a bounded domain of $\mathbb{R}^n (n \geqslant 1)$ with smooth boundary $\partial\Omega$ and let $T > 0$, α and β are non-positive constants and p,m are a positive constants.
The major purpose of this work is to use potential well techniques to investigate the existence and uniqueness of solution (the same way in the article see ([31]) for a problem with limits given by equations to non-linear partial derivatives. under certain conditions the solution also is stable over time. The fundamental tool in the proofs is Haraux and Zuazua ([19]) idea, which is centered on creating an adequate Lyapunov function .

3.2 preliminaries and Assumptions

The energy associated with our system is given by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{p}\|u\nabla u\|_p^p + \frac{1}{p}\|v\nabla v\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{m}\|u\|_m^m - \frac{1}{m}\|v\|_m^m, \tag{3.5}$$

the following functions are also introduced

$$J(t) = \frac{1}{p}\|u\nabla u\|_p^p + \frac{1}{p}\|v\nabla v\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{m}\|u\|_m^m - \frac{1}{m}\|v\|_m^m, \tag{3.6}$$

$$I(t) = \|u\nabla u\|_p^p + \|v\nabla v\|_p^p + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \|u\|_m^m - \|v\|_m^m. \tag{3.7}$$

We define the stable set by

$$\mathbf{D} = \left\{(u,v) \in \left(W_0^{1,p}(\Omega) \cap H_0^1(\Omega)\right)^2 : \quad I(t) > 0 \quad \text{and} \quad J(t) < d\right\}, \tag{3.8}$$

where d define by

$$d = \inf\left\{\sup_{(u,v)\in(W_0^{1,p}(\Omega)\cap H_0^{1,p}(\Omega))^2/(0,0)} J(\lambda(u,v))\right\}. \tag{3.9}$$

Assumptions
(i) Assume that

$$B = C^m\left[\left(\frac{2m}{m-2}\right)E(0)\right]^{\frac{m-2}{2}} < 1. \tag{3.10}$$

Where C is the Poincarare constant.
(ii) m satisfies

$$2 < p < m \leqslant \frac{2n}{n-2}, 2 < n. \tag{3.11}$$

Lemma 3.2.1. *[25] Let s be a number with $2 \leqslant s < \infty$ if $n \leqslant r$ and $2 \leqslant s \leqslant \frac{nr}{n-r}$ if $n > r$. Then there is a constant C depending on Ω and s such that $\|u\|_s \leqslant C\|\nabla u\|_{r'}$, $u \in W_0^{1,r}(\Omega)$.*

Theorem 3.2.1. *Let $(u_0, v_0) \in \left(W_0^{1,p}(\Omega) \cap H_0^1(\Omega)\right)^2$ and $(u_1, v_1) \in (L^2(\Omega))$ be given, suppose that* (3.10) *relationship is realized. Then there exists a local solution (u,v) to problem* (3.1)-(3.4) *which is satisfied*

$$(u,v) \in C\left([0,T]; (W_0^{1,p}(\Omega) \cap H_0^1(\Omega))^2\right), \tag{3.12}$$

$$(u_t, v_t) \in C\left([0,T]; (L^2(\Omega))^2\right). \tag{3.13}$$

3.3 Global solution and stabilization

Remark 3.3.1. *Using integration by parts in domain Ω, multiply the first equation of system* (3.1) *by element u_t and second equation by element v_t, we get*

$$\frac{d}{dt}E(t) = \alpha\|u_t\|_2^2 + \beta\|v_t\|_2^2 < 0, \tag{3.14}$$

for almost each t. Therefore,

$$E(t) \leqslant E(0) \quad \forall t \geqslant 0. \tag{3.15}$$

We will show that the set $\mathbf{D}$ is invariant. That is for some $t_0 > 0$ if $(u(t_0), v(t_0)) \in \mathbf{D}$, then $(u, v) \in \mathbf{D}$, $\forall t \geqslant t_0$ starting with the existence of the potential depth in the following lemma

Lemma 3.3.1. *[31] d is a positive constant*

PROOF. We have

$$\begin{aligned} J(\lambda(u,v)) = & \frac{\lambda^{2p}}{p}\|u\nabla u\|_p^p + \frac{\lambda^{2p}}{p}\|v\nabla v\|_p^p + \frac{\lambda^2}{2}\|\nabla u\|_2^2 + \frac{\lambda^2}{2}\|\nabla v\|_2^2 \\ & - \frac{\lambda^{2m}}{m}\|u\|_m^m - \frac{\lambda^{2m}}{m}\|v\|_m^m, \end{aligned} \tag{3.16}$$

we put

$$J(\lambda(u,v)) \geqslant G(\lambda), \tag{3.17}$$

where

$$G(\lambda) = \frac{\lambda^2}{2}\|\nabla u\|_p^p + \frac{\lambda^2}{2}\|\nabla v\|_p^p - \frac{\lambda^m}{m}\|u\|_m^m - \frac{\lambda^m}{m}\|v\|_m^m, , \tag{3.18}$$

differentiating the second term from the relationship (3.18) with respect to λ, to get

$$\frac{d}{d\lambda}G(\lambda) = \lambda\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \lambda^{m-1}\left(\|u\|_m^m + \|v\|_m^m\right), \tag{3.19}$$

so

$$\frac{d}{d\lambda}G(\lambda) = \lambda\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \lambda^{m-1}\left(\|u\|_m^m + \|v\|_m^m\right) = 0, \tag{3.20}$$

we are looking for solutions to equation (3.20)

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \frac{\lambda^{m-1}}{\lambda}\left(\|u\|_m^m + \|v\|_m^m\right) = 0, \tag{3.21}$$

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 = \lambda^{(m-2)}\left(\|u\|_m^m + \|v\|_m^m\right), \tag{3.22}$$

so

$$\lambda^{(m-2)} = \frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m}, \tag{3.23}$$

so, we have

$$\lambda_2 = \left(\frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m} \right)^{\frac{1}{(m-2)}}, \tag{3.24}$$

for $\lambda_1 = 0$
and

$$\lambda_2 = \left(\frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m} \right)^{\frac{1}{(m-2)}}, \tag{3.25}$$

then we have

$$\frac{d}{d\lambda} G(\lambda) = 0. \tag{3.26}$$

and, we have

$$\begin{aligned}
\frac{d^2}{d\lambda^2} G(\lambda) &= \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - (m-1)\lambda^{m-2} \left(\|u\|_m^m + \|v\|_m^m\right) \\
\frac{d^2}{d\lambda^2} G(\lambda_2) &= \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - (m-1) \left(\frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m} \right)^{\frac{m-2}{m-2}} \left(\|u\|_m^m + \|v\|_m^m\right) \\
&= (2-m) \left(\|u\|_m^m + \|v\|_m^m\right) < 0,
\end{aligned} \tag{3.27}$$

As

$$\frac{d}{d\lambda} G(\lambda_2) = 0, \quad \text{and} \quad G(\lambda_1) = 0, \tag{3.28}$$

and since

$$\frac{d^2}{d\lambda^2} G(\lambda_2) < 0 \tag{3.29}$$

we see that

$$\begin{aligned}
\sup_{\lambda \geqslant 0} J(\lambda) \geqslant \sup_{\lambda \geqslant 0} G(\lambda) = G(\lambda_2)& \\
&= \frac{1}{2} \left(\frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m} \right)^{\frac{2}{(m-2)}} \times \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&- \frac{1}{m} \left(\frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_m^m + \|v\|_m^m} \right)^{\frac{m}{(m-2)}} \times \left(\|u\|_m^m + \|v\|_m^m\right),
\end{aligned} \tag{3.30}$$

it follows from the Lemma (3.2.1) and assumptions (3.10), we get

$$\begin{aligned}
\|u\|_m^m + \|v\|_m^m &\leqslant C^m \left(\|\nabla u\|_2^m + \|\nabla v\|_2^m\right) \\
&\leqslant C^m \left(\|\nabla u\|_2^2 \|\nabla u\|_2^{m-2} + \|\nabla v\|_2^2 \|\nabla v\|_2^{m-2}\right) \\
&\leqslant \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) C^m \left[\left(\frac{2m}{m-2} \right) E(0) \right]^{\frac{m-2}{2}} \\
&\leqslant \|\nabla u\|_2^2 + \|\nabla v\|_2^2,
\end{aligned} \tag{3.31}$$

then we have

$$\sup_{\lambda\geqslant 0} J(\lambda) \geqslant \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \frac{1}{m}\left(\|u\|_m^m + \|v\|_m^m\right), \tag{3.32}$$

$$\geqslant \left(\frac{m-2}{2m}\right)\left(\|u\|_m^m + \|v\|_m^m\right), \tag{3.33}$$

if $\|u\|_m^m + \|v\|_m^m \geqslant 1$, we have

$$\sup_{\lambda\geqslant 0} J(\lambda) \geqslant \left(\frac{m-2}{2m}\right) = d > 0. \tag{3.34}$$

Lemma 3.3.2. *[31] **D** is a bounded neighborhood of 0 in* $W_0^{1,p}(\Omega)$

Proof. for $(u,v) \in \mathbf{D}$, and $(u,v) \neq (0,0)$, we find

$$\begin{aligned} J(t) = {} & \frac{1}{p}\|u\nabla u\|_p^p + \frac{1}{p}\|v\nabla v\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 \\ & - \frac{1}{m}\left(I(t) - \|u\nabla u\|_p^p - \|v\nabla v\|_p^p - \|\nabla u\|_2^2 - \|\nabla v\|_2^2\right) \\ & + \left(\frac{1}{p} - \frac{1}{m}\right)\left(\|u\nabla u\|_p^p + \|v\nabla v\|_p^p\right) \\ & + \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{m}I(t), \end{aligned} \tag{3.35}$$

so

$$\begin{aligned} J(t) \geqslant {} & \left(\frac{1}{p} - \frac{1}{m}\right)\left(\|u\nabla u\|_p^p + \|v\nabla v\|_p^p\right) \\ & + \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ J(t) \geqslant {} & \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \end{aligned} \tag{3.36}$$

it follows that

$$\begin{aligned} \|\nabla u\|_2^2 + \|\nabla v\|_2^2 & \leqslant \left(\frac{1}{\frac{1}{2} - \frac{1}{m}}\right) J(t) \\ & \leqslant \left(\frac{1}{\frac{1}{2} - \frac{1}{m}}\right) d = R. \end{aligned} \tag{3.37}$$

Consequently, $\forall (u,v) \in \mathbf{D}$, we have $(u,v) \in B$ where

$$B = \left\{ (u,v) \in \left(W_0^{1,p}(\Omega) \cap H_0^1(\Omega)\right)^2 \quad / \quad \sum_{i=1}^{2} \|\nabla u_i\|_2^2 < R \right\}. \tag{3.38}$$

This complete the proof.

