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Thesis

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Option: ***Applied functional analysis***

Study of the existence and uniqueness, stability, regularity of the solution of certain classes of parabolic and hyperbolic PDEs

Presented by:

Khaldi Aya

Publicly discussed:

In front of the Jury:

1.	**Hakim LAKHEL**	**Pro.**	**20 August 1955 University of Skikda**	**President**
2.	**Messaoud MAOUNI**	**Pro.**	**20 August 1955 University of Skikda**	**Supervisor**
3.	**Amar OUAOUA**	**M.C.A**	**20 August 1955 University of Skikda**	**Co-supervisor**
4.	**Amar DEBBOUCHE**	**Pro.**	**University of 8 May 1945 Guelma**	**Examiner**
5.	**Kamel SLIMANI**	**M.C.A**	**20 August 1955 University of Skikda**	**Examiner**

University year : 2022/2023

بسم الله الرحمن الرحيم

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Dedicace

With the help of Almighty God and the success of the Almighty, the completion of this research.

I dedicated this modest work to those who I would never manage to express my sincere love to them.

To the person whose name I bear so proudly and who I miss so badly that I shudder every time I find myself in the face of the difficulties without him, to the person that life took from me so early on to "My father may Allah rest his soul".

To my strength and support at all times, to the flame that light my life and guides me to the right path, to my gardian angel, to the spring of love and tenderness "My dear mother".

To my brother and my sisters along with their husbands and children.

To all my family and relatives.

To all my friends and colleagues.

To all those who are present in the heart but absent in the line.

Abstract

The aim of this thesis is the study of global existence and asymptotic behavior of solutions for three types of systems with variable exponents.

In the first type, we consider a nonlinear Kirchhoff type reaction diffusion equations with variable exponents and source terms, we prove with suitable assumptions on the variable exponents the global existence of the solution and stability result with positive initial energy. In the second type, we consider a nonlinear hyperbolic equation with multiple $\alpha(x)-$Laplacian and variable exponent nonlinearities, we prove with negative initial energy the blow up of solutions. In the third type, we study fractional p-Kirchhoff type hyperbolic equations with damping and source terms. We also prove the existence of weak solution, and the stability results with negative initial energy. The proofs of the existence of global solutions are based on Faedo-Galerkin approximation combined with the potential and Nihari's functional. Furthermore, the stability results being based on Komornik's inequality.

Keywords: Kirchhoff equation, Reaction diffusion equation, Variable exponent, Source term, Damping term, Positive initial energy, $\alpha(x)-$Laplacian, Negative initial energy, Blow up of solution, Fractional p-Kirchhoff equation, Faedo-Galerkin approximation, Potential functional, Komornik's inequality.

Résumé

L'objectif de cette thèse est l'étude de l'existence globale et du comportement asymptotique des solutions pour trois types de systèmes à exposants variables.

Dans le premier type, on considère des équations de réaction-diffusion de type Kirchhoff non linéaires avec des exposants et des termes sources variables, nous prouvons avec des hypothèses appropriées sur les exposants variables l'existence globale de la solution et le résultat de stabilité avec une énergie initiale positive. Dans le second type, on considère une équation hyperbolique non linéaire avec des multiples $\alpha(x)$-Laplaciennes et des non-linéarités exposants variables, on prouve avec une énergie initiale négative l'explosion des solutions. Dans le troisième type, nous étudions les équations hyperboliques d'ordre fractionnaires de type p-Kirchhoff avec amortissement et termes sources. Nous prouvons également l'existence de la solution faible, et les résultats de stabilité avec une énergie initiale négative. Les preuves de l'existence de solutions globales sont basées sur l'approximation de Faedo-Galerkin combinées avec le potentiel et la fonctionnelle de Nihari. De plus, les résultats de stabilité étant basés sur l'inégalité de Komornik.

Mots clés: Équation de Kirchhoff, Équation de réaction-diffusion, Exposant variable, Terme source, Terme d'amortissement, Énergie initiale positive, $\alpha(x)$-Laplacien, Énergie initiale négative, Explosion de la solution, Équation d'ordre fractionnaire de p-Kirchhoff, Approximation de Faedo-Galerkin, Fonctionnelle potentielle, Inégalité de Komornik.

ملخص

الهدف من هذه الأطروحة هو دراسة وجود الحل الكلي والسلوك المتقارب للحلول، لثلاثة أنواع من الأنظمة ذات الأسس المتغيرة.

حيث في النوع الأول، نعتبر معادلات التفاعل والانتشار من نوع كيرشوف غير الخطية ذات الأسس المتغيرة ومنبع، ونقوم بوضع فرضيات على الأسس المتغيرة لإثبات وجود الحل الكلي، مع اثبات استقرار الحل باستعمال دالة الطاقة الاولية الموجبة. في النوع الثاني، نقوم بدراسة مسألة زائدية غير خطية بمؤثر α (x)- لبلاس ذات أسس غير خطية متغيرة، باستعمال دالة الطاقة الأولية السالبة لإثبات تفجير الحلول. في النوع الثالث، نقوم بدراسة وجود الحلول لمسألة قطعية من نوع p– كيرشوف ذات مؤثرمن رتبة كسرية باستخدام طريقة Faedo-Galerkin ودالة نيهاري. مع دراسة استقرار الحلول بالاعتماد على متراجحة كومورنيك.

الكلمات المفتاحية: معادلة كيرشوف، معادلة تفاعل انتشار، الأس المتغير، الطاقة الأولية الإيجابية، α (x)- لبلاس، الطاقة الأولية السالبة، تفجير المحلول، p–كيرشوف ذات مؤثركسري، طريقةFaedo-Galerkin ، متراجحة كومورنيك.

Notations

- $\rightarrow$: designates the strong convergence .
- $\rightharpoonup$: indicates the weak convergence.
- $\hookrightarrow$: indicates the continuous embedding.
- ∇: stands for the gradient operator.
- div: is the divergence operator.
- $\frac{\partial}{\partial x}$: partial derivative.
- Δ_p: is the p-Laplace operator.
- Δ_p^s: is the fractional Laplace operator of order s.
- sp: denotes the spectrum of an operator.
- $\mathbb{N}$: the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, ...\}$.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}^n$: is the real space of dimension n.
- Ω: open set in $\mathbb{R}^n$.
- $\partial\Omega$: denote the boundary of domain Ω.
- $(.,.)$: denotes the scalar product.
- $C^m(\Omega)$: space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.
- $C^\infty(\Omega) = \bigcap\limits_{m\in\mathbb{N}} C^m(\Omega)$.
- $C_0^\infty(\Omega)$: the space of $C^\infty(\Omega)$ functions with compact support in Ω.
- $L^p(\Omega)$: Lebesgue space with norm $\|.\|_p$.

- $L^p_{loc}(\Omega)$: the set of locally integrable functions.
- $W^{k,p}(\Omega)$: Sobolev space with norm $\|.\|_{k,p}$.
- $W_0^{k,p}(\Omega)$: is the closure of $C^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
- $W^{-1,p'}(\Omega)$: is the dual of $W^{1,p}(\Omega)$.
- $H^m(\Omega) = W^{m,2}(\Omega)$.
- $L^{p(.)}(\Omega)$: Lebesgue space with variable exponent.
- $W^{k,p(.)}(\Omega)$: Sobolev space with variable exponent.
- $W^{s,p}(\Omega)$: fractional Sobolev space with norm $\|.\|_{s,p}$.
- $W_0^{s,p}(\Omega)$: denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$.
- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W_0^{s,2}(\mathbb{R}^n) = H_0^s(\mathbb{R}^n)$
- $P.V$: is an abbreviation for “in the principal value sense”.

General introduction

Partial differential equations (PDEs) involving variable exponents nonlinearities are a class of equations that arise in various areas of mathematics, physics, and engineers. For details, see ([45], [52]) . They are nonlinear equations in which the coefficients (or exponents) of the derivatives are allowed to vary with spatial cordinates. This makes them more complex than equations with constant coefficients. These type of equations are also used to describe several phenomena in mathematics such as fluid dynamics, elastic mechanics, nonlinear elasticity, electroheological fluid and so forth.

The study of PDEs with variable-exponents nonlinearities is an active research area that has attracted considerable attention in recent years due to its applications in different fields. The main challenge is to understand the qualitative properties of the solutions to these equations, such as the existence, uniqueness, and regularity of solutions. This involves developing techniques to estimate the size and smoothness of solutions in different regions of the domain. The study of this equations began in the $1960s$, and the fields has grown steadily since then. However, the concept of variable exponents dates back to the work of hardy and littlewood in the $1930s$, where they established inequalities involving variable exponents. In the $1960s$ and $1970s$, there was significant work done on nonlinear PDEs, and many equations studied incorporated variable-exponents. A notable example of this is the $p-$Laplacian equation, which is a nonlinear elliptic equation with variable exponents that was introduced in the $1950s$. In the $1980s$, the rigorous study of these PDEs has been facilitated by the developement of Lebesgue and Sobolev spaces with variable exponents. In the $1990s$, there was a surge of interest in PDEs with variable exponents due to their potential to model complex phenomena that cannot be accurately captured by equations with constant exponents. Research in this area has continued to grow, and variable exponent PDEs have become a major area of study in modern mathematics. For results of these studies, we refer the reader to ([6], [21], [22], [31] − [33], [58], [65]) .

Over the last four decades, The interest and focus of many researchers has shifted from the partial differential systems to the fractional systems, it's caused by both the intensive development of the theory of fractional calculus it self and by its applications in various fields. It is crucial to mention that fractional calculus is a generalization of ordinary differentiation and integration. Another aspect in the study of fractional systems is when involving fractional Laplacian and as we know, the fractional Laplacian is very common in the modern study of fractional differentiel systems because of its applications.

The difference between standard and fractional Laplacian can be explained from probabilistics interpretation. The standard Laplace operator represents the infinitesimal generator of a Brownian motion with continuous sample paths, thus for a particle in domain Ω, it must leave the domain via the boundary point on $\partial\Omega$. By contrast, the fractional Laplacian is the infinitesimal generator of a symetric s-stable Levy process with discontinuous sample paths, particles may "jump" out of the domain without touching any boundary points on $\partial\Omega$.

Overall, it is clear that fractional differential equations involving fractional Laplacian present a rich and fascinating area of research with both theoretical and practical significance.

These classes of equations had diverse applications in image processing, material science, and finance. For example, in image processing, variable-exponent PDEs are used to eliminate noise while preserving the edges of the image. In material science, they are used to simulate the behavior of composite materials with spatially varying properties. In finance, variable exponents non-linear partial differential equations are used to model the behavior of stock options.

Many methods and techniques have been proposed to deal with and among these nonlinear equations: variational methods, approximation theory, fixed point method, semi groups method, energy estimates, and numerical methods, ...etc.

This thesis is mainly devoted to the study of a nonlinear hyperbolic and parabolic systems with variable exponents and also hyperbolic systems involving fractional Laplacian. We use the Faedo Galerkin method to prove the existence of weak solution and using potential and Nihari's functionals to prove global existence of solution. Stability being based on Komornik's inequality. To be more precise, we are interested in the study of three classes of systems:

- The first class of system is a nonlinear Kirchhoff type reaction- diffusion with variable exponents and source term

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + |u|^{m(x)-2} u_t = |u|^{r(x)-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1}$$

  where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$ and the functions $M(s) = a + bs^\gamma$ with positive parameters a, b, γ. The result of this system can be seen as a generalization of the result obtained by Kirchhoff.

- The second class of systems are the nonlinear hyperbolic systems with $\alpha(x)$ $-$laplacian and variable exponent

$$
\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) - \gamma\Delta u_t = |u|^{p(x)-2}u, \text{ in } \Omega\times(0,T), \\ u(x,t) = 0 \qquad\qquad\qquad\qquad\qquad , x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \qquad\qquad x\in\Omega, \end{cases} \tag{2}
$$

where $\gamma > 0$, $0 < T < \infty$ and Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$. $m(.)$, $r(.)$, and $p(.)$ are given measurable functions on Ω. We study the asymptotic behavior of the solution.

- In the last class of systems, we are interested in the nonlinear fractional p-Kirchhoff type hyperbolic equation with variable exponent

$$
\begin{cases} u_{tt} + M\left(\int_\Omega |\nabla u|^p\,dx\right)(-\Delta)_p^s u + |u_t|^{m(x)-2}u_t = |u|^{r(x)-2}u, \text{ in } \Omega\times(0,T), \\ u(x,t) = 0 \qquad\qquad\qquad , x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \qquad x\in\Omega, \end{cases} \tag{3}
$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with a smooth boundary $\partial\Omega$. $M(s) = 1 + s$ and $m(.)$, $r(.)$ are given measurable functions on Ω. Where Ouaoua et al. studied the system (3) in the classical case .

This thesis is divided in four chapters, as follows:

- In the first chapter, We will discuss some preliminary materials that we will use throughout the thesis. We recall definitions and important results in the Lebesgue and Sobolev spaces with variable exponents, Fractional Sobolev spaces that have an essential role in the subsequent chapters.
- Our aim in the chapter two is to investigate the global existence and stability of solution for the problem (1) by using potential and Nihari's functionals. The stability being based on Komornik inequality.
- We devoted in chapter three to study of the blow up of solution for wave equation with multiple $\alpha(x)$ $-$laplacian the problem (2) by using the Lyapunov functional.
- For the last chapter, we study the existence of weak solution of the problem (3) by using Faedo Galerkin method, and also prove the stability results by Komornik inequality.

Chapter 1

Preliminaries

In this chapter, we will introduce and state without proofs some fundamental results and notations in functional analysis that will be utilized through this thesis.

1.1 Variable-exponent spaces

1.1.1 Variable-exponent Lebesgue space

Definition 1.1 [30] *Let X be a $\Bbbk-$vector space. A function $\varrho : X \to [0,\infty]$ is said to be left-continuous if the mapping $\lambda \to \varrho(\lambda x)$ is left-continuous on $[0,\infty)$, for every $x \in X$; that is*

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x), \forall x \in X.$$

Definition 1.2 [30] *Let X be a $\Bbbk$-vector space. A function $\varrho : X \to [0,\infty]$ is called a semimodular on X if the following properties hold:*

a- $\varrho(0) = 0$.

b- $\varrho(\lambda x) = \varrho(x)$, *for all $x \in X$ and $\lambda \in \Bbbk$, with $|\lambda| = 1$.*

c- *ϱ is convex.*

d- *ϱ is left-continuous.*

e- $\varrho(\lambda x) = 0$, *for all $\lambda > 0$ implies $x = 0$.*

A semimodular is called modular if

f- $\varrho(x) = 0$ *implies $x = 0$.*

A semimodular is called continuous if

g- *The mapping* $\lambda \rightarrow \varrho(\lambda x)$ *is continuous on* $[0, \infty)$ *for all* $x \in X$.

Theorem 1.1 [30] *Let* ϱ *be a semimodular on* X. *Then, the mapping* $\lambda \rightarrow \varrho(\lambda x)$ *is non-decreasing on* $[0, \infty)$ *for every* $x \in X$. *Moreover,*

$$\begin{aligned} \varrho(\lambda x) &= \varrho(|\lambda| x) \leq |\lambda| \varrho(x), \text{ for all } |\lambda| \leq 1, \\ \varrho(\lambda x) &= \varrho(|\lambda| x) \geq |\lambda| \varrho(x), \text{ for all } |\lambda| \geq 1. \end{aligned}$$

Definition 1.3 [30] *Let* (A, Σ, μ) *be a* σ*-finite, complete measure space. We define* $P(A, \mu)$ *to be the set of all* μ*-measurable functions* $p : \Omega \rightarrow [1, \infty]$. *The functions* $p \in P(A, \mu)$ *are called variable exponents on* A.
We set

$$p_1 = ess \inf_{y \in A} p(y) \text{ and } p_2 = ess \sup_{y \in A} p(y).$$

If $p_2 < +\infty$, *then we call* p *a bounded variable exponent. If* $p \in P(A, \mu)$, *then we define* $p' \in P(A, \mu)$ *by*

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \text{ where } \frac{1}{\infty} = 0.$$

The function p' *is called the dual variable-exponent of* p. *In the special case when* μ *is the* n*-dimensional Lebesgue measure and* Ω *is an open subset of* $\mathbb{R}^n$, *we abbreviate* $P(\Omega) = P(\Omega, \mu)$.

Definition 1.4 [30] *We define the Lebesgue space with variable-exponent* $p(.)$ *by*

$$L^{p(.)}(\Omega) = \left\{u : \Omega \rightarrow \mathbb{R}; \text{ measurable in } \Omega : \varrho_{p(.)}(\lambda u) < \infty, \text{ for some } \lambda > 0\right\}$$

where

$$\varrho_{p(.)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx$$

is a modular. We equip $L^{p(.)}(\Omega)$ *with the following Luxembourg-type norm:*

$$\|u\|_{p(.)} = \inf \left\{\lambda > 0 : \int_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Lemma 1.1 [30] *If* $p(x) = p$, *where* p *is constant. Then,*

$$\|u\|_{p(.)} = \|u\|_p = \lambda_0 = \left(\int_{\Omega} |u|^p\right)^{\frac{1}{p}}.$$

Definition 1.5 [30] *We say that a function $q : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω, if there exist $A > 0$ and $0 < \delta < 1$ such that*

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta.$$

Lemma 1.2 (Unit ball property) [30] *Let $p \in P(A, \mu)$ and $f \in L^{p(.)}(A, \mu)$. Then*

i- $\|f\|_{p(.)} \leq 1$ *if and only if* $\varrho_{p(.)}(f) \leq 1$.

ii- *If* $\|f\|_{p(.)} \leq 1$, *then* $\varrho_{p(.)}(f) \leq \|f\|_{p(.)}$.

iii- *If* $\|f\|_{p(.)} \geq 1$, *then* $\|f\|_{p(.)} \leq \varrho_{p(.)}(f)$.

iv- $\|f\|_{p(.)} \leq 1 + \varrho_{p(.)}(f)$.

Lemma 1.3 [30] *If $1 \leq p_1 \leq p(x) \leq p_2 < +\infty$ holds, then*

$$\min\left\{\|u\|_{p(.)}^{p_1}; \|u\|_{p(.)}^{p_2}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p_1}; \|u\|_{p(.)}^{p_2}\right\}$$

for any $u \in L^{p(.)}(\Omega)$.

