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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques

Option : ***Mathématique***

Etude théorique et numériques d'une EDP non linéaire de type parabolique

Présentée par :

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Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
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Presented by:

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Topic:

Theoretical and numerical study of a parabolic type nonlinear PDE

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UNIVERSITY OF 20 AUGUST 1955-SKIKDA, 2021.

بسم الله الرحمن الرحيم

Dédicace

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Á mes sœurs et fréres. Nezha, Hanane, Rafik, Mounir, Tarak, Rofia, Sami
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Abstract

The works presented in this thesis relate to the study of some nonlinear partial differential equations (PDE) of the parabolic type making a diverging operator composed with a Carathodory type function with nonlinear conditions on the edge. These equations are generally poorly posed in the context of weak solutions (i.e. in the sense of distributions), because in general we do not have uniqueness. More appropriate formulations then emerged: solutions called renormalized solutions. The original to put in appearance the functional card of the spaces of Lebesgue and Sobolev with variable exponent.
Keywords: Quasilinear parabolic equations; Nonlinear parabolic systems; Fixed point; Variable exponent; Truncation function; Renormalized solutions; Measure data; L^1 data .
AMS subject classification (2010): 35K59; 37C25; 35J70; 35D05.

Titre: **Etude théorique et numériques dune EDP non linéaire de type parabolique**

Résumé

Les travaux présentés dans cette thèse, portent sur l'étude de quelques équations aux dérivées partielles (EDP) non linéaire du type parabolique faisant composé un opérateur divergentielle avec une fonction de type Carathéodory avec des conditions non linéaires sur le bord.

Ces équations sont d'une façon générale mal posées dans le cadre de solutions faibles (i.e. au sens des distributions), car en général on n'a pas l'unicité. Des formulations plus appropriées ont alors vu le jour : les solutions appelées les solutions Rénormalisées.

L'originale de mettre en apparence le carde fonctionnel des espaces de Lebesgue et Sobolev à exposant variable.

Mots clés : Équations paraboliques quasi linéaires; Systémes paraboliques non linéaires; point fixe; exposant variable; fonction de troncature; solutions renormalisées; donnée mesure; donnée L^1.

ملخص

يركز العمل المقدم في هذه الأطروحة على دراسة بعض المعادلات التفاضلية الجزئية غير الخطية (EDP) من النوع المكافئ مما يجعل مؤثر متباعد مع الدالة تكون من نوع $Caratheodory$ بشروط غير خطية على الحافة. يتم وضع هذه المعادلات ضعيفة عموما بشكل سئ في سياق الحلول ضعيفة (بمعنى التوزيعات) ، لأننا بشكل عام لا يوجد وحدانية للحل. ثم ظهرت الحلول أكثر ملائمة: تسمى الحلول المعاد تنظيمها. علمنا يرتكز اساس على مساحات لوبيغ وسوبوليف بالأس المتغير .

Notations

- Ω : open in $\mathbb{R}^N$, $N \in \mathbb{N}^*$.
- $\partial\Omega$: topological boundary of Ω.
- $x = (x_1, x_2, ..., x_N)$: generic point of $\mathbb{R}^N$.
- $dx = dx_1 dx_2 ... dx_N$: measure of Lebesgue on Ω.
- $Q = \Omega \times]0, T[, \quad T > 0$.
- $\Sigma = \partial\Omega \times]0, T[$.
- $X \hookrightarrow Y$: The continuous (or compact) injection from X into Y.
- $\rightarrow$: designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- ∇u : gradient of u.
- div : is the divergence operator.
- $supp(u)$: support of a function u.
- χ_E :characteristic function of the set E.
- $\mathcal{D}(\Omega), \mathcal{D}(Q)$: space of differentiable functions and compact support in Q, Ω.
- $\mathcal{M}(Q)$: bounded Radon measurements space.
- $\mathcal{M}_0(Q)$: bounded Radon measurements space not loading sets of zero capacity.
- T_k : truncation function of level $k > 0$.
- $L^p(\Omega)$: space of the functions of power p-th integrable on for measurement dx; $\|u\|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$.
- $W^{1,p}(\Omega) = \{u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N\}; \|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$.
- $W_0^{1,p}(\Omega)$: adherence of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.
- $W^{-1,p}(\Omega)$: dual space of $W_0^{1,p}(\Omega)$.

If X is a Banach space

- $L^p(]0,T[;X) = \left\{ u :]0,T[\to X \text{ measurable } ; \int_0^T \|u\|_X^p \, dt < \infty \right\}$.
- $L^\infty(]0,T[;X) = \left\{ u :]0,T[\to X \text{ measurable}; ess - sup_{t\in(0,T)} \|u\|_X \, dt < \infty \right\}$.
- $C^k(]0,T[;X)$: space of functions k-times continuously differentiable from $]0,T[\to X$.
- $\mathcal{D}(]0,T[;X)$: continuously differentiable function space with compact support in $]0,T[$.

Let $p : \overline{\Omega} \longrightarrow [1, +\infty)$ be a continuous real-valued function

- $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$.
- $L^{p(.)}(\Omega) = \{u \text{ measurable function in } \Omega : \rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} \, dx\}$.
- $\|u\|_{L^{p(.)}(\Omega)} = \inf \left\{ \mu > 0, \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}$.
- $W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega), |\nabla u| \in L^{p(.)}(\Omega) \right\}$.
- $\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$.
- $W_0^{1,p(.)}(\Omega)$: adherence of $\mathcal{D}(\Omega)$ in $W^{1,p(.)}(\Omega)$.
- $W^{-1,p(.)}(\Omega)$: dual space of $W_0^{1,p(.)}(\Omega)$.
- $L^{p(.)}(Q) = \left\{ u :]0,T[\to \mathbb{R} \text{ measurable}; \int_Q |u(x,t)|^{p(x)} \, dxdt < \infty \right\}$.

•$\|u\|_{L^{p(\cdot)}(Q)} = \inf \left\{ \mu > 0, \int\limits_{Q} \left| \frac{u(x,t)}{\mu} \right|^{p(x)} dxdt \leq 1 \right\}$.

•$B(R)$ open ball centered at the point 0 with the radius R.

Contents

Introduction générale

Mathematics consists first of all a language, which makes it possible to transcribe quantitative problems: It is modeling. Once this transcription has been made, tools are available to understand and resolve the problems. Resulting from the phenomena of the real world which uses the laws of physics (mechanics, thermodynamics, electromagnetism, etc.), these laws are, generally, written in the form of balance sheets which are expressed mathematically by Differential Equations Ordinary or by Partial Differential Equations.
Partial differential equations (PDEs) are also used in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivatives and in image processing to restore degradations . PDEs probably appeared for the first time during the birth of rational mechanics during the 17th century (Newton, Leibniz ...). If only a few names have to be retained, one must cite that of Euler, then those of Navier and Stokes, for the equations of fluid mechanics, those of Fourier for the equation of heat, Maxwell for those of electromagnetism, Schrödinger and Heisenberg for the equations of quantum mechanics, and of course Einstein for the PDEs of the theory of relativity.
However, the systematic study of PDEs is much more recent, and it was only during the 20th century that mathematicians began to develop the necessary arsenal. A giant step was taken by Schwartz when he gave birth to the theory of distributions (around the 1950s), and progress at least comparable is due to Hörmander for the development of pseudodifferential calculus (in the early 1970s). It is certainly good to bear in mind that the study of EDPs remains a very active field of research at the start of the 21st century.
Besides, this research does not only have an impact in the applied sciences, but also plays a very important role in the current development of mathematics itself, both in geometry and in analysis. The primary concern of the mathematician faced with an equation to partial derivatives is to give it meaning in appropriate functional spaces and to demonstrate the existence and uniqueness of the solution.
The study of problems with variable exponent is a new and interesting topic which raises many mathematical difculties see [18],[44]. One of our motivations for studying (2.1),(1),(2),(4.1) comes from applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). Other important applications are related to image processing see [18] and elasticity [44].
The solutions framework Renormalized has been developed to try to overcome these drawbacks. Initially this notion was introduced for the first order equations and for the Boltzmann equations by Di Perna and Lions see [22]. It was then adapted to the case of elliptical equations to data L^1 in [33],[34] and in the case of parabolic equations with data L^1 in [12]. At the same

time, the notions of entropic solutions [16] and SOLA (solution obtained as approximation limit [20]) were born for the elliptic equations or parabolic with L^1 data.
These three notions are in fact equivalent (see for example [21] for the data L^1 or $L^1 + W^{-1,p'}$.
In [21] the authors give a notion of renormalized solution for the elliptic equations with second Radon bounded variation measurement member for which they demonstrate in particular the existence of a solution and a stability result.
Concerning the parabolic problems, the existence and uniqueness of the renormalized solution for parabolic equations with L^1-data in the classical Sobolev spaces see [5], [12] and [38] .
In a general way we have with the framework of the renormalized solutions of the results of existence, stability and, for certain equations, of uniqueness which are not accessible in the framework of solutions in the sense of distributions.
My research work concerns equations with nonlinear, parabolic partial derivatives with classical Lebesgue and Sobolev spaces and generalized Lebesgue and Sobolev spaces.
My thesis was devoted to the study of partial differential equations from a thermoviscoelasticity model. The peculiarity of these systems resides in the presence of a term "gradient squared", a term which is a priori in L^1. During the thesis we used the notion of renormalized solution which is adapted to the treatment of partial differential elliptic and parabolic equations with data L^1 and measure.
This thesis, which aims to contribute to the study of parabolic problems with boundary conditions of Dirichlet type in classical Sobolev spaces and generalized Sobolev spaces with L^1 or measure data.
We will present here results of existence of solutions for four nonlinear problems. After having built the key point of our study "motivation".
This thesis is organized as flows: In the first chapter, we present some definitions and reminders of the results necessary for the continuation of this work. After recalling some basic results on classical Sobolev spaces, we turn to the study of some properties of generalized Sobolev spaces. The first study was the work of Orlicz see [36] who introduced Lebesgue spaces in 1931 generalized $L^p(x)$. In the 1950s, the study of such spaces gained momentum with the work of Nakano (see [31]). Subsequently, Kovacik and Rakosnik see [27] in 1991 deepened the functional analysis of the spaces of Lebesgue and Sobolev with variable powers. They show in their works that the spaces of Lebesgue and Sobolev with constant powers and those with variable powers have several common properties, except one: the notion of continuity. Indeed the space $L^p(x)$ ($p(x)$ not constant) is not stable by translation.
We also present other definitions some basic knowledge on $p(.)-$parabolic capacity and properties of measures. We end this chapter by stating some classical results of integrations (Lebesgue dominated convergence theorem, Fatou's lemma, Vitali, ...).
Second chapter, is devoted to existence of entropy solution for quasilliear parabolic problem in bounded open subset Ω of $\mathbb{R}^N$, with data and u_0 in $L^1(\Omega)$. For this we use the Schauder fixed-point method. The results of the problem discussed can be applied to a variety of different fields in applied mathematics for example in elastic mechanics, image processing and electro-rheological fluid dynamics, etc..
In the third chapter, consists of two parts. In the first one we have studied (2.1) in the context of Sobolev space with variable exponents, in the case where $u = b(u)$ and $u_0 = b(u_0) \in L^1(\Omega)$

of the type

$$\begin{cases} (b(u))_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda\,|u|^{p(x)-2}\,u = f(x,t,u) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ and T is positive constant.
The second part we have solved the asymptotic behavrior of the problem

$$\begin{cases} (b_1(u))_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda\,|u|^{p(x)-2}\,u = f_1(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ (b_2(v))_t - \Delta v + = f_2(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = v = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b_1(u)(t=0) = b_1(u_0) & \text{in} \quad \Omega, \\ b_2(v)(t=0) = b_2(v_0) & \text{in} \quad \Omega. \end{cases} \tag{2}$$

The problem with the second member in L^1 this problem (2) can be seen as a generalization of the problem (1) and used the renormalized solution to obtain the result of existence of the problem (1) and (2).
The last problem, presented in last chapter, in this chapter we are interested in the study of the existence the existence of renormalized solutions u to the quasilinear parabolic problem with variable exponents and measure data involving the $p(x)-$Laplacian type operator

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda\,|u|^{p(x)-2}\,u = \mu & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ u(.,0) = u_0(.) & \text{in} \quad \Omega, \end{cases} \tag{3}$$

where $\lambda > 0$ and T is any positive constant, $\mu \in \mathcal{M}_0(Q)$ is a any measure with bounded variation over $Q = \Omega\times]0,T[$. The problem (3) we prove in clasical Sobolev space [37] in casz where $u = b(u)$, $b(u_0) \in L^1(\Omega)$ with $b : \mathbb{R} \to \mathbb{R}$ is a increasing C^1-function and $b(0) = 0$, for every μ is diffuse measure and also the purpose of this paper is to is to extend the results in [5] to the case of parabolic equations.

Chapter 1

Preliminaries

In this chapter, we collect several basic tools that will be needed throughout this work. The common link between all the results of this chapter is that they are preparatory for the main results, which are contained in the following chapters. In the following will be an open bounded $\mathbb{R}^N$ and we will use the measure of Lesbegue.

1.1 The Functional Spaces

1.1.1 Classic Sobolev and Lebesgue Spaces

Sobolev spaces are a ubiquitous tool in the study of elliptical and parabolic partial differential equations. Understanding them is therefore a necessary step before tackling the equations in question. We resume in this section certain statements by O. Kavian [26] and H. Brezis [17] on the subject, for a more complete presentation of Sobolev's spaces, we can consult the work of R.A. Adams [2].

For $1 \leq p < +\infty$, the space of Lebesgue $L^p(\Omega)$ is defined by:

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ \text{Measurable } \mathbb{R};\ \int_\Omega |u|^p dx < +\infty \right\},$$

provided with the standard

$$\|u\|_p = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}},$$

to make it a Banach space.

For $p = \infty$, we note

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ \text{Measurable}; ess - \sup_\Omega |u| < +\infty \right\},$$

with

$$ess - \sup_{\Omega} |u| = \inf\{C > 0; |u(x)| \leq C \ a.e. \text{in } \Omega\}.$$

•$L^\infty(\Omega)$ has the following standard: $\|u\|_\infty = ess - \sup_{\Omega} |u|$.

•$L^p(\Omega)$ is reflexive and separable for $1 < p < +1$ and its dual is isomorphic to $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

•$L^1(\Omega)$ is separable but not reflexive, its dual is $L^\infty(\Omega)$ on the other hand the dual of $L^\infty(\Omega)$ strictly contains $L^1(\Omega)$.

For $1 \leq p < +\infty$, the space of Sobolev $W^{1,p}(\Omega)$ is defined by:

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega); \nabla u \in (L^p(\Omega))^N\right\},$$

which provides the standard

$$\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p,$$

is a Banach space.

If $p = \infty$, we provide $W^{1,\infty}(\Omega)$ of the standard

$$\|u\|_{1,\infty} = \max(\|u\|_\infty, \|\nabla u\|_\infty).$$

We note for $1 \leq p < +\infty$,

$$W_0^{1,p}(\Omega) = \overline{D(\Omega)}^{W^{1,p}(\Omega)}.$$

As for$1 \leq p < +\infty$, the space $D(\Omega)$ is by definition dense in $W_0^{1,p}(\Omega)$, we can identify the dual of $W_0^{1,p}(\Omega)$ to a subspace of the distribution space$D'(\Omega)$ by :

$$W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))' \quad \left(p' = \frac{p}{p-1}\right).$$

In the manipulation of Sobolev's spaces, very often we call upon certain so-called Sobolev injections. We recall one of these injections given by the Rellich-Kondrachov theorem.

Theorem 1.1 (Rellich-Kondrachov)**.** *We assume Ω of class C^∞ and $p < N$. So*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \forall q \in [1, p^*[\ or \quad p^* = \frac{Np}{N-p}.$$

In particular, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for all $p \in [1, +\infty)$.

$L^p(0,T;X)$ spaces

In this section, we briefly present some useful results on the function spaces with values in a Banach space. Here and throughout suite X denotes a Banach space and $T > 0$. We define the following spaces:

$$C([0,T];X) = \{u : [0,T] \to X \text{ continues}\},$$

$$L^p(0,T;X) = \left\{ u : (0,T) \to X \text{ measurable; } \int_0^T \|u(t)\|_X^p \, dt < +\infty, \; 1 \leq p < +\infty \right\}.$$

Remark 1.1. *provided with the standard*

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}},$$

and

$$L^\infty(0,T;X) = \{u : (0,T) \to X \text{ measurable}; \exists C > 0 \;\; \|u(t)\|_X < C, a.e.t\},$$

provided with the standard

$$\|u\|_{L^\infty(0,T;X)} = \inf\{C > 0; \|u(t)\|_X \leq C \;, a.e.t\}.$$

For all $1 \leq p < \infty$, $L^p(0,T;X)$ is a Banach space and $C([0,T];X)$ is dense in $L^p(0,T;X)$. If $1 < p < \infty$ and if X is reflexive, then $L^p(0,T;X)$ is reflexive ([23]).