Theorem 3.3.1. *Suppose that* (3.10) *and* (3.11) *be given. If* $(u_0, v_0) \in \boldsymbol{D}$ *and* $(u_1, v_1) \in (L^2(\Omega))^2$, *so for all* $t \geqslant 0$ *the solution* $(u, v) \in \boldsymbol{D}$.

Proof. let $(u_0, v_0) \in \mathbf{D}$, we see that

$$I(t) = \|u_0 \nabla u_0\|_p^p + \|v_0 \nabla v_0\|_p^p + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 - \|u_0\|_m^m - \|v_0\|_m^m > 0. \tag{3.39}$$

As a result of continuity, there exist $T_m \leqslant T$ such that

$$I(t) = \|u \nabla u\|_p^p + \|v \nabla v\|_p^p + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \|u\|_m^m - \|v\|_m^m \geqslant 0 \quad \text{for} \quad t \in [0, T_m]. \tag{3.40}$$

This give

$$\begin{aligned} J(t) = & \frac{1}{p}\|u \nabla u\|_p^p + \frac{1}{p}\|v \nabla v\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 \\ & - \frac{1}{m}\left(\|u\|_m^m + \|v\|_m^m\right). \end{aligned} \tag{3.41}$$

From $I(t)$ we have

$$-\left(\|u\|_m^m + \|v\|_m^m\right) = I(t) - \|u \nabla u\|_p^p - \|v \nabla v|_p^p - \|\nabla u\|_2^2 - \|\nabla v\|_2^2. \tag{3.42}$$

Substituting the relationship (3.42) into the relationship (3.41), we find

$$\begin{aligned} J(t) = & \frac{1}{p}\|u \nabla u\|_p^p + \frac{1}{p}\|v \nabla v\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 \\ & - \frac{1}{m}\left(I(t) - \|u \nabla u\|_p^p - \|v \nabla v\|_p^p - \|\nabla u\|_2^2 - \|\nabla v\|_2^2\right) \\ = & \left(\frac{1}{p} - \frac{1}{m}\right)\left(\|u \nabla u\|_p^p + \|v \nabla v\|_p^p\right) \\ & + \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) + \frac{1}{m} I(t), \end{aligned} \tag{3.43}$$

so

$$\begin{aligned} J(t) \geqslant & \left(\frac{1}{p} - \frac{1}{m}\right)\left(\|u \nabla u\|_p^p + \|v \nabla v\|_p^p\right) \\ & + \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ J(t) \geqslant & \left(\frac{1}{2} - \frac{1}{m}\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \end{aligned} \tag{3.44}$$

so, we have

$$\begin{aligned} \|\nabla u\|_2^2 + \|\nabla v\|_2^2 & \leqslant \left(\frac{1}{\frac{1}{2} - \frac{1}{m}}\right) J(t) \\ & \leqslant \left(\frac{1}{\frac{1}{2} - \frac{1}{m}}\right) E(t) \\ & \leqslant \left(\frac{1}{\frac{1}{2} - \frac{1}{m}}\right) E(0). \end{aligned} \tag{3.45}$$

By apply Lemma (3.2.1) we have

$$\|u\|_m^m \leqslant C^m \|\nabla u\|_2^m \leqslant C^m \|\nabla u\|_2^2 \|\nabla u\|_2^{m-2}, \tag{3.46}$$

$$\|v\|_m^m \leqslant C^m \|\nabla v\|_2^m \leqslant C^m \|\nabla v\|_2^2 \|\nabla v\|_2^{m-2}, \tag{3.47}$$

from the relation (3.45), we get

$$\begin{aligned}
\|u\|_m^m + \|v\|_m^m &\leqslant C^m \left(\|\nabla u\|_2^2 \|\nabla u\|_2^{m-2} + \|\nabla v\|_2^2 \|\nabla v\|_2^{m-2}\right) \\
&\leqslant C^m \left(\|\nabla u\|_2^2 \left(E(0) \frac{2m}{m-2} \right)^{\frac{m-2}{2}} + \|\nabla v\|_2^2 \left(E(0) \frac{2m}{m-2} \right)^{\frac{m-2}{2}} \right) \\
&\leqslant \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) C^m \left[\left(\frac{2m}{m-2} \right) E(0) \right]^{\frac{m-2}{2}} \\
&< \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\
&< \|u \nabla u\|_p^p + \|v \nabla v\|_p^p + \|\nabla u\|_2^2 + \|\nabla v\|_2^2,
\end{aligned} \tag{3.48}$$

therefore, $I(t) > 0 \ \forall t \in [0, T_m]$, by taking the fact that

$$\lim_{t \to T_m} C^m \left[\left(\frac{2m}{m-2} \right) E(0) \right]^{\frac{m-2}{2}} \leqslant B < 1. \tag{3.49}$$

This demonstrate that the solution $(u, v) \in \mathbf{D}$, for all $t \in [0, T_m]$. By repeating this procedure T_m extends to T.

Theorem 3.3.2. *Suppose that* (3.10) *and* (3.11) *be given. If* $(u_0, v_0) \in \boldsymbol{D}$*,* $(u_1, v_1) \in (L^2(\Omega))^2$*, then the local solution* (u, v) *becomes global in time such that* $(u, v) \in G$ *where*

$$G = \begin{cases} (u, v) \in L^\infty \left(\mathbb{R}^+, \left(W_0^{1,p}(\Omega) \cap H_1^0(\Omega) \right)^2 \right), \\ (u_t, v_t) \in L^\infty \left(\mathbb{R}^+, (L^2(\Omega))^2 \right). \end{cases} \tag{3.50}$$

Proof. Using the relationships (3.5), (3.14) and (3.35), we get

$$E(0) \geqslant E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 \tag{3.51}$$

$$\begin{aligned}
E(0) \geqslant E(t) \geqslant &\left(\frac{1}{p} - \frac{1}{m} \right) \left(\|u \nabla u\|_p^p + \|v \nabla v\|_p^p \right) \\
&+ \left(\frac{1}{2} - \frac{1}{m} \right) \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{1}{m} I(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2,
\end{aligned} \tag{3.52}$$

since $I(t)$ is a positive, hence

$$\|u_t\|_2^2 + \|v_t\|_2^2 + \|u \nabla u\|_p^p + \|v \nabla v\|_p^p + \|\nabla u\ \|_2^2 + \|\nabla v\|_2^2 \leqslant C E(0), \tag{3.53}$$

such that C is a positive constant that only depends on p and m.

Theorem 3.3.3. *Suppose that* $m > p$ *and* (3.10) *are correct. If* $(u_0, u_1) \in \left(W_0^{1,p}(\Omega) \cap \in H_0^1(\Omega)\right)^2$, $(u_1, v_1) \in \left(L^2(\Omega)\right)^2$. *Then there are two positive constants* μ_1 *and* μ_2 *that are independent of* t *such that* $0 < E(t) \leqslant \mu_1 e^{-\mu_2 t}, \forall t \geqslant 0$.

Proof. We define Lyapunov's function as follows

$$L(t) = E(t) + \epsilon \int_\Omega u_t u + v_t v dx \qquad \forall \epsilon > 0. \tag{3.54}$$

We show that the relation L and E are equivalent, i.e. there are two positive constants N and M that depend on ϵ such that for $t \geqslant 0$

$$NE(t) \leqslant L(t) \leqslant ME(t). \tag{3.55}$$

From the Lemma (1.0.1), we have

$$L(t) \leqslant E(t) + \epsilon\left(\frac{1}{2\delta}\|u_t\|_2^2 + \delta\|u\|_2^2\right) + \epsilon\left(\frac{1}{2\delta}\|v_t\|_2^2 + \delta\|v\|_2^2\right), \tag{3.56}$$

by applying the Lemma (1.0.2), we find

$$L(t) \leqslant E(t) + \epsilon\left(\frac{1}{2\delta}\|u_t\|_2^2 + \delta C_1\|\nabla u\|_2^2\right) + \epsilon\left(\frac{1}{2\delta}\|v_t\|_2^2 + \delta C_2\|\nabla v\|_2^2\right), \tag{3.57}$$

from (3.52) we have

$$\begin{aligned} L(t) &\leqslant E(t) + \epsilon\left(\frac{1}{\delta}E(t) + \delta C_1\frac{2m}{m-2}E(t)\right) + \epsilon\left(\frac{1}{\delta}E(t) + \delta C_2\frac{2m}{m-2}E(t)\right) \\ &\leqslant E(t) + 2\epsilon\frac{1}{\delta}E(t) + \epsilon\frac{2\delta C_1 m}{m-2}E(t) + \epsilon\frac{2\delta C_2 m}{m-2}E(t), \end{aligned} \tag{3.58}$$

$$L(t) \leqslant M(t)E(t) \quad \text{such that} \quad M = 1 + 2\epsilon\frac{1}{\delta} + \epsilon\frac{2\delta C_1 m}{m-2} + \epsilon\frac{2\delta C_2 m}{m-2}. \tag{3.59}$$

On the otherwise, we have

$$\begin{aligned} L(t) &\geqslant E(t) - \epsilon\left(\frac{1}{2\delta}\|u_t\|_2^2 + \delta\|u\|_2^2\right) - \epsilon\left(\frac{1}{2\delta}\|v_t\|_2^2 + \delta\|v\|_2^2\right) \\ &\geqslant E(t) - \epsilon\left(\frac{1}{2\delta}\|u_t\|_2^2 + \delta C_1\|\nabla u\|_2^2\right) - \epsilon\left(\frac{1}{2\delta}\|v_t\|_2^2 + \delta C_2\|\nabla v\|_2^2\right) \\ &\geqslant E(t) - 2\epsilon\frac{1}{\delta}E(t) - \epsilon\frac{2\delta C_1 m}{m-2}E(t) - \epsilon\frac{2\delta C_2 m}{m-2}E(t), \end{aligned} \tag{3.60}$$

so

$$L(t) \geqslant N(t)E(t) \quad \text{such that} \quad N = 1 - 2\epsilon\frac{1}{\delta} - \epsilon\frac{2\delta C_1 m}{m-2} - \epsilon\frac{2\delta C_2 m}{m-2}. \tag{3.61}$$