Theorem 1.2 [30] *If $p \in P(A, \mu)$, then $L^{p(.)}(A, \mu)$ is a Banach space.*

Lemma 1.4 [30] *If $p : \Omega \to [1, \infty)$ is a measurable function with $p_2 < +\infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(.)}(\Omega)$.*

Lemma 1.5 (Young's inequality) [30] *Let $p, q, s \in P(\Omega)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{s(.)}}{s(y)} \leq \frac{(a)^{p(.)}}{p(.)} + \frac{(b)^{q(.)}}{q(.)}.$$

By taking $s = 1$, and $1 < p, q < +\infty$, then we have for any $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \forall a, b \geq 0,$$

where $C_\varepsilon = \frac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. For $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 1.6 (Hölder's inequality) [30] *Let $p, q, s \in P(\Omega)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e } y \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$ and

$$\|fg\|_{s(.)} \leq 2\|f\|_{p(.)}\|g\|_{q(.)}.$$

By taking $p = q = 2$, we have the Cauchy-Schwartz inequality.

1.1.2 Variable- exponent Sobolev spaces

Definition 1.6 [30] *Let $k \in \mathbb{N}$. We define the space $W^{k,p(.)}(\Omega)$ by*

$$W^{k,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } \partial_\alpha u \in L^{p(.)}(\Omega), \forall |\alpha| \leq k\right\}.$$

We define a semimodular on $W^{k,p(.)}(\Omega)$ by

$$\varrho_{W^{k,p(.)}(\Omega)}(u) = \sum_{0\leq|\alpha|\leq k} \varrho_{L^{p(.)}(\Omega)}(\partial_\alpha u).$$

This induces a norm given by

$$\begin{aligned}\|u\|_{W^{k,p(.)}(\Omega)} &= \inf\left\{\lambda > 0 : \varrho_{W^{k,p(.)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1\right\} \\ &= \sum_{0\leq|\alpha|\leq k} \|\partial_\alpha u\|_{p(.)}.\end{aligned}$$

For $k \in \mathbb{N}$, the space $W^{k,p(.)}(\Omega)$ is called Sobolev space and its elements are called functions. Clearly $W^{0,p(.)}(\Omega) = L^{p(.)}(\Omega)$ and

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

Theorem 1.3 [30] *Let $p \in P(\Omega)$. The space $W^{k,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive if $1 < p_1 \leq p_2 < +\infty$.*

Let $p \in P(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W_0^{k,p(.)}(\Omega)$ "with zero boundary trace" is the closure in $W^{k,p(.)}(\Omega)$ of the set of $W^{k,p(.)}(\Omega)-$ functions with compact support, i.e.,

$$W_0^{k,p(.)}(\Omega) = \overline{\{u \in W^{k,p(.)}(\Omega) : u = u_{\varkappa K} \text{ for a compact } K \subset \Omega\}}.$$

Remark 1.1 [30] *Let $p \in P(\Omega)$ and $k \in \mathbb{N}$. Then*

i- *The space $H_0^{k,p(.)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(.)}(\Omega)$.*

ii- $H_0^{k,p(.)}(\Omega) \subset W_0^{k,p(.)}(\Omega)$.

iii- *If p is log-Hölder continuous on Ω, then $W_0^{k,p(.)}(\Omega) = H_0^{k,p(.)}(\Omega)$.*

iv- *The dual of $W_0^{1,p(.)}(\Omega)$ is defined as $W^{-1,p'(.)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$.*

Theorem 1.4 [30] *Let $p \in P(\Omega)$. The space $W_0^{k,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive if $1 < p_1 \leq p_2 < +\infty$.*

Theorem 1.5 (Poincare inequality) [30] *Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(.)$ satisfies the Log-Hölder continuity property, then*

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)}, \text{ for all } u \in W_0^{1,p(.)}(\Omega)$$

where the positive constant C depends on $p(.)$ and Ω only. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}$.
If $p = 2$, then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Lemma 1.7 [30] *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.Assume that $p : \Omega \to (1, \infty)$ is a measurable function such that*

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \text{ for a.e } x \in \Omega.$$

If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ in $\overline{\Omega}$ with

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p_2 < n \\ \infty, & \text{if } p_2 \geq n \end{cases}.$$

Then the embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Corollary 1.1 [30] *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p : \overline{\Omega} \to (1, \infty)$ is a continuous function such that*

$$2 \leq p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2}, \ n \geq 3.$$

Then the embedding $H^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Lemma 1.8 (Komornik inequality) [28] *Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $C > 0$ such that*

$$\int_t^{\infty} G^{\alpha+1}(s)\, ds \leq C G^{\alpha}(0)\, G(s) \quad \forall t \in \mathbb{R}_+.$$

Then we have

$$G(t) \leq G(0) \left(\frac{C + \alpha t}{C + \alpha C} \right)^{-1/\alpha} \quad \forall t \geq C.$$

1.2 Fractional Sobolev space

Before we start defining the fractional Sobolev space mathematically. We have in literature, fractional Sobolev spaces are called also Aronszajin, Gagliardo or Slobodeckij spaces by the name of the ones who introduced them, almost simultaneously (see [5], [15], [65]).

Definition 1.7 [12] *Let Ω be an open set in $\mathbb{R}^n$ and for any $s \in (0,1)$ and $p \in [1, \infty)$. We define $W^{s,p}(\Omega)$ as follows*

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p} \right)^{\frac{1}{p}},$$

with

$$[u]_{s,p} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo (semi) norm of u.

Remark 1.2 [12] *The definition in (1) cannot be plainly extended to the case $s \geq 1$. Suppose that Ω is a connected open set in $\mathbb{R}^n$, then any measurable function $u : \Omega \to \mathbb{R}$ such that*

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy < +\infty,$$

is actually constant (see [8], proposition 2). This fact is a matter of scaling and it is stricly related to the following result that holds for any u in $W^{s,p}(\Omega)$

$$\lim_{s \to 1^-} (s-1)^{\frac{1}{p}} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy = C \int_{\Omega} |\nabla u|^p \, dx,$$

for a suitable positive constant C depending only on n and p.
When $s>1$ and it is not an integer we write $s=m+\sigma$, where m is an integer and $\sigma\in(0,1)$.
In this case the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u\in W^{m,p}(\Omega)$ whose distributional derivatives $D^{\alpha}u$, with $|\alpha|=m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$W^{s,p}(\Omega)=\{u\in W^{m,p}(\Omega):D^{\alpha}u\in W^{\sigma,p}(\Omega)\ \textit{for any}\ \alpha\ \textit{s.t}\ |\alpha|=\sigma\}$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)}=\left(\|u\|^{p}_{W^{m,p}(\Omega)}+\sum_{|\alpha|=m}\|D^{\alpha}u\|^{p}_{W^{\sigma,p}(\Omega)}\right)^{\frac{1}{p}}.$$

Clearly, if $s=m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Proposition 1.1 [11] *Let Ω be an open set of $\mathbb{R}^n$, $s\in(0,1)$ and $p\in[1,\infty)$, then*

1. *For $1\leq p<\infty$, $W^{s,p}(\Omega)$ is a Banach space.*
2. *For $1\leq p<\infty$, $W^{s,p}(\Omega)$ is a separable space.*
3. *For $1<p<\infty$, $W^{s,p}(\Omega)$ is a reflexive space.*
4. *For $1<p<\infty$, $W^{s,p}(\Omega)$ is uniformly convex space.*

Theorem 1.6 [12] *For any $s>0$, the space $C_0^{\infty}(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\Omega)$.*
Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^{\infty}(\Omega)$ in the norm $\|.\|_{W_0^{s,p}(\Omega)}$. Note that, in view of theorem 1.6, we have

$$W_0^{s,p}(\mathbb{R}^n)=W^{s,p}(\mathbb{R}^n),$$

but in general, for $\Omega\subset\mathbb{R}^n$, $W^{s,p}(\Omega)\neq W_0^{s,p}(\Omega)$, i.e $C_0^{\infty}(\Omega)$ is not dense in $W^{s,p}(\Omega)$.

Remark 1.3 [12] *For $s<0$ and $p\in(1,\infty)$, we can define $W^{s,p}(\Omega)$ as the dual space of $W^{-s,q}(\Omega)$ where $\frac{1}{p}+\frac{1}{q}=1$. Notice that, in this case the space $W^{s,p}(\Omega)$ is actually a space of distributions on Ω, since it is the dual of a space having $C_0^{\infty}(\Omega)$ as density subset.*

Corollary 1.2 (continue embeddings) [11] *Let $s\in(0,1)$ and $p\in\,]1,\infty[$. Let Ω be a Lipschitz open set of $\mathbb{R}^n$. Then we have:*

1. If $sp<n$, then $W^{s,p}(\Omega)\hookrightarrow L^q(\Omega)$ for every $q\leq np/(n-sp)$.

2. If $sp = n$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.

3. If $sp > n$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and more precisely,

$$W^{s,p}(\Omega) \hookrightarrow C^{0,s-n/p}(\Omega).$$

Theorem 1.7 (compact embedding) [11] *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. Let $s \in [0,1[$ and $p > 1$, and let $n \geq 1$. Then we have*

1. *If $sp < n$, then the embedding of $W^{s,p}(\Omega)$ into L^k is compact for every $k < \frac{np}{(n-sp)}$.*
2. *If $sp = n$, then the embedding of $W^{s,p}(\Omega)$ into L^q is compact for every $q < \infty$.*
3. *If $sp > n$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}$ is compact for every $\lambda < \frac{s-n}{p}$.*

1.2.1 The space $H^s(\Omega)$

In this part, we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W_0^{s,2}(\mathbb{R}^n)$ out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$.

Definition 1.8 [61] *Let Ω be an open set of $\mathbb{R}^n$, then we define*

$$H^s(\Omega) = W^{s,2}(\Omega) \text{ with } 0 < s < 1$$

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \text{ such that } \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega) \right\}.$$

The Sobolev spaces $H^s(\Omega)$ eqquiped with the following inner product and norm

$$\langle u, v \rangle_{H^s(\Omega)} = \int_\Omega u(x) v(x)\, dx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))}{|x-y|^{n+2s}} (v(x) - v(y))\, dxdy.$$

$$\|u\|_{H^s(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + [u]_{H^s(\Omega)}^2 \right)^{\frac{1}{2}},$$

with

$$[u]_{H^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)$$

It is a Hilbert space.

Theorem 1.8 (Poincare Inequality) [61] *Let Ω be an open sets of $\mathbb{R}^n$, let $s \in]0,1[$ and $p \in [1,+\infty[$ so there exists $C = C(n,s,p)$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|u\|_{W_0^{s,p}(\Omega)}, \quad \forall u \in W_0^{s,p}(\Omega).$$

Therefore, if Ω is bounded then $\|.\|_{W_0^{s,p}(\Omega)}$ is a norm of $W_0^{s,p}(\Omega)$ equivalent to $\|.\|_{W^{s,p}(\Omega)}$ with

$$\|u\|_{W_0^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\right)^{\frac{1}{p}}.$$

1.2.2 The fractional Laplacian operator

Definition 1.9 [61] *Let $s \in]0,1[$ and $p \in [1,+\infty[$, then we define the fractional p-Laplacian with*

$$\begin{aligned}(-\Delta)_p^s u(x) &= 2\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy \\ &= 2VP \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy.\end{aligned}$$

Remark 1.4 *If $p = 2$, we obtain the linear fractional Laplacian operator defined by*

$$(-\Delta)_p^s u(x) = 2C(n,s) \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy$$

with $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \frac{4^s \Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}} \frac{s}{\Gamma(1-s)},$$

where Γ is the usual Gamma function (see [9]).

Corollary 1.3 [61] $(-\Delta)_p^s \to (-\Delta)_p$ *when $s \to 1$. Where convergence is in the dual space of $W_0^{s,p}(\Omega)$.*

Theorem 1.9 *Let $s \in (0,1)$ and $p \in]1,+\infty[$ then*

$$(-\Delta)_p^s : W_0^{s,p}(\Omega) \to \left(W_0^{s,p}(\Omega)\right)'$$

is well defined, and further:

1. $\forall u, v \in W_0^{s,p}(\Omega)$ we have

$$\left\langle (-\Delta)_p^s u, v \right\rangle = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} (v(x)-v(y))\, dxdy.$$

2. $\forall u, v \in W_0^{s,p}(\Omega)$

$$\left\langle (-\Delta)_p^s u, v \right\rangle \leq \|u\|_{W_0^{s,p}}^{p-1} \|u\|_{W_0^{s,p}}$$

and consequently: $\left\| (-\Delta)_p^s u \right\|_{W^{-s,p'}} \leq \|u\|_{W_0^{s,p}}^{p-1}$.

Chapter 2

Global existence and stability of solution for a nonlinear Kirchhoff type reaction diffusion equation with variable exponents

The main prupose of this chapter is to prove with suitable assumptions on the variable exponents $r(.)$, $m(.)$ the global existence and the stability results for a nonlinear Kirchhoff type reaction-diffusion equation. The results appearing in this context have been published throught [25]

2.1 Introduction

In this chapter, we consider a class of Kirchhoff type reaction- diffusion equations with variable exponents and source terms

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + |u|^{m(x)-2} u_t = |u|^{r(x)-2} u, & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$ and $M(s) = a + bs^\gamma$ with positive parameters a, b, γ. Further, $r(.)$ and $m(.)$ are given measurable functions on Ω, satisfying

$$\begin{aligned} 2 &\leq q_1 \leq q(x) \leq q_2 < \frac{2n}{n-2} \text{ if } n \geq 3, \\ q(x) &\geq 2, \text{ if } n = 1, 2. \end{aligned} \tag{2.2}$$

We also assume that $m(.)$ and $r(.)$ satisfy the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|} \text{ for a.e. } x, y \in \Omega,\ |x-y| < \delta \text{ with } A > 0,\ 0 < \delta < 1. \tag{2.3}$$

Equation (2.1) appears in various physical contexts. In particular, this equation arises from the mathematical description of the reaction-diffusion or diffusion, heat transfer, and population dynamic processes (see [22]).

In the last few years, partial equations with different kinds of nonlocal terms have drawn more and more attention because of their wide applications in both physics and biology. For more results, we refer the reader to previous studies. We start with the hyperbolic equation with a nonlocal coefficient is

$$\varepsilon u_{tt}^{\varepsilon} + u_t^{\varepsilon} - M\left(\int_{\Omega} |\nabla u^{\varepsilon}|^p \, dx\right) \Delta_p u^{\varepsilon} = f\left(x, t, u^{\varepsilon}\right), \tag{2.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $M(s) = a + bs$, $a > 0$, $b > 0$ and $p > 1$. It is a potential model for damped small transversal vibrations of an elastic string with uniform density ε (see [17]). For $p = 2$, such nonlocal equations were first proposed by Kirchhoff in 1883 (see [27]) and therefore were usually refered to as Kirchhoff equations. In the case $\varepsilon = 0$, (2.4) becomes a Kirchhoff type parabolic equation

$$u_t - M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \Delta_p u = f\left(x, t, u\right). \tag{2.5}$$

Equation (2.5) can also be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium, and the nonlocal term M appearing in (2.5) can describe a possible change in the global state of the fluid or gas caused by its motion in the considered medium (see [13]). In [36], Li and Han studied the following p-Kirchhoff problem

$$u_t - \left(a + b\int_{\Omega} |\nabla u|^p \, dx\right) \Delta_p u = |u|^{q-1} u, \tag{2.6}$$

where a, b are two positive constants, $p > \max\left\{2n/(n+1), 1\right\}$, $2p < q < p^* - 1$, and p^* is the Sobolev conjugate of p. They proved the global existence and finite time blow-up of solutions. Also in [35], Haixia studied the same equation where the source term is a function depending on u and satisfying some conditions. He proved the blow-up of solutions and the results generalize some recent ones reported by Han and Li (see [20]). In [59], Polat studied a $1D$ problem and he established a blow-up result for the solution with vanishing initial energy of the equation

$$u_t - \Delta u_{xx} + |u|^{m-2} u_t = |u|^{r-2} u. \tag{2.7}$$

Ouaoua and Maouni in [54], considered the following nonlinear parabolic equation with $p(x)$ -Laplacian

$$u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) + \omega |u|^{m(x)-2} u_t = b |u|^{r(x)-2} u. \tag{2.8}$$

They proved a finite time blow-up result for the solutions in the case $\omega = 0$ and exponential growth in the case $\omega > 0$ with negative initial energy. Many authors have studied the existence and nonexistence of solutions for the problem with variable exponents or constants, see [2],[7], [9], [14], [16], [33], [34], [46], [42], [47], [50], [65], [66].

2.2 The main of results

In this section to state and prove the global existence of solution of system (2.1). We define the potential energy and Nehari's functionals as follow:

$$E(t) = E(u(t)) = \frac{a}{2}\|\nabla u(t)\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}dx, \tag{2.9}$$

$$I(t) = I(u(t)) = a\|\nabla u(t)\|_2^2 + b\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega |u(t)|^{r(x)}dx. \tag{2.10}$$

In the following, we consider $a = b = 1$ and this does not change the general result.

