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Common Fixed Points for Occasionally Weakly Biased Mappings in a Metric Space

Option : Applied functional analysis

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DEDICATION

Me Halima LAHCENE, I dedicate this work with the utmost respect, appreciation and thanks to:

- *my dear father Makhlouf, may God protect him,*
- *my late mother Warda Bensaci, may God have mercy on her,*
- *my brothers Skander and Madani,*
- *my sisters Karima, Yamina and Amira,*
- *the youngest Mohamed AHCENE,*
- *my friends,*
- *all my family that bears the name LAHCENE and BENSACI,*
- *and everyone who contributed to the realization of this dissertation.*

DEDICATION

Me Fairouz TABBAL, I dedicate this dissertation to:

- *my dear father Mohamed and my dear mother Fadila, may God protect them,*
- *my sisters Ibtissem and Salima,*
- *my brothers Kheireddine, Choaib and Moad,*
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Abstract

This work consists of two parts and is focused on studying the existence and uniqueness of common fixed points for mappings satisfying weak conditions in a dislocated metric space as well as a metric space.

In the first part of this work, in their paper [35], Wadkar et al. discussed the existence and uniqueness of fixed point for two pairs of weakly compatible mappings in a dislocated metric space which generalizes and improves similar fixed point results. However, this work contains a lot of mistakes in mathematics as well as in literature. The purpose of the first chapter is to repair and remedy this paper; i.e., we will correct the main result of [35] by adding some require conditions, removing certain undesirable things, rectify, change, ameliorate, modify and regulate some inevitable faults.

In the second and last part, we have introduced a new concept called occasionally weakly biased mappings of type (A) in order to prove a unique common fixed point theorem for four mappings on a metric space. This theorem improves the main result of [35].

Key words and phrases: Metric space, dislocated metric space, compatible mappings of type (A), biased mappings of type (A), weakly biased mappings of type (A), occasionally weakly biased mappings of type (A), unique common fixed points.

Résumé

Ce travail se compose de deux parties et se concentre sur l'étude de l'existence et de l'unicité de points fixes communs pour des applications satisfaisant des conditions faibles dans un espace métrique disloqué ainsi qu'un espace métrique.

Dans la première partie de ce travail, dans leur article [35], Wadkar et al. ont discuté l'existence et l'unicité du point fixe pour deux paires d'applications faiblement compatibles dans un espace métrique disloqué qui généralise et améliore des résultats similaires de point fixe. Cependant, ce travail contient beaucoup d'erreurs en mathématiques ainsi qu'en littérature. Le but du premier chapitre est de réparer et remédier cet article; c'est-à-dire, on va corriger le résultat principal de [35] en ajoutant des conditions nécessaires, en supprimant certaines choses indésirables, rectifier, changer, améliorer, modifier et réguler quelques erreurs inévitables.

Dans la deuxième et dernière partie, on a introduit un nouveau concept appelé « applications occasionnellement faiblement biaisées de type (A) » afin de prouver un théorème de point fixe commun et unique pour quatre applications dans un espace métrique. Ce théorème améliore le résultat principal de [35].

Mots et phrases clés: Espace métrique, espace métrique disloqué, applications compatibles de type (A), applications biaisées de type (A), applications faiblement biaisées de type (A), applications occasionnellement faiblement biaisées de type (A), points fixes communs uniques.

ملخص

يتكون هذا العمل من جزأين ويركز على دراسة وجود و وحدانية النقاط الثابتة المشتركة للتطبيقات التي تحقق الظروف الضعيفة في الفضاء المتري المخلوع و الفضاء المتري.

في الجزء الأول من هذا العمل، في ورقتهم [35] ودكار و شركائه ناقشوا وجود و وحدانية نقطة ثابتة لزوجين من التطبيقات المتوافقة بشكل ضعيف في فضاء متري مخلوع والتي تعمم وتحسن نتائج النقطة الثابتة المماثلة. ومع ذلك، يحتوي هذا العمل على الكثير من الأخطاء في الرياضيات وكذلك في الأدب. الغرض من الفصل الأول هو إصلاح وعلاج هذه الورقة؛ أي أننا سنصحح النتيجة الرئيسية لـ [35] بإضافة بعض الشروط المطلوبة، إزالة بعض الأشياء غير المرغوب فيها، تصحيح بعض الأخطاء التي لا مفر منها، تغييرها، تحسينها، تعديلها وتنظيمها.

في الجزء الثاني والأخير، قدمنا مفهومًا جديدًا يسمى التطبيقات المنحازة أحيانا بشكل ضعيف من النوع (أ) لإثبات نظرية النقطة الثابتة المشتركة الفريدة لأربعة تطبيقات في فضاء متري. هذه النظرية تحسن النتيجة الرئيسية لـ [35].

الكلمات والعبارات المفتاحية: فضاء متري، فضاء متري مخلوع، تطبيقات متوافقة من النوع (أ)، تطبيقات منحازة من النوع (أ)، تطبيقات منحازة ضعيفة من النوع (أ)، تطبيقات منحازة أحيانا بشكل ضعيف من النوع (أ)، نقاط ثابتة مشتركة فريدة.

Introduction

Fixed point theory is an interesting topic, with a wide number of applications in numerous branches of mathematics. Fixed point theorems concern a mapping f of a nonempty set X into itself that, under certain conditions, admits a fixed point, that is, a point $x \in X$ such that $f(x) = x$.

The knowledge of the existence of fixed points has pertinent applications in various branches of analysis and topology. It can be applied in several areas of mathematics and other fields, like game theory, mathematical economics, optimization theory, approximation theory, biology, chemistry, engineering, and physics and so on and so forth.

The early fixed point theorems were published between 1910 and 1955. The early fixed point theorems were established by Brouwer in 1912, Banach in 1922, Schauder in 1930, Kakutani in 1941, and Knaster and Tarski in 1955. Since then, many researchers established several results that use weaker conditions.