Lemma 1.1. *[39] [Aubin's type compactness lemma:] Let $X \subset B \subset Y$ be three Banach spaces such that the injection of X into $B$is compact. If F is bounded in $L^p(0,T;X)$ with $1 \leq p < N$ and if $F_t = \{f_t, f \in F\}$ is bounded in $L^1(0,T;Y)$ then F is relatively compact in $L^p(0,T;B)$.*

Another basic result of integration theory is Vitali's theorem, which is based on the following definition:

Definition 1.1. *[23] Let $1 \leq p < \infty$. We say that a sequence of functions $(f_n)_n$ of $L^p(0,T;X)$ is p-equi-integrable if it satisfies the following condition: $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall n \geq 1, \forall A \subset (0,T)$ measurable with $|A| < \delta$, we have*

$$\int_A \|f_n(t)\|_X^p \, dt < \varepsilon.$$

A function $f \in L^p(0,T;X)$ is p-equi-integrable .

Theorem 1.2. *[23][Vitali's theorem] Let $1 \leq p < \infty$. If $(f_n)_n$ is a sequence of $L^p(0,T;X)$ converging almost everywhere to f, then $f_n \to f$ in $L^p(0,T;X)$ if, and only if $(f_n)_n$ is p-equi-integrable.*

Theorem 1.3. *[25][Schauder fixed point] Let E be a Banach space, $R > 0$, $B_R = \{x \in E; \|x\| \leq R\}$ and f a compact application of B_R in B_R (that is to say f continuous and $\{f(x); x \in B_R\}$ relatively compact in E). Then f has a fixed point, that is to say that there exists $x \in B_R$ t.q. $f(x) = x$.*

1.1.2 Generalized Sobolev and Lebesgue Spaces

We recall some defnitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. To refer to Fan and Zhao [28] for further properties of Lebesgue-Sobolev spaces with variable exponents. Let $p : \overline{\Omega} \longrightarrow [1, +\infty)$ be a continuous real-valued function, let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(.) < N$. To denote the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx, \tag{1.1}$$

is finite. If the exponent is bounded, i.e, if $p^+ < +\infty$, then the expression

$$\|u\|_{L^{p(.)}(\Omega)} = \inf \left\{ \mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1 \right\}, \tag{1.2}$$

defines a norm in $L^{p(.)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(.)}(\Omega); \|.\|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for $x \in \Omega$. The following inequality will be used later:

$$\min\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\} \leq \int_\Omega |u(x)|^{p(x)}\, dx \leq \max\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\}. \tag{1.3}$$

Finally, the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{1.4}$$

for all u$\in L^{p(.)}(\Omega)$ and v $\in L^{p'(.)}(\Omega)$. Let

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega); |\nabla u| \in L^{p(.)}(\Omega)\right\}, \tag{1.5}$$

which is Banach space equipped with the following norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}. \tag{1.6}$$

The space $(W^{1,p(.)}(\Omega); \|.\|_{1,p(.)})$ is a separable and reflexive Banach space. The manipulation of the generalized Lebesgue and Sobolev spaces is plays and important role using the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. The results as follows:

Proposition 1.1. *If $u_n, u \in L^{p(.)}(\Omega)$ and $p^+ < +\infty$, the following properties hold true.*

(i) $\|u\|_{p(.)} > 1 \Longrightarrow \|u\|_{p(.)}^{p+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p-}$,
(ii) $\|u\|_{p(.)} < 1 \Longrightarrow \|u\|_{p(.)}^{p-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p+}$,
(iii) $\|u\|_{p(.)} < 1$ *(respectively* $= 1; > 1$*)*$\Longleftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*),*
(iv) $\|u_n\|_{p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)*$\Longleftrightarrow \rho_{p(.)}(u_n) < 1$*(respectively* $\longrightarrow +\infty$*),*
(v) $\rho_{p(.)}\left(\frac{u}{\|u\|_{p(.)}}\right) = 1.$

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the following notation

$$\rho_{1,p(.)} = \int_{\Omega} |u|^{p(x)}\, dx + \int_{\Omega} |\nabla u|^{p(x)}\, dx.$$

Proposition 1.2. *If $u \in W^{1,p(.)}(\Omega)$ and $p^+ < +\infty$, the following properties hold true.*

(i)$\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|_{1,p(.)}^{p+} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p-}$,
(ii)$\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|_{1,p(.)}^{p-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p+}$,
(iii)$\|u\|_{1,p(.)} < 1$ *(respectively* $= 1; > 1$*)*$\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively* $= 1; > 1$*).*

Extending a variable exponent $p : \overline{\Omega} \longrightarrow [1, +\infty)$ to $\overline{Q} = \overline{\Omega} \times [0, T]$ by setting $p(x, t) = p(x)$ for all $(x, t) \in \overline{Q}$. We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \longrightarrow \mathbb{R} \text{ mesurable such that} \int_{Q} |u(x, t)|^{p(x)}\, d(x, t) < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q)} = \inf \left\{ \mu > 0; \int_{Q} \left| \frac{u(x, t)}{\mu} \right|^{p(x)} d(x, t) \leq 1 \right\},$$

which share the same properties as $L^{p(.)}(\Omega)$.

1.1.3 The Inportance of Parabolic Capacity and Measures

The Parabolic Capacity

The relevant notion in the study of problems as (4.1) is the notion of parabolic $p(.)$-capacity. Let $Q = \Omega \times]0, T[$ for any fixed $T > 0$. We recall that for every $p > 1$ and every open subset $U \subset Q$, the $p(.)$-parabolic capacity of U is given by (see [35])

$$cap_{p(.)}(U) = \inf \left\{ \|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0, T), u \geq \mathcal{X}_U \text{ a.e in } Q \right\},$$

where

$$W_{p(.)}(0,T) = \left\{ u \in L^{p-}(0,T;V), \nabla u \in (L^{p(.)}(Q)^{\mathbb{N}}, u_t \in L^{(p-)'}(0,T;V') \right\},$$

being $V = W_0^{1,p(.)}(\Omega) \cap L^2(\Omega)$ and V' its dual space. As usual $W_{p(.)}(0,T)$ is endowed with the norm

$$\|u\|_{W_{p(.)}(0,T)} = \|u\|_{L^{p-}(0,T;V)} + \|\nabla u\|_{(L^{p(.)}(Q))^{\mathbb{N}}} + \|u_t\|_{L^{(p-)'}(0,T;V')}.$$

The $p(.)$-parabolic capacity *capp* is then extended to arbitrary Borel subsets $B \subseteq Q$ as

$$cap_{p(.)}(B) = \inf \left\{ cap_{p(.)}(U) : B \subseteq U \text{ and } U \subset \ Q \text{ is open } \right\}.$$

The Measures

Let $\mathcal{M}(Q)$ denotes the set of all Radon measures with bounded variation on Q. Moreover, as already mentioned, by $\mathcal{M}_0(Q)$ we will denote the set of all measures with bounded total variation over Q that do not charge the sets of zero $p(.)$-capacity, that is, if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$ for every Borel set $E \subset Q$ such that $cap_{p(.)}(E) = 0$.

In [35] the authors also proved the following decomposition theorem:

Theorem 1.4. *Let μ be a bounded measure on Q. If $\mu \in \mathcal{M}_0(Q)$, then there exists $(f;F;g_1;g_2)$ such that $f \in L^1(Q)$, $F \in (L^{p'(.)}(Q))^{\mathbb{N}}$, $g_1 \in L^{(p-)'}(0,T;W^{-1,p'(.)}(\Omega))$, $g_2 \in L^{(p-)}(0,T;V)$ and*

$$\int_Q \varphi d\mu = \int_Q f dxdt + \int_Q F \nabla\varphi dxdt + \int_0^T \langle g_1, \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g_2 \rangle \, dt, \tag{1.7}$$

for any $\varphi \in C_c^\infty([0,T] \times \Omega)$. Such a triplet (f, F, g_1, g_2) will be called a decomposition of μ.

Note that the decomposition of μ is not uniquely determined.
In the proof of that result the density will be used as an argument, and so the following preliminary result can be found, for instance, in [35].

Proposition 1.3. *Let $\mu \in M_0(Q)$. Then there exists a decomposition $(f;F;div(G);g)$ of μ in the sense of Theorem (1.4) and an approximation μ^ε of μ satisfying the following conditions:*

$$\mu^\varepsilon \in C_c^\infty(Q); \qquad \|\mu^\varepsilon\|_{L^1(Q)} \leq C, \tag{1.8}$$

$$\int_Q \mu^\varepsilon \varphi dxdt = \int_Q f^\varepsilon \varphi dxdt + \int_Q F^\varepsilon \nabla\varphi dxdt + \int_0^T \langle div(G^\varepsilon), \varphi \rangle \, dt$$

$$- \int_0^T \langle \varphi_t, g^\varepsilon \rangle \, dt, \ \ \forall \varphi \in C_c^\infty(Q), \tag{1.9}$$

and

$$\begin{array}{rcc} f^{\varepsilon} \in C_c^{\infty}(Q): & f^{\varepsilon} \to f & L^1(Q), \\ F^{\varepsilon} \in (C_c^{\infty}(Q))^N: & F^{\varepsilon} \to F & \left(L^{p'(.)}(Q)\right)^N, \\ G^{\varepsilon} \in (C_c^{\infty}(Q))^N: & G^{\varepsilon} \to G & \left(L^{p'(.)}(Q)\right)^N, \\ g^{\varepsilon} \in C_c^{\infty}(Q): & g^{\varepsilon} \to g & L^{(p-)}(]0,T[\,;V), \end{array}$$

as $\varepsilon \to 0$.

Here are some notations that be used throughout the paper. For any nonnegative real number k denoted by $T_k(r) = \min(k; \max(r; -k))$ the truncation function at level k. By using $\langle .,. \rangle$ mean the duality between suitable spaces in which functions are involved. In particular to consider both the duality between $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$ and the duality between $W_0^{1,p(.)}(\Omega) \cap L^{\infty}(\Omega)$ and $W^{-1,p'(.)}(\Omega) + L^1(\Omega)$. Note that the formulation of a renormalized solution does not depend on the decomposition of μ. The proof of this fact relies on the following result.

Lemma 1.2. *Let* $\mu \in \mathcal{M}_0(Q)$*, and let* $(f; F; g_1; g_2)$ *and* $(\widetilde{f}; \widetilde{F}; \widetilde{g_1}; \widetilde{g_2})$ *to be two different decompositions of* μ *according to Theorem* (1.4)*. Then we have* $(g_2 - \widetilde{g_2})_t = \widetilde{f} - f + \widetilde{F} - F + \widetilde{g_1} - g_1$ *in distributional sense,* $g_2 - \widetilde{g_2} \in C(]0,T[; L^1(\Omega))$ *and* $(g_2 - \widetilde{g_2})(0) = 0$*.*

Proof. See [35], Lemma 4.6. □

Chapter 2

Study of quaslinear parabolic problems with data in L^1

2.1 Introduction

In this chapter is devoted to present the results of the existence of a solution for a quasilinear parabolic problem with data in L^1, the main difficulty facing one who is interested in such problems is that the classical theories of existence, either using variational methods or compacite methods, are not applicable. Hence the need to use new techniques to prove the existence and uniqueness of solutions for such problems.
In the last few, different methods have been applied to study the existence of the weak solution of elliptic problems with L^1 under linear boundary conditions see [8], [3], [28] and [18]. The corresponding parabolic case equations have also been studied by many authors, see for instance [10], [19], [20] and [18].
Besides, partial differential equation (PDE) methods in image processing have proven to be fundamental tools for image diffusion and restoration. Go to [[1], [4]] and references therein.
The aim of this chapter, is to treat the existence of solution u for the following quasi-linear parabolic problem of the type

$$\begin{cases} u_t - div(A(u)\nabla u) + \lambda |u|^{p-2}u = f(t,x,u) & \text{in} \quad Q = [0,T]\times\Omega, \\ u = 0 & \text{on} \quad \Sigma = [0,T]\times\partial\Omega, \\ u(0,.) = u_0(.) & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

In the problem (2.1). Where $\lambda > 0$ and $T > 0$, Ω is a bounded open spatial domain in $\mathbb{R}^N$ ($N \geq 2$) with a lipschitz boundary denoted by $\partial\Omega$, and $u_0 \in L^1(\Omega)$. The function $\gamma(u) = \lambda|u|^{p-2}u$ such that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$ and the operator $A : \mathbb{R} \to \mathcal{M}_N(\mathbb{R})$ (or $\mathcal{M}_N(\mathbb{R})$ denotes the set of $N \times N$ matrices with real coefficients), such that satisfies the following assumption for some numbers $0 < \alpha < \beta < \infty$:

$$\forall s \in \mathbb{R}, A(s) = (a_{i,j}(s))_{i,j=1,\dots,N} \text{ where } a_{i,j} \in L^{\infty}(\mathbb{R}) \cap C(\mathbb{R}, \mathbb{R}), \tag{2.2}$$

$$\exists\ \alpha > 0,\ \text{such that } A(s)\xi.\xi \geq \alpha\, |\xi|^2\,, \quad \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \tag{2.3}$$

$$\exists\ \beta > 0,\ \text{such that } \|a_{i,j}\|_{L^{\infty}(\mathbb{R})} \leq \beta, \quad \forall i, j \in \{1, ..., N\}\,. \tag{2.4}$$

We will assume that $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Caratheòdory function such that the following hypotheses hold

$$|f(t, x, s)| \leq c(t, x) + \sigma |s|, \tag{2.5}$$

$$s f(t, x, s) \geq 0, \tag{2.6}$$

for almost every $(t, x) \in Q$, for every $s \in \mathbb{R}$, where c is a positive function in $L^2(Q)$ and $\sigma > 0$. In this work we are studying the existence of weak solution of the quasilinear parabolic problem (2.1) using the truncation technique and the Schauder fixed point theory see [4],[46].
This result generalizes an analog of this work which was made by N. Alaa and all [4] with an increase of γ but given L^1 and, on the other hand, to extend it to the case $f(t, x, u)$ in L^1 data.
To prove the main result, these are three steps: the first step, is to approximate the problem by the fixed point method. In the second step, is to estimate on the approximate solution.
In the last step, is to study the asymptotic behaviour of the approximate solution as n go to infinity we use the equi-integrable theorem.
The difficulty of this work lies in the fact that the variational method can not be used because f is in L^1.

2.2 The Main of Results

Before tackling the main problem, there must be a clear definition of weak solution to the quasilnear parabolic problem.

Definition 2.1. *Let $1 < p \leq N$ a fixed number with $p > 2 - \frac{1}{N}$. We call u a weak solution of the problem (2.1) in Q, if $u \in L^2([0, T], H_0^1(\Omega)) \cap C([0, T], L^1(\Omega))$, $u(0, .) = u_0$ for all $\varphi \in C_0^{\infty}(Q)$ we have*

$$\int_Q - u\varphi_t dxdt + \int_Q A(u)\nabla u \nabla \varphi dxdt + \int_Q \lambda\, |u|^{p-2}\, u\varphi dxdt = \int_Q f(t, x, u)\varphi dxdt, \tag{2.7}$$

where $f(t, x, u)$ and $\gamma(u) \in L^1(Q)$.

The main result of this paper is the following theorem:

Theorem 2.1. *Under the assumptions (2.2) − (2.6) satisfies, then for all $u_0 \in L^1(\Omega)$, there exists a weak solution u of problem (2.1) in the sense defined in (2.7).*

Now to prove our main result.

2.3 Proof of the Theorem

The proof of the theorem consists in the three steps in the first step we solve an approximate problem, in the second step is to estimates on the approximated solutions these estimates allow to go to the limit in the third step.

2.3.1 First step: Solving an approximated problem

For $n \in \mathbb{N}$ let us define the following approximation of $u_{n,0}$ and f_n, γ_n. Set

$$f_n(t,x,p) = \begin{cases} f(t,x,p) & \text{if } |f(t,x,p)| \leq n, \\ n \operatorname{sign}(f(t,x,p)) & \text{if } |f(t,x,p)| > n. \end{cases} \tag{2.8}$$

$$\gamma_n(p) = \begin{cases} \gamma(p) & \text{if } |\gamma(p)| \leq n, \\ 0 & \text{if } |\gamma(p)| > n. \end{cases} \tag{2.9}$$

And $(u_{n,0})_{n\in\mathbb{N}}$ be sequences in $L^2(\Omega)$ such that $(u_{n,0}) \to (u_0)$ in $L^1(\Omega)$.