Lemma 2.1 *Under the assumptions* (2.2)*, we have*

$$E'(t) = -\|u_t(t)\|_2^2 - \int_\Omega |u(t)|^{m(x)-2}|u_t(t)|^2 dx \le 0, \tag{2.11}$$

and

$$E(t) \le E(0).$$

Proof. We multiply the first equation of (2.1) by u_t and integrating over the domain Ω, we get

$$\int_\Omega u_t\left[u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + |u|^{m(x)-2}u_t = |u|^{r(x)-2}u\right]dx$$

$$\int_\Omega u_t^2 dx - \int_\Omega M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u u_t dx + \int_\Omega |u|^{m(x)-2}|u_t|^2 dx = \int_\Omega |u|^{r(x)-2}u u_t dx$$

$$\int_\Omega u_t^2 dx + \int_\Omega M\left(\int_\Omega |\nabla u|^2 dx\right)\nabla u \nabla u_t dx + \int_\Omega |u|^{m(x)-2}|u_t|^2 dx = \int_\Omega |u|^{r(x)-2}u u_t dx$$

$$\int_\Omega u_t^2 dx + \frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\right) + \int_\Omega |u|^{m(x)-2}|u_t|^2 dx = \frac{d}{dt}\int_\Omega \frac{|u|^{r(x)}}{r(x)}dx.$$

This implies

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx\right) = -\|u_t\|_2^2 - \int_\Omega |u|^{m(x)-2}|u_t|^2 dx.$$

Then

$$E'(t) = -\|u_t\|_2^2 - \int_\Omega |u|^{m(x)-2}|u_t|^2 dx \le 0.$$

Integrating (2.11) over $(0,t)$, we obtain

$$\int_0^t E'(s)\,ds = E(t) - E(0) \leq 0$$

$$E(t) \leq E(0).$$

■

Lemma 2.2 *Assume that the assumptions* (2.2) *hold, and* $r_1 > 2(\gamma+1)$, $I(0) > 0$ *and*

$$\beta_1 + \beta_2 < 1, \tag{2.12}$$

where

$$\begin{aligned} \beta_1 &:= \max\left\{ \alpha c_*^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}}, \alpha c_*^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}} \right\}, \\ \beta_2 &:= \max\left\{ (1-\alpha)\, c_*^{r_1}\left(\frac{2(\gamma+1)\,r_1}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_1-2(\gamma+1)}{2(\gamma+1)}}, (1-\alpha)\, c_*^{r_2}\left(\frac{2(\gamma+1)\,r_1}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_2-2(\gamma+1)}{2(\gamma+1)}} \right\} \end{aligned}$$

with $0 < \alpha < 1$ *and* c_* *is the best embedding constant of* $H_0^1(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$. *Then* $I(t) > 0$ *for all* $t \in [0,T]$.

Proof. Since $I(0) > 0$, then by continuity there exists T_* such that

$$I(t) \geq 0,\ \forall t \in [0,T_*]. \tag{2.13}$$

Now, we will prove that this inequality is strict. We have for all $t \in [0,T]$ that

$$\begin{aligned} E(t) &= E(u(t)) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \frac{1}{r_1}\int_\Omega |u|^{r(x)}\,dx \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \frac{1}{r_1}\left(\|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} - I(t)\right) \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \frac{1}{r_1}\|\nabla u\|_2^2 - \frac{1}{r_1}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{r_1}I(t) \\ &\geq \frac{r_1-2}{2r_1}\|\nabla u\|_2^2 + \frac{r_1-2(\gamma+1)}{2r_1(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{r_1}I(t). \end{aligned}$$

Using (2.13), we obtain

$$\frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1-2(\gamma+1)}{2r_1(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \leq E(t),\ \ \forall t \in [0,T_*]. \tag{2.14}$$

By the definition of $E(t)$, we find

$$\frac{r_1-2}{2r_1}\left\|\nabla u(t)\right\|_2^2 \leq E(t)$$

and

$$\frac{r_1-2(\gamma+1)}{2r_1(\gamma+1)}\left\|\nabla u(t)\right\|_2^{2(\gamma+1)} \leq E(t)$$

So using the nonincreasingness of E, we obtain

$$\left\|\nabla u(t)\right\|_2^2 \leq \frac{2r_1}{r_1-2}E(t) \leq \frac{2r_1}{r_1-2}E(0), \tag{2.15}$$

and

$$\left\|\nabla u(t)\right\|_2^{2(\gamma+1)} \leq \frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(t) \leq \frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0). \tag{2.16}$$

On the other hand, by Lemma 1.5, we have

$$\begin{aligned}\int_\Omega |u(t)|^{r(x)}dx &\leq \max\left\{\|u(t)\|_{r(.)}^{r_1}, \|u(t)\|_{r(.)}^{r_2}\right\} \\ &= \alpha\max\left\{\|u(t)\|_{r(.)}^{r_1}, \|u(t)\|_{r(.)}^{r_2}\right\} \\ &\quad +(1-\alpha)\max\left\{\|u(t)\|_{r(.)}^{r_1}, \|u(t)\|_{r(.)}^{r_2}\right\}.\end{aligned}$$

Hence, by the embedding of $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u(t)|^{r(x)}dx &\leq \alpha\max\left\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2}\right\} \\ &\quad +(1-\alpha)\max\left\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2}\right\} \\ &\leq \alpha\max\left\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1-2}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2-2}\right\}\times\|\nabla u(t)\|_2^2 \\ &\quad +(1-\alpha)\max\left\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1-2(\gamma+1)}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2-2(\gamma+1)}\right\}\times\|\nabla u(t)\|_2^{2(\gamma+1)}.\end{aligned}$$

By (2.15) and (2.16), we get

$$\|\nabla u(t)\|_2^{r_1-2} \leq \left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}}$$

and

$$\|\nabla u(t)\|_2^{r_1-2(\gamma+1)} \leq \left(\frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_1-2(\gamma+1)}{2(\gamma+1)}}.$$

Then

$$\begin{aligned}\int_{\Omega}|u(t)|^{r(x)}dx &\leq \alpha\max\left\{c_*^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}}, c_*^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\}\times\|\nabla u(t)\|_2^2 \\ &\quad +(1-\alpha)\max\left\{\begin{array}{c} c_*^{r_1}\left(\frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_1-2(\gamma+1)}{2(\gamma+1)}}, \\ c_*^{r_2}\left(\frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_2-2(\gamma+1)}{2(\gamma+1)}}\end{array}\right\}\times\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &\leq \max\left\{\alpha c_*^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}}, \alpha c_*^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\}\times\|\nabla u(t)\|_2^2 \\ &\quad +\max\left\{\begin{array}{c} (1-\alpha)c_*^{r_1}\left(\frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_1-2(\gamma+1)}{2(\gamma+1)}}, \\ (1-\alpha)c_*^{r_2}\left(\frac{2r_1(\gamma+1)}{r_1-2(\gamma+1)}E(0)\right)^{\frac{r_2-2(\gamma+1)}{2(\gamma+1)}}\end{array}\right\}\times\|\nabla u(t)\|_2^{2(\gamma+1)}.\end{aligned}$$

So,

$$\int_{\Omega}|u(t)|^{r(x)}dx \leq \beta_1\|\nabla u(t)\|_2^2+\beta_2\|\nabla u(t)\|_2^{2(\gamma+1)}, \quad \forall t\in[0,T_*]. \tag{2.17}$$

Since $\beta_1+\beta_2<1$, then

$$\int_{\Omega}|u(t)|^{r(x)}dx < \|\nabla u(t)\|_2^2+\|\nabla u(t)\|_2^{2(\gamma+1)}, \quad \forall t\in[0,T_*]. \tag{2.18}$$

This implies that $I(t)>0$, $\forall t\in[0,T_*]$.

Then, by repeating the above procedure, we can extend T_* to T. ■

2.2.1 The existence of weak solution

In this section, we are going to obtain the existence of the local solution to the problem (2.1). We will use Fadeo-Galerkin's approximation.

Theorem 2.1 *Assume that* (2.2) *holds. Suppose that* $u_0\in L^2(\Omega)$ *be given. We also assume that* $m(.)$ *and* $r(.)$ *satisfy the log-Hölder continuity condition. Then problem* (2.1) *has a weak local solution*

$$u\in L^{\infty}\left((0,T),H_0^1(\Omega)\right),\ u_t\in L^2\left((0,T),L^2(\Omega)\right).$$

Proof. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^1(\Omega)$ which constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k=span\{v_1,\ v_2,\ ...,\ v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. Applying the normalization, we have $\|v_l\|=1$ for any given integer k.

We conside the approximation solution

$$u_k(t)=\sum_{l=1}^{k}u_{lk}(t)v_l,$$

where u_k are the solution to the Cauchy problem

$$\begin{aligned}&\left(u_k^{'}(t)\,,\ v_l\right)-\left(M\left(\int_\Omega|\nabla u_k(t)|^2\,dx\right)\Delta u_k(t)\,,\ v_l\right)+\left(|u_k(t)|^{m(x)-2}u_k^{'}(t)\,,\ v_l\right)\\=&\left(|u_k(t)|^{r(x)-2}u_k(t)\,,\ v_l\right),\quad l=1,2,...,k\end{aligned}\tag{2.19}$$

$$u_k(0)=u_{0k}=\sum_{l=1}^{k}(u_k(0)\,,\ v_l)\,v_l\to u_0\text{ in }L^2(\Omega)\,.\tag{2.20}$$

Note that we can solve the system (2.19) and (2.20) by Picard's iterative method for ordinary differential equations.

Hence, there exists a solution in $[0,T_*)$ for some $T_*>0$ and we can extend this solution to the whole interval $[0,T]$ for any given $T>0$ by making use of the a priori estimates below.

Step 1 : **"The prior estimate"**

Multiplying equation (2.19) by $u_{lk}^{'}(t)$ and summing over l from 1 to k, we get

$$\sum_{l=1}^{k}u_{lk}^{'}(t)\left[\begin{array}{c}\left(u_k^{'}(t)\,,\ v_l\right)-\left(M\left(\int_\Omega|\nabla u_k(t)|^2\,dx\right)\Delta u_k(t)\,,\ v_l\right)\\+\left(|u_k(t)|^{m(x)-2}u_k^{'}(t)\,,\ v_l\right)=\left(|u_k(t)|^{r(x)-2}u_k(t)\,,\ v_l\right)\end{array}\right]$$

this implies

$$\begin{aligned}&\sum_{l=1}^{k}\left(\left|u_{lk}^{'}(t)\right|^2,\ v_l\right)+\sum_{l=1}^{k}\left(M\left(\int_\Omega|\nabla u_k(t)|^2\,dx\right)\nabla u_k^{'}(t)\,\nabla u_k(t)\,,\ v_l\right)\\&+\sum_{l=1}^{k}\left(|u_k(t)|^{m(x)-2}\left|u_{lk}^{'}(t)\right|^2,\ v_l\right)\\=&\sum_{l=1}^{k}\left(|u_k(t)|^{r(x)},\ v_l\right)\end{aligned}$$

so,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla u_k(t)\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|_2^{2(\gamma+1)}-\int_\Omega\frac{1}{r(x)}|u_k(t)|^{r(x)}\,dx\right)\\=&-\|u_{t,k}(t)\|_2^2-\int_\Omega|u_k(t)|^{m(x)-2}|u_{t,k}(t)|^2\,dx.\end{aligned}\tag{2.21}$$

Then,

$$E^{'}(u_k(t))=-\|u_{t,k}(t)\|_2^2-\int_\Omega|u_k(t)|^{m(x)-2}|u_{t,k}(t)|^2\,dx\leq 0.$$

Integrating (2.21) over $(0,t)$, gives the estimate

$$\begin{aligned}&\int_0^t\left[\frac{d}{dt}\left(\frac{1}{2}\|\nabla u_k(s)\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u_k(s)\|_2^{2(\gamma+1)}-\int_\Omega\frac{1}{r(x)}|u_k(s)|^{r(x)}\,dx\right)\right]ds\\=&-\int_0^t\|u_{t,k}(s)\|_2^2\,ds-\int_0^t\int_\Omega|u_k(s)|^{m(x)-2}|u_{t,k}(t)|^2\,dxds,\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\left\|\nabla u_k(t)\right\|_2^2+\frac{1}{2(\gamma+1)}\left\|\nabla u_k(t)\right\|_2^{2(\gamma+1)}-\int_{\Omega}\frac{1}{r(x)}\left|u_k(t)\right|^{r(x)}dx \\
& +\int_0^t\left\|u_{t,k}(s)\right\|_2^2ds+\int_0^t\int_{\Omega}\left|u_k(s)\right|^{m(x)-2}\left|u_{t,k}(t)\right|^2dxds \\
\leq\ & \frac{1}{2}\left\|\nabla u_k(0)\right\|_2^2+\frac{1}{2(\gamma+1)}\left\|\nabla u_k(0)\right\|_2^{2(\gamma+1)}-\int_{\Omega}\frac{1}{r(x)}\left|u_k(0)\right|^{r(x)}dx,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\left\|\nabla u_k(t)\right\|_2^2+\frac{1}{2(\gamma+1)}\left\|\nabla u_k(t)\right\|_2^{2(\gamma+1)}-\int_{\Omega}\frac{1}{r(x)}\left|u_k(t)\right|^{r(x)}dx \\
& +\int_0^t\left\|u_{t,k}(s)\right\|_2^2ds+\int_0^t\int_{\Omega}\left|u_k(s)\right|^{m(x)-2}\left|u_{t,k}(t)\right|^2dxds \\
\leq\ & E(0). \qquad (2.22)
\end{aligned}$$

Then, from (2.18) inequality (2.22) becomes

$$\begin{aligned}
& \frac{r_1-2}{2r_1}\sup_{t\in(0,T)}\left\|\nabla u_k(t)\right\|_2^2+\frac{r_1-2(\gamma+1)}{2r_1(\gamma+1)}\sup_{t\in(0,T)}\left\|\nabla u_k(t)\right\|_2^{2(\gamma+1)} \\
& +\int_0^t\left\|u_{t,k}(s)\right\|_2^2ds+\int_0^t\int_{\Omega}\left|u_k(s)\right|^{m(x)-2}\left|u_{t,k}(t)\right|^2dxds \\
\leq\ & E(0). \qquad (2.23)
\end{aligned}$$

From (2.23), we conclude that

$$\begin{cases} \{u_k\} \text{ is uniformly bounded in } L^{\infty}\left((0,T),\ H_0^1(\Omega)\right) \\ \{u_k'\} \text{ is uniformly bounded in } L^2\left((0,T),\ L^2(\Omega)\right) \end{cases}. \qquad (2.24)$$

Furthermore, it follows from Corollary 1.1 and (2.24) that

$$\begin{cases} \left\{|u_k|^{r(x)-2}u_k\right\} \text{ is uniformly bounded in } L^{\infty}\left((0,T),\ L^2(\Omega)\right) \\ \left\{|u_k|^{m(x)-2}u_k'\right\} \text{ is uniformly bounded in } L^{\infty}\left((0,T),\ L^2(\Omega)\right) \end{cases}. \qquad (2.25)$$

By (2.24) and (2.25), we infer that there exist a subsequence of u_k (denoted by the same symbol) and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^{\infty}\left((0,T),\ H_0^1(\Omega)\right), \\ u_k' \rightharpoonup u' \text{ weakly star in } L^2\left((0,T),\ L^2(\Omega)\right), \\ |u_k|^{r(x)-2}u_k \rightharpoonup \varphi \text{ weakly in } L^{\infty}\left((0,T),\ L^2(\Omega)\right), \\ |u_k|^{m(x)-2}u_k' \rightharpoonup \psi \text{ weakly in } L^{\infty}\left((0,T),\ L^2(\Omega)\right). \end{cases} \qquad (2.26)$$

By the Aubin-Lions compactness Lemma (see [40]), we conclude from (2.26) that

$$u_k \rightarrow u \text{ is strongly in } C\left([0,T],\ H_0^1(\Omega)\right)$$

which implies

$$u_k \to u \text{ everywhere in } \Omega \times [0,T]. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$\begin{cases} |u_k|^{r(x)-2} u_k \rightharpoonup |u|^{r(x)-2} u \text{ weakly in } L^{\infty}((0,T),\ L^2(\Omega)) \\ |u_k|^{m(x)-2} u_k' \rightharpoonup |u|^{m(x)-2} u' \text{ weakly in } L^{\infty}((0,T),\ L^2(\Omega)) \end{cases}. \tag{2.28}$$

Step 2 : Setting up $k \to +\infty$ and passing to the limit in (2.19), we obtain

$$\begin{aligned} &\left(u'(t),\ v_l\right) - \left(M\left(\int_\Omega |\nabla u(t)|^2 dx\right) \Delta u(t),\ v_l\right) + \left(|u(t)|^{m(x)-2} u'(t),\ v_l\right) \\ = &\left(|u(t)|^{r(x)-2} u(t),\ v_l\right), \quad l = 1, 2, ..., k. \end{aligned} \tag{2.29}$$

Since $\{v_l\}_{l=1}^{\infty}$ is a basis of $H_0^1(\Omega)$, we deduce that u satisfies equation (2.1).
From (2.26) and Lemma $3.1.7$ of [68] with $B = L^2(\Omega)$, we infer that

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } L^2(\Omega) \tag{2.30}$$

we get from (2.20) and (2.30) that $u(0) = u_0$. Thus, the proof of weak existence solution is complete. ■

2.2.2 Global existence of solution

Theorem 2.2 *Under the assumptions of Lemma* 2.2*, the local solution* u *of* (2.1) *is global.*

Proof. Based on the definition of E, we have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)} dx \\ &\geq \frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1 - 2(\gamma+1)}{2(\gamma+1)r_1}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{1}{r_1} I(t). \end{aligned}$$

By Lemma 2.2, we find

$$E(t) \geq \frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1 - 2(\gamma+1)}{2(\gamma+1)r_1}\|\nabla u(t)\|_2^{2(\gamma+1)}.$$

Consequently, a constant $C > 0$ exists such that

$$\|\nabla u(t)\|_2^2 \leq C E(t). \tag{2.31}$$

By Lemma 2.1, we obtain

$$\|\nabla u(t)\|_2^2 \leq C E(0). \tag{2.32}$$

This implies that the local solution u of problem (2.1) is global in time and bounded. ■

2.2.3 Stability of solution

In this section our main result is established based on Komornik's inequality. For this, we need the following Lemma.

Lemma 2.3 *Suppose that the assumptions of Lemma 2.2 hold, then there exists a positive constant c such that*

$$\int_{\Omega} |u(t)|^{m(x)} dx \leq cE(t). \tag{2.33}$$

Proof. Let c_* be the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{m(.)}(\Omega)$. Then, we have

$$\begin{aligned}
\int_{\Omega} |u(t)|^{m(x)} dx &= \max\left\{\|u(t)\|_{m(.)}^{m_1}, \|u(t)\|_{m(.)}^{m_2}\right\} \\
&\leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2}\right\} \\
&\leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1-2}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2-2}\right\} \times \|\nabla u(t)\|_2^2.
\end{aligned}$$

Using (2.15), we obtain the desired result

$$\int_{\Omega} |u(t)|^{m(x)} dx \leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1-2}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2-2}\right\} \times \frac{2r_1}{r_1-2}E(t).$$

This implies,

$$\int_{\Omega} |u(t)|^{m(x)} dx \leq cE(t).$$

■

Now, we state our main result.