Recently, the famous fixed point theorems cited above have been improved, extended, and generalized by several researchers in the following ways:

1. How to generalize these theorems?
2. How to extend these theorems to different various spaces?
3. How to extend these theorems to multi-valued mappings?
4. How to improve these theorems?
5. How to apply these theorems in various domains?

This work consists of two parts and is focused on studying the existence and uniqueness of common fixed points for mappings satisfying weak conditions.

In the first chapter, we will correct the main result of Wadkar et al. [35] by adding some require conditions, removing certain undesirable things, rectify, change, ameliorate, modify and regulate some inevitable things.

In the second and last part, we will prove a unique common fixed point theorem for four occasionally weakly biased mappings of type (A) on a metric space. This theorem improves the main result of [35].

At last a bibliography is given, which contains only those papers which have been referred to in this dissertation.

Contents

Chapter 1

A common fixed point theorem in dislocated metric space

In this chapter we will correct the main result of [35] by adding some require conditions, removing certain undesirable things, rectify, change, ameliorate, modify and regulate some inevitable things.

1.1 Introduction

According to [35], "fixed point theory is one of the most dynamic research subjects in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. The most impressing result in this direction was given by Banach, called the Banach contraction mapping principle: Every contraction in a complete metric space has a unique fixed point. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. This celebrated result has been generalized in many abstract spaces for distinct operators. In 2000, Hitzler and Seda [12] introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero. They also generalized the famous Banach contraction principle in this space. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. Aage and Salunke ([1], [2]), Isufati [14] established some important fixed point theorems in single and pair of mappings in dislocated metric space. Jha and Panthi [15] established a common fixed point theorem in dislocated metric spaces". In this paper [35], Wadkar et al. discussed the existence and uniqueness of fixed point for two pairs of weakly compatible mappings in dislocated metric space which generalizes and improves similar fixed point results. However, this work contains a lot of mistakes in mathematics as well as in literature. The purpose of this chapter is to repair and remedy this paper.

1.2 Preliminaries

Definition 1.1 *Let $\mathcal{X}$ be a non-empty set. A function $d : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ is said to be a dislocated metric (or a metric-like) on $\mathcal{X}$ if for any x, y, $z \in \mathcal{X}$, the following conditions hold:*

1. $d(x, y) = 0 \Rightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The pair $(\mathcal{X}, d)$ is then called a ***dislocated metric (metric-like) (d-metric) space****.*

Definition 1.2 *([12]) A sequence $\{X_n\}$ in a d-metric space $(\mathcal{X}, d)$ is called a Cauchy sequence if for $\epsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all m, $n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.*

Definition 1.3 *([12]) A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in \mathcal{X}$ such that $d(x_n, x) \to 0$ as $n \to \infty$.*

Definition 1.4 *([12]) A d-metric space $(\mathcal{X}, d)$ is called complete if every Cauchy sequence in it is convergent with respect to d.*

Definition 1.5 *([12]) Let (X, d) be a d-metric space. A mapping $T : \mathcal{X} \to \mathcal{X}$ is called contraction if there exist a number λ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$.*

Lemma 1.1 *([12]) Let $(\mathcal{X}, d)$ be a d-metric space. If $T : \mathcal{X} \to \mathcal{X}$ is a contraction function, then $\{T^n x_0\}$ is a Cauchy sequence for each $x_0 \in \mathcal{X}$.*

Lemma 1.2 *([12]) Limits in a d-metric space are unique.*

Definition 1.6 *([17]) Let A and S be mappings from a metric space $(\mathcal{X}, d)$ into itself, then A and S are said to be* ***weakly compatible*** *if they commute at their coincident point; that is, $Ax = Sx$, for some $x \in \mathcal{X}$ implies $ASx = SAx$.*

1.3 Existence and uniqueness of common fixed points

Theorem 1.1 *Let $(\mathcal{X}, d)$ be a complete d-metric space. Let A, B, S, $T : \mathcal{X} \to \mathcal{X}$ be continuous mappings satisfying,*

1. $T(\mathcal{X}) \subset A(\mathcal{X})$, $S(\mathcal{X}) \subset B(\mathcal{X})$.
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. $$d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(Ax, By) + \gamma d(Ax, Sx) + \eta d(By, Ty) + \delta d(Sx, By)$$

for all x, $y \in \mathcal{X}$ where α, β, γ, η, $\delta \geq 0$ and $0 \leq \alpha + \beta + \gamma + \eta + \delta < \frac{1}{2}$. Then A, B, S, and T have a unique common fixed point.

Proof. Existence: Using condition (1) we define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ by the rule

$$y_{2n} = Bx_{2n+1} = Sx_{2n}$$

and

$$y_{2n+1} = Ax_{2n+2} = Tx_{2n+1},$$

$n = 0, 1, \ldots$. If $y_{2n} = y_{2n+1}$ for some n then $Bx_{2n+1} = Tx_{2n+1}$. Therefore x_{2n+1} is a coincidence point of B and T also if $y_{2n+1} = y_{2n+2}$ for some n then $Ax_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of A and S. Assume that $y_{2n} \neq y_{2n+1}$ for all n then we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Ax_{2n}, Bx_{2n+1}) + \gamma d(Ax_{2n}, Sx_{2n}) + \eta d(Bx_{2n+1}, Tx_{2n+1}) \\
&\quad + \delta d(Sx_{2n}, Bx_{2n+1}) \\
&\leq \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n-1}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n}) + \eta d(y_{2n}, y_{2n+1}) + \delta d(y_{2n}, y_{2n}) \\
&\leq \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n-1}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n}) + \eta d(y_{2n}, y_{2n+1}) \\
&\quad + \delta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\
&\leq (\alpha + \beta + \gamma + \delta) d(y_{2n-1}, y_{2n}) + (\alpha + \eta + \delta) d(y_{2n}, y_{2n+1}) \\
&\leq \frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \eta - \delta} d(y_{2n-1}, y_{2n}).
\end{aligned}$$

$$d(y_{2n}, y_{2n+1}) \leq h d(y_{2n-1}, y_{2n})$$

where $\frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \eta - \delta} < 1$. This shows that

$$d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n) \leq \ldots \leq h^n d(y_0, y_1).$$