Remark

$$|f_n(t,x)| \leq n \text{ and } |\gamma_n(p)| \leq n,$$
$$\text{so} \quad \gamma_n,\ f_n \in L^\infty(Q) \hookrightarrow L^p(Q),\ p > n \geq 1.$$

The sequence of approximate problems is considered

$$\begin{cases} (u_n)_t - div(A(u_n)\nabla u_n) + \gamma_n(u_n) = f_n(t,x,u_n) & \text{in} \quad Q = [0,T] \times \Omega, \\ u_n = 0 & \text{on} \quad \Sigma = [0,T] \times \partial\Omega, \\ u_n(0,.) = u_{n,0}(.) & \text{in} \quad \Omega. \end{cases} \tag{2.10}$$

To show that for all $n \in \mathbb{N}^*$ and $f_n(t,x,u_n) \in L^2(Q)$, $u_{n,0} \in L^2(\Omega)$ there exists $u_n \in L^2([0,T], H_0^1(\Omega)) \cap C([0,T], L^2(\Omega))$ and $(u_n)_t \in L^2([0,T], H^{-1}(\Omega))$ verify for all $v \in L^2([0,T], H_0^1(\Omega))$, there have

$$\begin{aligned} &\int_0^T \langle (u_n)_t, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(u_n)\nabla u_n \nabla v dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(u_n) v dxdt = \int_0^T\int_\Omega f_n(t,x,u_n) v dxdt. \end{aligned} \tag{2.11}$$

To show the existence of a weak solution of the problem (2.10) by the classical Schauder's fixed point theorem. To show now that the nonlinear application F defined by

$$\begin{aligned} F:\ L^2([0,T], H_0^1(\Omega)) &\to \quad L^2([0,T], H_0^1(\Omega)) \\ v_n &\mapsto \quad F(v_n) = G \circ F_n(v_n) = v_n, \end{aligned}$$

solution of

$$\int_0^T \langle (v_n)_t, \varphi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T \int_\Omega A(v_n)\nabla v_n \nabla \varphi dxdt$$
$$+ \int_0^T \int_\Omega \gamma_n(v_n)\varphi dxdt = \int_0^T \int_\Omega f_n(t,x,v_n)\varphi dxdt, \forall \varphi \in L^2(]0,T[, H_0^1(\Omega)),$$

is completely continous application of $L^2([0,T]\,, H_0^1(\Omega))$ in $L^2([0,T]\,, H_0^1(\Omega))$.
Where the operator (F_n) is defined by

$$\begin{array}{llll} F_n: & L^2([0,T]\,, H_0^1(\Omega)) & \to & L^2([0,T]\,, H^{-1}(\Omega)) \\ & v_n & \mapsto & F_n(v_n) = (v_n)_t + div(A(v_n)\nabla v_n) = f_n(t,x,v_n) - \gamma_n(v_n) \quad = \widetilde{f}_n(v_n), \end{array}$$

is continous and compact (natural injection), and G is the Green's operator defined by :

$$\begin{array}{llll} G: & L^2([0,T]\,, H^{-1}(\Omega)) & \to & L^2([0,T]\,, H_0^1(\Omega)) \\ & \widetilde{f}_n(v_n) = w_n & \mapsto & G(w_n) = v_n, \end{array}$$

is continous because the operator of Green is isomorphism of $L^2([0,T]\,, H^{-1}(\Omega))$ in $L^2([0,T]\,, H_0^1(\Omega))$.
Therefore, the operator $F = G \circ F_n$ is completely continous.
The existence of a fixed point of $G \circ F_n$ is an immediate consequence of Schauder's fixed point theorem.
To apply the theorem of Schauder's, you have to choose a closed convex generally suitable a closed ball

$$C = \left\{ v \in L^2([0,T]\,, H_0^1(\Omega)) \text{ such that } \|v\|_{L^2([0,T],H_0^1(\Omega))} \leq M \right\},$$

where M is a constant to be determined subsequently, is therefore,

$$\begin{array}{llll} F: & L^2([0,T]\,, H_0^1(\Omega)) & \to & L^2([0,T]\,, H_0^1(\Omega)) \\ & v_n & \mapsto & F(v_n) = v_n, \end{array}$$

transforms the bounds of $L^2([0,T]\,, H_0^1(\Omega))$ into relatively compact sets in $L^2([0,T]\,, H_0^1(\Omega))$, the set C is a closed convex of $L^2([0,T]\,, H_0^1(\Omega))$ and bounded, so F is relatively compact.
To show that $R(F) = \{F(v_n),\ \forall v_n \in L^2([0,T], H_0^1(\Omega))\}$ is bounded in $L^2([0,T]\,, H_0^1(\Omega))$, as $F(v_n)$ is solution of the variational problem.

$$\int_0^T \langle (F(v_n))_t, \varphi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T \int_\Omega A(v_n))\nabla F(v_n) \nabla \varphi dxdt$$
$$+ \int_0^T \int_\Omega \gamma_n(v_n)\varphi dxdt = \int_0^T \int_\Omega f_n(t,x,v_n)\varphi dxdt, \forall \varphi \in L^2([0,T]\,, H_0^1(\Omega)). \tag{2.12}$$

We choose $F(v_n) = \varphi$ in (2.12), to obtain

$$\begin{aligned} &\int_0^T \langle F(v_n)_t, F(v_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(v_n)\nabla F(v_n)\nabla F(v_n)dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(v_n)F(v_n)dxdt = \int_0^T\int_\Omega f_n(t,x,v_n)F(v_n)dxdt. \end{aligned} \tag{2.13}$$

By using Cauchy-Schwarz inequality in (2.13), to have

$$\begin{aligned} \frac{1}{2}\|v_n(T)\|_2^2 - \frac{1}{2}\|v_n(0)\|_2^2 + \int_0^T\int_\Omega A(v_n)\left|\nabla F(v_n)\right|^2 dxdt \;\; &\leq \;\; \int_Q |\gamma_n(v_n)|\,|F(v_n)|\,dxdt \\ &\quad + \int_Q |f_n(t,x,v_n)|\,|F(v_n)|\,dxdt, \end{aligned}$$

by, using generalized Young's inequality and the hypothesis (2.3), to get

$$\begin{aligned} \alpha\int_0^T\int_\Omega |\nabla F(v_n)|^2\, dxdt \;\; &\leq \;\; \frac{1}{2}\|v_n(0)\|_2^2 + \|f_n(t,x,v_n)\|_{L^2(Q)}\,\|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\ &\quad + \|\gamma_n(v_n)\|_{L^2(Q)}\,\|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))} \\ &\leq \;\; \frac{1}{2}\|v_n(0)\|_2^2 + \frac{1}{2\varepsilon}\|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2}\|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 \\ &\quad + \frac{1}{2\varepsilon}\|\gamma_n(v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2}\|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 . \end{aligned}$$

So the conclusion is,

$$\begin{aligned} (\alpha-\varepsilon)\,\|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))}^2 \;\; &\leq \;\; \frac{1}{2}\|v_n(0)\|_2^2 + \frac{1}{2\varepsilon}\|f_n(t,x,v_n)\|_{L^2(Q)}^2 \\ &\quad + \frac{1}{2\varepsilon}\|\gamma_n(v_n)\|_{L^2(Q)}^2 . \end{aligned} \tag{2.14}$$

Therefore the sequence $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,,H_0^1(\Omega))$. Next is to show that $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n)\in R(F)\}$ is bounded in $L^2([0,T],H^{-1}(\Omega))$. to have

$$\begin{aligned} &\int_0^T \langle F(v_n)_t, F(v_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(v_n)\nabla F(v_n)\nabla F(v_n)dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(v_n)F(v_n)dxdt = \int_0^T\int_\Omega f_n(t,x,v_n)F(v_n)dxdt. \end{aligned}$$

By using hypothesis (2.2) and (2.4), we get

$$\begin{aligned} & \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\ \leq\ & \beta \|F(v_n)\|^2_{L^2([0,T],H_0^1(\Omega))} + \|f_n(t,x,v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\ & + \|\gamma_n(v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} . \end{aligned}$$

Eventually,

$$\begin{aligned} & \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \\ \leq\ & \beta \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} + \|f_n(t,x,v_n)\|_{L^2(Q)} + \|\gamma_n(v_n)\|_{L^2(Q)} . \end{aligned}$$

Therefore the sequence $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n) \in R(F)\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$.
As $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ and the sequence $(F(v_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$ according to the lemma of compactness then gives that $(F(v_n))_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T], L^2(\Omega))$, which gives the compactness of F. For (2.14), to have $F(C) \subset C$, it is enough to take

$$M = \frac{1}{2(\alpha-\varepsilon)} \|v_{n,0}\|_2^2 + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|f_n(t,x,v_n)\|^2_{L^2(Q)} + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|\gamma_n(v_n)\|^2_{L^2(Q)} .$$

Therefore the hypotheses of Schauder's fixed point theorem are satisfied consequently there exists at least one solution to the problem in the set C.

2.3.2 Second step: A priori estimates

Thise step is to proof the estimates of solution $(u_n)_{n\in\mathbb{N}}$ the problem (2.10).
For a given constant $k > 0$ is to define the truncated function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} -k & \text{for } s < -k, \\ s & \text{for } |s| \leq k, \\ k & \text{for } s > k. \end{cases}$$

For a function $u = u(x)$, $x \in \Omega$, so to define the truncated function $T_k u$ pointwise, i.e., for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$. Observe that

$$\lim_{k\to 0} \frac{1}{k} T_k(s) = sign(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases} \tag{2.15}$$

Let the function $\Phi_k : \mathbb{R} \to \mathbb{R}$ such that, $\Phi_k \geq 0$, $\Phi_k \in L^\infty(\mathbb{R})$ and $|\Phi_k(x)| \leq k\,|x|$,

$$\Phi_k(x) = \int_0^x T_k(s)ds.$$

(Φ_k it is the primitive function of T_k). To have

$$\langle v_t, T_k(v)\rangle = \frac{d}{dt}\left(\int_\Omega \Phi_k(v)dx\right) \in L^1(Q).$$

What implies that

$$\int_0^T \langle v_t, T_k(v)\rangle = \int_\Omega \Phi_k(v(T))dx - \int_\Omega \Phi_k(v(0))dx,$$

where $\langle .,. \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
To choose $v = T_k(u_n)$ as test function in (2.11), obtaining

$$\begin{cases} \int_\Omega \Phi_k(u_n(T))dx - \int_\Omega \Phi_k(u_n(0))dx + \int_0^T\int_\Omega A(u_n)\nabla u_n \nabla T_k(u_n)dxdt \\ +\int_Q \gamma_n(u_n)T_k(u_n)dxdt = \int_Q f_n(t,x,u_n)T_k(u_n)dxdt, \forall T_k(u_n) \in L^2([0,T], H_0^1(\Omega)). \end{cases}$$

By using hypothesis (2.3), we obtain

$$\int_0^T\int_\Omega A_n(u_n)\nabla u_n \nabla T_k(u_n)dxdt = \int_0^T\int_\Omega A(u_n)\nabla u_n \nabla u_n T_k'(u_n)dxdt \geq \alpha \int_0^T\int_\Omega |\nabla u_n|^2 T_k'(u_n)dxdt \geq 0,$$

and by $sf(t,x,s) \geq 0$, to have

$$\int_0^T\int_\Omega \gamma_n(u_n)T_k(u_n)dxdt \leq \int_\Omega \Phi_k(u_n(0))dx,$$

on the other hand, there is $\gamma_n(u_n) = \lambda|u_n|^{p-2}u_n \geq 0$ because $p > 1$ then,

$$\int_0^T\int_\Omega A(u_n)\nabla u_n \nabla T_k(u_n)dxdt \leq \int_\Omega \Phi_k(u_n(0))dx, \forall T_k(u_n) \in L^2([0,T], H_0^1(\Omega)).$$

And,

$$\int_Q f_n(t,x,u_n)T_k(u_n)dxdt \leq \int_\Omega \Phi_k(u_n(0))dx.$$

For all $t \in [0,T]$, the definite set Q_T by

$$Q_T = \{(t,x) \in Q : u_n > k\} \cup \{(t,x) \in Q : u_n < -k\} \cup \{(t,x) \in Q : -k \leq u_n \leq k\}.$$

By thise definition of Q_T, there is:

$$\begin{cases} \int_{Q_T} A_n(u_n)\nabla u_n \nabla u_n T_k'(u_n)dxdt = \int_{\{(t,x)\in Q:\ |u_n|\leq k\}} A(u_n)\nabla u_n \nabla u_n T_k'(u_n)dxdt \\ \leq \int_\Omega \Phi_k(u_n(0))dx, \end{cases}$$

so there is, $\forall k \in \mathbb{R}^+$,

$$\int_{\{(t,x)\in Q\,:\,|u_n|\leq k\}} |A(u_n)\nabla u_n \nabla u_n|\, dxdt \leq k \int_\Omega |u_{n,0}|\, dx. \tag{2.16}$$

There will be now prove that,

$$\int_{\{(t,x)\in Q\,:\,\ |u_n|\ \leq\ k\}} A(u_n)\nabla u_n \nabla u_n dxdt \leq k \left\| u_0 \right\|_{L^1(\Omega)},$$

by hypothesis (2.3), to obtain

$$\alpha \int_{\{(t,x)\in Q\,:\,\ |u_n|\ \leq\ k\}} |\nabla u_n|^2\, dxdt \leq k \left\| u_0 \right\|_{L^1(\Omega)}, \tag{2.17}$$

on the other hand, by (2.15), to obtain:

$$\int_{\{(t,x)\in Q:|u_n|>0\}} |\gamma_n(u_n)|\, dxdt \leq \left\| u_0 \right\|_{L^1(\Omega)}, \tag{2.18}$$

and,

$$\int_{\{(t,x)\in Q:|u_n|>0\}} |f_n(t,x,u_n)|\, dxdt \leq \left\| u_0 \right\|_{L^1(\Omega)}. \tag{2.19}$$

New to prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $C\left([0,T], L^1(\Omega)\right)$.

$$\int_0^T \left\langle (u_n)_t\,, T_k\left(u_n\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt \leq \int_{Q_T} \gamma_n(u_n)T_k(u_n)dxdt + \int_{Q_T} f_n(t,x,u_n)T_k(u_n)dxdt,$$

to have also, for every t in $[0,T]$

$$\int_\Omega \Phi_k(u_n(t))dx - \int_\Omega \Phi_k(u_n(0))dx \leq k\int_{\{(t,x)\in Q:|u_n|>k\}} |\gamma_n(u_n)|\, dxdt + k\int_{\{(t,x)\in Q:|u_n|>k\}} |f_n(t,x,u_n)|\, dxdt,$$

know that $\Phi_k(s) \geq |s| - 1$ we deduce that, for every t in $[0,T]$,

$$\begin{aligned}\int_\Omega |u_n(t)|\, dx \ &\leq\ \int_\Omega 1dx + k\int_{\{(t,x)\in Q:|u_n|>k\}} |\gamma_n(u_n)|\, dxdt \\ &+\ k\int_{\{(t,x)\in Q:|u_n|>k\}} |f_n(t,x,u_n)|\, dxdt + k\left\| u_{n,0} \right\|_{L^1(\Omega)} \leq meas(\Omega) + C\left\| u_0 \right\|_{L^1(\Omega)},\end{aligned}$$

which proves that u_n is bounded in $L^2([0,T], H_0^1(\Omega))$ and in $C([0,T], L^1(\Omega))$, on the other hand, to get

$$\int_\Omega \Phi_k(u_n(T))dx + \alpha \int_{Q_T} |\nabla T_k(u_n)|^2 \, dxdt \leq \int_\Omega \Phi_k(u_n(0))dx,$$

$\int_\Omega \Phi_k(u_n(T)) \geq 0$ and for all $s \geq 0, |\Phi_k(s)| \leq k|s|$, to have

$$\alpha \int_{Q_T} |\nabla T_k(u_n)|^2 \, dxdt \leq k \|u_0\|_{L^1(\Omega)}. \tag{2.20}$$

That $T_k(u_n)$ is bounded in $L^2([0,T], H_0^1(\Omega))$ for every $k > 0$.
Now to prove that $div(A(v_n)\nabla u_n)$ is bounded in $L^2([0,T], H^{-1}(\Omega))$. By using hypothesis (2.4) and Cauchy-Schwarz inequality, to get

$$\begin{aligned} |\langle -div(A(u_n)\nabla u_n, T_k(u_n)\rangle| &= \left| \int_{Q_T} A(u_n)\nabla u_n \nabla T_k(u_n) \, dxdt \right| \\ &\leq \beta \|\nabla u_n\|_{L^2(Q)} \|\nabla T_k(u_n)\|_{L^2(Q)} \\ &\leq C. \end{aligned}$$

Since

$$\begin{aligned} \|-div(A(u_n)\nabla u_n)\|^2_{L^2([0,T],H^{-1}(\Omega))} &= \int_0^T \|-div(A(u_n)\nabla u_n)\|^2_{H^{-1}(\Omega)} \, dt \\ &= \int_0^T \sup_{\|T_k(u_n)\|_{L^2\left([0,T],H_0^1(\Omega)\right)} \leq 1} |\langle -div(A(u_n)\nabla u_n), T_k(u_n)\rangle| \\ &\leq C. \end{aligned}$$

To know that $div(A(u_n)\nabla u_n)$ is bounded in $L^2([0,T], H^{-1}(\Omega))$.

Finally, denoting $(u_n)_t = f_n(t,x,u_n) + div(A(u_n)\nabla u_n) - \gamma_n(u_n)$ we observe that $f_n + div(A(u_n)\nabla u_n) + \gamma_n(u_n)$ is bounded in $L^2([0,T], H^{-1}(\Omega)) + L^1(Q)$ and by (2.17), $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$.

2.3.3 Third step: Passage to the limit

To show that $(u_n)_{n\in\mathbb{N}}$ the solution approache the problem (2.11) converges to the solution of the original problem (2.7). By the estimate (2.18) and (2.19), to see that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$ and $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$. The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ and also the sequence $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega)) + L^1(Q)$.