Now we have

$$\frac{d}{dt}L(t) = \frac{d}{dt}E(t) + \epsilon\int_\Omega u_{tt}u.dx + \epsilon\int_\Omega u_t^2 dx + \int_\Omega v_{tt}v.dx + \epsilon\int_\Omega v_t^2 dx, \tag{3.62}$$

from (3.1), we get

$$u_{tt} = \Delta u + div\left(\nabla u|\nabla u|^{p-2}|u|^p\right) - u|u|^{p-2}|\nabla u|^p + \alpha u_t + u|u|^{m-2}, \tag{3.63}$$
$$v_{tt} = \Delta v + div\left(\nabla v|\nabla v|^{p-2}|v|^p\right) - v|v|^{p-2}|\nabla v|^p + \alpha v_t + v|v|^{m-2}, \tag{3.64}$$

so, we have

$$\begin{aligned}\epsilon\int_\Omega u_{tt}udx &= \epsilon\int_\Omega\left(\Delta u + div\left(\nabla u|\nabla u|^{p-2}|u|^p\right) - u|u|^{p-2}|\nabla u|^p + \alpha u_t + u|u|^{m-2}\right)udx,\\ \epsilon\int_\Omega v_{tt}vdx &= \epsilon\int_\Omega\left(\Delta v + div\left(\nabla v|\nabla v|^{p-2}|v|^p\right) - v|v|^{p-2}|\nabla v|^p + \alpha v_t + v|v|^{m-2}\right)vdx,\end{aligned} \tag{3.65}$$

by applying the part integration relation, we find

$$\int_\Omega u\Delta uds = -\|\nabla u\|_2^2, \quad \int_\Omega v\Delta vds = -\|\nabla v\|_2^2, \tag{3.66}$$
$$\int_\Omega div\left(\nabla u|\nabla u|^{p-2}|u|^p\right)udx = -\|u\nabla u\|_p^p, \quad \int_\Omega div\left(\nabla v|\nabla v|^{p-2}|v|^p\right)vdx = -\|v\nabla v\|_p^p, \tag{3.67}$$
$$-\int_\Omega u|u|^{p-2}\|\nabla u\|^p udx = -\|u\nabla u\|_p^p, \quad -\int_\Omega v|v|^{p-2}|\nabla v|^p vdx = -\|v\nabla v\|_p^p, \tag{3.68}$$
$$\int_\Omega uu^{m-2}udx = \|u\|_m^m, \quad \int_\Omega vv^{m-2}vdx = \|v\|_m^m. \tag{3.69}$$

We apply the Young inequality in ([25]), we have

$$\epsilon\alpha\int_\Omega uu_tdx \leqslant \epsilon\alpha\left(\frac{1}{4\sigma}\|u_t\|_2^2 + \delta\|u\|_2^2\right), \tag{3.70}$$
$$\epsilon\beta\int_\Omega vv_tdx \leqslant \epsilon\beta\left(\frac{1}{4\sigma}\|v_t\|_2^2 + \delta\|v\|_2^2\right), \tag{3.71}$$

so, we get

$$\begin{aligned}\epsilon\int_\Omega u_{tt}udx &= \epsilon\int_\Omega\left(\Delta u + div\left(\nabla u|\nabla u|^{p-2}|u|^p\right) - u|u|^{p-2}|\nabla u|^p + \alpha u_t + u|u|^{m-2}\right)udx\\ &\leqslant \epsilon\left(-\|\nabla u\|_2^2 - 2\|u\nabla u\|_p^p + \|u\|_m^m + \alpha\left(\frac{1}{4\delta}\|u_t\|_2^2 + \delta\|u\|_2^2\right)\right)\\ &\leqslant \epsilon\left(-\|\nabla u\|_2^2 - 2\|u\nabla u\|_p^p + \|u\|_m^m + \alpha\left(\frac{1}{4\delta}\|u_t\|_2^2 + \delta C_1\|\nabla u\|_2^2\right)\right),\end{aligned} \tag{3.72}$$

$$\begin{aligned}\epsilon\int_\Omega v_{tt}vdx &= \epsilon\int_\Omega\left(\Delta v + div\left(\nabla v|\nabla v|^{p-2}|v|^p\right) - v|v|^{p-2}|\nabla v|^p + \beta v_t + v|v|^{m-2}\right)vdx\\ &\leqslant \epsilon\left(-\|\nabla v\|_2^2 - 2\|u\nabla v\|_p^p + \|v\|_m^m + \beta\left(\frac{1}{4\delta}\|v_t\|_2^2 + \delta\|v\|_2^2\right)\right)\\ &\leqslant \epsilon\left(-\|\nabla v\|_2^2 - 2\|v\nabla v\|_p^p + \|v\|_m^m + \beta\left(\frac{1}{4\delta}\|v_t\|_2^2 + \delta C_2\|\nabla v\|_2^2\right)\right),\end{aligned} \tag{3.73}$$

so the relation (3.62) becomes as follows

$$\begin{aligned}\frac{d}{dt}L(t) &\leqslant \frac{d}{dt}E(t) + \epsilon\|u_t\|_2^2 + \epsilon\|v_t\|_2^2 - \epsilon\|\nabla u\|_2^2 - \epsilon\|\nabla v\|_2^2 - 2\epsilon\|u\nabla u\|_p^p - 2\epsilon\|v\nabla v\|_p^p + \epsilon\|u\|_m^m + \epsilon\|v\|_m^m \\ &+ \epsilon\alpha\frac{1}{4\delta}\|u_t\|_2^2 + \epsilon\beta\frac{1}{4\delta}\|v_t\|_2^2 + \epsilon C_1\delta\|\nabla u\|_2^2 + \epsilon\beta\alpha C_2\delta\|\nabla v\|_2^2 \\ &\leqslant \alpha\|u_t\|_2^2 + \beta\|v_t\|_2^2 + \epsilon\|u_t\|_2^2 + \epsilon\|v_t\|_2^2 - \epsilon\|\nabla u\|_2^2 - \epsilon\|\nabla v\|_2^2 - 2\epsilon\|u\nabla u\|_p^p - 2\epsilon\|v\nabla v\|_p^p + \epsilon\|u\|_m^m + \epsilon\|v\|_m^m \\ &+ \epsilon\alpha\frac{1}{4\delta}\|u_t\|_2^2 + \epsilon\beta\frac{1}{4\delta}\|v_t\|_2^2 + \epsilon\alpha C_1\delta\|\nabla u\|_2^2 + \epsilon\beta C_2\delta\|\nabla v\|_2^2 \\ &\leqslant \left(\alpha + \epsilon + \epsilon\alpha\frac{1}{4\delta}\right)\|u_t\|_2^2 + \left(\beta + \epsilon + \epsilon\beta\frac{1}{4\delta}\right)\|v_t\|_2^2 + (\epsilon\alpha C_1\delta - \epsilon)\|\nabla u\|_2^2 + (\epsilon\beta C_2\delta - \epsilon)\|\nabla v\|_2^2 \\ &- 2\epsilon\|u\nabla u\|_p^p - 2\epsilon\|v\nabla v\|_p^p + \epsilon\|u\|_m^m + \epsilon\|v\|_m^m - \|u\nabla u\|_p^p - \|v\nabla v\|_p^p - \|u\|_m^m - \|v\|_m^m \\ &+ \|u\nabla u\|_p^p + \|v\nabla v\|_p^p + \|u\|_m^m + \|v\|_m^m. \end{aligned} \tag{3.74}$$

So

$$\frac{d}{dt}L(t) \leqslant \chi E(t) + \omega. \tag{3.75}$$

We chose a tiny enough ϵ such that

$$\omega = \|u\nabla u\|_p^p + \|v\nabla v\|_p^p + \|u\|_m^m + \|v\|_m^m,$$

$$\chi = Min\left(\alpha + \epsilon + \epsilon\alpha\frac{1}{4\delta}, \beta + \epsilon + \epsilon\beta\frac{1}{4\delta}, \epsilon\alpha C_1\delta - \epsilon, \epsilon\beta\delta C_2 - \epsilon, -2\epsilon, \epsilon\right) < 0.$$

As a result of utilizing (3.55), we get

$$\frac{d}{dt}L(t) \leqslant \frac{\chi}{N}L(t) + \omega. \tag{3.76}$$

We get the following estimate for the function L by integrating the preceding differential inequality (3.76) between 0 and t.

$$L(t) \leqslant C_3 e^{\frac{\chi}{M}t} - \frac{\omega M}{\chi}, \quad t \geqslant 0. \tag{3.77}$$

As a result of utilizing (3.55), we have

$$E(t) \leqslant k e^{\frac{\chi}{M}t} - \frac{\omega M}{\chi N}, \quad t \geqslant 0. \tag{3.78}$$

The proof is complete $\diamond$

Chapter 4

Existence and Uniqueness of solution for a fractional system of Klein-Gordon

In this chapter, We investigated the existence and uniqueness of solutions to a fractional system of Klein-Gordon equations with non-linear term in this research.