For any integer $q > 0$, we have

$$\begin{aligned}
d(y_n, y_{n+q}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+q-1}, y_{n+q}) \\
&\leq (1 + h + h^2 + \ldots + h^n) d(y_n, y_{n+1})
\end{aligned}$$

$$\Rightarrow d(y_n, y_{n+q}) \leq \frac{h^n}{1-h} d(y_0, y_1).$$

Since $0 < h < 1$, $h^n \to 0$ as $n \to \infty$, So we get $d(y_n, y_{n+q}) \to 0$. This implies that $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So there exist a point z in $\mathcal{X}$ such that $y_n \to z$ therefore the subsequences $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$, $\{Ax_{2n+2}\}$ converge to z.

Since $T(\mathcal{X}) \subset A(\mathcal{X})$ there exist a point u in $\mathcal{X}$ such that $z = Au$, so

$$\begin{aligned}
d(Su, z) &= d(Su, Tx_{2n+1} \\
&\leq \alpha d(Au, Tx_{2n+1}) + \beta d(Au, Bx_{2n+1}) + \gamma d(Au, Su) + \eta d(Bx_{2n+1}, Tx_{2n+1}) \\
&\quad + \delta d(Su, Bx_{2n+1}) \\
&= \alpha d(z, Tx_{2n+1}) + \beta d(z, Bx_{2n+1}) + \gamma d(z, Su) + \eta d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Su, Bx_{2n+1}).
\end{aligned}$$

Now taking limit as $n \to \infty$ we get

$$\begin{aligned} d(Su,z) &= \alpha d(z,z) + \beta d(z,z) + \gamma d(z,Su) + \eta d(z,z) + \delta d(Su,z) \\ &\leq (2\alpha + 2\beta + \gamma + 2\eta + \delta) d(Su,z) \end{aligned}$$

which is a contradiction since $2\alpha + 2\beta + \gamma + 2\eta + \delta < 1$. So we have $Su = Au = Z$.

Again since $S(\mathcal{X}) \subset B(\mathcal{X})$ there exist a point v in $\mathcal{X}$ such that $z = Bv$, We claim that $z = Tv$. If $Z \neq Tv$ then

$$\begin{aligned} d(z,Tv) &= d(Su,Tv) \\ &\leq \alpha d(Au,Tv) + \beta d(Au,Bv) + \gamma d(Au,Su) + \eta d(Bv,Tv) + \delta d(Bv,Su) \\ &= \alpha d(z,Tv) + \beta d(z,z) + \gamma d(z,z) + \eta d(z,Tv) + \delta d(z,z) \\ &\leq (\alpha + 2\beta + 2\gamma + \eta + 2\delta) d(z,Tv). \end{aligned}$$

A contradiction, since $\alpha + 2\beta + 2\gamma + \eta + 2\delta < 1$ so we get $z = Tv$. Hence we claim that $Su = Au = Tv = Bv = z$. Since S and A are weakly compatible so by definition $SAu = ASu \Rightarrow Sz = Az$.

Now we show that z is fixed point of S. If $Sz \neq z$ then

$$\begin{aligned} d(Sz,z) &= d(Sz,Tv) \\ &\leq \alpha d(Az,Tv) + \beta d(Az,Bv) + \gamma d(Az,Sz) + \eta d(Bv,Tv) + \delta d(Bv,Sz) \\ &= \alpha d(Sz,z) + \beta d(Sz,z) + \gamma d(Sz,Sz) + \eta d(z,z) + \delta d(z,Sz) \\ &\leq (\alpha + \beta + 2\gamma + 2\eta + \delta) d(Sz,z). \end{aligned}$$

This is a contradiction. So we have $Sz = z$. This implies that $Az = Sz = z$. Again the pair (T,B) is weakly compatible so by definition $TBv = BTv = Tz = Bz$.

Now we show that z is fixed point of T. If $Tz \neq z$ then

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq \alpha d(Az,Tz) + \beta d(Az,Bz) + \gamma d(Az,Sz) + \eta d(Bz,Tz) + \delta d(Bz,Sz) \\ &= \alpha d(z,Tz) + \beta d(z,Tz) + \gamma d(z,z) + \eta d(Tz,Tz) + \delta d(Tz,z) \\ &\leq (\alpha + \beta + 2\gamma + 2\eta + \delta) d(z,Tz) \end{aligned}$$

this is a contradiction. This implies that $z = Tz$ hence we have $Az = Bz = Sz = Tz = z$. This shows that z is a common fixed point of mappings A, B, S and T.

Uniqueness: Let $u \neq v$ be two common fixed points of mappings A, B, S and T then we have

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha d(Au,Tv) + \beta d(Au,Bv) + \gamma d(Au,Su) + \eta d(Bv,Tv) + \delta d(Bv,Su) \\ &= \alpha d(u,v) + \beta d(u,v) + \gamma d(u,u) + \eta d(v,v) + \delta d(v,u) \\ &\leq (\alpha + \beta + 2\gamma + 2\eta + \delta) d(u,v) \end{aligned}$$

a contradiction this shows that $d(u,v) = 0$. Since $(\mathcal{X}, d)$ is a dislocated metric space. So we have $u = v$. This establishes the theorem. ■

1.4 Illustrative example

Example 1.1 *Let* $\mathcal{X} = [0,1]$ *endowed with the d-metric* $\max\{x,y\}$. *Let mappings* A, B, S *and* T *be defined by* $Sx = 0$, $Ax = \frac{x}{2}$, $Bx = x$ *and* $Tx = \frac{x}{6}$. *Take* $\alpha = \beta = \gamma = \frac{1}{11}$ *and* $\eta = \delta = \frac{1}{9}$. *Then we have:*