Therefore, using Aubin-type compactness lemma (1.1), that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T], L^2(\Omega))$, thus we can deduce

$$u_n \to u \text{ in } L^2([0,T], L^2(\Omega)),$$

on the other hand $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ then, the possibility to extract a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$ such that:

$$u_n \rightharpoonup u \text{ weakly in } L^2([0,T], H_0^1(\Omega)),$$

and

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } (L^2([0,T], L^2(\Omega)))^{\mathbb{N}},$$

and $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$ and in $L^1(Q)$ can extract a subsequence, still denoted by $((u_n)_t)_{n\in\mathbb{N}}$ such that

$$(u_n)_t \rightharpoonup u_t \text{ weakly in } L^2([0,T], H^{-1}(\Omega)),$$

and either $u_{n,0}$ a sequence of $L^2(\Omega)$ such that

$$\|u_{n,0}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)},$$

and

$$u_{n,0} \longrightarrow u_0 \text{ strongly in } L^1(\Omega). \tag{2.21}$$

We will show that

$$\gamma_n(u_n) \to \gamma(u) \ \text{ strongly in } L^1(Q), \tag{2.22}$$

to have,

$$\begin{aligned} \|\gamma_n(u_n)\|_{L^1(Q)} &= \int_Q |\gamma_n(u_n)|\, dxdt \\ &\leq \int_{\{(t,x)\in Q:|u_n|>0\}} |\gamma_n(u_n)|\, dxdt \\ &\leq \|u_0\|_{L^1(\Omega)}. \end{aligned}$$

Then,

$$\sup \int_Q \gamma_n(u_n)\, dxdt < +\infty,$$

knowing that,

$$0 \leq \int_Q |\gamma_n(u_n)|\, dxdt, \text{ because } p > 1,$$

for each $(t,x) \in Q$, is to pose

$$\lim_{n\to+\infty} \inf \gamma_n(u_n) = \gamma(u),$$

by the Fateau's lemma, have $\gamma(u)$ in $L^1(Q)$.
As that

$$u_n \rightharpoonup u \text{ weakly in } L^2([0,T], L^2(\Omega)),$$

on the other hand, have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T], L^2(\Omega)))^{\mathbb{N}},$$

note that

$$\begin{aligned}&\int\limits_{\{(t,x)\in Q:|\gamma_n(u_n)|\le n\}} |\gamma_n(u_n) - \gamma(u)|\, dxdt \\ \le &\int\limits_Q |\gamma_n(u_n) - \gamma(u)|\, dxdt \to 0 \text{ when } n \to +\infty.\end{aligned}$$

So,

$$\gamma_n(u_n) \to \gamma(u) \text{ when } n \to +\infty \text{ on } \{(t,x) \in Q \ : |\gamma_n(u_n)| \le n\}.$$

For every $n \in \mathbb{N}$, to have

$$\begin{aligned} meas(\{(x,t) \in Q : |\gamma_n(u_n)| > n\}) &\le \frac{1}{n} \int\limits_Q |\gamma_n(u_n)|\, dxdt \\ &\le \frac{1}{n} \|\gamma_n(u_n)\|_{L^1(Q)} \\ &\le \frac{c}{n} \to 0 \text{ when } n \to +\infty, \end{aligned}$$

thus $\{(t,x) \in Q : |\gamma_n(u_n)| > n\}$ is the zero measurement set where $(\gamma_n(u_n))_{n\in\mathbb{N}}$ may not converge to $(\gamma(u))$, which shows that

$$\gamma_n(u_n) \to \gamma(u) \text{ almost everywhere in } Q.$$

For proof (2.22) we show that the sequence $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable.
Let $\delta > 0$ and $\mathbf{A}$ be a measurable subset belonging to $[0,T] \times \Omega$, is to define the following sets,

$$B_\delta = \{(t,x) \in Q : \ |u_n| \le \delta\}, \tag{2.23}$$
$$F_\delta = \{(t,x) \in Q : \ |u_n| > \delta\}, \tag{2.24}$$

$$\begin{aligned} \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt &= \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\, dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |\gamma_n(u_n)|\, dxdt \\ &\le \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\, dxdt + \|u_0\|_{L^1(\Omega)} \\ &\to 0 \text{ when } meas(\mathbf{A}) \to 0. \end{aligned}$$

Using the generalized Hölder's inequality and Poincaré inequality, to get

$$\begin{aligned}
\int_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt &\leq \left(\int_{\mathbf{A}} |\lambda|^2\, dxdt\right)^{\frac{1}{2}} \left(\int_{B_\delta} |u_n|^{(p-1)2}\, dxdt\right)^{\frac{1}{2}} \\
&\quad + \int_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\int_{B_\delta} |\nabla u_n|^2\, dxdt\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\frac{k}{\alpha}\left(\|u_0\|_{L^1(\Omega)}\right)\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, to obtain:

$$\gamma_n(u_n) \to \gamma(u) \text{ strongly in } L^1(Q). \tag{2.25}$$

Now we prove that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q),$$

there is,

$$\begin{aligned}
\|f_n(t,x,u_n)\|_{L^1(Q)} &= \int_Q |f_n(t,x,u_n)|\, dxdt \\
&\leq \int_{\{(t,x)\in Q: |u_n|>0\}} |f_n(t,x,u_n)|\, dxdt \\
&\leq \|u_0\|_{L^1(\Omega)},
\end{aligned}$$

then,

$$\sup \int_Q f_n(t,x,u_n) dxdt < +\infty.$$

By (2.6) knowing that, $0 \leq f_n(t,x,u_n)$ for each $(t,x) \in Q$, we pose

$$\lim_{n\to+\infty} \inf f_n(t,x,u_n) = f(t,x,u),$$

by the Fateau's lemma, there is $f(t,x,u)$ in $L^1(Q)$. As that

$$u_n \rightharpoonup u \text{ weakly in } L^2([0,T], L^2(\Omega)),$$

on the other hand, to have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T], L^2(\Omega)))^{\mathbb{N}},$$

note that,

$$\begin{aligned} &\int_{\{(t,x)\in Q:|f_n(t,x,u_n)|\leq n\}} |f_n(t,x,u_n) - f(t,x,u)|\, dxdt \\ \leq &\int_Q |f_n(t,x,u_n) - f(t,x,u)|\, dxdt \to 0 \text{ when } n \to +\infty. \end{aligned}$$

So,

$$f_n(t,x,u_n) \to f(t,x,u) \text{ when } n \to +\infty \text{ on } \{(t,x) \in Q \ : \ |f(t,x,u)| \leq n\}.$$

For every $n \in \mathbb{N}$, to have

$$\begin{aligned} meas(\{(x,t) \in Q : |f_n(t,x,u_n)| > n\}) &\leq \frac{1}{n}\int_Q |f_n(t,x,u_n)|\, dxdt \\ &\leq \frac{1}{n}\|f_n(t,x,u_n)\|_{L^1(Q)} \\ &\leq \frac{c}{n} \to 0 \text{ when } n \to +\infty, \end{aligned}$$

thus $\{(t,x) \in Q : |f_n(t,x,u_n)| > n\}$ is the zero measurement set where $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ may not converge to $(f(t,x,u))$, which shows that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ almost everywhere in } Q.$$

For proof (2.22) to show that the sequence $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable.
By the definitions of the sets (2.23) and (2.24), to get

$$\begin{aligned} \int_{\mathbf{A}} |f_n(t,x,u_n)|\, dxdt &= \int_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\, dxdt + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\, dxdt \\ &\leq \int_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\, dxdt + \|u_0\|_{L^1(\Omega)} \\ &\to 0 \text{ when } meas(\mathbf{A}) \to 0. \end{aligned}$$

Let $\delta > 0$ be large enough. Using the generalized Hölder's inequality and Poincaré inequality, there is

$$\int_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt = \int_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt,$$

therefore

$$\begin{aligned}
\int_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt &\leq \int_{\mathbf{A}\cap B_\delta} (c(x,t)+\sigma|u_n|)\,dxdt + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq \int_{\mathbf{A}} c(x,t)dxdt + \sigma \int_{Q} |\nabla T_\delta(u_n)|\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq \int_{\mathbf{A}} c(x,t)dxdt + \sigma\,(meas(\mathbf{A}))^{\frac{1}{2}} \left(\int_{Q_T} |\nabla T_\delta(u_n)|^2\,dxdt\right)^{\frac{1}{2}} \\
&\quad + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq K_1 + C_1\left(\frac{k}{\alpha}\|u_0\|_{L^1(\Omega)}\right)^{\frac{1}{2}} + \int_{\mathbf{A}\cap F_\delta} \frac{1}{|u_n|}|u_n f_n(t,x,u_n)|\,dxdt \\
&\leq K_2 + \int_{\mathbf{A}\cap F_\delta} \frac{1}{\delta}|u_n f_n(t,x,u_n)|\,dxdt \\
&\leq K_2 + \frac{1}{\delta}\left(\int_{\mathbf{A}\cap F_\delta} |u_n|^2\,dxdt\right)^{\frac{1}{2}} \left(\int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|^2\,dxdt\right)^{\frac{1}{2}} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, to get

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q). \tag{2.26}$$

Since $u_n \in C([0,T], L^2(\Omega))$, in order to see that $u \in C([0,T], L^1(\Omega))$, is only to prove that

$$u_n \to u \text{ in } C([0,T], L^1(\Omega)).$$

To do this fix $\tau \in [0,T]$. Choosing $T_k(u_n-u_m)1_{\{[0,\tau[\}}$ as test function in the weak formulation of u_n and $-T_k(u_n-u_m)1_{\{[0,\tau[\}}$ in that of u_m with $\tau \le T$, we get

$$\begin{aligned}
&\int_\Omega \Phi_k(u_n(\tau)-u_m(\tau))dx - \int_\Omega \Phi_k(u_n(0)-u_m(0))dx \\
&+\int_0^\tau\int_\Omega A(u_n-u_m)\nabla(u_n-u_m)\nabla T_k(u_n-u_m)dxdt \\
&+\int_0^\tau\int_\Omega \lambda\left[|u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right]T_k(u_n-u_m)dxdt \\
=\;&\int_0^\tau\int_\Omega \left(f_n(t,x,u_n)-f_n(t,x,u_m)\right)T_k(u_n-u_m)dxdt,
\end{aligned}$$

where Φ_k is the primitive of T_k such that $\Phi_k(0)=0$,

$$\begin{aligned}
\int_\Omega \Phi_k(u_n(\tau)-u_m(\tau))dx \;\le\; &\int_0^\tau\int_\Omega \lambda\left||u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right|dxdt \\
&+k\int_0^\tau\int_\Omega |f_n(t,x,u_n)-f_n(t,x,u_m)|\,dxdt \\
&+k\int_\Omega |u_{n,0}-u_{m,0}|\,dx.
\end{aligned}$$

Next, to divide this inequality by k and the Monotone convergence theorem and let k go to 0, to obtain

$$\begin{aligned}
\int_\Omega |u_n(\tau)-u_m(\tau)|\,dx \;\le\; &\int_0^\tau\int_\Omega \lambda\left||u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right|dxdt \\
&+\int_0^\tau\int_\Omega |f_n(t,x,u_n)-f_n(t,x,u_m)|\,dxdt \\
&+\int_\Omega |u_{n,0}-u_{m,0}|\,dx.
\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{\tau\in[0,T]}\int_{\Omega}|u_n(\tau)-u_m(\tau)|\,dx &\leq \int_0^{\tau}\int_{\Omega}\lambda\left||u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right|dxdt\\ &+\int_0^{\tau}\int_{\Omega}|f_n(t,x,u_n)-f_n(t,x,u_m)|\,dxdt\\ &+\int_{\Omega}|u_{n,0}-u_{m,0}|\,dx.\end{aligned}$$

Thus, it follows from (2.21), (2.26) and (2.25), that sequence (u_n) is a Cauchy sequence in $C\left([0,T],L^1(\Omega)\right)$ then $u_n\to u$ in $C([0,T],L^1(\Omega))$. Finally,

$$u\in C([0,T],L^1(\Omega)). \tag{2.27}$$

Chapter 3

Renormalized solution for quasilinear parabolic problem and nonlinear parabolic system in the Lebesgue-Sobolev Space with variable exponent and L^1 data

3.1 Introduction

In the present chapter, we establish the existence of a renormalized solution for a class of a quasilinear parabolic problem of type

$$\begin{cases} (b(u))_t - \mathrm{div}\mathcal{A}(x,t,\nabla u) + \gamma(u) = f(x,t,u) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

and we study the existence of renormalized solution for a class of nonlinear parabolic systeme with variable exponent and L^1 data. More precisely, we study the asymptotic behavrior of the problem

$$\begin{cases} (b_1(u))_t - \mathrm{div}\mathcal{A}(x,t,\nabla u) + \gamma(u) = f_1(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ (b_2(v))_t - \Delta v + = f_2(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = v = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b_1(u)(t=0) = b_1(u_0) & \text{in} \quad \Omega, \\ b_2(v)(t=0) = b_2(v_0) & \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

Where Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with lipshitz boundary $\partial\Omega$ and $Q = \Omega \times]0,T[$ for any fixed T is a positive real number. In the problem (3.1), let $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p^- \leq p^+ < N$. Let $-\mathrm{div}\mathcal{A}(x,t,\nabla u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary-Lions operator (see assumption (3.3)-(3.5)), respectively $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(u) = \lambda\,|u|^{p(x)-2}\,u$ is a continuous increasing function for $\lambda > 0$ and $\gamma(0) = 0$ such that $\gamma(u)$ is assumed to belong to $L^1(Q)$. The function $f : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (see assumptions (3.7)-(3.8)). And the function $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b(0) = 0$ (see (3.6)), the data $f(x,t,u)$ and $b(u_0)$ is in L^1. And in the problem (3.2), the operator $-\mathrm{div}\mathcal{A}(x,t,\nabla u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary lions operator (see assumptions (3.74)-(3.76)), respectively, $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(s) = \lambda\,|s|^{p(x)-2}\,s$ is a continuous increasing function for $\lambda > 0$ and $\gamma(0) = 0$ such that γ is assumed to belong to $L^1(Q)$. The function $f_i : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i = \overline{1,2}$ be a Carathéodory function (see assumptions (3.78)-(3.80)). Finally the function $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b_i(0) = 0$ (see (3.77)), the data f_i and $(b_1(u_0), b_2(v_0))$ is in $(L^1)^2$.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc (see [18]).

In the classical case ($p(.) = 2$ or $p(.) = p$ (a constant)), to recall that the notion of renormalized solutions was introduced by Di Perna and Lions [22] in their study of the Boltzmann equation. It has been studied by many authors under various conditions on the data the existence and uniqueness of the renormalized solution for parabolic equations with L^1-data in the classical Sobolev spaces (see [12], [38] and [5]).

In Sobolev space with variable exponents, the authors [42] have proved the existence of renormalized solutions for a class of nonlinear parabolic systems with variable exponents and, for the corresponding parabolic equations with L^1 data, the authors in [15] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponents and, in [46] have proved an existence and uniqueness results renormalized solutions and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data. On the other hand in [35] S. Ouaro and all obtains existence and uniqueness of entropy solutions to nonlinear parabolic equation with variable exponent and L^1-data. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

Problems of this type have been studied by serval a authors. In 2007 H. Redwane, studied the existence of solutions for a class of nonlinear parabolic systems see [38], in 2013 Youssef. B and all studied the existence of a renormalized solution for the nonlinear parabolic systems with unbounded nonlinearities see [5] agin in 2016 B . El Hamdaoui and all in [42] studied the renormalized solutions for nonlinear parabolic systems in the Lebesgue Sobolev Space with variable exponent and L^1 data.

In 2016 [32] authors proved the existence and uniqueness of renormalized solution of a reaction diffusion systems which has a variable exponent Laplacian term and could be applied to image denoising for the case of parabolic equations. The concept of renormalized solution in the context of variable exponent was for the first time studied by T. M. Bendahmane, P. Wittbold and A.Zimmermann [15]. C. Zhang and S. Zhou studied the renormalized and entropy solution for nonlinear parabolic equation with variable exponent and L^1 data. Moreover, they obtain

the equivalence of renormalized solution and entropy solution(see [46]).
The chapter is organized as follows:Section, we etablish the existence of renormalized solution for a class of a quasilinear parabolic problem.
Subsection 3.2.1, we give some assumptions. Subsection 3.2.2, we give the definition of a renormalized solution of (3.1), and we establish (Theorem (3.1)) the existence of such a solution. And in section 3.3, we study the existence of renormalized solution for a class of nonlinear parabolic systeme with variable exponent and L^1 data . Subsection 3.3.1, is devoted to specify the assumptions on, $\mathcal{A}(x,t,\xi)$, γ, b_1, b_2, f_1, f_2, $b_1(u_0)$ and $b_2(v_0)$ needed in the present study. Subsection 3.3.2 , to give the definition of a renormalized solution of (3.2), and we establish (Theorem (3.2)) the existence of such a solution.

3.2 Quasilinear parabolic problem

3.2.1 The Assumptions On The Data

The assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times]0,T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$

$$\mathcal{A}(x,t,\xi).\xi \geq \alpha \left|\xi\right|^{p(x)}, \tag{3.3}$$

$$|\mathcal{A}(x,t,\xi)| \leq \beta \left[L(x,t) + |\xi|^{p(x)-1}\right], \tag{3.4}$$

$$(\mathcal{A}(x,t,,\xi) - \mathcal{A}(x,t,\eta)).(\xi - \eta) > 0, \tag{3.5}$$

where $1 < p^- \leq p^+ < +\infty$, α, β are positives constants and L is a nonnegative function in $L^{p'(\cdot)}(Q)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$.

Let $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b(0) = 0$ and for any ρ, τ are positives constants such that

$$\rho \leq b'(s) \leq \tau, \ \ \forall s \in \mathbb{R}, \tag{3.6}$$

let $f : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for any $\sigma > 0$, there exists $c \in L^{p'(\cdot)}(Q)$ such that

$$|f(x,t,s)| \leq c(x,t) + \sigma|s|^{p(x)-1}, \tag{3.7}$$

for almost every $(x,t) \in (Q)$, $s \in \mathbb{R}$,

$$f(x,t,s)s \geq 0, \tag{3.8}$$

$$b(u_0) \in L^1(\Omega). \tag{3.9}$$

3.2.2 The Main of Results

In this section, we study the existence of renormalized solutions to problem (3.1).