Theorem 2.3 *Let the assumptions of Lemma 2.2 hold, then there exists a constant $C > 0$, such that the global solution of (2.1) satisfies*

$$E(t) \leq E(0)\left(\frac{C+qt}{C+qC}\right)^{\frac{-1}{q}}, \ \forall t \geq C.$$

Proof. Multiplying the first equation of (2.1) by $u(t)E^q(t)$ $(q > 0)$ and integrating over $\Omega \times (S,T)$, we obtain

$$\begin{aligned}
&\int_S^T \int_{\Omega} E^q(t)\left(u(t)u_t(t) - u(t)\left(M\left(\int_{\Omega}|\nabla u(t)|^2 dx\right)\Delta u(t) + |u(t)|^{m(x)-2}u_t(t)\right)\right) dxdt \\
= &\int_S^T E^q(t)\int_{\Omega} u(t)|u(t)|^{r(x)-2}u(t)\,dxdt.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_S^T \int_{\Omega} E^q(t)\left(u(t)u_t(t) + M\left(\int_{\Omega}|\nabla u(t)|^2 dx\right)|\nabla u(t)|^2 + u(t)|u(t)|^{m(x)-2}u_t(t)\right) dxdt \\
= &\int_S^T E^q(t)\int_{\Omega} |u(t)|^{r(x)} dxdt.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_S^T \int_\Omega E^q(t)\Big(u(t)\,u_t(t) + \left(1+\|\nabla u(t)\|_2^{2\gamma}\right)|\nabla u(t)|^2 + u(t)\,|u(t)|^{m(x)-2}\,u_t(t)\Big)\,dxdt \\
=\;& \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_S^T \int_\Omega E^q(t)\Big(u(t)\,u_t(t) + |\nabla u(t)|^2 + \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2 + u(t)\,|u(t)|^{m(x)-2}\,u_t(t)\Big)\,dxdt \\
=\;& \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.
\end{aligned}$$

We add and substract the term,

$$\int_S^T E^q(t)\int_\Omega \left(\beta_1|\nabla u(t)|^2 + \beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\right)dxdt$$

we find

$$\begin{aligned}
&\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt + \int_S^T E^q(t)\int_\Omega |\nabla u(t)|^2\,dxdt \\
&+\int_S^T E^q(t)\int_\Omega \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt \\
&+\int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}\,u_t(t)\,dxdt + \int_S^T E^q(t)\int_\Omega \beta_1|\nabla u(t)|^2\,dxdt \\
&+\int_S^T E^q(t)\int_\Omega \beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt - \int_S^T E^q(t)\int_\Omega \beta_1|\nabla u(t)|^2\,dxdt \\
&-\int_S^T E^q(t)\int_\Omega \beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt \\
=\;& \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt,
\end{aligned}$$

and use (2.17) to get

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u(t)|^2\,dxdt + (1-\beta_2)\int_S^T E^q(t)\int_\Omega \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt \\
&+\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt + \int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}\,u_t(t)\,dxdt \\
=\;& -\int_S^T E^q(t)\int_\Omega \Big(\beta_1|\nabla u(t)|^2 + \beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2 - |u(t)|^{r(x)}\Big)\,dxdt \\
\leq\;& 0. \qquad (2.34)
\end{aligned}$$

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u(t)|^2\,dxdt+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt\\
&+\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt+\int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}u_t(t)\,dxdt\\
\leq\;& 0.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\xi\int_S^T E^q(t)\int_\Omega\left(\frac{1}{2}|\nabla u(t)|^2+\frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2-\frac{|u(t)|^{r(x)}}{r(x)}\right)dxdt \qquad (2.35)\\
\leq\;& (1-\beta_1)\int_S^T E^q(t)\int_\Omega \frac{1}{2}|\nabla u(t)|^2\,dxdt+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \frac{\|\nabla u(t)\|_2^{2\gamma}}{2(\gamma+1)}|\nabla u(t)|^2\,dxdt\\
\leq\;& (1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u(t)|^2\,dxdt+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,dxdt.
\end{aligned}$$

Then

$$\begin{aligned}
&\xi\int_S^T E^q(t)\int_\Omega\left(\frac{1}{2}|\nabla u(t)|^2+\frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2-\frac{|u(t)|^{r(x)}}{r(x)}\right)dxdt\\
\leq\;& -\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt-\int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}u_t(t)\,dxdt,
\end{aligned}$$

where $\xi=\min\left((1-\beta_1),(1-\beta_2)\right).$ By (2.34), (2.35) and the definition of $E(t)$, we get

$$\xi\int_S^T E^{q+1}(t)\,dt\leq-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt-\int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}u_t(t)\,dxdt. \qquad (2.36)$$

We estimate the terms on the right-hand side of (2.36). For the first term, we use the Young inequality

$$XY\leq\frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,Y\geq 0,\ \varepsilon>0 \text{ and } \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1,$$

and get

$$-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,dxdt\leq\int_S^T E^q(t)\int_\Omega\left(\varepsilon c\,|u(t)|^2+c_\varepsilon\,|u_t(t)|^2\right)dxdt. \qquad (2.37)$$

We use again the above Young inequality to obtain

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega u(t)\,|u(t)|^{m(x)-2}u_t(t)\,dxdt\\
=\;& -\int_S^T E^q(t)\int_\Omega |u(t)|^{\frac{m(x)-2}{2}}u(t)\,u_t(t)\,|u(t)|^{\frac{m(x)-2}{2}}\,dxdt\\
\leq\;& \int_S^T E^q(t)\int_\Omega\left(\varepsilon c\,|u(t)|^{m(x)}+c_\varepsilon\,|u(t)|^{m(x)-2}u_t(t)^2\right)dxdt. \qquad (2.38)
\end{aligned}$$

By (2.37) and (2.38), inequality (2.36) becomes

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt &\le \int_S^T E^q(t)\int_\Omega\left(\varepsilon c\,|u(t)|^2+c_\varepsilon\,|u_t(t)|^2\right)dxdt\\
&\quad+\int_S^T E^q(t)\int_\Omega\left(\varepsilon c\,|u(t)|^{m(x)}+c_\varepsilon\,|u(t)|^{m(x)-2}u_t^2(t)\right)dxdt\\
&\le \varepsilon c\int_S^T E^q(t)\int_\Omega\left(|u(t)|^2+|u(t)|^{m(x)}\right)dxdt\\
&\quad+c_\varepsilon\int_S^T E^q(t)\int_\Omega\left(|u_t(t)|^2+|u(t)|^{m(x)-2}u_t^2(t)\right)dxdt. \qquad (2.39)
\end{aligned}$$

We use (2.31), Lemma 2.3, and definition of $E'(t)$ to obtain

$$\xi\int_S^T E^{q+1}(t)\,dt\le\varepsilon c\int_S^T E^q(t)\int_\Omega\left(|\nabla u(t)|^2+|u(t)|^{m(x)}\right)dxdt+c_\varepsilon\int_S^T E^q(t)\left(-E'(t)\right)dt$$

so,

$$\xi\int_S^T E^{q+1}(t)\,dt\le\varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\int_S^T E^q(t)\left(-E'(t)\right)dt. \qquad (2.40)$$

This implies

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt &\le \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\left(E^{q+1}(S)-E^{q+1}(T)\right)\\
&\le \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E^q(0)\,E(S). \qquad (2.41)
\end{aligned}$$

Choosing ε so small that $\xi>\varepsilon c$, we arrive at

$$\int_S^T E^{q+1}(t)\,dt\le cE^q(0)\,E(S).$$

Letting $T\to\infty$, we get

$$\int_S^\infty E^{q+1}(t)\,dt\le cE^q(0)\,E(S).$$

Komornik's inequality yields the result of stability

$$E(t)\le E(0)\left(\frac{c+qt}{c+qc}\right)^{\frac{-1}{q}},\ \forall t\ge c.$$

■

Chapter 3

Blow-up of solutions for wave equation with multiple $\alpha(x)$ −Laplacian and variable-exponent nonlinearities

The work in this chapter is devoted to the study of asymptotic behavior of solutions for wave equation with multiple $\alpha(x)$-Laplacian and variable-exponent nonlinearities with negative initial energy and under suitable assumptions on the nonlinearities. The results appearing in this context have been published throught [24]

3.1 Introduction

In this chapter, we consider the following wave equation with multiple $\alpha(x)$ −Laplacian and variable exponent nonlinearities

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) - \gamma\Delta u_t = |u|^{p(x)-2}u, \text{ in } \Omega\times(0,T), \tag{3.1}$$

$$u(x,t) = 0, \quad x\in\partial\Omega, t>0, \tag{3.2}$$

with the initial conditions

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x\in\Omega, \tag{3.3}$$

where $\gamma > 0$, $0 < T < \infty$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. $m(.)$, $r(.)$ and $p(.)$ are given measurable functions on Ω satisfying

$$2 \leq r_1 \leq r(x) \leq r_2 \leq m_1 \leq m(x) \leq m_2 \leq p_1 \leq p(x) \leq p_2 < r^*, \tag{3.4}$$

with

$$\begin{aligned} r_1 &: = ess \inf_{x\in\Omega} r(x), \ r_2 := ess \sup_{x\in\Omega} r(x), \\ m_1 &: = ess \inf_{x\in\Omega} m(x), \ m_2 := ess \sup_{x\in\Omega} m(x), \\ p_1 &: = ess \inf_{x\in\Omega} p(x), \ p_2 := ess \sup_{x\in\Omega} p(x), \end{aligned}$$

and

$$r^* = \begin{cases} \frac{Nm(x)}{ess \sup_{x\in\Omega}(N-m(x))}, & \text{if } m_2 < N \\ +\infty, & \text{if } m_2 \geq N \end{cases}.$$

We also assume that $m(.)$ satisfy the log-Hölder continuity conditions:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e } x, y \in \Omega, \text{ with } |x-y| < \delta, \ A > 0, \ 0 < \delta < 1. \tag{3.5}$$

Problems of this type arise in many different fields, such as physics, acoustics, electromagnetics, fluid mechanics, and so forth.

Many authors have studied problem (3.1) in case of constant and variable exponent nonlinearities see e.g., [19], [37], [38], [52].

In the case where $m(.), r(.)$ and $p(.)$ are constants, many problems similar or related to problem (3.1) has been exhaustively investigated in result of blow-up, global existence and stability have been established. Chen et al [10] considered the nonlinear p-Laplacian wave equation:

$$u_{tt} - \text{div}\left(|\nabla u|^{p-2} \nabla u\right) - \Delta u_t + g(x,t) = f(x), \text{ in } \Omega \times (0,T) \tag{3.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $2 \leq p < n$ and f, g are given functions. They proved the global existence, uniqueness under suitable conditions on the initial data and the functions f, g, they also discussed the long-time behavior of the solution. In [57], Erhan studied the following quasilinear hyperbolic equation:

$$u_{tt} - \text{div}\left(|\nabla u|^{m-2} \nabla u\right) - \Delta u_t + |u_t|^{q-1} u_t = |u|^{p-1} u, \tag{3.7}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ $(n \geq 1)$, $m > 0, p, q \geq 1$. He proved the decay estimates of the energy function by using Nakao's inequality and he also obtained the blow up of solutins and lifespan estimates in three different ranges of the initial energy. In [53], Ouaoua and Maouni considered the following equation:

$$u_{tt} - \text{div}\left(\frac{|\nabla u|^{2m-2} \nabla u}{\sqrt{1 + |\nabla u|^{2m}}}\right) - \omega \Delta u_t + \mu u_t = |u|^{p-2} u, \text{ in } \Omega \times (0,T) \tag{3.8}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. ω, μ and m, p are real numbers, they proved local existence and uniqueness of the solution by using the Faedo–Galerkin method and that the local solution is globally in time. They also proved that the solutions with some conditions exponentially decay. In [7], Benaissa and Mokeddem looked into the following equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \sigma(t) \operatorname{div}\left(|\nabla u_t|^{m-2} \nabla u_t\right) = 0, \text{ in } \Omega \times (0, T) \tag{3.9}$$

where σ is a positive function, p, $m \geq 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, they gave an energy decay estimate for the solution. In [48], the work of Messaoudi and Houari considered the nonlinear wave equation:

$$u_{tt} - \Delta u_t - \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{\beta-2} \nabla u_t\right) + a|u_t|^{m-2} u_t = b|u|^{p-2} u, \tag{3.10}$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. They proved under suitable conditions on α, β, m, $p > 2$ and for negative initial energy, a global non-existence theorem. Ye in [67], investigated the blow-up property of solutions of a quasilinear hyperbolic system. He proved that certain solutions with positive initial energy blow-up in finite time under suitable conditions and gave estimation for the solution.

In the case of variable exponents nonlinearities, Antonsev, Ferreira and Erhan in [1] considered a nonlinear plate Petrovesky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, \tag{3.11}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. They proved the local weak solutions by using the Banach contraction mapping principle. Then, they showed that the solution is global if $p(.) \geq q(.)$ and they proved that a solution with negative initial energy and $p(.) < q(.)$ blows up in finite time. In [56], Erhan considered the strongly damped nonlinear Klein- Gordon equation:

$$u_{tt} - \Delta u - \Delta u_t + m^2 u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, \tag{3.12}$$

where Ω is a bounded domain in $\mathbb{R}^n$. He obtained a nonexistence of solutions if variable exponents $p(.), q(.)$ and initial data satisfy some conditions. In [3], [4], Antontsev considered the equation:

$$u_{tt} - \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2} \nabla u\right) - \alpha\Delta u_t = b(x,t)|u|^{\sigma(x,t)-2} u, \text{ in } \Omega \times (0,T), \tag{3.13}$$

where $\alpha > 0$ is a constant, a, b, p, σ are given functions and Ω is a bounded domain in $\mathbb{R}^n$. Under appropriate conditions on the initial data and the functions a, b, p, σ, he proved some blow-up results for certain solutions with non positive initial energy. And discussed the same equation and proved the local and global existence of a weak solution under suitable conditions on a, b, p, σ. In [47], Messaoudi and Talahmeh considered the following equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(x)-2}\nabla u\right) + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u, \text{ in } \Omega \times (0,T), \tag{3.14}$$

where a, b is a nonnegative constant. They proved a finite-time blow-up result of the solution with negative initial energy as well as for certain solutions with positive initial energy. In [38], the cas where $m(x) = 2$ and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy. In [44], Messaoudi and Al.Smail the case where $b = 0$ and $a = 1$ of the same equation (3.14). They proved the decay estimates for the solution under suitable assumptions on the variable exponents m, r and the initial data. They also gave two numerical applications to illustrate your theoretical results.

3.2 The main of results

Theorem 3.1 *[4] Let $u_0 \in W_0^{1,m(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ and assume that the exponents m, r, p satisfy conditions (3.4) and (3.5). Then problem (3.1) has a unique weak solution such that*

$$\begin{aligned} u &\in L^\infty\left((0,T), W_0^{1,m(.)}(\Omega)\right), \\ u_t &\in L^\infty\left((0,T), L^2(\Omega)\right), \\ u_{tt} &\in L^\infty\left((0,T), W_0^{-1,m'(.)}(\Omega)\right), \end{aligned}$$

where $\frac{1}{m(.)} + \frac{1}{m'(.)} = 1$.

In this section to state and prove our result, we define the potential energy function by the following:

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)} dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx. \tag{3.15}$$

Lemma 3.1 *Assume that u be a solution of (3.1). Then, we have*

$$E'(t) = -\int_\Omega |\nabla u_t|^{r(x)} dx - \gamma\int_\Omega |\nabla u_t|^2 dx \le 0,\ t \in [0,T] \tag{3.16}$$

and

$$E(t) \le E(0). \tag{3.17}$$

Proof. we multiply the first equation of (3.1) by u_t and integrate over the domain Ω to get

$$\int_\Omega u_t\left[u_{tt} - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{r(x)-2}\nabla u_t\right) - \gamma\Delta u_t\right] dx = \int_\Omega u_t|u|^{p(x)-2}u dx$$

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega}u_t^2dx-\int_{\Omega}\operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right)\frac{d}{dt}udx+\int_{\Omega}|\nabla u_t|^{r(x)-2}\nabla u_t^2dx+\gamma\int_{\Omega}\nabla u_t^2dx=\int_{\Omega}\frac{d}{dt}u\,|u|^{p(x)-2}udx.$$

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega}u_t^2dx+\int_{\Omega}|\nabla u|^{m(x)-2}\nabla u\frac{d}{dt}\nabla udx+\int_{\Omega}|\nabla u_t|^{r(x)}dx+\gamma\int_{\Omega}\nabla u_t^2dx=\frac{d}{dt}\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}dx.$$

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega}u_t^2dx+\frac{d}{dt}\int_{\Omega}\frac{1}{m(x)}|\nabla u|^{m(x)}dx+\int_{\Omega}|\nabla u_t|^{r(x)}dx+\gamma\int_{\Omega}\nabla u_t^2dx=\frac{d}{dt}\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}dx.$$

This implies that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega}u_t^2dx+\int_{\Omega}\frac{1}{m(x)}|\nabla u|^{m(x)}dx-\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}dx\right)=-\int_{\Omega}|\nabla u_t|^{r(x)}dx-\gamma\int_{\Omega}\nabla u_t^2dx.$$

So,

$$E(t)=\frac{1}{2}\int_{\Omega}u_t^2dx+\int_{\Omega}\frac{1}{m(x)}|\nabla u|^{m(x)}dx-\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}dx$$

and

$$E'(t)=-\int_{\Omega}|\nabla u_t|^{r(x)}dx-\gamma\int_{\Omega}\nabla u_t^2dx\leq 0.$$

Integrating (3.16) over $(0,t)$, we obtain

$$\int_0^t E'(s)\,ds=E(t)-E(0)\leq 0$$

$$E(t)\leq E(0).$$

■

3.2.1 Blow up of solution

Let $H(t)=-E(t)$, using equation (3.15) and (3.17), we have

$$H(0)\leq H(t)\leq\frac{1}{p_1}\int_{\Omega}|u|^{p(x)}dx, \tag{3.18}$$

for any $t\geq 0$. Let

$$L(t)=H^{(1-\sigma)}(t)+\varepsilon\int_{\Omega}uu_tdx, \tag{3.19}$$

where ϵ and σ are constants.