1. $([0,1], d)$ *is a complete d-metric space,*
2. $T\mathcal{X} = \left[0, \frac{1}{6}\right] \subset A\mathcal{X} = \left[0, \frac{1}{2}\right]$ *and* $S\mathcal{X} = \{0\} \subset B\mathcal{X} = [0,1]$,
3. *mappings* A, B, S *and* T *are continuous,*
4. A *and* S *as well as* B *and* T *are weakly compatible, and for all* x, $y \in [0,1]$ *we have*
5.

$$\begin{aligned} d(Sx,Ty) &= \frac{y}{6} \\ &\leq \frac{1}{11}\max\left\{\frac{x}{2},\frac{y}{6}\right\} + \frac{1}{11}\max\left\{\frac{x}{2},y\right\} + \frac{1}{11}\times\frac{x}{2} + \frac{1}{9}\times y + \frac{1}{9}\times y \\ &= \frac{1}{11}\max\left\{\frac{x}{2},\frac{y}{6}\right\} + \frac{1}{11}\max\left\{\frac{x}{2},y\right\} + \frac{x}{22} + \frac{2y}{9} \\ &= \alpha d(Ax,Ty) + \beta d(Ax,By) + \gamma d(Ax,Sx) + \eta d(By,Ty) + \delta d(Sx,By). \end{aligned}$$

So, mappings A, B, S *and* T *satisfy all the conditions of the above theorem and* $x = 0$ *is the unique common fixed point of the four mappings.*

Chapter 2

A unique common fixed point theorem for occasionally weakly biased mappings of type (A)

In this chapter, we will prove a unique common fixed point theorem for four occasionally weakly biased mappings of type (A) on a metric space. This theorem improves the main result of [35].

2.1 Introduction

According to Kumari et al. [23] the notion of dislocated metric (d-metric) space was introduced by Hitzler in [11]. Using this notion, some researchers studied the existence of common fixed points under different conditions (see for example [15], [22], [18], [6], [23], [28], [32], [34], [5], [9], [10], [24], [25], [35]).

In 1982, Sessa in [33] introduced the concept of weak commutativity. After that, in 1986, Jungck [16] generalized the concept of weak commutativity by introducing the notion of compatible mappings. In 1995, Jungck and Pathak [20] gave a generalization of the concept of compatible mappings called biased mappings. Again, the same authors [20], introduced the concept of weakly biased mappings which represents a convenient generalization of biased mappings. In 2012, in [8], we introduced the concept of occasionally weakly biased mappings which is a legitimate generalization of weakly biased mappings given by Jungck and Pathak in [20]. Let us return back to 1993, Jungck et al. [19] introduced the concept of compatible mappings of type (A) which is equivalent to compatible mappings under the continuity condition. After two years, Pathak et al. [30] generalized the above notion by giving the concept of biased mappings of type (A). Also and in the same paper [30], the authors gave the definition of weakly g-biased of type (A). In 1996, the notion of compatible mappings was again generalized in [17] by Jungck himself. In 2008, Al-Thagafi and Shahzad [4] introduced the notion of occasionally weakly compatible (owc) mappings as a generalization of weakly compatible mappings. While the paper [4] was under review, Jungck and Rhoades [21] used the concept of owc and proved several results under different contractive conditions (see [3]). Since then, a lot of important common fixed point theorems of commuting, weakly commuting, compatible, biased, biased of type (A), weakly compatible, weakly biased, occasionally weakly compatible, weakly biased of type (A) and occasionally weakly biased mappings under various contractive conditions have been obtained by several authors. Recently, in 2022, in [7] we introduced the concept of weakly f-biased of type (A), and the concepts of occasionally weakly f-biased of type (A) and occasionally weakly g-biased of type (A), and we showed that the two last new definitions coincide with our concepts; occasionally weakly f-biased and occasionally weakly g-biased respectively given in [8]. In this chapter, we will prove a unique common fixed point theorem for four occasionally weakly biased mappings of type (A) on a metric space. Our result improves the one's of Wadkar et al. [35].

2.2 Preliminaries

Definition 2.1 *(see chapter 1) Let $\mathcal{X}$ be a non-empty set. A function $d : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ is said to be a dislocated metric (or a metric-like) on $\mathcal{X}$ if for any x, y, $z \in \mathcal{X}$, the following conditions hold:*

1. $d(x, y) = 0 \Rightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The pair $(\mathcal{X}, d)$ is then called a ***dislocated metric (metric-like) space****.*

Example 2.1 *If $\mathcal{X} = [0, \infty)$, then $d(x, y) = x + y$ defines a dislocated metric on $\mathcal{X}$.*

Example 2.2 *Let $\mathcal{X} = [0, \infty)$ define the distance function $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ by $d(x, y) = \max\{x, y\}$. Clearly $\mathcal{X}$ is a dislocated metric space.*

Definition 2.2 *Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be* ***commuting*** *if and only if*

$$fgx = gfx$$

for all x in $\mathcal{X}$.

Definition 2.3 *([33]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called* ***weakly commuting*** *if and only if*

$$d(fgx, gfx) \leq d(fx, gx)$$

for all x in $\mathcal{X}$.

Definition 2.4 *([16]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called* ***compatible*** *if and only if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 2.5 *([20]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is* ***g-biased*** *if and only if whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ and fx_n, $gx_n \to t \in \mathcal{X}$, then*

$$\alpha d(gfx_n, gx_n) \leq \alpha d(fgx_n, fx_n)$$

if $\alpha = \lim inf$ and $\alpha = \lim sup$.