Definition 3.1. *Let* $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$ *and* $b(u_0) \in L^1(\Omega)$. *A measurable function* u *defined on* Q *is a renormalized solution of problem* (3.1) *if,*

$$T_k(u) \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega)) \text{ for any } k > 0\ ,\ \gamma(u), f(x,t,u) \in L^1(Q)\,, \tag{3.10}$$

$$\text{and } b(u) \in L^\infty\left(]0,T[;L^1(\Omega)\right) \cap L^{q^-}(]0,T[;W_0^{1,q(.)}(\Omega)), \tag{3.11}$$

for all continuous functions $q(x)$ *on* $\overline{\Omega}$ *satisfying* $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ *for all* $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int\limits_{\{n\leq|u|\leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt = 0, \tag{3.12}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ *which is piecewise* C^1 *and such that* S' *has compact support on* $\mathbb{R}$, *we have,*

$$(B_S(u))_t - div(\mathcal{A}(x,t,\nabla u)S'(u)) + S''(u)\mathcal{A}(x,t,\nabla u)\nabla u + \gamma(u)S'(u) \tag{3.13}$$

$$= f(x,t,u)S'(u) \text{ in } \mathcal{D}'(Q),$$

$$B_S(u)(t=0) = S(b(u_0)) \text{ in } \Omega, \tag{3.14}$$

where $B_S(z) = \int_0^z b'(r)S'(r)dr$.

The following remarks are concerned with a few comments on definition (3.1).

Remark 3.1. *Note that, all terms in* (3.13) *are well defined. Indeed, let* $k > 0$ *such that* $supp(S') \subset [K,K]$, *we have* $B_S(u)$ *belongs to* $L^\infty(Q)$ *because*

$$|B_S(u)| \leq \int_0^u |b'(r)S'(r)|dr \leq \tau\|S'\|_{L^\infty(\mathbb{R})},$$

and $S(u) = S(T_k(u)) \in L^{p-}(]0,T[;W_0^{1;p(.)}(\Omega))$ *and* $\frac{\partial B_S(u)}{\partial t} \in \mathcal{D}'(Q)$. *The term* $S'(u)\mathcal{A}(x,t,\nabla T_k(u))$ *identifes with* $S'(T_k(u))\mathcal{A}(x,t,\nabla(T_k(u)))$ *a.e. in* Q, *where* $u = T_k(u)$ *in* $\{|u| \leq k\}$, *assumptions* (3.4) *imply that*

$$\begin{aligned}&|S'(T_k(u))\mathcal{A}(x,t,\nabla T_k(u))| \\ &\leq \beta\,\|S'\|_{L^\infty(\mathbb{R})}\left[L(x,t) + |\nabla(T_k(u))|^{p(x)-1}\right] \text{ a.e in } Q.\end{aligned} \tag{3.15}$$

Using (3.4) *and* (3.10), *it follows that* $S'(u)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. *The term* $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u)$ *identifes with* $S''(u)\mathcal{A}(t,x,\nabla(T_k(u)))\nabla T_k(u)$ *and in view of* (3.4), (3.10) *and* (3.15), *to obtain* $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u) \in L^1(Q)$ *and* $S'(u)\gamma(u) \in L^1(Q)$.

Finally $f(x,t,u)\ S'(u) = f(x,t,T_k(u))S'(u)$ *a.e in* Q. *Since* $|T_k(u)| \leq k$ *and* $S'(u) \in L^\infty(Q)$, $c(x,t) \in L^{p'(.)}(Q)$, *to obtain from* (3.7) *that* $f(x,t,T_k(u))S'(u) \in L^1(Q)$.

Also $\frac{\partial B_S(u)}{\partial t} \in L^{(p^-)'}(]0,T[;W^{-1,p'(.)}(\Omega)) + L^1(Q)$ *and* $B_S(u) \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$, *which implies that* $B_S(u) \in C(]0,T[;L^1(\Omega))$.

3.2.3 The Existence Theorem

Theorem 3.1. *Let* $b(u_0) \in L^1(\Omega)$, *assume that* (3.3)-(3.9) *hold true, then there exists at least one renormalized solution* u *of problem* (3.1) *(in the sens of Definition* (3.1) *).*

Proof. of Theorem (3.1): The above theorem is to be proved in 5 steps.

- **Step 1: Approximate problem and a priori estimates.**
  Let us define the following approximation of b and f for $\varepsilon > 0$ fixed

$$b_\varepsilon(r) = T_{\frac{1}{\varepsilon}}(b(r)) \text{ a.e in } \Omega \text{ for } \varepsilon > 0,\ \ \forall r \in \mathbb{R}, \tag{3.16}$$

$$b_\varepsilon(u_0^\varepsilon) \text{ are a sequence of } C_c^\infty(\Omega) \text{ functions such that} \tag{3.17}$$

$$b_\varepsilon(u_0^\varepsilon) \to b(u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0,$$

$$f^\varepsilon(x,t,r) = f(x,t,T_{\frac{1}{\varepsilon}}(r)), \tag{3.18}$$

in view of (3.7) and (3.8), there exist $c_\varepsilon \in L^{p'(.)}(Q)$ and $\sigma_\varepsilon > 0$ such that

$$|f^\varepsilon(x,t,s)| \leq c_\varepsilon(x,t) + \sigma_\varepsilon |s|^{p(x)-1}, \tag{3.19}$$

for almost every $(x,t) \in (Q)$, $\ s \in \mathbb{R}$,

$$f^\varepsilon(x,t,s)s \geq 0. \tag{3.20}$$

Let us now consider the approximate problem

$$(b_\varepsilon(u^\varepsilon))_t - div\mathcal{A}(x,t,\nabla u^\varepsilon) + \gamma(u^\varepsilon) = f^\varepsilon(x,t,u^\varepsilon) \text{ in } Q, \tag{3.21}$$

$$u^\varepsilon = 0 \text{ on }]0,T[\times \partial\Omega, \tag{3.22}$$

$$b_\varepsilon(u^\varepsilon)(t=0) = b_\varepsilon(u_0^\varepsilon) \text{ in } \Omega. \tag{3.23}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ of (3.21)-(3.23) is an easy task (see [30]).

We choose $T_k(u^\varepsilon)\chi_{(0,t)}$ as a test function in (3.21), to have

$$\int_\Omega B_k^\varepsilon(u^\varepsilon)(t)dx + \int_0^t\int_\Omega \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) + \int_0^t\int_\Omega \gamma(u^\varepsilon)\,T_k(u^\varepsilon)dxds \tag{3.24}$$

$$= \int_0^t \int_\Omega f^\varepsilon(x,t,u^\varepsilon) T_k(u^\varepsilon) dx ds + \int_\Omega B_k^\varepsilon(u_0^\varepsilon) dx,$$

for almost every t in $(0,T)$, and where

$$B_k^\varepsilon(r) = \int_0^r T_k(s) \frac{\partial b_\varepsilon(s)}{\partial s} ds.$$

Under the definition of $B_k^\varepsilon(r)$ the inequality

$$0 \le \int_\Omega B_k^\varepsilon(u_0^\varepsilon)(t) dx \le k \int_\Omega |b_\varepsilon(u_0^\varepsilon)| dx, \ \ k > 0.$$

Using (3.3), $f^\varepsilon(x,t,u^\varepsilon)T_k(u^\varepsilon) \ge 0$ and we have $\gamma(u^\varepsilon) = \lambda |u^\varepsilon|^{p(x)-1} u^\varepsilon \ge 0$ because $1 < p^- \le p(x) \le +\infty$ and the definition of $B_k^\varepsilon(r)$ in (3.24), to obtain

$$\int_\Omega B_k^\varepsilon(u^\varepsilon)(t) dx + \alpha \int_{E_k} |\nabla u^\varepsilon|^{p(x)} \, dx ds \le k \left\| b_\varepsilon(u_0^\varepsilon) \right\|_{L^1(\Omega)}, \tag{3.25}$$

where $E_k = \{(x,t) \in Q : |u^\varepsilon| \le k\}$, using $B_k^\varepsilon(u^\varepsilon)(t) \ge 0$ and inequality (1.3) in (3.25), to get

$$\begin{aligned} &\alpha \int_0^T \min\left\{ \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p-}, \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p+} \right\} \\ \le \ &\alpha \int_{\{(x,t)\in Q: \ |u^\varepsilon| \le k\}} |\nabla u^\varepsilon|^{p(x)} \, dx dt \le C, \end{aligned} \tag{3.26}$$

then is $T_k(u^\varepsilon)$ is bounded in $L^{p-}(]0,T[\,; W_0^{1,p(x)}(\Omega))$.

Also, to obtain

$$k \int_{\{(t,x)\in Q: |u^\varepsilon| > k\}} |\gamma(u^\varepsilon)| \, dx dt \le k \left\| b_\varepsilon(u_0^\varepsilon) \right\|_{L^1(\Omega)}, \tag{3.27}$$

and

$$k \int_{\{(x,t)\in Q: |u^\varepsilon| > k\}} |f^\varepsilon(x,t,u^\varepsilon)| \, dx dt \le k \left\| b_\varepsilon(u_0^\varepsilon) \right\|_{L^1(\Omega)}. \tag{3.28}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(b_\varepsilon(u^\varepsilon))$ as test function in (3.21). Reasoning as above, using that $\nabla T_{k,1}(s) = \nabla s \chi_{\{k \leq |s| \leq k+1\}}$ and appling young's inequality, to obtain

$$\begin{aligned}
\alpha \int_{\{k\leq |b_\varepsilon(u^\varepsilon)|\leq k+1\}} b'_\varepsilon(u^\varepsilon)\left|\nabla(u^\varepsilon)\right|^{p(x)} dxdt \;\leq\; & k \int_{\left|b_\varepsilon(u_0^\varepsilon)\right|>k} \left|b_\varepsilon(u_0^\varepsilon)\right| dx \\
& + Ck \int_{|b_\varepsilon(u^\varepsilon)|>k} |\gamma(u^\varepsilon)|\, dxdt \\
& + Ck \int_{|b_\varepsilon(u^\varepsilon)|>k} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt \leq C_1,
\end{aligned}$$

inequality (1.3) implies that

$$\begin{aligned}
& \int_0^T \alpha \chi_{\{k\leq |b_\varepsilon(u^\varepsilon)|\leq k+1\}} \min\left\{ \|\nabla(b_\varepsilon(u^\varepsilon))\|^{p^-}_{L^{p(x)}(\Omega)}, \|\nabla(b_\varepsilon(u^\varepsilon))\|^{p^+}_{L^{p(x)}(\Omega)} \right\} \\
\leq\; & \alpha \int_{\{k\leq |b_\varepsilon(u^\varepsilon)|\leq k+1\}} b'_\varepsilon(u^\varepsilon)\left|\nabla(u^\varepsilon)\right|^{p(x)} dxdt \leq C_1. \qquad (3.29)
\end{aligned}$$

On sait que the propriete of $B_k^\varepsilon(u^\varepsilon)$, $(B_k^\varepsilon(u^\varepsilon) \geq 0, B_k^\varepsilon(u^\varepsilon)) \geq \rho(|s|-1)$, to obtain

$$\begin{aligned}
\int_\Omega |B_k^\varepsilon(u^\varepsilon)(t)|\, dx \;\leq\; & k \int_\Omega |b_\varepsilon(u^\varepsilon)(t)|\, dx \leq \rho\left(\int_\Omega |1|\, dx + k\left\|b_\varepsilon(u_0^\varepsilon)\right\|_{L^1(\Omega)} \right) \\
\leq\; & \rho\left(meas(\Omega) + k\left\|b_\varepsilon(u_0^\varepsilon)\right\|_{L^1(\Omega)} \right). \qquad (3.30)
\end{aligned}$$

From the estimation (3.26), (3.29), (3.30) and the properites of B_k^ε and $b_\varepsilon(u_0^\varepsilon)$, we deduce that

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \qquad (3.31)$$

and

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^{p^-}(]0,T[\,;W_0^{1,p(x)}(\Omega)), \qquad (3.32)$$

by Lemma 2.1 in [15] and by (3.29), (3.30) and si $2 - \dfrac{1}{N+1} < p(.) < N$, to obtain

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^{q^-}(]0,T[\,;W_0^{1,q(x)}(\Omega)), \qquad (3.33)$$

for all continuous variable exponents $q \in C(\overline{\Omega})$ satisfying

$$1 \leq q(x) < \frac{N(p(x)-1)+p(x)}{N+1},$$

for all $x \in \overline{\Omega}$.

And

$$T_k(u^\varepsilon) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right). \tag{3.34}$$

By (3.27) and (3.28) , we may conclude that

$$\gamma(u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{3.35}$$

and

$$f^\varepsilon(x,t,u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{3.36}$$

independently of ε.

Proceeding as in [12], [13] that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S' \subset [-k,k]$)

$$S(u^\varepsilon) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{3.37}$$

and

$$\left(S(u^\varepsilon)\right)_t \text{ is bounded in } L^1(Q) + L^{(p^-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right). \tag{3.38}$$

In fact, as a consequence of (3.34), by Stampacchia's Theorem, we obtain (3.37). To show that (3.38) holds true, we multiply the equation (3.21) by $S'(u^\varepsilon)$, to obtain

$$\begin{aligned}\left(B_S(u^\varepsilon)\right)_t &= div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) - \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla\left(S'(u^\varepsilon)\right) \\ &\quad -\gamma(u^\varepsilon)S'(u^\varepsilon) + f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{3.39}$$

Since supp(S') and supp(S'') are both included in $[-k;k]$; u^ε may be replaced by $T_k(u^\varepsilon)$ in $\{|u^\varepsilon| \leq k\}$. To have

$$\begin{aligned}&|S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)| \\ &\leq \beta\,\|S'\|_{L^\infty}\left[L(x,t) + |\nabla T_k(u^\varepsilon)|^{p(x)-1}\right].\end{aligned} \tag{3.40}$$

As a consequence, each term in the right hand side of (3.39) is bounded either in $L^{(p^-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and to obtain (3.38).

Now an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(s) = T_{n+1}(s) - T_n(s) = \begin{cases} 0 & \text{if } |s| \leq n, \\ (|s|-n)\,sign(s) & \text{if } n \leq |s| \leq n+1, \\ sign(s) & \text{if } |s| \geq n. \end{cases}$$

Remark that $||\theta_n||_{L^\infty} \leq 1$ for any $n \geq 1$ and that $\theta_n(s) \to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^\varepsilon)$ in (3.21) leads to

$$\begin{aligned}\int_\Omega \widetilde{\theta_n}(u^\varepsilon)(t)\,dx &+ \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla\left(\theta_n(u^\varepsilon)\right)dxdt + \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt \\ &= \int_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx,\end{aligned} \tag{3.41}$$

where $\widetilde{\theta_n}(r)(t) = \int_0^r \theta_n(s)\dfrac{\partial b_\varepsilon(s)}{\partial s}ds$,

for almost any t in $]0,T[$ and where $\widetilde{\theta_n}(r) = \int\limits_0^r \theta_n(s)ds \geq 0$. Hence, dropping a nonnegative term

$$\int\limits_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \tag{3.42}$$

$$\begin{aligned}
&\leq \int\limits_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt + \int\limits_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int\limits_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx \\
&\leq \int\limits_{\{|u^\varepsilon|\geq n\}} |\gamma(u^\varepsilon)|\,dxdt + \int\limits_{\{|u^\varepsilon|\geq n\}} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt + \int\limits_{\{|b_\varepsilon(u_0^\varepsilon)|\geq n\}} |b_\varepsilon(u_0^\varepsilon)|\,dx.
\end{aligned}$$

- **Step 2: The limit of the solution of the approximated problem.**
  Arguing again as in [[12],[13],[14]] estimate (3.37) and (3.38) implies that, for a subsequence still indexed by ε,

$$u^\varepsilon \text{ converge almost every where to } u \text{ in } Q, \tag{3.43}$$

using (3.21) ,(3.34) and (3.40), to get

$$T_k(u^\varepsilon) \text{ converge weakly to } T_k(u) \text{ in } L^{p-}\left(]0,T[\,,W_0^{1,p(.)}(\Omega)\right), \tag{3.44}$$

$$\chi_{\{|u^\varepsilon|\leq k\}}\mathcal{A}(x,t,\nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N, \tag{3.45}$$

as ε tends to 0 for any $k>0$ and any $n\geq 1$ and where for any $k>0$, η_k belongs to $\left(L^{p'(.)}(Q)\right)^N$. Since $\gamma(u^\varepsilon)$ is a continuous incrassing function, from the monotone convergence theorem and (3.27) and by (3.43), to obtain that

$$\gamma(u^\varepsilon) \text{ converge weakly to } \gamma(u) \text{ in } L^1(Q). \tag{3.46}$$

Now establish that $b(u)$ belongs to $L^\infty(]0,T[\,;L^1(\Omega))$. Indeed using (3.24) and $|B_k^\varepsilon(s)| \geq |s|-1$ leads to

$$\begin{aligned}
\int\limits_\Omega |b_\varepsilon(u^\varepsilon)|(t)dx \;\; &\leq \;\; meas(\Omega) + k\,\|f^\varepsilon(x,t,u^\varepsilon)\|_{L^1(Q)} + k\,\|\gamma(u^\varepsilon)\|_{L^1(Q)} \\
&\quad + k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}.
\end{aligned}$$