Lemma 3.2 *Suppose that $u(x,t)$ is a regular solution of (3.1) under conditions (3.2)-(3.3), and the initial energy satisfies $E(0)<0$. If $\sigma<\min\left\{\frac{m_1-2}{p_1},\frac{m_1-2}{p_2},\frac{m_1-r_1}{p_1(r_2-1)},\frac{m_1-r_2}{p_1(r_2-1)},\frac{m_1-r_1}{p_2(r_2-1)},\frac{m_1-r_2}{p_2(r_2-1)},1\right\}$, then there exists a positive constant c such that*

$$L'(t)\geq c\varepsilon\left(H(t)+\int_{\Omega}|\nabla u|^{m(x)}dx+\int_{\Omega}u_t^2dx\right). \tag{3.20}$$

Proof. Differentiating (3.19), the following equality can be obtained:

$$L^{'}(t) = (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \int_{\Omega} u_t^2 dx + \varepsilon \int_{\Omega} u u_{tt} dx. \tag{3.21}$$

Using Eqs. (3.1)-(3.3) and Green's first formula, we obtain

$$\begin{aligned} L^{'}(t) &= (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \int_{\Omega} u_t^2 dx \qquad (3.22)\\ &+\varepsilon \int_{\Omega} u \left[\operatorname{div}\left(|\nabla u|^{m(x)-2} \nabla u\right) + \operatorname{div}\left(|\nabla u_t|^{r(x)-2} \nabla u_t\right) + \gamma \Delta u_t + |u|^{p(x)-2} u\right] dx. \end{aligned}$$

$$\begin{aligned} &= (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \int_{\Omega} |\nabla u|^{m(x)-2} \nabla u \nabla u dx \qquad (3.23)\\ &-\varepsilon \int_{\Omega} |\nabla u_t|^{r(x)-2} \nabla u_t \nabla u dx - \varepsilon\gamma \int_{\Omega} \nabla u \nabla u_t dx + \varepsilon \int_{\Omega} |u|^{p(x)} dx. \end{aligned}$$

$$\begin{aligned} &= (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \int_{\Omega} |\nabla u|^{m(x)} dx \\ &-\varepsilon \int_{\Omega} |\nabla u_t|^{r(x)-2} \nabla u_t \nabla u dx - \varepsilon\gamma \int_{\Omega} \nabla u \nabla u_t dx + \varepsilon \int_{\Omega} |u|^{p(x)} dx. \end{aligned}$$

We can obtain the following inequalities from Young's inequality and Hölder inequality:

$$\int_{\Omega} \nabla u \nabla u_t dx \leq \frac{1}{4k} \int_{\Omega} |\nabla u|^2 dx + k \int_{\Omega} |\nabla u_t|^2 dx. \tag{3.24}$$

$$\begin{aligned} \int_{\Omega} |\nabla u_t|^{r(x)-2} \nabla u_t \nabla u dx &\leq \frac{1}{r_1} \int_{\Omega} \delta^{r(x)} |\nabla u|^{r(x)} dx \qquad (3.25)\\ &+\frac{r_2-1}{r_2} \int_{\Omega} \delta^{-\frac{r(x)}{r(x)-1}} |\nabla u_t|^{r(x)} dx. \end{aligned}$$

Hence,

$$\begin{aligned} L^{'}(t) &\geq (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \int_{\Omega} |\nabla u|^{m(x)} dx \qquad (3.26)\\ &-\frac{\varepsilon}{r_1} \int_{\Omega} \delta^{r(x)} |\nabla u|^{r(x)} dx - \varepsilon \frac{r_2-1}{r_2} \int_{\Omega} \delta^{-\frac{r(x)}{r(x)-1}} |\nabla u_t|^{r(x)} dx \\ &-\frac{\varepsilon\gamma}{4k} \int_{\Omega} |\nabla u|^2 dx - \varepsilon\gamma k \int_{\Omega} |\nabla u_t|^2 dx + \varepsilon \int_{\Omega} |u|^{p(x)} dx. \end{aligned}$$

Taking $k = M_1 H^{-\sigma}(t)$ and $\delta^{-\frac{r(x)}{r(x)-1}} = M_2 H^{-\sigma}(t)$, and using $H(t) > 0$, we obtain

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{m(x)} dx \\ & - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & - \varepsilon \frac{r_2-1}{r_2} M_2 H^{-\sigma}(t) \int_\Omega |\nabla u_t|^{r(x)} dx \\ & - \frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx - \varepsilon\gamma M_1 H^{-\sigma}(t) \int_\Omega |\nabla u_t|^2 dx \\ & + \varepsilon \int_\Omega |u|^{p(x)} dx. \end{aligned} \tag{3.27}$$

Let $M = \max\left\{M_1, \frac{r_2-1}{r_2} M_2\right\}$, using the energy functional (3.16), we have

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{m(x)} dx \\ & - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & - \varepsilon \max\left\{M_1, \frac{r_2-1}{r_2} M_2\right\} H^{-\sigma}(t) \left[\int_\Omega |\nabla u_t|^{r(x)} dx + \gamma \int_\Omega |\nabla u_t|^2 dx\right] \\ & - \frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx. \end{aligned}$$

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{m(x)} dx \\ & - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & - \varepsilon M H^{-\sigma}(t) \left[\int_\Omega |\nabla u_t|^{r(x)} dx + \gamma \int_\Omega |\nabla u_t|^2 dx\right] \\ & + \varepsilon \int_\Omega |u|^{p(x)} dx. \end{aligned}$$

Then,

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx \\ & - \varepsilon \int_\Omega |\nabla u|^{m(x)} dx - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & - \frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx. \end{aligned} \tag{3.28}$$

From the definition of $H(t)$, it follows that there is a constant k such that

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + kH(t) + \varepsilon \int_\Omega u_t^2 dx \\ & -\varepsilon \int_\Omega |\nabla u|^{m(x)} dx - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx \\ & -k\left[-\frac{1}{2}\int_\Omega u_t^2 dx - \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx\right]. \end{aligned} \tag{3.29}$$

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + kH(t) + \varepsilon \int_\Omega u_t^2 dx \\ & -\varepsilon \int_\Omega |\nabla u|^{m(x)} dx - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx \\ & -\frac{k}{2}\int_\Omega u_t^2 dx + k\int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} dx - k\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \end{aligned}$$

So, we have

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + kH(t) + \varepsilon \int_\Omega u_t^2 dx \\ & -\varepsilon \int_\Omega |\nabla u|^{m(x)} dx - \frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega |u|^{p(x)} dx \\ & -\frac{k}{2}\int_\Omega u_t^2 dx + \frac{k}{m_2}\int_\Omega |\nabla u|^{m(x)} dx - \frac{k}{p_1}\int_\Omega |u|^{p(x)} dx \end{aligned}$$

then we get,

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + kH(t) + \left(\varepsilon + \frac{k}{2}\right) \int_\Omega u_t^2 dx \\ & +\left(\frac{k}{m_2} - \varepsilon\right) \int_\Omega |\nabla u|^{m(x)} dx + \left(\varepsilon - \frac{k}{p_1}\right) \int_\Omega |u|^{p(x)} dx \\ & -\frac{\varepsilon M_2^{1-r_1}}{r_1} H^{\sigma(r_2-1)}(t) \int_\Omega |\nabla u|^{r(x)} dx \\ & -\frac{\varepsilon\gamma}{4M_1} H^{\sigma}(t) \int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{3.30}$$

Using $W^{1,m(.)}(\Omega) \hookrightarrow H^1(\Omega)$, $W^{1,m(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, $m(x) \geq r(x)$, and inequality (3.18), after simple calculation, it can be concluded

$$\begin{aligned} H^{\sigma}(t)\int_{\Omega}|\nabla u|^2\,dx &\leq \left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|u|^{p(x)}\,dx\right)^{\sigma}\int_{\Omega}|\nabla u|^2\,dx \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|\nabla u|^{p(x)}\,dx\right)^{\sigma}\int_{\Omega}|\nabla u|^2\,dx \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma}\left[\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1}{m_1}}+\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2}{m_1}}\right]\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{2}{m_1}} \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1}{m_1}}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{2}{m_1}} \\ &\quad +C\left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2}{m_1}}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{2}{m_1}} \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1+2}{m_1}}+C\left(\frac{1}{p_1}\right)^{\sigma}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2+2}{m_1}} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} H^{\sigma(r_2-1)}(t)\int_{\Omega}|\nabla u|^{r(x)}\,dx &\leq \left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|u|^{p(x)}\,dx\right)^{\sigma(r_2-1)}\int_{\Omega}|\nabla u|^{r(x)}\,dx \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|\nabla u|^{p(x)}\,dx\right)^{\sigma(r_2-1)}\int_{\Omega}|\nabla u|^{r(x)}\,dx \\ &\leq C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_1}{m_1}} \\ &\quad +C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_1}{m_1}} \\ &\quad +C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_2}{m_1}} \\ &\quad +C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_2}{m_1}}. \end{aligned} \tag{3.32}$$

For any constants $z \geq 0$ and $M > 0$, the algebric inequality

$$z^{\upsilon} \leq z+1 \leq \left(1+\frac{1}{M}\right)(z+M), \quad (0<\upsilon<1) \tag{3.33}$$

holds. Using known condition $\sigma < \min\left\{\frac{m_1-2}{p_1}, \frac{m_1-2}{p_2}, \frac{m_1-r_1}{p_1(r_2-1)}, \frac{m_1-r_2}{p_1(r_2-1)}, \frac{m_1-r_1}{p_2(r_2-1)}, \frac{m_1-r_2}{p_2(r_2-1)}\right\}$. Hence from (3.33) the following inequalities can be acquired:

$$\begin{aligned} \left(\int_{\Omega}|\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1+2}{m_1}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx+H(0)\right) \\ &\leq \left(1+\frac{1}{H(0)}\right)\left(\int_{\Omega}|\nabla u|^{m(x)}\,dx+H(t)\right) \end{aligned} \tag{3.34}$$

and the same for the other inequalities

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2+2}{m_1}} \leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.35}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_1}{m_1}} \leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.36}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_1}{m_1}} \leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.37}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_1(r_2-1)+r_2}{m_1}} \leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.38}$$

$$\left(\int_\Omega |\nabla u|^{m(x)}\,dx\right)^{\frac{\sigma p_2(r_2-1)+r_2}{m_1}} \leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.39}$$

From inequalities (3.31) and (3.32), we can obtain that there exists two positive constants N_1 and N_2 such that

$$\frac{\varepsilon\gamma}{4M_1}H^{\sigma}(t)\int_\Omega |\nabla u|^2\,dx \leq \varepsilon N_1\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right) \tag{3.40}$$

and

$$\frac{\varepsilon M_2^{1-r_1}}{r_1}H^{\sigma(r_2-1)}(t)\int_\Omega |\nabla u|^{r(x)}\,dx \leq \varepsilon N_2\left(\int_\Omega |\nabla u|^{m(x)}\,dx + H(t)\right). \tag{3.41}$$

with

$$\begin{aligned} N_1 &= \frac{\gamma}{2M_1}C\left(\frac{1}{p_1}\right)^{\sigma}\left(1+\frac{1}{H(0)}\right). \\ N_2 &= \frac{4M_2^{1-r_1}}{r_1}C\left(\frac{1}{p_1}\right)^{\sigma(r_2-1)}\left(1+\frac{1}{H(0)}\right). \end{aligned}$$

Consequently, taking $k = \frac{1}{2}(N_1+N_2+m_2N_1+m_2N_2+m_2+p_1)\varepsilon$, we obtain from (3.30) that there exists a positive constant c such that

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t)+kH(t)+\left(\varepsilon+\frac{k}{2}\right)\int_\Omega u_t^2dx \\ & +\left(\frac{k}{m_2}-\varepsilon\right)\int_\Omega |\nabla u|^{m(x)}\,dx+\left(\varepsilon-\frac{k}{p_1}\right)\int_\Omega |u|^{p(x)}\,dx \\ & -\frac{\varepsilon\gamma}{4M_1}H^{\sigma}(t)\int_\Omega |\nabla u|^2\,dx-\frac{\varepsilon M_2^{1-r_1}}{r_1}H^{\sigma(r_2-1)}(t)\int_\Omega |\nabla u|^{r(x)}\,dx \end{aligned}$$

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t)+kH(t)+\left(\varepsilon+\frac{k}{2}\right)\int_\Omega u_t^2dx \\ & +\left(\frac{k}{m_2}-\varepsilon\right)\int_\Omega |\nabla u|^{m(x)}\,dx+\left(\varepsilon-\frac{k}{p_1}\right)\int_\Omega |u|^{p(x)}\,dx \\ & -\varepsilon N_1\left(\int_\Omega |\nabla u|^{m(x)}\,dx+H(t)\right)-\varepsilon N_2\left(\int_\Omega |\nabla u|^{m(x)}\,dx+H(t)\right) \end{aligned}$$

$$
\begin{aligned}
L'(t) &\geq (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + (k-\varepsilon N_1 - \varepsilon N_2) H(t) \\
&+ \left(\varepsilon + \frac{k}{2}\right) \int_\Omega u_t^2 dx + \left(\varepsilon - \frac{k}{p_1}\right) \int_\Omega |u|^{p(x)} dx \\
&+ \left(\frac{k}{m_2} - \varepsilon - \varepsilon N_1 - \varepsilon N_2\right) \int_\Omega |\nabla u|^{m(x)} dx.
\end{aligned}
$$

$$
\begin{aligned}
L'(t) &\geq (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) \\
&+ c\varepsilon \left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^{m(x)} dx \right].
\end{aligned} \tag{3.42}
$$

Taking $0 < \varepsilon < \frac{1-\sigma}{M}$, we can get the following formula from (3.19):

$$
\begin{aligned}
L(0) &= H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0(x) u_1(x) dx \\
&> 0.
\end{aligned} \tag{3.43}
$$

Using inequality (3.42), it holds

$$
L'(t) \geq c\varepsilon \left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^{m(x)} dx \right]. \tag{3.44}
$$

After integral, we can get $L(t) \geq L(0) > 0, (\forall t \geq 0)$. ■

Theorem 3.2 *Suppose* $\gamma > 0$ *and* $2 \leq r_1 \leq r(x) \leq r_2 \leq m_1 \leq m(x) \leq m_2 \leq p_1 \leq p(x) \leq p_2 < r^*$, *where* r^* *is the critical Sobolev index in* $W^{1,m(.)}(\Omega)$. *If the initial energy* $E(0) < 0$, *then any regular solutions of equation* (3.1)-(3.3) *must blow up in finite time.*

Proof. Firstly, it is proved that when $0 < \sigma < \frac{m_1-2}{2m_1}$, there exists a positive constant C such that

$$
L^{\frac{1}{1-\sigma}}(t) \leq C \left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^{m(x)} dx \right]. \tag{3.45}
$$

In fact, using (3.19), we get

$$
\begin{aligned}
L^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx \right]^{\frac{1}{1-\sigma}} \\
&\leq C(\varepsilon, \sigma) \left[H(t) + \left(\int_\Omega u u_t dx \right)^{\frac{1}{1-\sigma}} \right]
\end{aligned} \tag{3.46}
$$

Furthermore, by using Hölder's inequality and Young's inequality, we can get

$$
\left(\int_\Omega u u_t dx \right)^{\frac{1}{1-\sigma}} \leq \left(\int_\Omega u^2 dx \right)^{\frac{1}{2(1-\sigma)}} \left(\int_\Omega u_t^2 dx \right)^{\frac{1}{2(1-\sigma)}}
$$

we have the embedding $L^{m(.)}(\Omega) \hookrightarrow L^2(\Omega)$, then

$$\|u\|_2 \leq c\|u\|_{m(.)}$$

this implies that

$$\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \leq c\left(\int_\Omega u^{m(x)} dx\right)^{\frac{1}{m_1}}.$$

So, we get

$$\left[\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}\right]^{\frac{1}{1-\sigma}} \leq c\left[\left(\int_\Omega u^{m(x)} dx\right)^{\frac{1}{m_1}}\right]^{\frac{1}{1-\sigma}}$$

then

$$\left(\int_\Omega u^2 dx\right)^{\frac{1}{2(1-\sigma)}} \leq c\left(\int_\Omega u^{m(x)} dx\right)^{\frac{1}{m_1(1-\sigma)}}$$

$$\left(\int_\Omega uu_t dx\right)^{\frac{1}{1-\sigma}} \leq C\left(\int_\Omega |u|^{m(x)} dx\right)^{\frac{1}{m_1(1-\sigma)}}\left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2(1-\sigma)}}$$

by Young's inequality we get,

$$\left(\int_\Omega uu_t dx\right)^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_\Omega |u|^{m(x)} dx\right)^{\frac{\mu}{m_1(1-\sigma)}} + \left(\int_\Omega u_t^2 dx\right)^{\frac{\theta}{2(1-\sigma)}}\right]$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. Taking $\theta=2(1-\sigma)$, we have $\mu=\frac{2(1-\sigma)}{1-2\sigma}$, by Poincare's inequality, it follows

$$\left(\int_\Omega uu_t dx\right)^{\frac{1}{1-\sigma}} \leq C\left[\left(\int_\Omega |\nabla u|^{m(x)} dx\right)^{\frac{2}{m_1(1-\sigma)}} + \int_\Omega u_t^2 dx\right]. \tag{3.47}$$

If $0<\sigma<\frac{m_1-2}{2m_1}$, then we get $0<\frac{2}{m_1(1-\sigma)}<1$. From (3.18), (3.33), and (3.47), we obtain

$$\begin{aligned}\left(\int_\Omega uu_t dx\right)^{\frac{1}{1-\sigma}} &\leq C\left(\int_\Omega |\nabla u|^{m(x)} dx + H(0) + \int_\Omega u_t^2 dx\right) \\ &\leq C\left(\int_\Omega |\nabla u|^{m(x)} dx + H(t) + \int_\Omega u_t^2 dx\right).\end{aligned} \tag{3.48}$$

From (3.46), (3.48), we get

$$L^{\frac{1}{1-\sigma}}(t) \leq C\left(\int_\Omega |\nabla u|^{m(x)} dx + H(t) + \int_\Omega u_t^2 dx\right), \tag{3.49}$$

where C is only related to σ, ε.