Definition 2.6 *([20]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is* ***weakly g-biased*** *if and only if $fp = gp$ implies*

$$d(gfp, gp) \leq d(fgp, fp).$$

Definition 2.7 *([8]) Let f and g be self-mappings of a set $\mathcal{X}$. The pair (f, g) is said to be* ***occasionally weakly f-biased*** *and* ***g-biased****, respectively, if and only if, there exists a point p in $\mathcal{X}$ such that $fp = gp$ implies*

$$d(fgp, fp) \leq d(gfp, gp),$$
$$d(gfp, gp) \leq d(fgp, fp),$$

respectively.

Definition 2.8 *([19]) Self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be* ***compatible of type*** *(A) if*

$$\lim_{n\to\infty} d(gfx_n, ffx_n) = 0, \lim_{n\to\infty} d(fgx_n, ggx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that fx_n and $gx_n \to t \in \mathcal{X}$.

Definition 2.9 *([30]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be* ***g-biased*** *and* ***f-biased of type*** *(A), respectively, if, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ and fx_n, $gx_n \to t \in \mathcal{X}$,*

$$\alpha d(ggx_n, fx_n) \leq \alpha d(fgx_n, gx_n),$$
$$\alpha d(ffx_n, gx_n) \leq \alpha d(gfx_n, fx_n),$$

respectively, where $\alpha = \lim inf_{n\to\infty}$ and if $\alpha = \lim sup_{n\to\infty}$.

Definition 2.10 *([30]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be* ***weakly g-biased of type*** *(A) if $fp = gp$ implies*

$$d(ggp, fp) \leq d(fgp, gp).$$

Definition 2.11 *([17])(see chapter 1) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called* ***weakly compatible*** *if and only if f and g commute on the set of coincidence points.*

Definition 2.12 *([4]) Two self-mappings f and g of a set $\mathcal{X}$ are* ***occasionally weakly compatible*** *if and only if, there is a point t in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute.*

Definition 2.13 *([7]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be* ***weakly f-biased of type*** *(A) if $fp = gp$ implies*

$$d(ffp, gp) \leq d(gfp, fp).$$

Definition 2.14 *([7]) Let f and g be self-mappings of a non-empty set $\mathcal{X}$. The pair (f, g) is said to be **occasionally weakly f-biased of type** (A) and **occasionally weakly g-biased of type** (A), respectively, if and only if, there exists a point p in $\mathcal{X}$ such that $fp = gp$ implies*

$$d(ffp, gp) \leq d(gfp, fp),$$
$$d(ggp, fp) \leq d(fgp, gp),$$

respectively.

In addition that weakly f-biased of type (A) and weakly g-biased of type (A) are occasionally weakly f-biased of type (A) and occasionally weakly g-biased of type (A), respectively, it is also clear from the definitions that if f and g are occasionally weakly compatible or weakly compatible then f, g are both occasionally weakly f-biased and g-biased of type (A). Therefore, occasionally weakly compatible and weakly compatible mappings are subclasses of occasionally weakly biased of type (A) mappings. The next example confirms.

Example 2.3 *Let $\mathcal{X} = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$. Define $f,\ g : \mathcal{X} \longrightarrow \mathcal{X}$ by*

$$fx = \begin{cases} 2x & \text{if } x \in [0,1] \\ \dfrac{3}{x} & \text{if } x \in (1, \infty), \end{cases} \quad gx = \begin{cases} \dfrac{1}{3} & \text{if } x \in [0,1] \\ \dfrac{x}{12} & \text{if } x \in (1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \left\{\dfrac{1}{6} - \dfrac{1}{6n}\right\}$ in $\mathcal{X}$ then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \dfrac{1}{3}$ and

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \frac{1}{3} \neq 0,$$

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = \frac{1}{3} \neq 0 \text{ and } \lim_{n\to\infty} d(gfx_n, ffx_n) = \frac{1}{3} \neq 0,$$

thus, f and g are neither compatible nor compatible of type (A).

We have $fx = gx$ if and only if $x = \dfrac{1}{6}$ or $x = 6$ and

$$0 = d\left(gg\left(\frac{1}{6}\right), f\left(\frac{1}{6}\right)\right) \leq d\left(fg\left(\frac{1}{6}\right), g\left(\frac{1}{6}\right)\right) = \frac{1}{3};$$

that is, f and g are occasionally weakly g-biased of type (A). However,

$$\frac{1}{2} = d\left(ff(6), g(6)\right) \nleq d\left(gf(6), f(6)\right) = \frac{1}{6},$$

then, f and g are not weakly f-biased of type (A).

On the other hand we have

$$fg\left(\frac{1}{6}\right) = \frac{2}{3} \neq \frac{1}{3} = gf\left(\frac{1}{6}\right),$$
$$fg(6) = 1 \neq \frac{1}{3} = gf(6),$$

that is, f and g are neither occasionally weakly compatible nor weakly compatible mappings.

In their paper [35], Wadkar et al. discussed the existence and uniqueness of fixed point for two pairs of weakly compatible mappings in dislocated metric space which generalizes and improves similar fixed point results.

Theorem 2.1 *(see chapter 1) Let $(\mathcal{X}, d)$ be a complete d-metric space. Let A, B, S, $T : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous mappings satisfying,*

1. $T(\mathcal{X}) \subset A(\mathcal{X})$, $S(\mathcal{X}) \subset B(\mathcal{X})$
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(Ax, By) + \gamma d(Ax, Sx) + \eta d(By, Ty) + \delta d(By, Sx)$

for all x, $y \in \mathcal{X}$, where α, β, γ, η, $\delta \geq 0$ and $2\alpha + \beta + \gamma + \eta + 2\delta < 1$. Then A, B, S, and T have a unique common fixed point.

We will improve the above theorem by removing the completeness of the space, the continuity of the all mappings and the inclusions between the range spaces, using the concept of occasionally weakly biased mappings of type (A) which is more general than the weak compatible concept.