Using (3.27) and (3.17),(3.28), to have u belongs to $L^\infty(]0,T[\,;L^1(\Omega))$. Now in a position to exploit (3.42). Since u^ε is bounded in $L^\infty(]0,T[\,;L^1(\Omega))$, to get

$$\lim_{n\to+\infty}\left(\sup_\varepsilon(meas\,\{|u^\varepsilon|\geq n\})\right) = 0. \tag{3.47}$$

The equi-integrability of the sequence $f^\varepsilon(x,t,u^\varepsilon)$ in $L^1(Q)$. We shall now prove that $f^\varepsilon(x,t,u^\varepsilon)$ converges to $f(x,t,u)$ strongly in $L^1(Q)$, by using Vitali's theorem. Since $f^\varepsilon(x,t,u^\varepsilon) \to f(x,t,u)$ a.e in Q it suffices to prove that $f^\varepsilon(x,t,u^\varepsilon)$ are equi-integrable in Q. Let $\delta > 0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$G_\delta = \{(x,t) \in Q : \ |u_n| \le \delta\}, \tag{3.48}$$
$$F_\delta = \{(x,t) \in Q : \ |u_n| > \delta\}. \tag{3.49}$$

Using the generalized Hölder's inequality and Poincaré inequality, to have

$$\int_{\mathbf{A}} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt = \int_{\mathbf{A}\cap G_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt + \int_{\mathbf{A}\cap F_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt,$$

therfore

$$\begin{aligned}
\int_{\mathbf{A}} |f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt &\leq \int_{\mathbf{A}\cap G_\delta} \left(c_\varepsilon(x,t) + \sigma_\varepsilon |u_n|^{p(x)-1}\right) dxdt \\
&+ \int_{\mathbf{A}\cap F_\delta} |f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt \\
&\leq \int_{\mathbf{A}} c_\varepsilon(x,t)dxdt + \sigma_\varepsilon \int_{Q} |\nabla T_\delta(u^\varepsilon)|^{p(x)-1}\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_\delta} |f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt \\
&\leq \int_{\mathbf{A}} c_\varepsilon(x,t)dxdt + \sigma_\varepsilon \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) (meas(\mathbf{Q})+1)^{\frac{1}{p^-}} \\
&\quad \left(\int_{Q_T} |\nabla T_\delta(u^\varepsilon)|^{(p(x)-1)p'(x)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&+ \int_{\mathbf{A}\cap F_\delta} |f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt \\
&\leq K_1 + C_2 \left(\frac{k}{\alpha}\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \\
&+ \int_{\mathbf{A}\cap F_\delta} \frac{1}{|u^\varepsilon|} |u^\varepsilon f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt \\
&\leq K_2 + \int_{\mathbf{A}\cap F_\delta} \frac{1}{\delta} |u^\varepsilon f^{\varepsilon}(x,t,u^{\varepsilon})|\,dxdt \\
&\leq K_2 + \frac{1}{\delta}\left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \left(\int_{\mathbf{A}\cap F_\delta} |u^\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{p^-}} \\
&\quad \left(\int_{\mathbf{A}\cap F_\delta} |f^{\varepsilon}(x,t,u^{\varepsilon})|^{p'(x)(p(x)-1)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&\rightarrow 0 \text{ when } meas(\mathbf{A}) \rightarrow \mathbf{0}.
\end{aligned}$$

Which shows that $f^{\varepsilon}(x,t,u^{\varepsilon})$ is equi-integrable. By using Vitali's theorem, to get

$$f^{\varepsilon}(x,t,u^{\varepsilon}) \rightarrow f(x,t,u) \text{ strongly in } L^1(Q). \tag{3.50}$$

Using (3.46), (3.50) and the equi-integrability of the sequence $|b_\varepsilon(u_0^\varepsilon)|$ in $L^1(\Omega)$, to deduce that

$$\lim_{n\to+\infty}\left(\sup_{\varepsilon}\int_{\{n\le|u^\varepsilon|\le n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\right)=0. \tag{3.51}$$

- **Step 3: Strong convergence.**
  The specifie time regularization of $T_k(u)$ (for fixed $k\ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequaence in $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$ such that$\|v_0^\mu\|_{L^\infty(\Omega)}\le k,\ \forall\mu>0$, and $v_0^\mu\to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\|v_0^\mu\|_{L^{p(.)}(\Omega)}\to 0$ as $\mu\to+\infty$.

  For fixed $k\ge 0$ and $\mu>0$, let us consider the unique solution $T_k(u)_\mu\in L^\infty(\Omega)\cap L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right)$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t}+\mu\left(T_k(u)_\mu-T_k(u)\right)=0 \text{ in } \mathcal{D}'(Q), \tag{3.52}$$

$$T_k(u)_\mu(t=0)=v_0^\mu. \tag{3.53}$$

  The behavior of $T_k(u)_\mu$ as $\mu\to+\infty$ is investigated in [18] and we just recall here that (3.52)-(3.53) imply that

$$T_k(u)_\mu\to T_k(u) \text{ strongly in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right) \text{ a.e in } Q \text{ as } \mu\to+\infty, \tag{3.54}$$

  with $\|T_k(u)_\mu\|_{L^\infty(\Omega)}\le k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t}\in L^{(p^-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$.
  The main estimate is the following

  Lemma 3.1. *Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r)=r$ for $r\le k$, and suppS' is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle\frac{\partial B_S(u^\varepsilon)}{\partial t},(T_k(u^\varepsilon)_\mu-T_k(u))\right\rangle dt\ge 0,$$

  where here $\langle .,.\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'(.)}(\Omega)$ and $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$, and where $B_S(z)=\int_0^t b'(r)S'(r)dr$.

  Proof. See[14], Lemma 1. □

- **Step 4:** Here to prove that the weak limit η_k and we prove the weak L^1 convergence of the "truncted" energy $\mathcal{A}(x,t,\nabla T_k(u^\varepsilon))$ as ε tends to 0. In order to show this result recall the lemma below.

Lemma 3.2. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla T_k(u^\varepsilon)dxdt \le \int_Q \eta_k \nabla T_k(u)dxdt, \tag{3.55}$$

$$\lim_{\varepsilon\to 0}\int_Q \left[\mathcal{A}\left(x,t,\nabla u^\varepsilon_{\chi_{\{|u^\varepsilon|\le k\}}}\right)-\mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\le k\}}}\right)\right] \tag{3.56}$$

$$\times\left[\nabla u^\varepsilon_{\chi_{\{|u^\varepsilon|\le k\}}}-\nabla u_{\chi_{\{|u|\le k\}}}\right]dxdt=0,$$

$\eta_k = \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\le k\}}}\right)$ *a.e in Q, for any $k\ge 0$, as ε tends to 0.*

$$\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla T_k(u^\varepsilon) \rightharpoonup \mathcal{A}\left(x,t,\nabla u\right)\nabla T_k(u) \text{ weakly in } L^1(Q). \tag{3.57}$$

Proof. Let us introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that, for any $n\ge 1$

$$\begin{cases} S_n(r)=r \text{ if } |\mathrm{r}|\le n,\\ \operatorname{supp}\left(S_n'\right)\subset\left[-(n+1),(n+1)\right],\\ \|S_n''\|_{L^\infty(\mathbb{R})}\le 1.\end{cases} \tag{3.58}$$

For fixed $k\ge 0$, we consider the test function $S_n'(u^\varepsilon)\left(T_k(u_\varepsilon)-(T_k(u))_\mu\right)$ in (3.21), we use the definition (3.58) of S_n' and we definie $W_\mu^\varepsilon = T_k(u_\varepsilon)-(T_k(u))_\mu$, to get

$$\int_0^T \left\langle (B_S(u^\varepsilon))_t, W_\mu^\varepsilon\right\rangle dt + \int_Q S_n'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla W_\mu^\varepsilon dxdt \tag{3.59}$$

$$+\int_Q S_n''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon dxdt + \int_Q \gamma(u^\varepsilon)S_n'(v^\varepsilon)W_\mu^\varepsilon dxdt$$

$$= \int_Q f^\varepsilon(x,t,u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt.$$

Now pass to the limit in (3.59) as $\varepsilon\to 0$, $\mu\to+\infty$, $n\to+\infty$ for k real number fixed. In order to perform this task, to prove below the following results for any $k\ge 0$:

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T \left\langle (B_S(u^\varepsilon))_t, S_n'(u^\varepsilon)W_\mu^\varepsilon\right\rangle dt \ge 0 \text{ for any } n\ge k, \tag{3.60}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon dxdt = 0, \tag{3.61}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{3.62}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f^\varepsilon(x,t,u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1. \tag{3.63}$$

Proof of (3.60). In view of the definition W_μ^ε, we apply lemma (3.1) with $S = S_n$ for fixed $n \geq k$. As a consequence, (3.60) hold true. □

Proof of (3.61). For any $n \geq 1$ fixed, we have $supp(S_n'') \subset [-(n+1), -n] \cup [n, n+1]$, $\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q)} \leq 2k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$, to get

$$\begin{aligned}
&\left|\int_Q S_n''(u^\varepsilon)\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon W_\mu^\varepsilon dxdt\right| \\
\leq\ & 2k \int_{\{n\leq |u^\varepsilon|\leq n+1\}} \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon dxdt,
\end{aligned} \tag{3.64}$$

for any $n \geq 1$, by (3.47) it possible to etablish (3.61) □

Proof of (3.62). For fixed $n \geq 1$ and in view (3.46) . Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon\ dxdt = \int_Q \gamma(u)S_n'(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt. \tag{3.65}$$

Appealing now to (3.54) and passing to the limit as $\mu \to +\infty$ in (3.65) allows to conclude that (3.62) holds true. □

Proof of (3.63). By (3.18), (3.50) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\int_Q f^\varepsilon(x,t,u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon\ dxdt = \int_Q f(x,t,u)S_n'(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt,$$

using (3.54) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (3.63). □

Now turn back to the proof of Lemma (3.2), due to (3.60)-(3.63), we are in a position to pass to the limit-sup when $\varepsilon \to 0$, then to the limit-sup when $\mu \to +\infty$ and then to the limit as $n \to +\infty$ in (3.59). Using the definition of W_μ^ε, we deduce that for any $k \geq 0$,

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla\left(T_k(u^\varepsilon) - T_k(u)_\mu\right)dxdt \leq 0.$$

Since $\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u^\varepsilon) = \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)$ fo $k \le n$, the above inequality implies that for $k \le n$,

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \tag{3.66}$$
$$\le \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(t,x,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt.$$

Due to (3.45), to have

$$\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon) \to \eta_{n+1}S_n'(u) \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N \text{ as } \varepsilon\to 0,$$

and the strong convergence of $T_k(u)_\mu$ to $T_k(u)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu\to+\infty$, to get

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt \tag{3.67}$$
$$= \int_Q S_n'(u)\eta_{n+1}\nabla T_k(u)dxdt = \int_Q \eta_{n+1}\nabla T_k(u)dxdt,$$

as soon as $k \le n$, since $S_n'(s) = 1$ for $|s| \le n$. Now, for $k \le n$, we have

$$S_n'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon\to 0$, to obtain

$$\eta_{n+1}\chi_{\{|u|\le k\}} = \eta_k\chi_{\{|u|\le k\}} \text{ a.e in } Q - \{|u| = k\} \text{ for } k \le n.$$

Recalling (3.66) and (3.67) allows to conclude that (3.55) holds true. □

Proof of (3.56). Let $k \ge 0$ be fixed. We use the monotone character (3.5) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, to obtain

$$I^\varepsilon = \int_Q \left[\left(\mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}) - \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\right)\right. \tag{3.68}$$
$$\left.\times\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)\right] dxdt \ge 0.$$

Inequality (3.68) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$I_1^\varepsilon = \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}dxdt,$$
$$I_2^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u\chi_{\{|u|\le k\}}dxdt,$$
$$I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)dxdt.$$

To pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that $u^\varepsilon = T_k(u^\varepsilon)$ and $\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} = \nabla T_k(u^\varepsilon)$ a.e in Q, assume that k is such that $\chi_{\{|u^\varepsilon|\leq k\}}$ almost everywhere converges to $\chi_{\{|u|\leq k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [16]). Using (3.55), to obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \\ &\leq \int_Q \eta_k \nabla T_k(u)dxdt. \end{aligned} \tag{3.69}$$

In view of (3.44) and (3.45),

$$\begin{aligned} \lim_{\varepsilon\to 0} I_2^\varepsilon &= -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\left(\nabla T_k(u)\right)dxdt \\ &= -\int_Q \eta_k \left(\nabla T_k(u)\right)dxdt. \end{aligned} \tag{3.70}$$

As a consequence of (3.44), for all $k > 0$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u \chi_{\{|u|\leq k\}})\left(\nabla T_k(u^\varepsilon) - \nabla T_k(u)\right)dxdt = 0. \tag{3.71}$$

Taking the limit-sup as $\varepsilon \to 0$ in (3.68) and using (3.69), (3.70) and (3.71) show that (3.56) holds true. □

Proof of (3.57). Using (3.56) and the usual Minty argument applies it follows that (3.57) holds true.

□

- **Step 5:** In this step we prove that u satisfies (3.12)-(3.14). For any fixed $n \geq 0$ one has

$$\begin{aligned} &\int_{\{n\leq |u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \\ &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(u^\varepsilon)dxdt - \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(u^\varepsilon)dxdt. \end{aligned}$$

According to (3.45) and (3.57) one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 1$, to obtain

$$\lim_{\varepsilon\to 0} \int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \tag{3.72}$$
$$= \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(u)dxdt - \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_n(u)dxdt$$
$$= \int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt.$$

Taking that limit as n tends to $+\infty$ in (3.72) and using the estimate (3.47), that u satisfies (3.12).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\mathrm{supp}(S') \subset [-k,k]$. Pontwise multiplication of that approximate equation (3.21) by $S'(u^\varepsilon)$ leads to

$$\begin{aligned}&(B_S(u^\varepsilon))_t - div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) \\ &+S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon) + \gamma(u^\varepsilon)S'(u^\varepsilon) = f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{3.73}$$

In what follows to pass to the limit as ε tends to 0 in each term of (3.73). Since S is bounded, and $S(u^\varepsilon)$ converges to $S(u)$ a.e in Q and in $L^\infty(Q)*$-weak, then $(S(u^\varepsilon))_t$ converges to $(S(u))_t$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\mathrm{supp}(S') \subset [-k,k]$, $S'(u^\varepsilon)\mathcal{A}(t,x,\nabla u^\varepsilon) = S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|u^\varepsilon|\leq k\}}$ a.e in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (3.57) of Lemma(3.2) imply that $S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(.)}(Q)\right)^N$ as ε tends to 0, because $S'(u) = 0$ for $|u| \geq k$ a.e in Q. The pointwise convergence of u^ε to u, the bounded character of S', S'' and (3.57) of Lemma (3.2) allow to conclude that

$$S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) \to S''(u)\mathcal{A}(x,t,\nabla u)\nabla T_k(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \to 0$. The use of (3.46) to obtain that $\gamma(u^\varepsilon)S'(u^\varepsilon)$ converges to $\gamma(u)S'(u)$ in $L^1(Q)$, and we use (3.18), (3.44) and we obtain that $f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon)$ converges to $f(x,t,u)S'(u)$ in $L^1(Q)$. As a consequence of the above convergence result, The position to pass to the limit as ε tends to 0 in equation (3.73) and to conclude that u satisfies (3.13). It remains to show that $S(u)$ satisfies the initial condition (3.14). To this end, firstly remark that, S being bounded, $S(u^\varepsilon)$ is bounded in $L^\infty(Q)$, $B_S(u^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.73) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p-)'}(]0,T[;W^{-1,p'(.)}(\Omega))$. As a consequence, an Aubin's type lemma ([39], Corollary 4) implies that $B_S(u^\varepsilon)$ lies in a compact set of $C(]0,T[;L^1(\Omega))$. It follows that, on the one hand, $B_S(u^\varepsilon)(t=0)$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$ Due to(3.17), to conclude that (3.14) holds true. As a conclusion of **Step 3** and **Step 5**, the proof of Theorem (3.1) is complete.

□

3.3 Nonlinear parabolic system

3.3.1 The Assumptions On The Data

This paper, we assume that the following assumptions hold true :
Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times]0, T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$

$$\mathcal{A}(x,t,\xi).\xi \geq \alpha\, |\xi|^{p(x)}, \tag{3.74}$$

$$|\mathcal{A}(x,t,\xi)| \leq \beta \left[L(x,t) + |\xi|^{p(x)-1}\right], \tag{3.75}$$

$$(\mathcal{A}(x,t,,\xi) - \mathcal{A}(x,t,\eta)).(\xi - \eta) > 0, \tag{3.76}$$

where $1 < p^- \leq p^+ < +\infty$, α, β are positives constants and L is a nonnegative function in $L^{p'(.)}(Q)$, $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$.
Let $b_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1- function lipchizienne with $b_i(0) = 0$ and for any ρ, τ are positives constants and for $i = \overline{1,2}$ such that

$$\rho \leq b_i'(s) \leq \tau, \ \ \forall s \in \mathbb{R}, \tag{3.77}$$

$f_i : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for any $k > 0$, there exists $\sigma_k > 0$, $c_k \in L^1(Q)$ such that

$$|f_1(x,t,s_1,s_2)| \leq c_k(x,t) + \sigma_k |s_2|^2, \tag{3.78}$$

for almost every $(x,t) \in (Q)$, for every s_1 such that $|s_1| \leq k$, and for every $s_2 \in \mathbb{R}$.
For any $k > 0$, there exists $\zeta_k > 0$ and $G_k \in L^{p'(.)}(Q)$ such that

$$|f_2(x,t,s_1,s_2)| \leq G_k(x,t) + \zeta_k |s_1|^{p(x)-1}, \tag{3.79}$$

for almost every $(x,t) \in (Q)$, for every s_2 such that $|s_2| \leq k$, and for every $s_1 \in \mathbb{R}$.

$$f_1(x,t,s_1,s_2)s_1 \geq 0 \text{ and } f_2(x,t,s_1,s_2)s_2 \geq 0, \tag{3.80}$$

$$(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2. \tag{3.81}$$

3.3.2 The Main of Results

In this section, we study the existence of renormalized solutions to problem (3.2).