Taking $\sigma<\min\left\{\frac{m_1-2}{p_1}, \frac{m_1-2}{p_2}, \frac{m_1-r_1}{p_1(r_2-1)}, \frac{m_1-r_2}{p_1(r_2-1)}, \frac{m_1-r_1}{p_2(r_2-1)}, \frac{m_1-r_2}{p_2(r_2-1)}, \frac{m_1-2}{2m_1}, 1\right\}$, by inequality (3.45) and (3.20) in Lemma 3.2, it follows that there exists a constant $\zeta>0$ such that

$$L'(t) \geq \zeta L^{\frac{1}{1-\sigma}}(t) \tag{3.50}$$

for any $t \geq 0$. Integrating the above formula with respect to t on $[0, t]$, we get

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{\sigma}{1-\sigma}}(0) - \frac{\sigma\zeta}{1-\sigma}t} \tag{3.51}$$

for any $t \geq 0$. Hence there exists $T^* \leq \frac{1-\sigma}{\sigma\zeta L^{\frac{\sigma}{1-\sigma}}(0)}$ such that $\lim\limits_{t\to T^*} L(t) = \infty$, that means the regular solution $u(x,t)$ must blow up in finite time. ∎

Chapter 4

Asymptotic behavior of solution for a fractional p-Kirchhoff type hyperbolic equation with variable exponents

In this chapter, we establish the existence, and stability results for the fractional p-Kirchhoff type hyperbolic equation [23].

4.1 Introduction

In this work, we investigate to study the following fractional p-Kirchhoff type hyperbolic equation with variable exponent and zero Dirichlet boundary value condition

$$\begin{aligned} u_{tt} + M\left(\int_{\Omega} |\nabla u|^{p}\, dx\right)(-\Delta)_{p}^{s} u + |u_t|^{m(x)-2} u_t &= |u|^{r(x)-2} u, \text{ in } \Omega \times (0,T), && (4.1)\\ u(x,t) &= 0,\ x \in \partial\Omega,\ t > 0, && (4.2) \end{aligned}$$

with the initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{4.3}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. $M(s) = 1 + s$ and $m(.)$, $r(.)$ are given measurable functions on Ω, satisfying

$$\begin{aligned} 2 &\leq r_1 \leq r(x) \leq r_2 < 2\frac{n-1}{n-2},\ \text{if } n \geq 3, && (4.4)\\ r(x) &\geq 2,\ \text{if } n = 1, 2, \end{aligned}$$

and

$$\begin{aligned} 2 &\leq m_1 \leq m(x) \leq m_2 < \frac{2n}{n-2}, \text{ if } n \geq 3, \\ m(x) &\geq 2, \text{ if } n = 1, 2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} r_1 &: = ess\inf_{x\in\Omega} r(x), \; r_2 := ess\sup_{x\in\Omega} r(x) \\ m_1 &: = ess\inf_{x\in\Omega} m(x), \; m_2 := ess\sup_{x\in\Omega} m(x). \end{aligned}$$

We also assume that $m(.)$ and $r(.)$ satisfy the log-Hölder continuity condition

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e } x, y \in \Omega, \text{ with } |x-y| < \delta, \; A > 0, \; 0 < \delta < 1. \tag{4.6}$$

The non-local fractional p-Laplacian operator $(-\Delta)_p^s u$ with $s \in (0,1)$ and $p \in [1, +\infty[$ is defined as

$$\begin{aligned} (-\Delta)_p^s u &= 2\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} dy \\ &= 2VP \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} dy. \end{aligned} \tag{4.7}$$

The fractional Laplacian as a generalization of the integer order Laplace opeartor has been studied in classical monographs such as [63],[29]. In recent years, a great deal of attention has been devoted to investigate the study of matheamtical nonlinear models of hyperbolic, parabolic and elliptic equation involved with non-local fractional operators and gained enormous reputation amid reaserchers because of their wide applications in different fields such as digit image processing [18], obstacle problem [62], phase transitions [12], dynamic systems [26] and so on. For any further and detailed information, we refer the interested reader to ([39], [41], [60]) and the references cited therein.

The present work is about the existence of weak solutions, and stability results of solutions on fractional Sobolev space for a fractional p-Kirchhoff type hyperbolic equations involving the fractional Laplacian with boundary conditions.

This problem can be regarded as the fractional setting of the problem in [51] where the study is about the global existence and stability of solution for a p-Kirchhoff type hyperbolic equation with variable exponents on classical Sobolev space and it also involving the classical Laplacian operator with Dirichlet boundary conditions. By comparison between the two problems, it appears that the fractional problem is more interesting by his non-local property.

To the best of our knowledge, many authors have focused on the study of the existence of elliptic and parabolic equations involving fractional p-Laplacian, see for example ([49], [55]) and the references therein where the authors have used the direct method of variation and montain pass theorem to study the existence and multiplicity of solutions.

However, to the author best knowledge there are no papers to deal with the global existence and stability results for problem like (4.1). The aim difficulty of this problem arises from the fact of working with this new-local fractional p-Laplacian operator.

4.2 The main of results

In order to state and prove our result in this section, we define the potential energy function and the Nehari's functional, respectively by the following:

$$\begin{aligned} E(t) &= E(u(t)) = \frac{1}{2}\|u_t\|_2^2 + \frac{a}{p}\|u\|^p_{W_0^{s,p}(\Omega)} + \frac{b}{p}\|\nabla u\|_p^p\|u\|^p_{W_0^{s,p}(\Omega)} \\ &\quad - \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx. \end{aligned} \tag{4.8}$$

$$I(t) = I(u(t)) = a\|u\|^p_{W_0^{s,p}(\Omega)} + b\|\nabla u\|_p^p\|u\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega |u|^{r(x)}dx. \tag{4.9}$$

In the following, we consider $a = b = 1$ and this does not change the general result.

Lemma 4.1 *Under the assumptions* (4.4)-(4.6), *we have*

$$E'(t) = -\int_\Omega |u_t(t)|^{m(x)}dx \leq 0,\ t \in [0,T] \tag{4.10}$$

and

$$E(t) \leq E(0). \tag{4.11}$$

Proof. We multiply the first equation of (4.1) by u_t and integrating over the domain Ω, we get

$$\begin{aligned} &\int_\Omega u_{tt}u_t dx + \int_\Omega M\left(\int_\Omega |\nabla u|^p dx\right)(-\Delta)_p^s u(x)u_t dx + \int_\Omega |u_t|^{m(x)-2}u_t u_t dx \\ =&\int_\Omega |u|^{r(x)-2}uu_t dx \end{aligned}$$

$$\begin{aligned} &\int_\Omega u_{tt}u_t dx + M\left(\int_\Omega |\nabla u|^p dx\right)\int_\Omega (-\Delta)_p^s u(x)u_t dx + \int_\Omega |u_t|^{m(x)-2}u_t^2 dx \\ =&\int_\Omega |u|^{r(x)-2}uu_t dx, \end{aligned}$$

$$\int_\Omega \frac{d}{dt}u_t u_t dx + 2M\left(\int_\Omega |\nabla u|^p dx\right)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\frac{d}{dt}u(x)dxdy$$

$$+\int_{\Omega}|u_t|^{m(x)-2}u_t^2dx=\int_{\Omega}|u|^{r(x)-2}u\frac{d}{dt}udx,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}u_t^2dx+M\left(\int_{\Omega}|\nabla u|^p dx\right)\int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\frac{d}{dt}(u(x)-u(y))dxdy$$

$$+\int_{\Omega}|u_t|^{m(x)}dx=\frac{d}{dt}\int_{\Omega}\frac{1}{r(x)}|u|^{r(x)}dx,$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega}u_t^2dx+\left(1+\int_{\Omega}|\nabla u|^p dx\right)\left(\frac{1}{p}\frac{d}{dt}\|u\|^p_{W_0^{s,p}(\Omega)}\right)+\int_{\Omega}|u_t|^{m(x)}dx\\ =\;&\frac{d}{dt}\int_{\Omega}\frac{1}{r(x)}|u|^{r(x)}dx,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\frac{1}{p}\frac{d}{dt}\|u\|^p_{W_0^{s,p}(\Omega)}+\frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p\|u\|^p_{W_0^{s,p}(\Omega)}+\int_{\Omega}|u_t|^{m(x)}dx\\ =\;&\frac{d}{dt}\int_{\Omega}\frac{1}{r(x)}|u|^{r(x)}dx.\end{aligned}$$

This implies

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{p}\|u\|^p_{W_0^{s,p}(\Omega)}+\frac{1}{p}\|\nabla u\|_p^p\|u\|^p_{W_0^{s,p}(\Omega)}-\int_{\Omega}\frac{1}{r(x)}|u|^{r(x)}dx\right)\\ =\;&-\int_{\Omega}|u_t|^{m(x)}dx.\end{aligned}$$

Then

$$E'(t)=-\int_{\Omega}|u_t|^{m(x)}dx\leq 0.$$

Integrating (4.10) over $(0,t)$, we obtain

$$\begin{aligned}\int_0^t E'(s)ds&=E(t)-E(0)\leq 0\\ E(t)&\leq E(0).\end{aligned}$$

■

Lemma 4.2 *Let the assumptions of (4.4)-(4.6) hold, and $r_1>2p$, $I(0)>0$ and*

$$\beta_1+\beta_2<1, \tag{4.12}$$

where

$$\beta_1 := \max\left\{\alpha c_*^{r_1}(n,s,p)\left(\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_1-p}{p}},\alpha c_*^{r_2}(n,s,p)\left(\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_2-p}{p}}\right\},$$

$$\beta_2 := \max\left\{(1-\alpha)c_*^{r_1}(n,s,p)\left(\frac{c^p(n,s,p)pr_1}{r_1-p}E(0)\right)^{\frac{r_1-2p}{2p}},(1-\alpha)c_*^{r_2}(n,s,p)\left(\frac{c^p(n,s,p)pr_1}{r_1-p}E(0)\right)^{\frac{r_2-2p}{2p}}\right.$$

with $0 < \alpha < 1$, $c_*(n,s,p)$ *is the best embedding constant of* $W_0^{s,p}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$. *Then* $I(t) > 0$ *for all* $t \in [0,T]$.

Proof. Since $I(0) > 0$, then by continuity there exists T_* such that

$$I(t) \geq 0, \text{ for all } t \in [0,T]. \tag{4.13}$$

Now, we have for all $t \in [0,T]$ that

$$\begin{aligned} J(t) &= J(u(t)) = \frac{1}{p}\|u\|^p_{W_0^{s,p}} + \frac{1}{p}\|\nabla u\|_p^p \|u\|^p_{W_0^{s,p}} - \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\ &\geq \frac{1}{p}\|u\|^p_{W_0^{s,p}} + \frac{1}{p}\|\nabla u\|_p^p \|u\|^p_{W_0^{s,p}} - \frac{1}{r_1}\left(\|u\|^p_{W_0^{s,p}} + \|\nabla u\|_p^p \|u\|^p_{W_0^{s,p}} - I(t)\right) \\ &\geq \frac{r_1-p}{pr_1}\|u\|^p_{W_0^{s,p}} + \frac{r_1-p}{pr_1}\|\nabla u\|_p^p \|u\|^p_{W_0^{s,p}} + \frac{1}{r_1}I(t), \end{aligned}$$

using (4.12), we obtain

$$\frac{r_1-p}{pr_1}\|u\|^p_{W_0^{s,p}} + \frac{r_1-p}{pr_1}\|\nabla u\|_p^p \|u\|^p_{W_0^{s,p}} \leq J(t) \leq E(t). \tag{4.14}$$

We have $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega)$

$$\|u\|^p_{W_0^{s,p}} \leq c^p(n,s,p)\|u\|_p^p \leq c^p(n,s,p)\|\nabla u\|_p^p.$$

So,

$$\|u\|^p_{W_0^{s,p}} \leq c^p(n,s,p)\|\nabla u\|_p^p.$$

Then,

$$\begin{aligned} \frac{r_1-p}{pr_1}\|u\|^p_{W_0^{s,p}} + c^{-p}(n,s,p)\frac{r_1-p}{pr_1}\|u\|^p_{W_0^{s,p}}\|u\|^p_{W_0^{s,p}} &\leq E(t) \\ \frac{r_1-p}{pr_1}\|u\|^p_{W_0^{s,p}} + c^{-p}(n,s,p)\frac{r_1-p}{pr_1}\|u\|^{2p}_{W_0^{s,p}} &\leq E(t). \end{aligned} \tag{4.15}$$

By Lemma 4.1, we get

$$\|u\|^p_{W_0^{s,p}} \leq \frac{pr_1}{r_1-p}E(t) \leq \frac{pr_1}{r_1-p}E(0). \tag{4.16}$$

$$\|u\|^{2p}_{W_0^{s,p}} \leq c^p(n,s,p)\frac{pr_1}{r_1-p}E(t) \leq c^p(n,s,p)\frac{pr_1}{r_1-p}E(0). \tag{4.17}$$

On the other hand, by Lemma 1.5, we have

$$\begin{aligned} \int_\Omega |u|^{r(x)}dx &\leq \max\left\{\|u(t)\|^{r_1}_{r(.)}, \|u(t)\|^{r_2}_{r(.)}\right\} \\ &= \alpha \max\left\{\|u(t)\|^{r_1}_{r(.)}, \|u(t)\|^{r_2}_{r(.)}\right\} \\ &\quad + (1-\alpha)\max\left\{\|u(t)\|^{r_1}_{r(.)}, \|u(t)\|^{r_2}_{r(.)}\right\}. \end{aligned}$$

Hence, by the embedding of $W_0^{s,p}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u|^{r(x)}\,dx &\leq \alpha \max\left\{c_*^{r_1}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1}, c_*^{r_2}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_2}\right\}\\ &\quad + (1-\alpha)\max\left\{c_*^{r_1}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1}, c_*^{r_2}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_2}\right\}\\ &\leq \alpha \max\left\{c_*^{r_1}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1-p}, c_*^{r_2}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_2-p}\right\}\times\|u(t)\|_{W_0^{s,p}(\Omega)}^{p}\\ &\quad + (1-\alpha)\max\left\{c_*^{r_1}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1-2p}, c_*^{r_2}(n,s,p)\,\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_2-2p}\right\}\times\|u(t)\|_{W_0^{s,p}(\Omega)}^{2p}.\end{aligned}$$

By (4.16) and (4.17), we get

$$\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1-p} \leq \left(\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_1-p}{p}}$$

and

$$\|u(t)\|_{W_0^{s,p}(\Omega)}^{r_1-2p} \leq \left(c^p(n,s,p)\,\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_1-2p}{2p}}.$$

Then,

$$\begin{aligned}\int_\Omega |u|^{r(x)}\,dx &\leq \alpha\max\left\{c_*^{r_1}(n,s,p)\left(\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_1-p}{p}}, c_*^{r_2}(n,s,p)\left(\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_2-p}{p}}\right\}\times\|u(t)\|_{W_0^{s,p}(\Omega)}^{p}\\ &\quad + (1-\alpha)\{c_*^{r_1}(n,s,p)\left(c^p(n,s,p)\,\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_1-2p}{2p}},\\ &\quad c_*^{r_2}(n,s,p)\left(c^p(n,s,p)\,\frac{pr_1}{r_1-p}E(0)\right)^{\frac{r_2-2p}{2p}}\}\times\|u(t)\|_{W_0^{s,p}(\Omega)}^{2p}.\end{aligned}$$

So,

$$\int_\Omega |u|^{r(x)}\,dx \leq \beta_1\|u(t)\|_{W_0^{s,p}(\Omega)}^{p} + \beta_2\|u(t)\|_{W_0^{s,p}(\Omega)}^{2p}, \text{ for all } t\in[0,T_*]. \tag{4.18}$$

Since $\beta_1+\beta_2<1$, then

$$\int_\Omega |u|^{r(x)}\,dx < \|u(t)\|_{W_0^{s,p}(\Omega)}^{p} + \|u(t)\|_{W_0^{s,p}(\Omega)}^{2p}, \text{ for all } t\in[0,T_*]. \tag{4.19}$$

This implies that $I(t)>0$, for all $t\in[0,T_*]$. By repeating the above procedure, we can extend T_* to T. ■

4.2.1 The existence of weak solution

Theorem 4.1 (Existence of weak solution) *Assume that the assumptions* (4.4)-(4.6) *hold. Let* $(u_0,u_1)\in W_0^{s,p}(\Omega)\times L^2(\Omega)$ *be given, we also assume that* $m(.)$ *and* $r(.)$ *satisfy the log-Hölder* *continuity condition. Then problem* (4.1) *has a weak local solution*

$$\begin{aligned} u &\in L^{\infty}\left((0,T), W_0^{s,p}(\Omega)\right) \\ u_t &\in L^{\infty}\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}\left(\Omega \times (0,T)\right) \\ u_{tt} &\in L^2\left((0,T), W^{-s,p'}(\Omega)\right). \end{aligned}$$

Proof. We will use the Faedo-Galerkin method of approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $W_0^{s,p}(\Omega)$ which forms a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = span\{v_1, v_2, ..., v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. After normalization, we have $\|v_l\| = 1$ and for any given integer k, we consider the approximation solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t) v_l,$$

where u_k are the solutions to the Cauchy problem

$$\begin{aligned} &\left(u_k''(t), v_l\right) + \left(M\left(\int_{\Omega} |\nabla u_k|^p dx\right)(-\Delta)_p^s u_k(t), v_l\right) + \left(\left|u_k'(t)\right|^{m(x)-2} u_k', v_l\right) \\ = &\left(|u_k(t)|^{r(x)-2} u_k(t), v_l\right), \quad l = 1, 2, ..., k. \end{aligned} \tag{4.20}$$

$$u_k(0) = u_{0k} = \sum_{i=1}^{k} (u_k(0), v_l) v_l \to u_0 \text{ in } W_0^{s,p}(\Omega). \tag{4.21}$$

$$u_k'(0) = u_{1k} = \sum_{i=1}^{k} \left(u_k'(0), v_l\right) v_l \to u_1 \text{ in } L^2(\Omega). \tag{4.22}$$

Note that we can solve the system (4.20)-(4.22) by Picard's iterative method for ordinary differential equations. Hence, there exists a solution in $[0, T_*]$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of the priori estimates below.