4.1 Introduction

We consider a non-linear hyperbolic system of fractional Klein-Gordon equations, of one-dimension defined as the following

$$\begin{cases} u_{tt} - D^{\alpha}u - \beta_1 u_t - (|v|^2 u_x)_x + u|v_x|^2 = 0 \quad \text{in} \quad I \times (0,T), \\ v_{tt} - D^{\alpha}v - \beta_2 v_t - (|u|^2 v_x)_x + v|u_x|^2 = 0 \quad \text{in} \quad I \times (0,T), \end{cases} \tag{4.1}$$

with boundary conditions

$$u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \Gamma \times (0,T), \tag{4.2}$$

$$u_t(x,t) = v_t(x,t) = 0 \qquad \text{on} \quad \Gamma \times (0,T), \tag{4.3}$$

and Initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \qquad \text{on} \quad I, \tag{4.4}$$

$$u_t(x,0) = u_1(x), \quad v_t(x,0) = v_1(x) \qquad \text{on} \quad I. \tag{4.5}$$

Where I is a domain of $\mathbb{R}$ and let $T > 0$, $0 < \alpha \leqslant 4$, β_1 and β_2 are non-positive constants, and $D^{\alpha} \equiv (-\partial^2/\partial x^2)^{\frac{\alpha}{2}}$.
Define (D^{α}) as ([27]) and ([17]) by

$$(D^{\alpha}v)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{-\alpha-1} v(z)dz,$$

where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$ denotes Euler's Gamma function.
A group of authors have studied the non-linear fractional Klein-Gordon equation and we mention among them
Gan, C. L., Xiao, T. and Zhang [16], considered the non-linear fractional Klein-Gordon maxwell system

$$\begin{cases} (-\Delta)^p u + V(x)u - (2\omega + \phi)\phi u = K(x)f(u), & \text{in} \quad \mathbb{R}^3, \\ (\Delta)^p \phi = (\omega + \phi)u^2 & \text{in} \quad \mathbb{R}^3, \quad \text{where} \quad p \in (3/4.1) \end{cases} \tag{4.6}$$

$(-\Delta)^p$ stands for the fractional Laplacian, $\omega > 0$ is a constant, V is vanishing potential and K is smooth function. The existence of positive solution is proved by mountain pass theorem.
M. Chatzakou, M. Ruzhansky & N. Tokmagambetov [11], considered the non-linear fractional Klein-Gordon equation

$$\begin{cases} u_{tt}(t,x) + R^s u(t,x) + m(x)u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{G}, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \mathbb{G} \end{cases} \tag{4.7}$$

where m is a non-negative and possibly singular function/distribution, $\mathbb{G} = \mathbb{R}^d$ to be the Euclidean space, and $R = -\Delta$ to be the Laplacian on $\mathbb{R}^d$. They proved that it has a very weak solution, and the consistency with the classical solution are also proved.
A. Altybay, M.Ruzhansky, M. SEBIH & N. Tokmagambetov [3], considered the non-linear fractional Klein-Gordon equation

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\alpha u(t,x) + m(x)u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \mathbb{R} \end{cases} \tag{4.8}$$

The function m is supposed to be non-negative and singularand $\alpha > 0$, $d \in \mathbb{N}$. They proved that the solution is very weak well-posed. The uniqueness is proved in some appropriate sense. Moreover, they proved the consistency of the very weak solution with classical solutions when they exist.
Our goal in this article is to prove that the problem (4.1)-(4.5) has a weak and unique solution by using the Glarkin method, by following tha smae method as in articles [17], [5].

4.2 Main result

The energy associated with our system is given by

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + \|uv_x\|_2^2 + \|vu_x\|_2^2 + \|D^{\frac{\alpha}{2}}u\|_2^2 + \|D^{\frac{\alpha}{2}}v\|_2^2, \tag{4.9}$$

we look for weak solutions of problem (1) with initial data $u(x,0) = u_0$ and $v(x,0) = v_0$ in V such that

$$V = L^\infty\left(]0,T[, L^2(\Omega)\right) \cap L^2\left(]0,T[, H_0^1(\Omega)\right). \tag{4.10}$$

Theorem 4.2.1. *[5] Let $\frac{5}{2} < \alpha \leqslant 4$, $T > 0$, and $u_0(x), v_0(x) \in H_0^1(I)$. Then the problem* 4.1-4.5 *has an unique weak solution $(u,v) \in V$. Moreover (u,v) satisfies the following regularity properties*

$$u, v \in L^\infty(0,T; H_0^1(I)) \cap L^\infty(0,T; H^{\frac{\alpha}{2}}(I)), \tag{4.11}$$
$$u_t, v_t \in L^\infty(0,T; L^2(I)). \tag{4.12}$$

4.3 Global Existence

To simplify the writing, we will put

$$u(x,t) = u, \quad v(x,t) = v.$$

Suppose (u,v) is a weak solution of (1). We consider the following approximate problem

$$\int_\Omega \{u_{tt}^m - \beta_1 u_t^m + D^\alpha u^m\} \phi dx + \int_\Omega |v^m|^2 u_x^m \phi_x dx + \int_\Omega u^m |v_x^m| \phi dx = 0, \tag{4.13}$$
$$\int_\Omega \{v_{tt}^m - \beta_2 v_t^m + D^\alpha v^m\} \phi dx + \int_\Omega |u^m|^2 v_x^m \phi_x dx + \int_\Omega v^m |u_x^m| \phi dx = 0, \tag{4.14}$$

for $t \in]0,T[$ and $\phi \in H^1(I,]0,T[)$, let us put

$$2E^m(t) = \|u_t^m\|_2^2 + \|v_t^m\|_2^2 + \|u^m v_x^m\|_2^2 + \|v^m u_x^m\|_2^2 + \|D^\alpha u^m\|_2^2 + \|D^\alpha v^m\|_2^2.$$

Then multiplying (4.13) by u_t^m, (4.14) by v_t^m, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [u_t^m]^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega [D^{\frac{\alpha}{2}} u^m]^2 dx - \beta_1 \int_\Omega [u_t^m]^2 dx + \int_\Omega |v^m|^2 u_x^m (u_t^m)_x dx + \int_\Omega u^m |v_x^m| u_t^m dx = 0, \tag{4.15}$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega [v_t^m]^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega [D^{\frac{\alpha}{2}} v^m]^2 dx - \beta_2 \int_\Omega [v_t^m]^2 dx + \int_\Omega |u^m|^2 v_x^m (v_t^m)_x dx + \int_\Omega v^m |u_x^m| v_t^m dx = 0, \tag{4.16}$$

by adding the two last equalities we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t^m\|_2^2 + \|v_t^m\|_2^2 + \|D^{\frac{\alpha}{2}} u^m\|_2^2 + \|D^{\frac{\alpha}{2}} v^m\|_2^2 + \|uv_x\|_2^2 + \|vu_x\|_2^2\right) = \beta_1 \|u_t\|_2^2 + \beta_2 \|v_t\|_2^2, \tag{4.17}$$

so

$$\frac{d}{dt}E(t) \leqslant 0, \tag{4.18}$$

then

$$E(t) \leqslant E(0), \tag{4.19}$$

so

$$\|u_t^m\|_2^2 + \|v_t^m\|_2^2 + \|D^{\frac{\alpha}{2}} u^m\|_2^2 + \|D^{\frac{\alpha}{2}} v^m\|_2^2 + \|uv_x\|_2^2 + \|vu_x\|_2^2 \leqslant 2E(0), \tag{4.20}$$

from (4.20), that a solution is a bounded.

$$\begin{cases} u^m, v^m & \text{is bounded in} \quad L^\infty(0,T; H_0^1(I)), \\ u^m, v^m & \text{is bounded in} \quad L^\infty(0,T; H^{\frac{\alpha}{2}}(I)), \\ u^m, v^m & \text{is bounded in} \quad L^2(0,T; L^2(I)), \\ u_t^m, v_t^m & \text{is bounded in} \quad L^\infty(0,T; L^2(I)). \end{cases} \tag{4.21}$$

Then it is sufficient in order to apply approximation Galerkin's procedure. Hence, we can extract a sub-sequence of (u^m) and a subsequence of (v^m), denoted by same symbols, such that

$$\begin{cases} u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T; H_0^1(I)), \\ u^m \to u \quad \text{and} \quad v^m \to v \quad \text{weak star in} \quad L^\infty(0,T; L^2(I)), \\ u_t^m \to u_t \quad \text{and} \quad v_t^m \to v_t \quad \text{weak star in} \quad L^\infty(0,T; L^2(I)). \end{cases} \tag{4.22}$$

Now we prove that the non-linears terms are also bounded.
By the Holder inequality, the embedding $H_0^1(I) \hookrightarrow L^6(I)$ and (4.21), we obtain

$$\|u^m |v_x^m|^2\|_2^2 \leqslant \|u^m\|_{L^6(I)}^2 \|v_x^m\|_{L^6(I)}^4 \leqslant C_1, \tag{4.23}$$

$$\||v^m|^2 u_x^m\|_2^2 \leqslant \|v^m\|_{L^6(I)}^4 \|u_x^m\|_{L^6(I)}^2 \leqslant C_2 \quad \forall (u^m, v^m) \quad \text{in} \quad H^2(I). \tag{4.24}$$

Therefore

$$(u^m |v_x^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T; L^2(I)), \tag{4.25}$$

$$(|v^m|^2 u_x^m) \quad \text{is bounded in} \quad L^\infty(0,T; L^2(I)). \tag{4.26}$$

Analogously

$$(v^m |u_x^m|^2) \quad \text{is bounded in} \quad L^\infty(0,T; L^2(I)), \tag{4.27}$$

$$(|u^m|^2 v_x^m) \quad \text{is bounded in} \quad L^\infty(0,T; L^2(I)). \tag{4.28}$$

From (4.25), (4.26), (4.27) and (4.28), we get

$$\begin{cases} u^m |v_x^m|^2 \to \chi_1 & \text{weak star in} \quad L^\infty(0,T; L^2(I)), \\ v^m |u_x^m|^2 \to \chi_2 & \text{weak star in} \quad L^\infty(0,T; L^2(I)), \\ |v^m|^2 u_x^m \to \chi_3 & \text{weak star in } L^\infty(0,T; L^2(I)), \\ |u^m|^2 v_x^m \to \chi_4 & \text{weak star in } L^\infty(0,T; L^2(I)). \end{cases} \tag{4.29}$$

From (4.22) and Aubin-Lions compactness Lemma in ([9]), we obtain

$$u^m \to u, \quad v^m \to v \quad \text{strongly in} \quad L^\infty(0,T; L^2(I)), \tag{4.30}$$

since u_x^m and v_x^m are bounded, then we have

$$\begin{cases} u^m|v_x^m|^2 \to u|v_x|^2 & \text{strongly in} \quad L^2(0,T;L^2(I)), \\ v^m|u_x^m|^2 \to v|u_x|^2 & \text{strongly in} \quad L^2(0,T;L^2(I)), \\ |u^m|^2 v_x^m \to |u|^2 v_x & \text{strongly in} \quad L^2(0,T;L^2(I)), \\ |v^m|^2 u_x^m \to |v|^2 u_x & \text{strongly in} \quad L^2(0,T;L^2(I)). \end{cases} \tag{4.31}$$