Definition 3.2. *Let* $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$ *and* $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$. *A measurable functions* $(u,v) \in (C(]0,T[; L^1(\Omega)))^2$ *is a renormalized solution of the problem* (3.2) *if* ,

$$T_k(u) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega)), T_k(v) \in L^2(]0,T[; H_0^1(\Omega)) \text{ for any } k > 0 \ , \tag{3.82}$$
$$\gamma(u) \in L^1(Q) \ \text{ and } f_i(x,t,u,v) \in (L^1(Q))^2, \ \forall i = \overline{1,2},$$

$$b_1(u) \in L^{\infty}\left(]0,T[; L^1(\Omega)\right) \cap L^{q^-}(]0,T[; W_0^{1,q(.)}(\Omega)) \quad (3.83)$$
$$\text{and} \qquad b_2(v) \in L^{\infty}\left(]0,T[; L^1(\Omega)\right) \cap L^2(]0,T[; H_0^1(\Omega)),$$

for all continuous functions $q(x)$ on $\overline{\Omega}$ satisfying $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ for all $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int_{\{n\le|u|\le n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt + \lim_{n\to\infty} \int_{\{n\le|v|\le n+1\}} |\nabla v|^2 dxdt = 0, \quad (3.84)$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support on $\mathbb{R}$, to have,

$$(B_S^1(u))_t - div(\mathcal{A}(x,t,\nabla u)S'(u)) + S''(u)\mathcal{A}(x,t,\nabla u)\nabla u + \gamma(u)S'(u) \quad (3.85)$$
$$= f_1(x,t,u,v)S'(u) \text{ in } \mathcal{D}'(Q),$$
$$(B_S^2(v))_t - div(\nabla v S'(v)) + S''(v)\nabla v = f_2(x,t,u,v)S'(v) \text{ in } \mathcal{D}'(Q), \quad (3.86)$$
$$B_S^1(u)(t=0) = S(b_1(u_0)) \text{ in } \Omega, \quad (3.87)$$
$$B_S^2(v)(t=0) = S(b_2(v_0)) \text{ in } \Omega, \quad (3.88)$$

where $B_S^i(z) = \int_0^z b_i'(r)S'(r)dr$, for $i = \overline{1,2}$.

The following remarks are concerned with a few comments on definition (3.2).

Remark 3.2. *Note that, all terms in* (3.85) *are well defined. Indeed, let $k > 0$ such that $supp(S') \subset [K,K]$, we have $B_S^i(u)$ belongs to $L^\infty(Q)$ for all $i = \overline{1,2}$ because*

$$|B_S^1(u)| \le \int_0^u |b_1'(r)S'(r)|dr \le \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$|B_S^2(v)| \le \int_0^v |b_2'(r)S'(r)|dr \le \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$S(u) = S(T_k(u)) \in L^{p-}(]0,T[; W_0^{1,p(.)}(\Omega)), S(v) = S(T_k(v)) \in L^2(]0,T[; H_0^1(\Omega))$$

and $\frac{\partial B_S^i(u)}{\partial t} \in (\mathcal{D}'(Q))^2$ for $i = \overline{1,2}$. The term $S'(u)\mathcal{A}(x,t,\nabla T_k(u))$ identifes with $S'(T_k(u))\mathcal{A}(x,t,\nabla(T_k(u$ a.e. in Q, where $u = T_k(u)$ in $\{|u| \le k\}$, assumptions (3.75) *imply that*

$$|S'(T_k(u))\mathcal{A}(x,t,\nabla T_k(u))| \quad (3.89)$$
$$\le \beta \|S'\|_{L^\infty(\mathbb{R})} \left[L(x,t) + |\nabla(T_k(u))|^{p(x)-1}\right] \text{ a.e in } Q.$$

Using (3.75) *and* (3.82), *it follows that $S'(u)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. The term $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u)$ identifes with $S''(u)\mathcal{A}(t,x,\nabla(T_k(u)))\nabla T_k(u)$ and in view of* (3.75), (3.82) *and* (3.89), *to obtain $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u) \in L^1(Q)$ and $S'(u)\gamma(u) \in L^1(Q)$. Finally $f_1(x,t,u,v)$ $S'(u) =$*

$f_1(x,t,T_k(u),v)S'(u)$ *a.e in* Q *. Since* $|T_k(u)| \leq k$ *and* $S'(u) \in L^\infty(Q)$*,* $c_k(x,t) \in L^1(Q)$*, to obtain from* (3.78) *that* $f_1(x,t,T_k(u),v)S'(u) \in L^1(Q)$*, and* $f_2(x,t,u,v)\ S'(v) = f_2(x,t,u,T_k(v))S'(v)$ *a.e in* Q*. Since* $|T_k(v)| \leq k$ *and* $S'(v) \in L^\infty(Q)$*,* $G_k(x,t) \in L^{p'(.)}(Q)$ *, to obtain from* (3.79) *that* $f_2(x,t,u,T_k(v))S'(v) \in L^1(Q)$*. Also* $\frac{\partial B_S^1(u)}{\partial t} \in L^{(p^-)'}(]0,T[;W^{-1,p'(.)}(\Omega)) + L^1(Q)$ *and* $B_S^1(u) \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$*, and* $\frac{\partial B_S^2(v)}{\partial t} \in L^2(]0,T[;H^{-1}(\Omega)) + L^1(Q)$ *and* $B_S^2(v) \in L^2(]0,T[;H_0^1(\Omega)) \cap L^\infty(Q)$*, which implies that* $(B_S^1(u), B_S^2(v)) \in (C(]0,T[;L^1(\Omega)))^2$.

The Existence Theorem

Theorem 3.2. *Let* $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$*, assume that* (3.74)-(3.81) *hold true, then there exists at least one renormalized solution* $(u,v) \in (C(]0,T[,L^1(\Omega)))^2$ *of Problem* (3.2) *(in the sens of Definition* (3.2) *).*

Proof. of Theorem (3.2) The above theorem is to be proved in fourth steps.

- **Step 1: Approximate problem and a priori estimates.** Let us define the following approximation of b_i and f_i for $\varepsilon > 0$ fixed and for $i = \overline{1,2}$

$$b_\varepsilon^i(r) = T_{\frac{1}{\varepsilon}}(b_i(r)) \text{ a.e in } \Omega \text{ for } \varepsilon > 0,\ \ \forall r \in \mathbb{R}, \tag{3.90}$$

$$b_\varepsilon^i(u_0^\varepsilon) \text{ are a sequence of } (C_c^\infty(\Omega))^2 \text{ functions such that} \tag{3.91}$$
$$(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon)) \to (b_1(u_0), b_2(v_0)) \text{ in } (L^1(\Omega))^2 \text{ as } \varepsilon \text{ tends to } 0.$$

$$\begin{aligned} f_1^\varepsilon(x,t,r_1,r_2) &= f_1(x,t,T_{\frac{1}{\varepsilon}}(r_1),r_2), \\ f_2^\varepsilon(x,t,r_1,r_2) &= f_2(x,t,r_1,T_{\frac{1}{\varepsilon}}(r_2)), \end{aligned} \tag{3.92}$$

in view of (3.78), (3.79) and (3.80), there exist $G_k^\varepsilon \in L^{p'(.)}(Q)$, $c_k^\varepsilon \in L^1(Q)$ and $\sigma_k^\varepsilon, \zeta_k^\varepsilon > 0$ such that

$$|f_1^\varepsilon(x,t,s_1,s_2)| \leq c_k^\varepsilon(x,t) + \sigma_k^\varepsilon|s_2|^2, \tag{3.93}$$

$$|f_2^\varepsilon(x,t,s_1,s_2)| \leq G_k^\varepsilon(x,t) + \zeta_k^\varepsilon|s_1|^{p(x)-1}, \tag{3.94}$$

for almost every $(x,t) \in (Q)$, $s_1, s_2 \in \mathbb{R}$,

$$f_1^\varepsilon(x,t,s_1,s_2)s_1 \geq 0 \text{ and } f_2^\varepsilon(x,t,s_1,s_2)s_2 \geq 0. \tag{3.95}$$

Let us now consider the approximate problem:

$$\left(b_\varepsilon^1(u^\varepsilon)\right)_t - div\mathcal{A}(x,t,\nabla u^\varepsilon) + \gamma\left(u^\varepsilon\right) = f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ in } Q, \tag{3.96}$$

$$\left(b_\varepsilon^2(v^\varepsilon)\right)_t - \Delta v^\varepsilon = f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ in } Q, \tag{3.97}$$

$$u^\varepsilon = v^\varepsilon = 0 \text{ on }]0,T[\times\partial\Omega, \tag{3.98}$$

$$b_\varepsilon^1(u^\varepsilon)\,(t=0) = b_\varepsilon^1(u_0^\varepsilon) \text{ in } \Omega, \tag{3.99}$$

$$b^2_\varepsilon(v^\varepsilon)\,(t=0) = b^2_\varepsilon(v^\varepsilon_0) \text{ in } \Omega. \tag{3.100}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ and $v^\varepsilon \in L^2(]0,T[;H^1_0(\Omega))$ of (3.96)-(3.100) is an easy task (see [30]). We choose $T_k(u^\varepsilon)\chi_{(0,t)}$ as a test function in (3.96), to get

$$\begin{aligned}\int_\Omega B_k^{1,\varepsilon}(u^\varepsilon)(t)dx \quad &+ \quad \int_0^t\int_\Omega \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) + \int_0^t\int_\Omega \gamma\,(u^\varepsilon)\,T_k(u^\varepsilon)dxds \\ &= \quad \int_0^t\int_\Omega f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)T_k(u^\varepsilon)dxds + \int_\Omega B_k^{1,\varepsilon}(u_0^\varepsilon)dx, \end{aligned}\tag{3.101}$$

for almost every t in $(0,T)$, and where

$$B_k^{i,\varepsilon}(r) = \int_0^r T_k(s)\frac{\partial b^i_\varepsilon(s)}{\partial s}ds.\forall i=\overline{1,2}.$$

Under the definition of $B_k^{i,\varepsilon}(r)$ the inequality

$$0 \le \int_\Omega B_k^{1,\varepsilon}(u_0^\varepsilon)(t)dx \le k\int_\Omega |b^1_\varepsilon(u^\varepsilon_0)|dx, \;\; k>0.$$

Using (3.74), $f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)T_k(u^\varepsilon) \ge 0$, and we have $\gamma(u^\varepsilon) = \lambda|u^\varepsilon|^{p(x)-1}u^\varepsilon \ge 0$ because $1 < p^- \le p(x) \le +\infty$ and the definition of $B_k^\varepsilon(r)$ in (3.101), to obtain

$$\int_\Omega B_k^{1,\varepsilon}(u^\varepsilon)(t)dx + \alpha\int_{E_k} |\nabla u^\varepsilon|^{p(x)}\,dxds \le k\left\|b^1_\varepsilon(u^\varepsilon_0)\right\|_{L^1(Q)}, \tag{3.102}$$

where $E_k = \{(x,t)\in Q : |u^\varepsilon| \le k\}$, using $\overline{B}_k^{1,\varepsilon}(u^\varepsilon)(t) \ge 0$ and inequality (1.3) in (3.102), to get

$$\begin{aligned}\alpha\int_0^T \min\left\{\|\nabla T_k(u^\varepsilon)\|^{p^-}_{L^{p(x)}(\Omega)}, \|\nabla T_k(u^\varepsilon)\|^{p^+}_{L^{p(x)}(\Omega)}\right\} \;&\le\; \alpha \int_{\{(x,t)\in Q:\,|u^\varepsilon|\le k\}} |\nabla u^\varepsilon|^{p(x)}\,dxdt \\ &\le\; C, \end{aligned}\tag{3.103}$$

then is $T_k(u^\varepsilon)$ is bounded in $L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega))$.

Similarly, we choose $T_k(v^\varepsilon)\chi_{(0,t)}$ as a test function in (3.97), to get

$$\int_\Omega B_k^{2,\varepsilon}(v^\varepsilon)(t)dx + \alpha\int_{F_k} |\nabla v^\varepsilon|^2\,dxds \le k\left\|b^2_\varepsilon(v^\varepsilon_0)\right\|_{L^1(Q)}, \tag{3.104}$$

where $F_k = \{(x,t) \in Q : |v^\varepsilon| \leq k\}$, then is $T_k(v^\varepsilon)$ is bounded in $L^2(]0,T[\,;H_0^1(\Omega))$. Adding (3.102) and (4.52), one gets

$$\int_\Omega B_k^{1,\varepsilon}(u^\varepsilon)(t)dx + \int_\Omega B_k^{2,\varepsilon}(v^\varepsilon)(t)dx \leq k \left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}. \tag{3.105}$$

Also, to obtain

$$k \int_{\{(t,x)\in Q:|u^\varepsilon|>k\}} |\gamma(u^\varepsilon)|\, dxdt \leq k \left\|b_\varepsilon(u_0^\varepsilon)\right\|_{L^1(Q)}. \tag{3.106}$$

Hence

$$k \int_{\{(x,t)\in Q:|u^\varepsilon|>k\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt \;+\; k \int_{\{(x,t)\in Q:|v^\varepsilon|>k\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt \leq k \left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}. \tag{3.107}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(b_\varepsilon^1(u^\varepsilon))$ as test function in (3.96). Reasoning as above, by $\nabla T_{k,1}(s) = \nabla s \chi_{\{k\leq|s|\leq k+1\}}$ and the young's inequality, to obtain

$$\begin{aligned} \alpha \int_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon)\, |\nabla(u^\varepsilon)|^{p(x)}\, dxdt \;&\leq\; k \int_{\{\left|b_\varepsilon^1(u_0^\varepsilon)\right|>k\}} \left|b_\varepsilon^1(u_0^\varepsilon)\right| dx \\ &+\; Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)|>k\}} |\gamma(u^\varepsilon)|\, dxdt \\ &+\; Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)|>k\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt \\ &\leq\; C_1, \end{aligned}$$

inequality (1.3) implies that

$$\begin{aligned} &\int_0^T \alpha \chi_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} \min\left\{ \left\|\nabla(b_\varepsilon^1(u^\varepsilon))\right\|_{L^{p(x)}(\Omega)}^{p-}, \left\|\nabla(b_\varepsilon^1(u^\varepsilon))\right\|_{L^{p(x)}(\Omega)}^{p+} \right\} \\ &\leq\; \alpha \int_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon)\, |\nabla(u^\varepsilon)|^{p(x)}\, dxdt \leq C_1. \end{aligned} \tag{3.108}$$

Similarly, we choose $T_k(b_\varepsilon^2(v^\varepsilon))$ as test function in (3.97), to have

$$\begin{aligned} \int_{\{|b_\varepsilon^2(v^\varepsilon)|\leq k\}} b'_{2,\varepsilon}(v^\varepsilon)\, |\nabla(v^\varepsilon)|^2\, dxdt \;&\leq\; k \int_{\{\left|b_\varepsilon^2(v_0^\varepsilon)\right|>k\}} \left|b_\varepsilon^2(v_0^\varepsilon)\right| dx \\ &+Ck \int_{\{|b_\varepsilon^2(v^\varepsilon)|>k\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt \leq C_2, \end{aligned}$$

we know that properties of $B_k^{i,\varepsilon}(u^\varepsilon)$, $(B_k^{i,\varepsilon}(r^\varepsilon) \geq 0,\ \ B_k^{i,\varepsilon}(r^\varepsilon)) \geq \rho(|r|-1)$, for all $i = \overline{1,2}$, to obtain

$$\int_\Omega \left|B_k^{1,\varepsilon}(u^\varepsilon)(t)\right| dx \ + \ \int_\Omega \left|B_k^{2,\varepsilon}(v^\varepsilon)(t)\right| dx \leq k\int_\Omega \left|b_\varepsilon^1(u^\varepsilon)(t)\right| dx + k\int_\Omega \left|b_\varepsilon^2(v^\varepsilon)(t)\right| dx$$
$$\leq \ \rho\left(2meas(\Omega) + k\left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}\right). \quad (3.109)$$

From the estimation (3.103), (4.52), (3.108) , (3.109) and the properites of $B_k^{i,\varepsilon}$ and $b_\varepsilon^1(u_0^\varepsilon)$, $b_\varepsilon^2(v_0^\varepsilon)$, we deduce that

$$b_\varepsilon^1(u^\varepsilon) \text{ and } b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \quad (3.110)$$

$$u^\varepsilon \text{ and } v^\varepsilon \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \quad (3.111)$$

and

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega)), \quad (3.112)$$

and

$$b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^2(]0,T[\,;H_0^1(\Omega)), \quad (3.113)$$

by (3.108), (3.109) and Lemma 2.1 in [15] by and if

$$2 - \frac{1}{N+1} < p(.) < N,$$

to obtain

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{q-}(]0,T[\,;W_0^{1,q(x)}(\Omega)), \quad (3.114)$$

for all continuous variable exponents $q \in C(\overline{\Omega})$ satisfying

$$1 \leq q(x) < \frac{N(p(x)-1)+p(x)}{N+1}$$

, for all $x \in \Omega$.