2.3 Existence and uniqueness of a common fixed point

Theorem 2.2 *Let f, g, h and k be self-mappings of a metric space $\mathcal{X}$ satisfying the following condition*

$$d(fx, gy) \leq \alpha d(hx, ky) + \beta d(fx, hx) + \gamma d(gy, ky) + \eta d(hx, gy) + \delta d(fx, ky) \tag{2.1}$$

for all x, $y \in \mathcal{X}$, where α, β, γ, η, $\delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$. If the pair (f, h) as well as (g, k) is occasionally weakly h-biased of type (A) and occasionally weakly k-biased of type (A), respectively, then f, g, h and k have a unique common fixed point.

Proof. By hypotheses, there are two points u and v in $\mathcal{X}$ such that $fu = hu$ implies $d(hhu, fu) \leq d(fhu, hu)$ and $gv = kv$ implies $d(kkv, gv) \leq d(gkv, kv)$.

First, we are going to prove that $fu = gv$. From inequality (2.1) we have

$$\begin{aligned} d(fu, gv) &\leq \alpha d(hu, kv) + \beta d(fu, hu) + \gamma d(gv, kv) + \eta d(hu, gv) \\ &\quad + \delta d(fu, kv) \\ &= [\alpha + \eta + \delta] d(fu, gv) \\ &< d(fu, gv) \end{aligned}$$

a contradiction, hence $fu = gv$.

Now, we assert that $ffu = fu$. If not, then the use of condition (2.1) gives

$$\begin{aligned} d(ffu, fu) = d(ffu, gv) &\leq \alpha d(hfu, kv) + \beta d(ffu, hfu) + \gamma d(gv, kv) \\ &\quad + \eta d(hfu, gv) + \delta d(ffu, kv) \\ &= \alpha d(hhu, fu) + \beta d(ffu, hhu) + \eta d(hhu, fu) \\ &\quad + \delta d(ffu, fu) \\ &\leq \alpha d(hhu, fu) + \beta [d(ffu, fu) + d(fu, hhu)] \\ &\quad + \eta d(hhu, fu) + \delta d(ffu, fu) \\ &= [\alpha + \beta + \eta] d(hhu, fu) + [\beta + \delta] d(ffu, fu) \\ &\leq [\alpha + 2\beta + \eta + \delta] d(ffu, fu) \\ &< d(ffu, fu) \end{aligned}$$

a contradiction, thus $ffu = fu$ and so $hfu = fu$.

Suppose that $ggv \neq gv$. Using inequality (2.1) we obtain

$$\begin{aligned} d(gv, ggv) = d(fu, ggv) &\leq \alpha d(hu, kgv) + \beta d(fu, hu) + \gamma d(ggv, kgv) \\ &\quad + \eta d(hu, ggv) + \delta d(fu, kgv) \\ &= \alpha d(gv, kkv) + \gamma d(ggv, kkv) + \eta d(gv, ggv) \\ &\quad + \delta d(gv, kkv) \\ &\leq \alpha d(gv, kkv) + \gamma [d(ggv, gv) + d(gv, kkv)] \\ &\quad + \eta d(gv, ggv) + \delta d(gv, kkv) \\ &= [\alpha + \gamma + \delta] d(gv, kkv) + [\gamma + \eta] d(gv, ggv) \\ &\leq [\alpha + 2\gamma + \eta + \delta] d(gv, ggv) \\ &< d(gv, ggv) \end{aligned}$$

which implies that $ggv = gv$ and so $kgv = gv$; i.e., $gfu = fu$ and $kfu = fu$. Put $fu = hu = gv = kv = w$, therefore w is a common fixed point of mappings f, g, h and k.

Finally, let w and t be two distinct common fixed points of mappings f, g, h and k. Then, $w = fw = gw = hw = kw$ and $t = ft = gt = ht = kt$. From (2.1) we have

$$d(ft, gw) \leq \alpha d(ht, kw) + \beta d(ft, ht) + \gamma d(gw, kw) + \eta d(ht, gw) + \delta d(ft, kw);$$

i.e.,

$$\begin{aligned} d(t, w) &\leq \alpha d(t, w) + \eta d(t, w) + \delta d(t, w) \\ &= [\alpha + \eta + \delta] d(t, w) \\ &< d(t, w) \end{aligned}$$

which is a contradiction, this implies that $t = w$. ■

2.4 Illustrative Example

Example 2.4 *Let $\mathcal{X} = [0, 31)$ with usual metric $d(x, y) = |x - y|$. Define*

$$fx = \begin{cases} \frac{x}{2} \text{ if } x \in [0,1] \\ \frac{1}{5} \text{ if } x \in (1,31), \end{cases} \quad gx = \begin{cases} 0 \text{ if } x \in [0,1] \\ \frac{1}{4} \text{ if } x \in (1,31), \end{cases}$$

and

$$hx = \begin{cases} 30x \text{ if } x \in [0,1] \\ 3 \text{ if } x \in (1,31), \end{cases} \quad kx = \begin{cases} \frac{x^2}{3} \text{ if } x \in [0,1] \\ 30 \text{ if } x \in (1,31). \end{cases}$$

First it is clear to see that f and h are occasionally weakly h-biased of type (A) and g and k are occasionally weakly k-biased of type (A). Take $\alpha = \frac{3}{4}$ and $\beta = \gamma = \eta = \delta = \frac{1}{25}$, we get

1. *for $x,\ y \in [0,1]$ we have $fx = \frac{x}{2}$, $gy = 0$, $hx = 30x$ and $ky = \frac{y^2}{3}$ and*

$$\begin{aligned} & \alpha d(hx,ky) + \beta d(fx,hx) + \gamma d(gy,ky) + \eta d(hx,gy) + \delta d(fx,ky) \\ = \ & \frac{3}{4}\left|30x - \frac{y^2}{3}\right| + \frac{119x}{50} + \frac{y^2}{75} + \frac{1}{25}\left|\frac{x}{2} - \frac{y^2}{3}\right| \\ \geq \ & \frac{x}{2} = d(fx,gy), \end{aligned}$$