Then, there exists a subsequences of u^m and v^m, which we will denote by u^m, v^m respectively, such that

$$\begin{cases} u^m|v_x^m|^2 \to u|v_x|^2 & \text{almost everywhere in} \quad (0,T)\times I, \\ v^m|u_x^m|^2 \to v|u_x|^2 & \text{almost everywhere in} \quad (0,T)\times I, \\ |u^m|^2 v_x^m \to |u|^2 v_x & \text{almost everywhere in} \quad (0,T)\times I, \\ |v^m|^2 u_x^m \to |v|^2 u_x & \text{almost everywhere in} \quad (0,T)\times I. \end{cases} \tag{4.32}$$

From Lemma (1.0.5) in ([32]) and (4.32) we deduce

$$\begin{cases} u^m|v_x^m|^2 \to u|v_x|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(I)), \\ v^m|u_x^m|^2 \to v|u_x|^2 & \text{weakly in} \quad L^\infty(0,T;L^2(I)), \\ |u^m|^2 v_x^m \to |u|^2 v_x & \text{weakly in} \quad L^\infty(0,T;L^2(I)), \\ |v^m|^2 u_x^m \to |v|^2 u_x & \text{weakly in} \quad L^\infty(0,T;L^2(I)). \end{cases} \tag{4.33}$$

By the last formula (4.33) and (4.29) we get

$$\begin{aligned} \chi_1 &= u|v_x|^2, \\ \chi_2 &= v|u_x|^2, \\ \chi_3 &= |v|^2 u_x, \\ \chi_4 &= |u|^2 v_x. \end{aligned} \tag{4.34}$$

Taking $\phi = 1$ in (4.13) become

$$(u_{tt}^m, 1) - \beta_1(u_t^m, 1) + (D^\alpha u^m, 1) + (u^m|v_x^m|^2, 1) = 0, \tag{4.35}$$

$$|(u_{tt}^m, 1)| = \left|\beta_1(u_t^m, 1) - (u|v_x^m|^2, 1) - (D^\alpha u^m, 1)\right|.$$

Using the Cauchy Schwartz inequality in [7] , we have

$$\|u_{tt}^m\|_{L^1(I)} \leqslant |\beta_1|\|u_t^m\|_2 m^{\frac{1}{2}}(I) + \|u^m|v_x^m|^2\|_2 m^{\frac{1}{2}}(I) + \|D^\alpha u^m\|_{L^1(I)},$$

such that, $m(I)$ is a measure of I.
Since, the measure of I is finite, and from (4.25),(4.26), (4.27), (4.28), (4.33) and (4.22), we obtain

$$\|u_{tt}^m\|_{L^1(I)} \leqslant C_1. \tag{4.36}$$

Analogously

$$\|v_{tt}^m\|_{L^1(I)} \leqslant C_2. \tag{4.37}$$

Then

$$\begin{cases} u_{tt}^m & \text{is bounded in} \quad L^\infty(0,T;L^1(I)), \\ v_{tt}^m & \text{is bounded in} \quad L^\infty(0,T;L^1(I)). \end{cases} \tag{4.38}$$

Similarly we have

$$\begin{cases} u_{tt}^m \to u_{tt} & \text{weakly star in} \quad L^\infty(0,T;L^1(I)), \\ v_{tt}^m \to v_{tt} & \text{weakly star in} \quad L^\infty(0,T;L^1(I)). \end{cases} \tag{4.39}$$

Now we prove that $D^\alpha u^m$ converge strongly to $D^\alpha u$ in $L^1(]0,T[;L^2(I))$. In the same way (see ([17]), we get

$$\begin{aligned} &\|D^\alpha u^m - D^\alpha u\|_{L^1(]0,T[;L^2(I))} \leqslant \\ &\|\frac{\partial^2 u^m}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))} \leqslant \|\frac{\partial^2 u^m}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))} + \|\frac{\partial^2 u}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))}, \end{aligned} \tag{4.40}$$

analougously

$$\begin{aligned} &\|D^\alpha v^m - D^\alpha u\|_{L^1(]0,T[;L^2(I))} \leqslant \\ &\|\frac{\partial^2 v^m}{\partial x^2} - \frac{\partial^2 v}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))} \leqslant \|\frac{\partial^2 v^m}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))} + \|\frac{\partial^2 v}{\partial x^2}\|_{L^1(]0,T[;H^{\frac{\alpha}{2}}(I))}, \end{aligned} \tag{4.41}$$

and since the term $\frac{\partial^2}{\partial x^2}$ is linear, approach problem converges weakly to a limit point, then the existence holds.
The proof is complete.

4.4 Uniqueness

Now we will prove the uniqueness of the solution. Let (u,v) and (u_1,v_1) two solutions of (4.1), we assume that $U = u - u_1$ and $V = v - v_1$ satisfy

$$U_{tt} - D^\alpha U - \beta_1 U_t - (|v|^2 u_x - |v_1|^2 u_{1x})_x + (u|v_x|^2 - u_1|v_{1x}|^2) = 0 \quad \text{in} \quad I \times (0,T), \tag{4.42}$$

$$V_{tt} - D^\alpha V - \beta_2 V_t - (|u|^2 v - |u_1|^2 v_{1x})_x + (v|u_x|^2 - v_1|u_{1x}|^2) = 0 \quad \text{in} \quad I \times (0,T), \tag{4.43}$$

with

$$U(0) = V(0) = 0 \quad U_t(0) = V_t(0) = 0. \tag{4.44}$$

Multiplying (4.42) by $U(t)$ and (4.43) by $V(t)$ and integrating over I, we get

$$\begin{aligned} &\int \big(U_{tt}U - D^\alpha UU - \beta_1 U_t U - (|v|^2 u_x - |v_1|^2 u_{1x})_x U\big)\, dx \\ &+ \int \big((u|v_x|^2 - u_1|v_{1x}|^2)U\big)\, dx = 0 \quad \text{in} \quad I \times (0,T), \end{aligned} \tag{4.45}$$

$$\begin{aligned} &\int \big(V_{tt}V - D^\alpha VV - \beta_2 V_t V - (|u|^2 v_x - |u_1|^2 v_{1x})_x V\big)\, dx \\ &+ \int \big((v|u_x|^2 - v_1|u_{1x}|^2)V\big)\, dx = 0 \quad \text{in} \quad I \times (0,T), \end{aligned} \tag{4.46}$$

so

$$\int U_{tt}U dx - |D^{\frac{\alpha}{2}}U|_2^2 - \frac{1}{2}\beta_1\frac{d}{dt}|U|_2^2 + \int(|v|^2u_x - |v_1|^2u_{1x})U_x dx$$
$$+ \int(u|v_x|^2 - u_1|v_{1x}|^2)U dx = 0 \quad \text{in} \quad I \times (0,T), \tag{4.47}$$
$$\int V_{tt}V dx - |D^{\frac{\alpha}{2}}V|_2^2 - \frac{1}{2}\beta_2\frac{d}{dt}|V|_2^2 + \int(|u|^2v_x - |u_1|^2v_{1x})V_x dx$$
$$+ \int(v|u_x|^2 - v_1|u_{1x}|^2)V dx = 0 \quad \text{in} \quad I \times (0,T), \tag{4.48}$$

so

$$\|D^{\frac{\alpha}{2}}U\|_2^2 - \frac{1}{2}\beta_1\frac{d}{dt}\|U\|_2^2 = \int U_{tt}U dx + \int(|v|^2u_x - |v_1|^2u_{1x})U_x dx$$
$$+ \int(u|v_x|^2 - u_1|v_{1x}|^2)U dx \quad \text{in} \quad I \times (0,T), \tag{4.49}$$
$$\|D^{\frac{\alpha}{2}}V\|_2^2 - \frac{1}{2}\beta_2\frac{d}{dt}\|V\|_2^2 = \int V_{tt}V dx + \int(|u|^2v_x - |u_1|^2v_{1x})V_x dx$$
$$+ \int(v|u_x|^2 - v_1|u_{1x}|^2)V dx \quad \text{in} \quad I \times (0,T). \tag{4.50}$$

We know that

$$\begin{aligned} u|v_x|^2 - u_1|v_{1x}|^2 &= u(|v_x|^2 - |v_{1x}|^2) + u|v_{1x}|^2 - u_1|v_{1x}|^2 \\ &= u(|v_x|^2 - |v_{1x}|^2) + |v_{1x}|^2(u - u_1) \\ &= u(|v_x|^2 - |v_{1x}|^2) + |v_{1x}|^2U, \end{aligned} \tag{4.51}$$

in the same way we find that

$$\begin{aligned} v|u_x|^2 - v_1|u_{1x}|^2 &= v(|u_x|^2 - |u_{1x}|^2) + v|u_{1x}|^2 - v_1|u_{1x}|^2 \\ &= v(|u_x|^2 - |u_{1x}|^2) + |u_{1x}|^2(v - v_1) \\ &= v(|u_x|^2 - |u_{1x}|^2) + |u_{1x}|^2V, \end{aligned} \tag{4.52}$$

so, we have

$$\|D^{\frac{\alpha}{2}}U\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|U\|_2^2 = \int U_{tt}U dx + \int(|v|^2u_x - |v_1|^2u_{1x})U_x dx + \int u(|v_x|^2 - |v_{1x}|^2)U dx$$
$$+ \int |v_{1x}|^2U^2 dx \quad \text{in} \quad I \times (0,T), \tag{4.53}$$
$$\|D^{\frac{\alpha}{2}}V\|_2^2 + \frac{1}{2}\beta_2\frac{d}{dt}\|V\|_2^2 = \int V_{tt}V dx + \int(|u|^2v_x - |u_1|^2v_{1x})V_x dx + \int v(|u_x|^2 - |u_{1x}|^2)V dx$$
$$+ \int |u_{1x}|^2V^2 dx \quad \text{in} \quad I \times (0,T), \tag{4.54}$$

by the Cauchy-Schwartz inequality (in [7]) and $H^1(I) \hookrightarrow C(I)$, we get