Step 1: "The prior estimate".

Multiplying equation (4.20) by $u_{lk}'(t)$ and summing over l from 1 to k, we get

$$\begin{aligned} &\sum_{l=1}^{k} u_{lk}'(t)\left[\left(u_k''(t), v_l\right) + \left(M\left(\int_{\Omega} |\nabla u_k|^p dx\right)(-\Delta)_p^s u_k(t), v_l\right)\right. \\ &+ \left(\left|u_k'(t)\right|^{m(x)-2} u_k', v_l\right) \\ = &\left.\left(|u_k(t)|^{r(x)-2} u_k(t), v_l\right)\right]. \end{aligned}$$

This implies

$$\begin{aligned}&\sum_{l=1}^{k}\left(u_k''(t),v_l\right)u_{lk}'(t)+\sum_{l=1}^{k}\left(M\left(\int_\Omega|\nabla u_k|^p\,dx\right)(-\Delta)_p^s u_k(t),v_l\right)u_{lk}'(t)\\&+\sum_{l=1}^{k}\left(\left|u_k'(t)\right|^{m(x)-2}u_k',v_l\right)u_{lk}'(t)\\=&\sum_{l=1}^{k}\left(|u_k(t)|^{r(x)-2}u_k(t),v_l\right)u_{lk}'(t).\end{aligned}$$

So,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\left\|u_k'\right\|_2^2+\frac{1}{p}\|u_k\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u_k\|_p^p\|u_k\|_{W_0^{s,p}}^p-\int_\Omega\frac{1}{r(x)}|u_k|^{r(x)}\,dx\right)\\=&-\int_\Omega\left|u_k'\right|^{m(x)}dx.\end{aligned}\tag{4.23}$$

Then,

$$E'(u_k(t))=-\int_\Omega\left|u_k'\right|^{m(x)}dx\le 0.$$

Integrating (4.23) over $(0,t)$, gives the estimate

$$\begin{aligned}&\int_0^t\left[\frac{d}{dt}\left(\frac{1}{2}\left\|u_k'\right\|_2^2+\frac{1}{p}\|u_k\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u_k\|_p^p\|u_k\|_{W_0^{s,p}}^p-\int_\Omega\frac{1}{r(x)}|u_k|^{r(x)}\,dx\right)\right]ds\\=&-\int_0^t\int_\Omega\left|u_k'\right|^{m(x)}dxds.\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\left\|u_k'\right\|_2^2+\frac{1}{p}\|u_k\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u_k\|_p^p\|u_k\|_{W_0^{s,p}}^p-\int_\Omega\frac{1}{r(x)}|u_k|^{r(x)}\,dx+\int_0^t\int_\Omega\left|u_k'\right|^{m(x)}dxds\\\le&\frac{1}{2}\left\|u_k'(0)\right\|_2^2+\frac{1}{p}\|u_k(0)\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u_k(0)\|_p^p\|u_k(0)\|_{W_0^{s,p}}^p-\int_\Omega\frac{1}{r(x)}|u_k(0)|^{r(x)}\,dx.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{2}\left\|u_k'\right\|_2^2+\frac{1}{p}\|u_k\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u_k\|_p^p\|u_k\|_{W_0^{s,p}}^p-\int_\Omega\frac{1}{r(x)}|u_k|^{r(x)}\,dx+\int_0^t\int_\Omega\left|u_k'\right|^{m(x)}dxds\\\le&E(0).\end{aligned}\tag{4.24}$$

Then, from (4.19), inequality (4.24) becomes

$$\begin{aligned}&\frac{1}{2}\sup_{t\in(0,T)}\left\|u_k'(t)\right\|_2^2+\frac{r_1-p}{pr_1}\sup_{t\in(0,T)}\|u_k(t)\|_{W_0^{s,p}}^p+\frac{r_1-p}{pr_1}\sup_{t\in(0,T)}\|\nabla u_k(t)\|_p^p\|u_k(t)\|_{W_0^{s,p}}^p\\&+\int_0^t\int_\Omega\left|u_k'(s)\right|^{m(x)}dxds\\\le&E(0).\end{aligned}\tag{4.25}$$

From (4.25), we conclude that

$$\begin{cases} \{u_k\} \text{ is uniformly bounded in } L^\infty\left((0,T), W_0^{s,p}(\Omega)\right) \\ \{u_k'\} \text{ is uniformly bounded in } L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}\left(\Omega \times (0,T)\right) \end{cases}. \tag{4.26}$$

Since $\{u_k'\}$ is uniformly bounded in $L^{m(.)}(\Omega \times (0,T))$, then $\left\{\left|u_k'\right|^{m(x)-2} u_k'\right\}$ is bounded in $L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T))$. Hence, up to a subsequence, $\left|u_k'\right|^{m(x)-2} u_k' \to \Phi$ weakly in $L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T))$. We have to show that $\Phi = \left|u'\right|^{m(x)-2} u'$.

Furthermore, it follows from Corollary 1.1 and (4.26) that

$$\left\{|u_k|^{r(x)-2} u_k\right\} \text{ is uniformly bounded in } L^2\left((0,T), W^{-s,p'}(\Omega)\right). \tag{4.27}$$

By (4.26) and (4.27), we infer that there exists a subsequence of u_k (still denoted by the same symbol) and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^\infty\left((0,T), W_0^{s,p}(\Omega)\right). \\ u_k' \rightharpoonup u' \text{ weakly star in } L^\infty\left((0,T), L^2(\Omega)\right) \text{ and weakly star in } L^{m(.)}\left(\Omega \times (0,T)\right). \\ |u_k|^{r(x)-2} u_k \rightharpoonup \Psi \text{ weakly in } L^2\left((0,T), W^{-s,p'}(\Omega)\right). \end{cases} \tag{4.28}$$

By the Aubin-Lions compactness Lemma [40], we conclude from (4.28) that

$$u_k \to u \text{ strongly in } C\left([0,T], W_0^{s,p}(\Omega)\right), \tag{4.29}$$

which implies

$$u_k \to u \text{ everywhere in } [0,T] \times \Omega. \tag{4.30}$$

It follows from (4.28) and (4.30) that

$$|u_k|^{r(x)-2} u_k \rightharpoonup |u|^{r(x)-2} u \text{ weakly in } L^2\left((0,T), W^{-s,p'}(\Omega)\right). \tag{4.31}$$

Step 2: Letting $k \to \infty$ and passing to the limit in (4.20), we obtain

$$\begin{aligned} &\left(u''(t), v_l\right) + \left(M\left(\int_\Omega |\nabla u|^p\, dx\right)(-\Delta)_p^s u(t), v_l\right) + \left(\left|u'(t)\right|^{m(x)-2} u'(t), v_l\right) \\ = &\left(|u(t)|^{r(x)-2} u(t), v_l\right),\ l = 1, 2, ..., k. \end{aligned} \tag{4.32}$$

Since $\{v_l\}_{l=1}^\infty$ is a basis of $W_0^{s,p}(\Omega)$, we deduce that u satisfies equation (4.1).

From (4.28) and Lemma 3.1.7 of [68] with $B = W_0^{s,p}(\Omega)$ and $B = L^2(\Omega)$, respectively, we infer that

$$\begin{cases} u_k(0) \rightharpoonup u(0) \text{ weakly in } W_0^{s,p}(\Omega) \\ u_k'(0) \rightharpoonup u'(0) \text{ weakly star in } L^2(\Omega) \end{cases}. \tag{4.33}$$

We get from (4.21) and (4.22) that $u(0) = u_0(0)$, $u'(0) = u_1(0)$. Thus, the proof is complete. ■

4.2.2 Global existence of solution

Theorem 4.2 *Under the assumptions of Lemma 4.1, the local solution of problem (4.1) is global.*

Proof. We have from definition of E,

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|u\|_{W_0^{s,p}(\Omega)}^p + \frac{1}{p}\|\nabla u\|_p^p \|u\|_{W_0^{s,p}(\Omega)}^p - \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|u\|_{W_0^{s,p}(\Omega)}^p + \frac{C^{-p}(n,s,p)}{p}\|u\|_{W_0^{s,p}(\Omega)}^{2p} - \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{r_1-p}{pr_1}\|u\|_{W_0^{s,p}(\Omega)}^p + C^{-p}(n,s,p)\frac{r_1-p}{pr_1}\|u\|_{W_0^{s,p}(\Omega)}^{2p} + \frac{1}{r_1}I(t). \end{aligned}$$

By Lemma 4.2, we find

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{r_1-p}{pr_1}\|u\|_{W_0^{s,p}(\Omega)}^p + C^{-p}(n,s,p)\frac{r_1-p}{pr_1}\|u\|_{W_0^{s,p}(\Omega)}^{2p}.$$

So that

$$\|u_t\|_2^2 + \|u\|_{W_0^{s,p}(\Omega)}^p \leq C(n,s,p)E(t). \tag{4.34}$$

By Lemma 4.1, we obtain

$$\|u_t\|_2^2 + \|u\|_{W_0^{s,p}(\Omega)}^p \leq C(n,s,p)E(0). \tag{4.35}$$

This implies that the local solution is global in time. ■

4.2.3 Stability result

In this section our result is based on Komornik's inequality. For this, we need the following Lemma:

Lemma 4.3 *Suppose that the assumptions of (4.4)-(4.6) and $m_1 > p$ hold, then there exists a positive constant $c(n,s,p)$ such that*

$$\int_\Omega |u|^{m(x)}dx \leq c(n,s,p)E(t). \tag{4.36}$$

Proof. We have

$$\begin{aligned} \int_\Omega |u|^{m(x)}dx &= \max\left\{\|u(t)\|_{m(.)}^{m_1}, \|u(t)\|_{m(.)}^{m_2}\right\} \\ &\leq \max\left\{c_*^{m_1}(n,s,p)\|u(t)\|_{W_0^{s,p}(\Omega)}^{m_1}, c_*^{m_2}(n,s,p)\|u(t)\|_{W_0^{s,p}(\Omega)}^{m_2}\right\} \\ &\leq \max\left\{c_*^{m_1}(n,s,p)\|u(t)\|_{W_0^{s,p}(\Omega)}^{m_1-p}, c_*^{m_2}(n,s,p)\|u(t)\|_{W_0^{s,p}(\Omega)}^{m_2-p}\right\} \times \|u(t)\|_{W_0^{s,p}(\Omega)}^p \end{aligned}$$

such that $c_*(n,s,p)$ is the best constant of the embedding of $W_0^{s,p}(\Omega) \hookrightarrow L^{m(.)}(\Omega)$.
By using (4.16), we obtain

$$\int_\Omega |u|^{m(x)}\,dx \leq c(n,s,p)\,E(t).$$

■

Now, we state our main result.

Theorem 4.3 *Let the assumptions of* (4.4)-(4.6) *hold, then there exists a constant such that*

$$\begin{cases} E(t) \leq \frac{\eta}{(1+t)^{\frac{2}{m_2-2}}},\ t\geq 0 \text{ if } m_2>2 \\ E(t) \leq \eta e^{-\zeta t},\ \text{for all } t\geq 0 \text{ if } m_2=2 \end{cases}.$$

Where η *is constant only related to* $c,\gamma,c(n,s,p)$.

Proof. Multiplying the first equation of (4.1) by $u(t)E^q(t)$ $(q>0)$ and integrating over $\Omega\times(S,T)$, we obtain

$$\int_S^T\int_\Omega E^q(t)\left[u(t)u_{tt}(t)+u(t)\left(M\left(\int_\Omega|\nabla u(t)|^p\,dx\right)(-\Delta)_p^s u(t)+|u_t(t)|^{m(x)-2}u_t(t)\right)\right]dxdt$$
$$=\int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.$$

This implies

$$\int_S^T\int_\Omega E^q(t)\left(u(t)u_{tt}(t)+M\left(\int_\Omega|\nabla u(t)|^p\,dx\right)(-\Delta)_p^s u(t)\,u(t)+u(t)|u_t(t)|^{m(x)-2}u_t(t)\right)dxdt$$
$$=\int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.$$

Then

$$\int_S^T\int_\Omega E^q(t)\left((u(t)u_t(t))_t-|u_t(t)|^2\right)dxdt+\|u(t)\|^p_{W_0^{s,p}}\int_S^T E^q(t)\,dt$$
$$+\|\nabla u(t)\|_p^p\|u(t)\|^p_{W_0^{s,p}}\int_S^T E^q(t)\,dt+\int_S^T\int_\Omega E^q(t)u(t)|u_t(t)|^{m(x)-2}u_t(t)\,dxdt$$
$$=\int_S^T E^q(t)\int_\Omega|u(t)|^{r(x)}\,dxdt.$$

We add and substract the term

$$\int_S^T E^q(t)\left[\beta_1\|u(t)\|^p_{W_0^{s,p}}+\beta_2\|\nabla u(t)\|_p^p\|u(t)\|^p_{W_0^{s,p}}+(2+\beta_1+\beta_2)\int_\Omega|u_t(t)|^2\,dx\right]dt$$

we find

$$\begin{aligned}
&\int_S^T \int_\Omega E^q(t)\left((u(t)u_t(t))_t - |u_t|^2\right)dxdt + \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt \\
&+\|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt + \int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m(x)-2}\,dxdt \\
&+\beta_1 \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt + \beta_2 \|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt \\
&+(2+\beta_1+\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \\
&-\beta_1 \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt - \beta_2 \|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} \int_S^T E^q(t)\,dt \\
&-(2+\beta_1+\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \\
=\;& \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.
\end{aligned}$$

and use (4.18) to get

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\left[\|u(t)\|^p_{W_0^{s,p}} + \int_\Omega |u_t(t)|^2\,dx\right]dt \\
&+(1-\beta_2)\int_S^T E^q(t)\left[\|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} + \int_\Omega |u_t(t)|^2\,dx\right]dt \\
&+\int_S^T E^q(t)\int_\Omega \left[(u(t)u_t(t))_t - (3-\beta_1-\beta_2)|u_t(t)|^2\right]dxdt \\
&+\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m(x)-2}\,dxdt \\
=\;& -\int_S^T E^q(t)\left[\beta_1\|u(t)\|^p_{W_0^{s,p}} + \beta_2\|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} - \int_\Omega |u(t)|^{r(x)}\,dx\right]dt \le 0
\end{aligned} \tag{4.37}$$

It is clear that

$$\begin{aligned}
&\gamma\int_S^T E^q(t)\left[\int_\Omega \frac{1}{2}|u_t(t)|^2\,dx + \frac{1}{p}\|u(t)\|^p_{W_0^{s,p}} + \frac{1}{p}\|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}} - \int_\Omega \frac{|u(t)|^{r(x)}}{r(x)}dx\right]dt \\
\le\;& (1-\beta_1)\int_S^T E^q(t)\left[\int_\Omega \frac{|u_t(t)|^2}{2}dx + \frac{\|u(t)\|^p_{W_0^{s,p}}}{p}\right]dt \\
&+(1-\beta_2)\int_S^T E^q(t)\left[\int_\Omega \frac{|u_t(t)|^2}{2}dx + \frac{\|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}}}{p}\right]dt \\
\le\;& (1-\beta_1)\int_S^T E^q(t)\left[\int_\Omega |u_t(t)|^2\,dx + \|u(t)\|^p_{W_0^{s,p}}\right]dt \\
&+(1-\beta_2)\int_S^T E^q(t)\left[\int_\Omega |u_t(t)|^2\,dx + \|\nabla u(t)\|_p^p \|u(t)\|^p_{W_0^{s,p}}\right]dt.
\end{aligned} \tag{4.38}$$

Then,

$$\begin{aligned}
&\gamma\int_S^T E^q(t)\left[\int_\Omega \frac{1}{2}|u_t(t)|^2dx+\frac{1}{p}\|u(t)\|_{W_0^{s,p}}^p+\frac{1}{p}\|\nabla u(t)\|_p^p\|u(t)\|_{W_0^{s,p}}^p-\int_\Omega\frac{|u(t)|^{r(x)}}{r(x)}dx\right]dt\\
\leq\ & -\int_S^T E^q(t)\int_\Omega\left[(u(t)u_t(t))_t-(3-\beta_1-\beta_2)|u_t(t)|^2\right]dxdt\\
&-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m(x)-2}dxdt
\end{aligned}$$

where $\gamma=\min((1-\beta_1),(1-\beta_2))$. By (4.37), (4.38) and the definition of $E(t)$, we get

$$\begin{aligned}
\gamma\int_S^T E^{q+1}(t)\,dt \leq\ & -\int_S^T E^q(t)\int_\Omega (u(t)u_t(t))_t\,dxdt\\
&-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m(x)-2}dxdt\\
&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2dxdt. \qquad (4.39)
\end{aligned}$$

Using the definition of $E(t)$ and the following expression

$$\begin{aligned}
\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)u_t(t)\,dx\right) =\ & qE^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)\,dx\\
&+E^q(t)\int_\Omega (u(t)u_t(t))_t\,dx.
\end{aligned}$$

Inequality (4.39), becomes

$$\begin{aligned}
\gamma\int_S^T E^{q+1}(t)\,dt \leq\ & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)\,dxdt\\
&-\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega u(t)u_t(t)\,dx\right)dt\\
&-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m(x)-2}dxdt\\
&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2dxdt. \qquad (4.40)
\end{aligned}$$

We estimate the terms in the right-hand side of (4.40) as follow:
By (4.10) and Young's inequality, we obtain

$$\begin{aligned}
&q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)\,dxdt\\
\leq\ & q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\int_\Omega\left[\frac{1}{p}|u(t)|^p+\frac{p-1}{p}|u_t(t)|^{\frac{p}{p-1}}\right]dxdt \qquad (4.41)
\end{aligned}$$

Since, $1 \leq \frac{p}{p-1} < 2$, by the embedding of $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $L^2(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, we have

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dxdt \\ \leq\ & q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\frac{c(n,s,p)}{p}\|u\|_{W_0^{s,p}}^p + c\frac{p-1}{p}\int_\Omega |u_t|^2\,dx\right]dt \end{aligned}$$

by (4.19) we find

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dxdt \\ \leq\ & c^p(n,s,p)\int_S^T E^{q-1}(t)\left(-E'(t)\right)dt \\ \leq\ & c^p(n,s,p)\,E^{q+1}(S) - c^p(n,s,p)\,E^{q+1}(T). \\ \leq\ & c^p(n,s,p)\,E^q(0)\,E(S) \\ \leq\ & c^p(n,s,p)\,E(S). \end{aligned} \tag{4.42}$$

For the second term, we have

$$\begin{aligned} & -\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt \\ \leq\ & \left|E^q(t)\int_\Omega u(S)\,u_t(S)\,dx - E^q(t)\int_\Omega u(T)\,u_t(T)\,dx\right| \\ \leq\ & E^q(t)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right| + E^q(t)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ \leq\ & c^p(n,s,p)\,E^{q+1}(S) + c^p(n,s,p)\,E^{q+1}(T) \\ \leq\ & c^p(n,s,p)\,E^q(0)\,E(S) \\ \leq\ & c^p(n,s,p)\,E(S). \end{aligned} \tag{4.43}$$

For the third term, we use the following Young inequality

$$XY \leq \frac{\varepsilon}{\lambda_1}X^{\lambda_1} + \frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,Y \geq 0,\ \varepsilon > 0 \text{ and } \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1,$$

with $\lambda_1(x) = m(x)$, $\lambda_2(x) = \frac{m(x)}{m(x)-1}$.