2. *for $x,\ y \in (1,31)$, we have $fx = \frac{1}{5}$, $gy = \frac{1}{4}$, $hx = 3$, $ky = 30$ and*

$$\begin{aligned} \frac{1}{20} &= d(fx,gy) \\ &\leq \alpha d(hx,ky) + \beta d(fx,hx) + \gamma d(gy,ky) + \eta d(hx,gy) + \delta d(fx,ky) \\ &= \frac{11427}{500}, \end{aligned}$$

3. *for $x \in [0,1]$, $y \in (1,31)$, we have $fx = \frac{x}{2}$, $gy = \frac{1}{4}$, $hx = 30x$, $ky = 30$ and*

$$\begin{aligned} & \alpha d(hx,ky) + \beta d(fx,hx) + \gamma d(gy,ky) + \eta d(hx,gy) + \delta d(fx,ky) \\ = \ & \frac{45}{2}(1-x) + \frac{59x}{50} + \frac{119}{100} + \frac{1}{100}|120x - 1| + \frac{1}{50}(60 - x) \\ \geq \ & \left|\frac{x}{2} - \frac{1}{4}\right| = d(fx,gy), \end{aligned}$$

4. *finally, for $x \in (1,31)$, $y \in [0,1]$, we have $fx = \frac{1}{5}$, $gy = 0$, $hx = 3$, $ky = \frac{y^2}{3}$ and*

$$\begin{aligned} & \alpha d(hx,ky) + \beta d(fx,hx) + \gamma d(gy,ky) + \eta d(hx,gy) + \delta d(fx,ky) \\ = \ & \frac{1}{4}\left(9 - y^2\right) + \frac{y^2}{75} + \frac{29}{125} + \frac{1}{25}\left|\frac{1}{5} - \frac{y^2}{3}\right| \\ \geq \ & \frac{1}{5} = d(fx,gy), \end{aligned}$$

so, all hypotheses of the above theorem are satisfied and 0 *is the unique common fixed point of mappings* f, g, h *and* k.

Note that Theorem 2.1 of Wadkar et al. [35] is not applicable because the space is not complete, the four mappings are discontinuous and $f\mathcal{X} = \left[0, \frac{1}{2}\right] \nsubseteq \left[0, \frac{1}{3}\right] \cup \{30\} = k\mathcal{X}$.

2.5 Some results

Corollary 2.1 *Let* f *and* h *be self-mappings of a metric space* $\mathcal{X}$ *satisfying the following condition*

$$d(fx, fy) \leq \alpha d(hx, hy) + \beta d(fx, hx) + \gamma d(fy, hy) + \eta d(hx, fy) + \delta d(fx, hy) \tag{2.2}$$

for all x, $y \in \mathcal{X}$, *where* α, β, γ, η, $\delta \geq 0$ *and* $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$. *If the pair* (f, h) *is occasionally weakly* h*-biased of type* (A), *then* f *and* h *have a unique common fixed point.*

Proof. By hypotheses, there exists a point u in $\mathcal{X}$ such that $fu = hu$ implies $d(hhu, fu) \leq d(fhu, hu)$.

First, we confirm that $ffu = fu$. Suppose that $ffu \neq fu$, the use of inequality (2.2) gives

$$\begin{aligned} d(ffu, fu) &\leq \alpha d(hfu, hu) + \beta d(ffu, hfu) + \gamma d(fu, hu) + \eta d(hfu, fu) + \delta d(ffu, hu) \\ &= \alpha d(hhu, fu) + \beta d(ffu, hhu) + \eta d(hhu, fu) + \delta d(ffu, fu), \end{aligned}$$

using the triangle inequality we get

$$\begin{aligned} d(ffu, fu) &\leq \alpha d(hhu, fu) + \beta[d(ffu, fu) + d(fu, hhu)] + \eta d(hhu, fu) + \delta d(ffu, fu) \\ &= [\alpha + \beta + \eta] d(hhu, fu) + [\beta + \delta] d(ffu, fu), \end{aligned}$$

since f and h are occasionally weakly h-biased of type (A), we get

$$\begin{aligned} d(ffu, fu) &\leq [\alpha + \beta + \eta] d(fhu, hu) + [\beta + \delta] d(ffu, fu) \\ &= [\alpha + 2\beta + \eta + \delta] d(ffu, fu) \\ &< d(ffu, fu) \end{aligned}$$

which is a contradiction, hence $ffu = fu$ which implies that $hfu = fu$ because f and h are occasionally weakly h-biased of type (A). Putting $fu = hu = t$, therefore t is a common fixed point of f and h.

Now, let t and w be two distinct common fixed points of both f and h. Using inequality (2.2) we obtain

$$d(ft, fw) \leq \alpha d(ht, hw) + \beta d(ft, ht) + \gamma d(fw, hw) + \eta d(ht, fw) + \delta d(ft, hw);$$

i.e.,

$$\begin{aligned} d(t, w) &\leq \alpha d(t, w) + \eta d(t, w) + \delta d(t, w) \\ &= [\alpha + \eta + \delta] d(t, w) \\ &< d(t, w) \end{aligned}$$

a contradiction, thus, $w = t$. ■

Corollary 2.2 *Let f, g and h be self-mappings of a metric space $\mathcal{X}$ satisfying the following condition*

$$d(fx, gy) \leq \alpha d(hx, hy) + \beta d(fx, hx) + \gamma d(gy, hy) + \eta d(hx, gy) + \delta d(fx, hy) \tag{2.3}$$

for all x, $y \in \mathcal{X}$, where α, β, γ, η, $\delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$. If f and h as well as g and h are occasionally weakly h-biased of type (A), then f, g and h have a unique common fixed point.

Proof. Since f and h as well as g and h are occasionally weakly h-biased of type (A), then, there exist two points u and v in $\mathcal{X}$ such that $fu = hu$ implies $d(hhu, fu) \leq d(fhu, hu)$ and $gv = hv$ implies $d(hhv, gv) \leq d(ghv, hv)$.