$$\|D^{\frac{\alpha}{2}}U\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|U\|_2^2 \leqslant \max\{|U|\}\int |U_{tt}|dx + \||v|^2u_x\|_2\|U\|_2 - \||v_1|^2u_{1x}\|_2\|U_x\|_2 + \|u|v_x|^2\|_2\|U\|_2 - \|u|v_{1x}|^2\|_2\|U\|_2 + \int |v_{1x}|^2U^2dx \quad \text{in} \quad I\times(0,T), \tag{4.55}$$

$$\|D^{\frac{\alpha}{2}}V\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|V\|_2^2 \leqslant \max\{|V|\}\int |V_{tt}|dx + \||u|^2v_x\|_2\|V\|_2 - \||u_1|^2v_{1x}\|_2\|V_x\|_2 + \|v|u_x|^2\|_2\|V\|_2 - \|v|u_{1x}|^2\|_2\|V\|_2 + \int |u_{1x}|^2V^2dx \quad \text{in} \quad I\times(0,T), \tag{4.56}$$

so from (4.39), we get

$$\|D^{\frac{\alpha}{2}}U\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|U\|_2^2 \leqslant c_1 + \|v_{1x}U\|_2^2dx \quad \text{in} \quad I\times(0,T), \tag{4.57}$$

$$\|D^{\frac{\alpha}{2}}V\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|V\|_2^2 \leqslant c_2 + \|u_{1x}V\|_2^2 \quad \text{in} \quad I\times(0,T), \tag{4.58}$$

then, we find

$$\|D^{\frac{\alpha}{2}}U\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|U\|_2^2 \leqslant c_1 + c_3\|U\|_2^2dx \quad \text{in} \quad I\times(0,T), \tag{4.59}$$

$$\|D^{\frac{\alpha}{2}}V\|_2^2 + \frac{1}{2}\beta_1\frac{d}{dt}\|V\|_2^2 \leqslant c_2 + c_4\|V\|_2^2 \quad \text{in} \quad I\times(0,T), \tag{4.60}$$

by the Gronowall's Lemma in [6] we get

$$U = V = 0 \qquad \text{on} \quad (0,T).$$

Chapter 5

Approach numeric by finite volume method for fractional klein-Gordon system in one-dimension

In this chapter, we will look for the approximate solution to the one-dimensional fractional Klein-Gordon equation by adopting the finite volume method, then we show that the apprixi-mate solution is stable.

5.1 Introduction

In this paper, we consider a non-linear hyperbolic system of fractional Klein-Gordon equations, of one-dimension defined as the following

$$\begin{cases} u_{tt} - D^\alpha u - \beta_1 u_t - (|v|^2 u_x)_x + u|v_x|^2 = 0 \quad \text{in} \quad (0,1)\times(0,T), \\ v_{tt} - D^\alpha v - \beta_2 v_t - (|u|^2 v_x)_x + v|u_x|^2 = 0 \quad \text{in} \quad (0,1)\times(0,T), \\ u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \Gamma\times(0,T), \\ u_t(x,t) = v_t(x,t) = 0 \qquad \text{on} \quad \Gamma\times(0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \qquad \text{on} \quad (0,1), \\ u_t(x,0) = u_1(x), \quad v_t(x,0) = v_1(x) \qquad \text{on} \quad (0,1). \end{cases} \tag{5.1}$$

Where $(0,1)$, is a domain of $\mathbb{R}$ and let $T > 0$, $0 < \alpha \leqslant 4$, β_1 and β_2 are non-positive constants, and $D^\alpha \equiv (-\partial^2/\partial x^2)^{\frac{\alpha}{2}}$.
Define D^α as [20] and [22]:

$$(D^\alpha v)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{\alpha-1} v(z)dz,$$

where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$ denotes Euler's Gamma function.

Remark 5.1.1. ***Trapez formula***

$$\int_a^b f \simeq (b-a).\frac{f(a)+f(b)}{2}. \tag{5.2}$$

5.2 Discretization by finite volumes

Definition 5.2.1. *[14] (**admessible mesh**) Anadmessible mesh of* $(0,1)$*, denoted by* τ*, is given by family* $(K_i)_{i=1,..,N}$*,* $N \in \mathbb{N}^*$*, such that* $K_i = (x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}})$*, and a family* $(x_i)_{i=0,..,N+1}$ *such that*

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < ... < x_{i-\frac{1}{2}} < ... < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1.$$

One sets

$$\begin{aligned}
&h_i = m(K_i) = x_{i+\frac{1}{2}} - x_{1-\frac{1}{2}}, i = 1,..,N, \textit{and therefore} \sum_{i=1}^{N} h_i = 1,\\
&h_i^- = x_i - x_{i-\frac{1}{2}}, i = 1,..,N,\\
&h_i^+ = x_{i+\frac{1}{2}} - x_i, i = 1,..,N,\\
&h_{i+\frac{1}{2}} = x_{i+1} - x_i, i = 1,..,N,\\
&h_{i-\frac{1}{2}} = x_i - x_{i-1}, i = 1,..,N,\\
&size(\tau) = h = max h_i, i = 1,..,N.
\end{aligned} \tag{5.3}$$

A weak solution $(u(x,t), v(x,t))$ is defined on $(0,T) \times (0,1)$, we introduce a mean value $(u_t(t), v_t(t))$ and $(u_{tt}(t), v_{tt}(t)$ of a solution, that one assumes exists in the following meaning.

$$u_i(t) = \frac{1}{h_i}\int_{K_i} u(x,t)dx, i \in N, \qquad v_i(t) = \frac{1}{h_i}\int_{K_i} v(x,t)dx, i \in N, \tag{5.4}$$

$$\frac{\partial u_i}{\partial t}(t) = \frac{1}{h_i}\int_{K_i} \frac{\partial u}{\partial t}(x,t)dx, i \in N, \qquad \frac{\partial v_i}{\partial t}(t) = \frac{1}{h_i}\int_{K_i} \frac{\partial v}{\partial t}(x,t)dx, i \in N, \tag{5.5}$$

$$\frac{\partial^2 u_i}{\partial t^2}(t) = \frac{1}{h_i}\int_{K_i} \frac{\partial^2 u}{\partial t^2}(x,t)dx, i \in N, \qquad \frac{\partial^2 v_i}{\partial t^2}(t) = \frac{1}{h_i}\int_{K_i} \frac{\partial^2 v}{\partial t^2}(x,t)dx, i \in N. \tag{5.6}$$

Now we go to approach $(D^\alpha u(x), D^\alpha v(x))$ on K_i (namely $x \in K_i$) by quadrature formula. Anew quadrature formula has been proposed which use weight functions. This formula has the form given below

$$(D^\alpha u)(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-z)^{\alpha-1} u(z)dz = \frac{1}{\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}(x-sh)^{\alpha-1}u(sh),$$

and

$$(D^\alpha v)(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-z)^{\alpha-1} v(z)dz = \frac{1}{\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}(x-sh)^{\alpha-1}v(sh),$$

where sh are nodes of a quadrature formula and $w_{s,i}$ are weight functions with $\sum_{s=0}^{i} w_{s,i} = 1$ Itegrate equation (5.1) on K_i in order to obtain

$$\int_{K_i} u_{tt}dx - \int_{K_i} D^{\alpha}udx - \int_{K_i} \beta_1 u_t dx - \int_{K_i} (|v|^2 u_x)_x dx + \int_{K_i} u|v_x|^2 dx = 0, \tag{5.7}$$

$$\int_{K_i} v_{tt}dx - \int_{K_i} D^{\alpha}vdx - \int_{K_i} \beta_2 v_t dx - \int_{K_i} (|u|^2 v_x)_x dx + \int_{K_i} v|u_x|^2 dx = 0, \tag{5.8}$$

we use the Trapez formula, and some calculations, we find

$$\begin{aligned} &h_i \frac{\partial^2 u_i}{\partial t^2}(t) - \frac{1}{\Gamma(-\alpha)} \int_{K_i} \int_0^x (x-z)^{\alpha-1} u(z)dzdx - \beta_1 h_i \frac{\partial u_i}{\partial t}(t) - |v(x_{i+\frac{1}{2}})|^2 u_x(x_{i+\frac{1}{2}}) \\ &+ |v(x_{i-\frac{1}{2}})|^2 u_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(u(x_{i+\frac{1}{2}})|v_x(x_{i+\frac{1}{2}})|^2 + u(x_{i-\frac{1}{2}})|v_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.9}$$

$$\begin{aligned} &h_i \frac{\partial^2 v_i}{\partial t^2}(t) - \frac{1}{\Gamma(-\alpha)} \int_{K_i} \int_0^x (x-z)^{-\alpha-1} v(z)dzdx - \beta_2 h_i \frac{\partial v_i}{\partial t}(t) - |u(x_{i+\frac{1}{2}})|^2 v_x(x_{i+\frac{1}{2}}) \\ &+ |u(x_{i-\frac{1}{2}})|^2 v_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(v(x_{i+\frac{1}{2}})|u_x(x_{i+\frac{1}{2}})|^2 + v(x_{i-\frac{1}{2}})|u_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.10}$$

namely,

$$\begin{aligned} &h_i \frac{\partial^2 u_i}{\partial t^2}(t) - \frac{1}{\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \int_{K_i} (x-x_s)^{\alpha-1} dx - \beta_1 h_i \frac{\partial u_i}{\partial t}(t) - |v(x_{i+\frac{1}{2}})|^2 u_x(x_{i+\frac{1}{2}}) \\ &+ |v(x_{i-\frac{1}{2}})|^2 u_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(u(x_{i+\frac{1}{2}})|v_x(x_{i+\frac{1}{2}})|^2 + u(x_{i-\frac{1}{2}})|v_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.11}$$

$$\begin{aligned} &h_i \frac{\partial^2 v_i}{\partial t^2}(t) - \frac{1}{\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} v(x_s,t) \int_{K_i} (x-x_s)^{-\alpha-1} dx - \beta_2 h_i \frac{\partial v_i}{\partial t}(t) - |u(x_{i+\frac{1}{2}})|^2 v_x(x_{i+\frac{1}{2}}) \\ &+ |u(x_{i-\frac{1}{2}})|^2 v_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(v(x_{i+\frac{1}{2}})|u_x(x_{i+\frac{1}{2}})|^2 + v(x_{i-\frac{1}{2}})|u_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.12}$$