By (4.10) and Lemma 4.1, we have

$$\begin{aligned} & -\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt \\ \leq\ & \int_S^T E^q(t)\left(\varepsilon c\int_\Omega |u(t)|^{m(x)}\,dx + c_\varepsilon\int_\Omega |u_t(t)|^{m(x)}\,dx\right)dt \\ \leq\ & \varepsilon c(n,s,p)\int_S^T E^{q+1}(t)\,dt + c_\varepsilon\int_S^T E^q(t)\left(-E'(t)\right)dt \\ \leq\ & \varepsilon c(n,s,p)\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \end{aligned} \tag{4.44}$$

For the last term of (4.40), we have

$$\begin{aligned}
&(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \\
\leq\;& (3-\beta_1-\beta_2)\int_S^T E^q(t)\left[\int_{\Omega^-}|u_t(t)|^2\,dx+\int_{\Omega^+}|u_t(t)|^2\,dx\right]dt \\
\leq\;& c\int_S^T E^q(t)\left[\left(\int_{\Omega^-}|u_t(t)|^{m_2}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega^+}|u_t(t)|^{m_1}\,dx\right)^{\frac{2}{m_1}}\right]dt \\
\leq\;& c\int_S^T E^q(t)\left[\left(\int_{\Omega}|u_t(t)|^{m_2}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega}|u_t(t)|^{m_1}\,dx\right)^{\frac{2}{m_1}}\right]dt \\
\leq\;& c\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt+c\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt
\end{aligned}\tag{4.45}$$

First, if we use Young inequality with $\lambda_1=\frac{q+1}{q}$ and $\lambda_2=q+1$, we have

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\int_S^T\left(-E'(t)\right)^{\frac{2q+1}{m_2}}dt$$

we take $q=\frac{m_2}{2}-1$ to find

$$\begin{aligned}
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt &\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\int_S^T\left(-E'(t)\right)dt \\
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt &\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E(S).
\end{aligned}\tag{4.46}$$

On the other hand, we have:

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E(S).\tag{4.47}$$

Indeed,

if $m_1=2$ then

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt\leq cE(S)\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E(S).$$

If $m_1>2$, we use the Young's inequality $\lambda_1=\frac{m_1}{m_1-2}$ and $\lambda_2=\frac{m_1}{2}$ to obtain

$$\begin{aligned}
\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt &\leq \varepsilon c\int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt+c_\varepsilon\int_S^T\left(-E'(t)\right)dt \\
&\leq \varepsilon c\int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt+c_\varepsilon E(S).
\end{aligned}$$

We notice that $q\frac{m_1}{m_1-2} = q+1+\frac{m_1-m_2}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}} dt &\le \varepsilon c\left(E(S)\right)^{\frac{m_1-m_2}{m_1-2}} \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c\left(E(0)\right)^{\frac{m_1-m_2}{m_1-2}} \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S).\end{aligned}$$

So, we substituting (4.46) and (4.47) in (4.45), we obtain

$$(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.48}$$

By insert (4.42), (4.43), (4.44) and (4.48) in (4.40), we arrive at

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le \varepsilon c(n,s,p)\int_S^T E^{\frac{m_2}{2}}(t)\,dt + \left(c^p(n,s,p)+c_\varepsilon\right)E(S).$$

Choosing ε small enough for that

$$(\gamma - c(n,s,p))\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le \left(c^p(n,s,p)+c\right)E(S).$$

We take $\gamma - c(n,s,p) > 0$, we obtain

$$\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le \eta E(S),$$

where η is only related to $c,\ \gamma,\ c(n,s,p)$. So, by taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m_2}{2}}(t)\,dt \le \eta E(S).$$

By Komornik's integral inequality yields the result. ■

Conclusion

In conclusion, in our thesis we have been treated three classes of systems supplemented with Dirichlet boundary conditions. As a model of a nonlinear parabolic and hyperbolic systems with variable exponents and hyperbolic system involving fractional Laplacian. We have been studied in the first class of system the global existence and stability of solution for a nonlinear Kirchhoff type reaction diffusion equation with variable exponents. In the second class we have been showed that the solution with negative initial energy of wave equation with multiple $\alpha(x)$ −Laplacian and variable exponent blow up in finite time. Then, in last class we have been obtained the existence of weak solution, stability of solution for a new model of fractional p-Kirchhoff type hyperbolic equation with variable exponents.

These systems have been treated by the technique of Faedo-Galerkin approximation and some analysis tools in order to prove the global existence and by Komornik's inequality to prove the stability results.

Finally, these studies can extend more general boundary value systems involving fractional Laplacian with variable exponents and find the appropriate numerical methods. We can also try to find an application of these models in image processing.

Bibliography

[1] S. Antontsev, J. Ferreira, E. Pişkin. *Existence and Blow up of Petrovsky Equation Solutions with Strong Damping and Variable Exponents*. Electronic Journal of Differential Equations, 2021 (6), 1-18, (2021).

[2] S. Antontsev, S. Shmarev. *Evolution PDEs with Nonstandard Growth Conditions*: *Existence, Uniqueness, Localization, Blow-up*. Atlantis Studies in Differential Equations 4. Springer, Berlin, (2015).

[3] S. Antontsev. *Wave equation with p(x,t)-Laplacian and damping term*: *Blow-up of solutions*. Comptes Rendus Mécanique, 339 (12), 751-755, (2011).

[4] S. Antontsev. *Wave equation with p(x,t)-Laplacian and damping term*: *Existence and blow-up*. Differential Equations & Applications, 3 (4), 503-525, (2011).

[5] N. Aronszajn. *Boundary values of functions with finite Dirichlet integral*. Tech. Report of Univ. of Kansas, Vol. 14, 77–94, (1955).

[6] J. Ball. *Remarks on blow up and nonexistence theorem for nonlinear evolution equations*. Quart J Math, 28, 473-486, (1977).

[7] A. Benaissa, S. Mokeddem. *Decay estimates for the wave equation of p-Laplacian type with dissipation of m-Laplacian type*. Mathematical Methods in the Applied Sciences, 30 (2), 237-247, (2007).

[8] H. Brezis, *How to recognize constant functions. Connections with Sobolev spaces,* Uspekhi Mat. Nauk, Vol. 57, No. 4, 59–74 (2002); translation in: Russian Math. Surveys, Vol. 57,No. 4, 693–708, (2002).

[9] H. Chen, G. Liu. *Global existence, uniform decay and exponential growth for a class of semi-linear wave equations with strong damping*. Acta Math. Sci, Ser. B, Engl. Ed. 33 (2013), 41–58, (2013).

[10] C. Chen, H. Yao, L. Shao. *Global existence, uniqueness, and asymptotic behavior of solution for p-Laplacian type wave equation*. Journal of Inequalities and Applications, 2010 (1), 216760, (2010).

[11] F. Demengel, G. Demengel. *Functional spaces for the theory of elliptic partial differential equations.* Universitext, (2012).

[12] E. Di nezza, G. Palatucci, E. Valdinoci. *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. math. 136, pp.521-573.

[13] Y. Fu, M. Xiang. *Existence of solutions for parabolic equations of Kirchhoff type involving variable exponent*. Appl. Anal. 95 (2016), 524–544, (2016).

[14] Q. Gao, F. Li, Y. Wang. *Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation*. Cent. Eur. J. Math. 9 (2011), 686–698, (2011).

[15] E. Gagliardo. *Proprietà di alcune classi di funzioni in più variabili*. Ric. Mat, Vol. 7, 102–137, (1958).

[16] S. Ghegal, I. Hamchi, S.A. Messaoudi. *Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities*. Appl. Anal. 99 (2020), 1333–1343, (2020)

[17] M. Ghisi, M. Gobbino. *Hyperbolic-parabolic singular perturbation for middly degenerate Kirchhoff equations*: *Time-decay estimates*. J. Differ. Equations, 245 (2008), 2979–3007, (2008)

[18] G. Gilboa, S. Osher. *Nonlocal operators with applications to image processing*. Multi scale Model. Simul, 7 (2008), no. 3, 1005–1028, (2008).

[19] B. Guo, W.J. Gao. *Blow-up of solutions to quasilinear hyperbolic equations with p(x,t)-Laplacian and positive initial energy*. Comptes Rendus Mécanique, 342 (9), 513-519, (2014).

[20] Y. Han, Q. Li. *Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy*. Comput. Math. Appl, 75 (2018), 3283–3297, (2018).

[21] A. Haraux, E. Zuazua. *Decay estimates for some semilinear damped hyperbolic problems.* Arch Ration Mech Anal. 150, 191-206, (1988).

[22] Z. Jiang, S. Zheng, X. Song. *Blow-up analysis for a nonlinear diffusion equation with nonlinear boundary conditions.* Appl. Math. Lett, 17, 193–199, (2004).

[23] A. Khaldi, M. Maouni, A. Ouaoua. *Asymptotic behavior for a fractional p-kirchhoff type hyperbolic equation with variable exponent.* Accepting for reviewing in International Journal of Applied and Computational Mathematics.

[24] A. Khaldi, A. Ouaoua, M. Maouni. *Blow up of solutions for wave equation with multiple* $\alpha(x)$ *−laplacian and variable exponent nonlinearities.* Turkish Journal of Mathematics, 47(3), 1039-1050, (2023).

[25] A. Khaldi, A. Ouaoua, M. Maouni. *Global existence and stability of solution for a nonlinear Kirchhoff type reaction- diffusion equation with variable exponents*. Mathematica Bohemica, (2021).

[26] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*. North-Holland mathematics studies, Elsevier, 204 (2006).

[27] G. Kirchhoff. *Vorlesungen über mathematische Physik*. 1. Band: Mechanik. Teubner, Leipzig, German, (1883).

[28] V. Komornik. *Exact Controllability and Stabilization*. The Multiplier Method. Research in Applied Mathematics 36. Wiley, Chichester, (1994).

[29] N.S. Landkof. *Foundations of Modern Potential Theory*. Grundlehren Math. Wiss, Springer, Berlin, vol. 180, (1972).

[30] D. Lars, P. Harjulehto, P. Hasto, M. Ruzicka. *Lebesgue and Sobolev Spaces with Variable Exponents*. Lecture Notes 18 in Mathematics, vol. 2017, (2011).

[31] H. A. Levine, S. Ro Park. *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation.* J. Math. Anal. Appl. 228, 181–205,(1998).

[32] H. A. Levine, J. Serrin. *A global nonexistence theorem for quasilinear evolution equation with dissipation*. Arch. Rational Mech. Anal. 137, 341–361,(1997).

[33] H.A. Levine. *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + F(u)$. Trans. Am. Math. Soc. 192 (1974), 1–21, (1974).

[34] H.A. Levine. *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*. SIAM J. Math. Anal. 5 (1974), 138–146, (1974).

[35] H. Li. *Blow-up of solutions to a p-Kirchhoff-type parabolic equation with general nonlinearity*. J. Dyn. Control Syst, 26 (2020), 383–392, (2020).

[36] J. Li, Y. Han. *Global existence and finite time blow-up of solutions to a nonlocal p-Laplace equation*. Math. Model. Anal. 24 (2019), 195–217, (2019).

[37] X. Li, B. Guo, M. Liao. *Asymptotic stability of solutions to quasilinear hyperbolic equations with variable sources*. Computers & Mathematics with Applications, 79 (4), 1012-1022, (2020).

[38] F. Li, F. Liu. *Blow-up of solutions to a quasilinear wave equation for high initial energy*. Comptes Rendus Mécanique, 346 (5), 402-407, (2018).

[39] M. Liao, Q. Liu, H. Ye. *Global existence and blow up of weak solutions for a class of fractional p-Laplacian evolution equations*. Advances in nonlinear Analysis, (2020).

[40] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Gauthier-Villars, Paris, (1969).

[41] J.N. Mazon, J.D Rossi, J. Teldo. *Fractional p-Laplacian evolution equations. Journal de mathématiques Pures et appliquées*, (2016).

[42] S.A. Messaoudi, A.A. Talahmeh. *Blow up in a semilinear pseudo-parabolic equation with variable exponents*. Ann. Univ. Ferrara, Sez. VII, Sci. Mat, 65 (2019), 311–326, (2019).

[43] S.A. Messaoudi, A.A. Talahmeh. *On wave equation: review and recent results.* Arabian Journal of Mathematics, 113-145, (2018).

[44] S.A. Messaoudi, H.A. Jamal, A.A. Talahmeh. *Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities*. Computers & Mathematics with Applications, 76 (8): 1863-1875, (2018).

[45] S.A. Messaoudi, A.A. Talahmeh. *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities*. Applicable Analysis, 96 (9), 1509-1515, (2017).

[46] S.A. Messaoudi, A.A. Talahmeh. *Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities.* Math. Methods Appl. Sci, 40 (2017), 6976–6986, (2017).

[47] S.A. Messaoudi, A.A. Talahmeh, H.A. Jamal. *Nonlinear damped wave equation: Existence and blow-up*. Comput. Math. Appl. 74 (2017), 3024–3041, (2017).

[48] S.A. Messaoudi, B.S. Houari. *Global non-existence of solutions of a class of wave equations with non-linear damping and source terms*. Mathematical Methods in the Applied Sciences, 27 (14), 1687-1696, (2004).

[49] M. Musso, Y. Sire, J.C. Wei, Y.Q. Zheng, Y.F. Zhou. *Infinite time blow-up for the fractional heat equation with critical exponent*. Mathematische Annalen, 375 (2019), no. 1–2, 361–424, (2019).

[50] K. Ono. *Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings*. J. Differ. Equations, 137 (1997), 273–301, (1997).

[51] A. Ouaoua, A. Khaldi, M. Maouni. *Global existence and stability of solution for a p-Kirchhoff type hyperbolic equation with variable exponents*. Boletim da SociedadeParanaense de Matematica, 1-12, (2020).

[52] A. Ouaoua, A. Khaldi, M. Maouni. *Existence and Stability results of a nonlinear Timoshenko equation with damping and source terms*. Theoretical and Applied Mechanics, 48 (1), 53-66, (2020).

[53] A. Ouaoua, M. Maouni, A. Khaldi. *Exponential decay of solutions with Lp -norm for a class to semilinear wave equation with damping and source terms*. Open Journal of Mathematical Analysis, 4 (2), 123-131, (2020).

[54] A. Ouaoua, M. Maouni. *Blow-up, exponential growth of solution for a nonlinear parabolic equation with $p(x)$-Laplacian*. Int. J. Anal. Appl, 17 (2019), 620–629, (2019).

[55] N. Pan, B.L. Zhang, J. Cao. *Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian*. Nonlinear Anal. Real World Appl, 37 (2017), 56–70, (2017).

[56] E. Pişkin. *Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon equation with variable exponents.* Honam Mathematical Journal, 40 (4), 771-783, (2018).

[57] E. Pişkin. *On the decay and blow up of solutions for a quasilinear hyperbolic equations with nonlinear damping and source terms*. Boundary Value Problems,127, (2015).

[58] E. Pişkin, N, Polat. *On the decay of solutions for a nonlinear Petrovsky equations*. Math. Sci. Letters 3(1), 43-47, (2014).

[59] N. Polat. *Blow up of solution for a nonlinear reaction diffusion equation with multiple nonlinearities*. Int. J. Sci. Technol. 2 (2007), 123–128, (2007).

[60] P. Pucci, M. Xiang, B. Zhang. *A diffusion problem of Kirchhoff type involving the nonocal fractional p-laplacian*. Discrete contiu. Dyn. Syst. 37, 4035-4051, (2017).

[61] V.P. Radulescu, G.M. Bisci, R. Servadei. *Variational methods for a nonlocal fractional problem.* University Printing House, Cambridge CB2 8BS, United Kingdom, 2016.

[62] L.E. Silvestre. *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Thesis (Ph.D.)-The University of Texas at Austin. 2005. 95 pp, (2005).

[63] E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, (1970).

[64] L. N. Slobodeckij. *Generalized Sobolev spaces and their applications to boundary value problems of partial differential equations.* Leningrad. Gos. Ped. Inst. Ucep. Zap, Vol. 197, 54–112, (1958).

[65] E. Vitillaro. *Global nonexistence theorems for a class of evolution equations with dissipation*. Arch. Ration. Mech. Anal. 149 (1999), 155–182, (1999).

[66] S.T. Wu, L.-Y. Tsai. *Blow-up solutions for some non-linear wave equations of Kirchhoff type with some dissipation*. Nonlinear Anal, Theory Methods Appl, Ser. A, 65 (2006), 243–264, (2006).

[67] Y. Ye. *Global non existence of solutions for systems of quasilinear hyperbolic equations with damping and source terms*. Boundary Value Problems, 2014 (1), 251, (2014).

[68] S. Zheng. *Nonlinear Evolution Equations*. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics 133. Chapman & Hall/CRC, Boca Raton, pp. 103 (2004).