First, we are going to prove that $fu = gv$. From inequality (2.3) we have

$$\begin{aligned} d(fu, gv) &\leq \alpha d(hu, hv) + \beta d(fu, hu) + \gamma d(gv, hv) + \eta d(hu, gv) + \delta d(fu, hv) \\ &= [\alpha + \eta + \delta] d(fu, gv) \\ &< d(fu, gv) \end{aligned}$$

a contradiction, hence $fu = hu = gv = hv$.

Now, we assert that $ffu = fu$. If not, then the use of condition (2.3) gives

$$\begin{aligned} d(ffu, fu) = d(ffu, gv) &\leq \alpha d(hfu, hv) + \beta d(ffu, hfu) + \gamma d(gv, hv) \\ &\quad + \eta d(hfu, gv) + \delta d(ffu, hv) \\ &= \alpha d(hhu, fu) + \beta d(ffu, hhu) + \eta d(hhu, fu) + \delta d(ffu, fu) \\ &\leq \alpha d(hhu, fu) + \beta [d(ffu, fu) + d(fu, hhu)] + \eta d(hhu, fu) \\ &\quad + \delta d(ffu, fu) \\ &= [\alpha + \beta + \eta] d(hhu, fu) + [\beta + \delta] d(ffu, fu) \\ &\leq [\alpha + 2\beta + \eta + \delta] d(ffu, fu) \\ &< d(ffu, fu) \end{aligned}$$

a contradiction, thus $ffu = fu$ and so $hfu = fu$.

Suppose that $ggv \neq gv$. Using inequality (2.3) we obtain

$$\begin{aligned} d(gv, ggv) = d(fu, ggv) &\leq \alpha d(hu, hgv) + \beta d(fu, hu) + \gamma d(ggv, hgv) \\ &\quad + \eta d(hu, ggv) + \delta d(fu, hgv) \\ &= \alpha d(gv, hhv) + \gamma d(ggv, hhv) + \eta d(gv, ggv) + \delta d(gv, hhv) \\ &\leq \alpha d(gv, hhv) + \gamma [d(ggv, gv) + d(gv, hhv)] + \eta d(gv, ggv) \\ &\quad + \delta d(gv, hhv) \\ &= [\alpha + \gamma + \delta] d(gv, hhv) + [\gamma + \eta] d(gv, ggv) \\ &\leq [\alpha + 2\gamma + \eta + \delta] d(ggv, gv) \\ &< d(ggv, gv) \end{aligned}$$

which implies that $ggv = gv$ and so $hgv = gv$; i.e., $gfu = fu$. Put $fu = hu = gv = hv = w$, therefore w is a common fixed point of mappings f, g and h.

Finally, let w and t be two distinct common fixed points of mappings f, g and h. Then, $w = fw = gw = hw$ and $t = ft = gt = ht$. By (2.3) we have

$$d(ft, gw) \leq \alpha d(ht, hw) + \beta d(ft, ht) + \gamma d(gw, hw) + \eta d(ht, gw) + \delta d(ft, hw);$$

i.e.,

$$\begin{aligned} d(t, w) &\leq \alpha d(t, w) + \eta d(t, w) + \delta d(t, w) \\ &= [\alpha + \eta + \delta] d(t, w) \\ &< d(t, w) \end{aligned}$$

which is a contradiction, this implies that $t = w$. ■

2.6 A unique common fixed point for a sequence of mappings

Theorem 2.3 *Let h, k and $\{f_n\}_{n=1,2,...}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition*

$$d(f_n x, f_{n+1} y) \leq \alpha d(hx, ky) + \beta d(f_n x, hx) + \gamma d(f_{n+1} y, ky) + \eta d(hx, f_{n+1} y) + \delta d(f_n x, ky) \quad (2.4)$$

for all x, $y \in \mathcal{X}$, where α, β, γ, η, $\delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$. If (f_n, h) as well as (f_{n+1}, k) is occasionally weakly h-biased of type (A) and occasionally weakly k-biased of type (A), respectively, then, h, k and $\{f_n\}_{n=1,2,...}$ have a unique common fixed point.

Proof. Putting $n = 1$, we get that mappings f_1, f_2, h and k satisfy the hypotheses of Theorem 2.2, then they have a unique common fixed point w.

Now, letting $n = 2$, we get that mappings f_2, f_3, h and k have a unique common fixed point t. Assume that $t \neq w$, using inequality (2.4) we get

$$d(f_2 w, f_3 t) \leq \alpha d(hw, kt) + \beta d(f_2 w, hw) + \gamma d(f_3 t, kt) + \eta d(hw, f_3 t) + \delta d(f_2 w, kt);$$

i.e.,

$$\begin{aligned} d(w, t) &\leq \alpha d(w, t) + \beta d(w, w) + \gamma d(t, t) + \eta d(w, t) + \delta d(w, t) \\ &= [\alpha + \eta + \delta] d(w, t) \\ &< d(w, t) \end{aligned}$$

which is a contradiction, hence $t = w$.

Continuing in this manner, we clearly see that w is the unique common fixed point of mappings h, k and $\{f_n\}_{n=1,2,...}$. ■

Conclusion

In this dissertation, in the first chapter we could correct the main result of Wadkar et al. [35] by adding some require conditions, removing certain undesirable things, rectify, change, ameliorate, modify and regulate some inevitable things. In the second and last chapter, we could improve the main results of the same paper by removing some conditions. In other words, we could find a unique common fixed point with neither continuity nor completeness and inclusions, under the new concept of occasionally weakly biased of type (A) mappings.

We finish our conclusion by Browder's and Hazewinkel's (editor's preface to the book of Istratescu [13]) quotations respectively.

> "The theory of fixed points is one of the most powerful tools of modern mathematics".
>
> "Fixed point and fixed point theorems have always been a major theoretical tool as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control and functional analysis. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed point methods a major weapon in the arsenal of the applied mathematician".

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