implies

$$\begin{aligned} &h_i \frac{\partial^2 u_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^{\alpha} - (x_{i-\frac{1}{2}} - x_s)^{\alpha} \right) - \beta_1 h_i \frac{\partial u_i}{\partial t}(t) \\ &- |v(x_{i+\frac{1}{2}})|^2 u_x(x_{i+\frac{1}{2}}) + |v(x_{i-\frac{1}{2}})|^2 u_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(u(x_{i+\frac{1}{2}})|v_x(x_{i+\frac{1}{2}})|^2 + u(x_{i-\frac{1}{2}})|v_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.13}$$

$$\begin{aligned} &h_i \frac{\partial^2 v_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} v(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^{\alpha} - (x_{i-\frac{1}{2}} - x_s)^{\alpha} \right) - \beta_2 h_i \frac{\partial v_i}{\partial t}(t) \\ &- |u(x_{i+\frac{1}{2}})|^2 v_x(x_{i+\frac{1}{2}}) + |u(x_{i-\frac{1}{2}})|^2 v_x(x_{i-\frac{1}{2}}) + \frac{h_i}{2} \left(v(x_{i+\frac{1}{2}})|u_x(x_{i+\frac{1}{2}})|^2 + v(x_{i-\frac{1}{2}})|u_x(x_{i-\frac{1}{2}})|^2 \right), \end{aligned} \tag{5.14}$$

implies

$$
\begin{aligned}
&h_i \frac{\partial^2 u_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) - \beta_1 h_i \frac{\partial u_i}{\partial t}(t) \\
&- \left| \frac{v(x_{i+1}) + v(x_i)}{2} \right|^2 \left(\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \right) + \left| \frac{v(x_i) + v(x_{i-1})}{2} \right|^2 \left(\frac{u(x_i) - u(x_{i-1})}{h_{i-\frac{1}{2}}} \right) \\
&+ \frac{h_i}{2} \left(\left(\frac{u(x_{i+1}) + u(x_i)}{2} \right) \left| \frac{v(x_{i+1}) - v(x_i)}{h_{i+\frac{1}{2}}} \right|^2 + \left(\frac{u(x_i) + u(x_{i-1})}{2} \right) \left| \frac{v(x_i) - v(x_{i-1})}{h_{i-\frac{1}{2}}} \right|^2 \right),
\end{aligned} \tag{5.15}
$$

$$
\begin{aligned}
&h_i \frac{\partial^2 v_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} v(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) - \beta_2 h_i \frac{\partial v_i}{\partial t}(t) \\
&- \left| \frac{u(x_{i+1}) + u(x_i)}{2} \right|^2 \left(\frac{v(x_{i+1}) - v(x_i)}{h_{i+\frac{1}{2}}} \right) + \left| \frac{u(x_i) + u(x_{i-1})}{2} \right|^2 \left(\frac{v(x_i) - v(x_{i-1})}{h_{i-\frac{1}{2}}} \right) \\
&+ \frac{h_i}{2} \left(\left(\frac{v(x_{i+1}) + v(x_i)}{2} \right) \left| \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \right|^2 + \left(\frac{v(x_i) + v(x_{i-1})}{2} \right) \left| \frac{u(x_i) - u(x_{i-1})}{h_{i-\frac{1}{2}}} \right|^2 \right).
\end{aligned} \tag{5.16}
$$

implies

$$
\begin{aligned}
&h_i \frac{\partial^2 u_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) - \beta_1 h_i \frac{\partial u_i}{\partial t}(t) \\
&- \frac{1}{4h_{i+\frac{1}{2}}} \left| v(x_{i+1}) + v(x_i) \right|^2 \left(u(x_{i+1}) - u(x_i) \right) + \frac{1}{4h_{i-\frac{1}{2}}} \left| v(x_i) + v(x_{i-1}) \right|^2 \left(u(x_i) - u(x_{i-1}) \right) \\
&+ \frac{h_i}{4h^2_{i+\frac{1}{2}}} \left(u(x_{i+1}) + u(x_i) \right) \left| v(x_{i+1}) - v(x_i) \right|^2 + \frac{h_i}{4h^2_{i+\frac{1}{2}}} \left(u(x_i) + u(x_{i-1}) \right) \left| v(x_i) - v(x_{i-1}) \right) ,
\end{aligned} \tag{5.17}
$$

$$
\begin{aligned}
&h_i \frac{\partial^2 v_i}{\partial t^2}(t) - \frac{1}{\alpha\Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} v(x_s,t) \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) - \beta_2 h_i \frac{\partial v_i}{\partial t}(t) \\
&- \frac{1}{4h_{i+\frac{1}{2}}} \left| u(x_{i+1}) + u(x_i) \right|^2 \left(v(x_{i+1}) - v(x_i) \right) + \frac{1}{4h_{i-\frac{1}{2}}} \left| u(x_i) + u(x_{i-1}) \right|^2 \left(v(x_i) - v(x_{i-1}) \right) \\
&+ \frac{h_i}{4h^2_{i+\frac{1}{2}}} \left(v(x_{i+1}) + v(x_i) \right) \left| u(x_{i+1}) - u(x_i) \right|^2 + \frac{h_i}{4h^2_{i+\frac{1}{2}}} \left(v(x_i) + v(x_{i-1}) \right) \left| u(x_i) - u(x_{i-1}) \right) .
\end{aligned} \tag{5.18}
$$

As x_i is the midpoint of k_i, one has

$$
|u_i(t) - u(x_i,t)| \leqslant c_1 h^2, \quad |v_i(t) - v(x_i,t)| \leqslant c_2 h^2,
$$

the numerical flux, we will have

$$h_i\frac{\partial^2 u_i}{\partial t^2}(t)-\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}w_{s,i}u(x_s,t)\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)-\beta_1 h_i\frac{\partial u_i}{\partial t}(t)$$
$$-\frac{1}{4h_{i+\frac{1}{2}}}\left|v_{i+1}+v_i\right|^2(u_{i+1}-u_i)+\frac{1}{4h_{i-\frac{1}{2}}}\left|v_i+v_{i-1}\right|^2(u_i-u_{i-1})$$
$$+\frac{h_i}{4h^2_{i+\frac{1}{2}}}(u_{i+1}+u_i)\left|v_{i+1}-v_i\right|^2+\frac{h_i}{4h^2_{i+\frac{1}{2}}}(u_i+u_{i-1})\left|v_i-v_{i-1}\right|^2, \tag{5.19}$$
$$h_i\frac{\partial^2 v_i}{\partial t^2}(t)-\frac{1}{\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}w_{s,i}v(x_s,t)\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)-\beta_2 h_i\frac{\partial v_i}{\partial t}(t)$$
$$-\frac{1}{4h_{i+\frac{1}{2}}}\left|u_{i+1}+u_i\right|^2(v_{i+1}-v_i)+\frac{1}{4h_{i-\frac{1}{2}}}\left|u_i+u_{i-1}\right|^2(v_i-v_{i-1})$$
$$+\frac{h_i}{4h^2_{i+\frac{1}{2}}}(v_{i+1}+v_i)\left|u_{i+1}-u_i\right|^2+\frac{h_i}{4h^2_{i+\frac{1}{2}}}(v_i+v_{i-1})\left|u_i-u_{i-1}\right|^2, \tag{5.20}$$

so

$$\frac{\partial^2 u_i}{\partial t^2}(t)-\beta_1\frac{\partial u_i}{\partial t}(t)=\frac{1}{h_i\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}w_{s,i}u(x_s,t)\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)$$
$$+\frac{1}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}+v_i\right|^2(u_{i+1}-u_i)-\frac{1}{4h_ih_{i-\frac{1}{2}}}\left|v_i-v_{i-1}\right|^2(u_i-u_{i-1})$$
$$-\frac{1}{4h^2_{i+\frac{1}{2}}}(u_{i+1}+u_i)\left|v_{i+1}-v_i\right|^2-\frac{1}{4h^2_{i+\frac{1}{2}}}(u_i+u_{i-1})\left|v_i-v_{i-1}\right|^2, \tag{5.21}$$
$$\frac{\partial^2 v_i}{\partial t^2}(t)-\beta_2\frac{\partial v_i}{\partial t}(t)=\frac{1}{h_i\alpha\Gamma(-\alpha)}\sum_{s=0}^{i}w_{s,i}v(x_s,t)\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)$$
$$+\frac{1}{4h_ih_{i+\frac{1}{2}}}\left|u_{i+1}+u_i\right|^2(v_{i+1}-v_i)-\frac{1}{4h_ih_{i-\frac{1}{2}}}\left|u_i+u_{i-1}\right|^2(v_i-v_{i-1})$$
$$-\frac{1}{4h^2_{i+\frac{1}{2}}}(v_{i+1}+v_i)\left|u_{i+1}-u_i\right|^2-\frac{1}{4h^2_{i+\frac{1}{2}}}(v_i+v_{i-1})\left|u_i-u_{i-1}\right|^2. \tag{5.22}$$

We find an ordinary second-order system defined by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t)-\beta_1\frac{\partial u}{\partial t}(t)=F(t,u,v), \\ \frac{\partial^2 v}{\partial t^2}(t)-\beta_2\frac{\partial v}{\partial t}(t)=G(t,u,v), \\ u(0)=u_0, \\ v(0)=v_0. \end{cases} \tag{5.23}$$

Stability for Explicit Schemes

Theorem 5.2.1. *Let the assumptions* (i) *and* (ii) *holds*
(i) $u^0, v^0 \in L^\infty([0,1])$;
(ii) *a condition C-F-L (Courant- Freidrichs-Lewy)*

$$\Delta t \leqslant \frac{inf_{i\in\mathbb{N}} h_i}{L_{m_1}}.$$

Where L_{m_1} *is a Lipschiz constant.*
Where u_i^{n+1}, v_i^{n+1} *verifies*

$$\begin{aligned}(1-\beta_1\Delta t)u_i^{n+1} &= \frac{(\Delta t)^2}{h_i\alpha\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}u_i^n\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)\\
&+\left(\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2\right)u_{i-1}^n\\
&+\left(-\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2\right)u_i^n\\
&+\left(-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2+2-\beta_1\Delta t\right)u_i^n\\
&+\left(+\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2\right)u_{i+1}^n-u_i^{n-1},\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}(1-\beta_2\Delta t)v_i^{n+1} &= \frac{(\Delta t)^2}{h_i\alpha\Gamma(-\alpha)}\sum_{s=0}^{i} w_{s,i}v_i^n\left((x_{1+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)\\
&+\left(\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|u_i^n-u_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|u_i^n-u_{i-1}^n\right|^2\right)v_{i-1}^n\\
&+\left(-\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|u_{i+1}^n+u_i^n\right|^2-\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|u_i^n-u_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|u_{i+1}^n-u_i^n\right|^2\right)v_i^n\\
&+\left(-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|u_i^n-u_{i-1}^n\right|^2+2-\beta_2\Delta t\right)v_i^n\\
&+\left(+\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|u_{i+1}^n+u_i^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|u_{i+1}^n-u_i^n\right|^2\right)v_{i+1}^n-v_i^{n-1},\end{aligned} \tag{5.25}$$

and

$$u_i^0=\frac{1}{h_i}\int_{K_i}u_o(x)dx,\quad v_i^0=\frac{1}{h_i}\int_{K_i}v_o(x)dx,\quad i\in\mathbb{N}, \tag{5.26}$$

verifies
$A_1 \leqslant u_i^n \leqslant B_1$ *and* $A_2 \leqslant v_i^n \leqslant B_2$ *for all* $n, i \in \mathbb{N}$,
and

$$\|u_i^n\|_\infty \leqslant c_1^{N_{max}} \|u_0\|_\infty \leqslant B_1, \quad \|v_i^n\|_\infty \leqslant c_2^{N_{max}} \|v_0\|_\infty \leqslant B_2. \tag{5.27}$$

Proof. According to assumptions $A_1 \leqslant u_0 \leqslant B_1$ and $A_2 \leqslant v_0 \leqslant B_2$, a.e, and a definition of u_i^0, v_i^0, we see that $\bar{A}_1 \leqslant u_i^0 \leqslant \bar{B}_1$ and $\bar{A}_2 \leqslant v_i^0 \leqslant \bar{B}_2$, for all $i \in \mathbb{N}$.
Let us show that this property is still true in the rank $n+1$. We have

$$\begin{aligned}
&\frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{(\Delta t)^2} - \beta_1 \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h_i \alpha \Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u_i^n \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) \\
&+ \frac{1}{4h_i h_{i+\frac{1}{2}}} \left| v_{i+1}^n + v_i^n \right|^2 \left(u_{i+1}^n - u_i^n \right) - \frac{1}{4h_i h_{i-\frac{1}{2}}} \left| v_i^n - v_{i-1}^n \right|^2 \left(u_i^n - u_{i-1}^n \right) \\
&- \frac{1}{4h_{i+\frac{1}{2}}^2} \left(u_{i+1}^n + u_i^n \right) \left| v_{i+1}^n - v_i^n \right|^2 - \frac{1}{4h_{i+\frac{1}{2}}^2} \left(u_i^n + u_{i-1}^n \right) \left| v_i^n - v_{i-1}^n \right|^2,
\end{aligned} \tag{5.28}$$

so

$$\begin{aligned}
&\frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{(\Delta t)^2} - \beta_1 \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h_i \alpha \Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u_i^n \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) \\
&+ \left(\frac{1}{4h_i h_{i-\frac{1}{2}}} \left| v_i^n - v_{i-1}^n \right|^2 - \frac{1}{4h_{i+\frac{1}{2}}^2} \left| v_i^n - v_{i-1}^n \right|^2 \right) u_{i-1}^n \\
&+ \left(-\frac{1}{4h_i h_{i+\frac{1}{2}}} \left| v_{i+1}^n + v_i^n \right|^2 - \frac{1}{4h_i h_{i-\frac{1}{2}}} \left| v_i^n - v_{i-1}^n \right|^2 - \frac{1}{4h_{i+\frac{1}{2}}^2} \left| v_{i+1}^n - v_i^n \right|^2 - \frac{1}{4h_{i+\frac{1}{2}}^2} \left| v_i^n - v_{i-1}^n \right|^2 \right) u_i^n \\
&+ \left(+\frac{1}{4h_i h_{i+\frac{1}{2}}} \left| v_{i+1}^n + v_i^n \right|^2 - \frac{1}{4h_{i+\frac{1}{2}}^2} \left| v_{i+1}^n - v_i^n \right|^2 \right) u_{i+1}^n,
\end{aligned} \tag{5.29}$$

so

$$\begin{aligned}
(1 - \beta_1 \Delta t) u_i^{n+1} &= \frac{(\Delta t)^2}{h_i \alpha \Gamma(-\alpha)} \sum_{s=0}^{i} w_{s,i} u_i^n \left((x_{1+\frac{1}{2}} - x_s)^\alpha - (x_{i-\frac{1}{2}} - x_s)^\alpha \right) \\
&+ \left(\frac{(\Delta t)^2}{4h_i h_{i-\frac{1}{2}}} \left| v_i^n - v_{i-1}^n \right|^2 - \frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2} \left| v_i^n - v_{i-1}^n \right|^2 \right) u_{i-1}^n \\
&+ \left(-\frac{(\Delta t)^2}{4h_i h_{i+\frac{1}{2}}} \left| v_{i+1}^n + v_i^n \right|^2 - \frac{(\Delta t)^2}{4h_i h_{i-\frac{1}{2}}} \left| v_i^n - v_{i-1}^n \right|^2 - \frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2} \left| v_{i+1}^n - v_i^n \right|^2 \right) u_i^n \\
&+ \left(-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2} \left| v_i^n - v_{i-1}^n \right|^2 + 2 - \beta_1 \Delta \right) u_i^n \\
&+ \left(+\frac{(\Delta t)^2}{4h_i h_{i+\frac{1}{2}}} \left| v_{i+1}^n + v_i^n \right|^2 - \frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2} \left| v_{i+1}^n - v_i^n \right|^2 \right) u_{i+1}^n - u_i^{n-1},
\end{aligned} \tag{5.30}$$

Let us exiplicit $\left((x_{i+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\right)$ (see [18]). Then, we have

$$\begin{aligned}&(x_{i+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha\\=&\frac{(x_{i+\frac{1}{2}}-x_s)^\alpha-(x_{i-\frac{1}{2}}-x_s)^\alpha}{(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}})}(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}})\\=&\alpha(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}})^{\alpha-1}(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}})=\alpha h_i^\alpha.\end{aligned}\tag{5.31}$$

Therfore

$$\begin{aligned}(1-\beta_1\Delta t)u_i^{n+1}&=\frac{(\Delta t)^2}{\Gamma(-\alpha)}\sum_{s=0}^{i}w_{s,i}u_s^nh_i^{\alpha-1}\\+&\left(\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2\right)u_{i-1}^n\\&+\left(-\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2\right)u_i^n\\&+\left(-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2+2-\beta_1\Delta t\right)u_i^n\\&+\left(+\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2\right)u_{i+1}^n-u_i^{n-1},\end{aligned}\tag{5.32}$$

hence

$$(1-\beta_1)u_i^{n+1}=\Psi_1u_{i-1}^n+\Psi_2u_i^n+\Psi_3u_{i+1}^n+(\Delta t)^2\sum_{s=0}^{i}\psi(\alpha,i,s)u_s^n-u_i^{n-1}.\tag{5.33}$$

Where

$$\begin{aligned}\psi(\alpha,i,s)&=\frac{1}{\Gamma(\alpha)}w_{i,s}h_i^{\alpha-1},\\\Psi_1&=\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2,\\\Psi_2&=-\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_ih_{i-\frac{1}{2}}}\left|v_i^n-v_{i-1}^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2,\\&-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_i^n-v_{i-1}^n\right|^2+2-\beta_1\Delta t,\\\Psi_3&=+\frac{(\Delta t)^2}{4h_ih_{i+\frac{1}{2}}}\left|v_{i+1}^n+v_i^n\right|^2-\frac{(\Delta t)^2}{4h_{i+\frac{1}{2}}^2}\left|v_{i+1}^n-v_i^n\right|^2.\end{aligned}$$

We have

$$\Psi_1 \leqslant Lm_1 \frac{(\Delta t)^2}{4h_i},$$
$$\Psi_2 \leqslant Lm_2 \frac{(\Delta t)^2}{4h_i},$$
$$\Psi_3 \leqslant Lm_3 \frac{(\Delta t)^2}{4h_i},$$

$$\text{and} \quad \sum_{s=0}^{N_{\max}} \psi(\alpha, i, s) \leqslant 1,$$

then, the relation (5.33) give

$$|1-\beta_1||u_i^{n+1}| \leqslant |\Psi_1||u_{i-1}^n| + |\Psi_2||u_i^n| + |\Psi_3||u_{i+1}^n| + (\Delta t)^2 |\sum_{s=0}^{i} \psi(\alpha, i, s)||u_s^n| + |u_i^{n-1}|,$$

and

$$|1-\beta_1||u_i^{n+1}| \leqslant (|\Psi_1| + |\Psi_2| + |\Psi_3| + (\Delta t)^2 |\sum_{s=0}^{i} \psi(\alpha, i, s)|)\|u^n\|_\infty + \|u^{n-1}\|_\infty,$$

and a recurrence assumption implies

$$\|u_i^{n+1}\|_\infty \leqslant c_2 \|u_i^n\|_\infty + \|u_i^{n-1}\|_\infty,$$

and give us

$$\|u^n\|_\infty \leqslant c_1^{N_{\max}} \|u_0\|_\infty \leqslant B_1,$$

in the same way, we have

$$\|v^n\|_\infty \leqslant c_2^{N_{\max}} \|v_0\|_\infty \leqslant B_2.$$

Conclusion and perspectives

This work made it possible to make a rather important contribution to the qualitative study of solutions to three nonlinear problems of Klein-Gordon equations. Based on the Faedo-Galerkin method, we analyzed the question of the local existence and uniqueness of weak solutions to hyperbolic problems with the source terms in the second part of this thesis. In the third chapter of this thesis, using the well-potential method, we have studied the global existence of solutions as well as constructing the Lyapunov function to stabilize both problems. In the fourth chapter, we studied the Klein-Gordon fractional problem using the Galerkin method and proved its existence and uniqueness. We finished the work by solving the last chapter using the finite volume method. About the work expected to be accomplished in the future, the completion of the fifth chapter by completing the simulation using the Matlab program, I also solved the problems of the first and second chapter using numerical methods, including the finite difference method as well as the finite element method.

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