And

$$T_k(u^\varepsilon) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \quad (3.115)$$

and

$$T_k(v^\varepsilon) \text{ is bounded in } L^2\left(]0,T[;H_0^1(\Omega)\right). \quad (3.116)$$

By (3.106) and (3.107), we may conclude that

$$\gamma(u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \quad (3.117)$$

and

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ and } f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \quad (3.118)$$

independently of ε.
Proceeding as in [12], [13] that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S' \subset [-k, k]$),

$$S(u^\varepsilon) \text{ is bounded in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{3.119}$$

and

$$S(v^\varepsilon) \text{ is bounded in } L^2\left(]0,T[;H_0^1(\Omega)\right), \tag{3.120}$$

and

$$(S(u^\varepsilon))_t \text{ is bounded in } L^1(Q) + L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right), \tag{3.121}$$

and

$$(S(v^\varepsilon))_t \text{ is bounded in } L^1(Q) + L^2\left(]0,T[;H^{-1}(\Omega)\right). \tag{3.122}$$

In fact, as a consequence of (3.115), by Stampacchia's Theorem, we obtain (3.119). To show that (3.121) holds true, we multiply the equation (3.96) by $S'(u^\varepsilon)$ and the equation (3.97) by $S'(v^\varepsilon)$, to obtain

$$\begin{aligned}\left(B_S^1(u^\varepsilon)\right)_t &= div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) - \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(S'(u^\varepsilon)) \\ &-\gamma(u^\varepsilon)S'(u^\varepsilon) + f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{3.123}$$

And

$$\begin{aligned}\left(B_S^2(v^\varepsilon)\right)_t &= div(S'(v^\varepsilon)\nabla v^\varepsilon) - \nabla(S'(v^\varepsilon)) \\ &+f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{3.124}$$

Since supp(S') and supp(S'') are both included in $[-k;k]$; u^ε may be replaced by $T_k(u^\varepsilon)$ in $\{|u^\varepsilon| \le k\}$. To have

$$\begin{aligned}&|S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)| \\ &\le \beta\|S'\|_{L^\infty}\left[L(x,t) + |\nabla T_k(u^\varepsilon)|^{p(x)-1}\right],\end{aligned} \tag{3.125}$$

as a consequence, each term in the right hand side of (3.123) is bounded either in $L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and obtain (3.121).

Now we look for an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \ge 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(s) = T_{n+1}(s) - T_n(s) = \begin{cases} 0 & \text{if } |s| \le n, \\ (|s|-n)\,sign(s) & \text{if } n \le |s| \le n+1, \\ sign(s) & \text{if } |s| \ge n. \end{cases}$$

Remark that $||\theta_n||_{L^\infty} \le 1$ for any $n \ge 1$ and that $\theta_n(s) \to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^\varepsilon)$ in (3.96) leads to

$$\int_\Omega \widetilde{\theta_n}(u^\varepsilon)(t)\,dx + \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(\theta_n(u^\varepsilon))\,dxdt + \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt$$
$$= \int_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx, \qquad (3.126)$$

where $\widetilde{\theta_n}(r)(t) = \int_0^r \theta_n(s)\dfrac{\partial b_\varepsilon^i(s)}{\partial s}ds$, for all $i = \overline{1,2}$,
for almost any t in $]0,T[$ and where $\widetilde{\theta_n}(r) = \int\limits_0^r \theta_n(s)ds \geq 0$. Hence, dropping a nonnegative term

$$\int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \qquad (3.127)$$

$$\leq \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt + \int_Q f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx$$
$$\leq \int_{\{|u^\varepsilon|\geq n\}} |\gamma(u^\varepsilon)|\,dxdt + \int_{\{|u^\varepsilon|\geq n\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt + \int_{\{|b_\varepsilon^1(u_0^\varepsilon)|\geq n\}} \left|b_\varepsilon^1(u_0^\varepsilon)\right|dx.$$

Similarly, we take test function $\theta_n(v^\varepsilon)$ in (3.97) leads to

$$\int_{\{n\leq|v^\varepsilon|\leq n+1\}} |\nabla v^\varepsilon|^2 dxdt \qquad (3.128)$$

$$\leq \int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\theta_n(v^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(v_0^\varepsilon)\,dx \leq \int_{\{|v^\varepsilon|\geq n\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt$$
$$+ \int_{\{|b_\varepsilon^2(v_0^\varepsilon)|\geq n\}} \left|b_\varepsilon^2(v_0^\varepsilon)\right|dx.$$

Next, we study the convergence of $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $C(]0,T[;L^1(\Omega))$.

Lemma 3.3. *Both $(u^{\varepsilon n})_{n\in\mathbb{N}}$ and $(v^{\varepsilon n})_{n\in\mathbb{N}}$ are Cauchy sequences in $C(]0,T[;L^1(\Omega))$.*

Proof. Let ε_n and ε_m two positive integers. It follows frome (3.96) and (3.97) that

$$\int_\Omega \frac{\partial b^1_{\varepsilon_n}(u^{\varepsilon_n}-u^{\varepsilon_m})}{\partial t}\varphi dx + \int_0^t\int_\Omega (\mathcal{A}(x,t,\nabla u^{\varepsilon_n}) - \mathcal{A}(x,t,\nabla u^{\varepsilon_m}))\nabla\varphi dxdt$$
$$+\int_0^t\int_\Omega \lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]\phi dxds$$
$$=\int_0^t\int_\Omega [f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})]\varphi dxds, \tag{3.129}$$

and

$$\int_\Omega \frac{\partial b^2_{\varepsilon_n}(v^{\varepsilon_n}-v^{\varepsilon_m})}{\partial t}\phi dx + \int_0^t\int_\Omega (\nabla v^{\varepsilon_n} - \nabla v^{\varepsilon_m})\nabla\phi dxdt \tag{3.130}$$
$$=\int_0^t\int_\Omega [f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})]\phi dxds,$$

where $\varphi \in L^\infty(]0,T[;W^{1,p(.)}(\Omega))$ and $\phi \in L^2(]0,T[;H^1_0(\Omega))$. To do this fix $\tau \in [0,T]$. Taking $\varphi = \frac{1}{k}T_k(u^{\varepsilon_n}-u^{\varepsilon_m})1_{\{[0,\tau[\}}$ in (3.129) and $\phi = \frac{1}{k}T_k(v^{\varepsilon_n}-v^{\varepsilon_m})1_{\{[0,\tau[\}}$ in (3.130), one gets

$$\frac{1}{k}\int_\Omega B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau)-u^{\varepsilon_n}(\tau))dx - \frac{1}{k}\int_\Omega B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(0)-u^{\varepsilon_m}(0))dx$$
$$+\int_0^\tau\int_\Omega \frac{1}{k}(\mathcal{A}(x,t,\nabla u^{\varepsilon_n}) - \mathcal{A}(x,t,\nabla u^{\varepsilon_m}))\nabla T_k(u^{\varepsilon_n}-u^{\varepsilon_m})dxdt \tag{3.131}$$
$$+\int_0^\tau\int_\Omega \frac{\lambda}{k}\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]T_k(u^{\varepsilon_n}-u^{\varepsilon_m})dxds$$
$$=\int_0^t\int_\Omega \frac{1}{k}[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})]T_k(u^{\varepsilon_n}-u^{\varepsilon_m})dxds,$$

and

$$\frac{1}{k}\int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau))dx-\frac{1}{k}\int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(0)-v^{\varepsilon_m}(0))dx$$
$$+\frac{1}{k}\int_0^t\int_{\Omega}\nabla(v^{\varepsilon_n}-v^{\varepsilon_m})\nabla T_k(v^{\varepsilon_n}-v^{\varepsilon_m})dxdt \tag{3.132}$$
$$=\int_0^t\int_{\Omega}\frac{1}{k}\left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]T_k(v^{\varepsilon_n}-v^{\varepsilon_m})dxds,$$

where

$$B_k^{i,\varepsilon_n}(r)=\int_0^r T_k(s)\frac{\partial b_{\varepsilon_n}^i(s)}{\partial s}ds.\ \forall i=\overline{1,2},$$

adding (3.131) and (3.132), we get

$$\frac{1}{k}\int_{\Omega} B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau)-u^{\varepsilon_m}(\tau))dx+\frac{1}{k}\int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau))dx$$
$$\leq\int_0^\tau\int_{\Omega}\lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n}-|u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt+$$
$$\int_0^\tau\int_{\Omega}\left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt+$$
$$\int_0^\tau\int_{\Omega}\left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt+$$
$$\int_{\Omega}\left|b_{\varepsilon_n}^1(u_0^{\varepsilon_n}-u_0^{\varepsilon_m})\right|dx+\int_{\Omega}\left|b_{\varepsilon_n}^2(v_0^{\varepsilon_n}-v_0^{\varepsilon_m})\right|dx,$$

since $B_k^{i,\varepsilon_n}(r) \geq \rho \int_0^r T_k(s)ds \geq \rho\left(|s|-1\right).\forall i=\overline{1,2}$

$$\int_\Omega |u^{\varepsilon_n}(\tau)-u^{\varepsilon_m}(\tau)|\,dx+\int_\Omega |v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau)|\,dx$$

$$\leq 2k\ \ meas(\Omega)+\int_0^\tau\int_\Omega k\lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n}-|u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt$$

$$+k\int_0^\tau\int_\Omega \left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$

$$+k\int_0^\tau\int_\Omega \left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$

$$+k\int_\Omega \left|b^1_{\varepsilon_n}(u_0^{\varepsilon_n}-u_0^{\varepsilon_m})\right|dx+k\int_\Omega \left|b^2_{\varepsilon_n}(v_0^{\varepsilon_n}-v_0^{\varepsilon_m})\right|dx,$$

letting ε_n, $\varepsilon_m\to\infty$ and them $k\to 0$, to obtain

$$\sup_{\tau\in[0,T]}\int_\Omega |u^{\varepsilon_n}(\tau)-u^{\varepsilon_m}(\tau)|\,dx+\sup_{\tau\in[0,T]}\int_\Omega |v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau)|\,dx$$

$$\leq\int_0^\tau\int_\Omega k\lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n}-|u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt$$

$$+k\int_0^\tau\int_\Omega \left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$

$$+k\int_0^\tau\int_\Omega \left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$

$$+k\int_\Omega \left|b^1_{\varepsilon_n}(u_0^{\varepsilon_n}-u_0^{\varepsilon_m})\right|dx+k\int_\Omega \left|b^2_{\varepsilon_n}(v_0^{\varepsilon_n}-v_0^{\varepsilon_m})\right|dx.$$

□

- **Step 2: The limit of the solution of the approximated problem.** Arguing again as in [[12],[13],[14]] estimates (3.119), (3.121), (3.120) and (3.122) imply that, for a subsequence still indexed by ε,

$$(u^\varepsilon,v^\varepsilon)\text{ converge almost every where to }(u,v),\tag{3.133}$$

using (3.96), (3.115), (3.116) and (3.125), to get

$$T_k(u^\varepsilon)\text{ converge weakly to }T_k(u)\text{ in }L^{p-}\left(]0,T[\,;W_0^{1,p(.)}(\Omega)\right),\tag{3.134}$$

and

$$T_k(v^\varepsilon) \text{ converge weakly to } T_k(v) \text{ in } L^2\left(]0,T[\,;H_0^1(\Omega)\right), \tag{3.135}$$

$$\chi_{\{|u^\varepsilon|\leq k\}}\mathcal{A}(x,t,\nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N, \tag{3.136}$$

as ε tends to 0 for any $k>0$ and any $n\geq 1$ and where for any $k>0$, η_k belongs to $\left(L^{p'(.)}(Q)\right)^N$. Since $\gamma(u^\varepsilon)$ is a continuous incrassing function, from the monotone convergence theorem and (3.106) and by (3.133), to obtain that

$$\gamma(u^\varepsilon) \text{ converge weakly to } \gamma(u) \text{ in } L^1(Q). \tag{3.137}$$

We now establish that $(b_1(u),b_2(v))$ belongs to $(L^\infty(]0,T[\,;L^1(\Omega)))^2$. Indeed using (3.101) and $\left|B_k^{i,\varepsilon}(s)\right|\geq\rho(|s|-1)$, $\forall i=\overline{1,2}$, leads to

$$\begin{aligned}\int_\Omega \left|b_\varepsilon^1(u^\varepsilon)\right|(t)dx+\int_\Omega\left|b_\varepsilon^2(v^\varepsilon)\right|(t)dx &\leq \rho(2meas(\Omega)\\ &+ \|(f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon),f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon))\|_{(L^1(Q))^2}\\ &+ k\|\gamma(u^\varepsilon)\|_{L^1(Q)}\\ &+ k\left\|(b_\varepsilon^1(u_0^\varepsilon),b_\varepsilon^2(v_0^\varepsilon))\right\|_{(L^1(\Omega))^2}).\end{aligned}$$

By lemma (3.3) and (3.127), (3.128), we conclude that there exist two subsequences of u^{ε_n} and v^{ε_n}, still denoted by themselves for convenience, such that u^{ε_n} converges to a function u in $C(]0,T[;L^1(\Omega))$, v^{ε_n}converges to a function v in $C(]0,T[;L^1(\Omega))$. Using (3.106) and (3.91),(3.107), we have $(b_1(u),b_2(v))$ belongs to $(L^\infty(]0,T[\,;L^1(\Omega)))^2$. We are now in a position to exploit (3.127) and (3.128). Since $(u^\varepsilon,v^\varepsilon)$ is bounded in $(L^\infty(]0,T[\,;L^1(\Omega)))^2$, to get

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\{|u^\varepsilon|\geq n\}\right)=0. \tag{3.138}$$

and

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\{|v^\varepsilon|\geq n\}\right)=0. \tag{3.139}$$

The equi-integrability of the sequence $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ in $(L^1(Q))^2$. We shall now prove that $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ converges to $f_i(x,t,u,v)$ strongly in $(L^1(Q))^2$, for all $i=\overline{1,2}$ by using Vitalis theorem. Since $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\to f_i(x,t,u,v)$ a.e in Q it suffices to prove that $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ are equi-integrable in Q. Let $\delta_1>0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$\begin{aligned}G_{\delta_1} &= \{(x,t)\in Q:\ |u_n|\leq\delta_1\}, &(3.140)\\ F_{\delta_1} &= \{(x,t)\in Q:\ |u_n|>\delta_1\}. &(3.141)\end{aligned}$$

Using the generalized Hölder's inequality and Poincaré inequality, to have

$$\int_{\mathbf{A}}|f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt=\int_{\mathbf{A}\cap G_{\delta_1}}|f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt+\int_{\mathbf{A}\cap F_{\delta_1}}|f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt,$$

therfore

$$\begin{aligned}
\int_{\mathbf{A}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt &\le \int_{\mathbf{A}\cap G_{\delta_1}} \left(c_{k,\varepsilon}(x,t)+\sigma_{k,\varepsilon}\,|v^\varepsilon|^2\right)dxdt \\
&+ \int_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} c_{k,\varepsilon}(x,t)dxdt + \sigma_{k,\varepsilon}\int_Q |\nabla T_{\delta_1}(v^\varepsilon)|^2\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} c_{k,\varepsilon}(x,t)dxdt + \sigma_{k,\varepsilon}\,(meas(\mathbf{Q})+1)^{\frac{1}{2}} \\
&\quad \left(\int_{Q_T} |\nabla T_{\delta_1}(v^\varepsilon)|^2\,dxdt\right)^{\frac{1}{2}} + \int_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_1 + C_2\left(\frac{k}{\alpha}\left\|b_\varepsilon^2(v_0^\varepsilon)\right\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \\
&+ \int_{\mathbf{A}\cap F_{\delta_1}} \frac{1}{|u^\varepsilon|}\,|u^\varepsilon f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_2 + \int_{\mathbf{A}\cap F_{\delta_1}} \frac{1}{\delta_1}\,|u^\varepsilon f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_2 + \frac{1}{\delta_1}\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\left(\int_{\mathbf{A}\cap F_{\delta_1}} |u^\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{p^-}} \\
&\quad \left(\int_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|^{p'(x)(p(x)-1)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&\to 0 \text{ when } meas(\mathbf{A})\to \mathbf{0}.
\end{aligned}$$

Which shows that $f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ is equi-integrable. By using Vitali's theorem, to get

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\to f_1(x,t,u,v) \text{ strongly in } L^1(Q). \tag{3.142}$$

Now we prove that

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\to f_2(x,t,u,v) \text{ strongly in } L^1(Q). \tag{3.143}$$

Let $\delta_2 > 0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$G_{\delta_2} = \{(x,t) \in Q : \ |v_n| \leq \delta_2\}, \tag{3.144}$$
$$F_{\delta_2} = \{(x,t) \in Q : \ |v_n| > \delta_2\}. \tag{3.145}$$

Using the generalized Hölder's inequality and Poincaré inequality, to get

$$\int\limits_{\mathbf{A}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt = \int\limits_{\mathbf{A}\cap G_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt + \int\limits_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\, dxdt,$$

therfore

$$\begin{aligned}
\int_{\mathbf{A}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt &\le \int_{\mathbf{A}\cap G_{\delta_2}} \left(G_k^\varepsilon(x,t)+\xi_k^\varepsilon\,|u^\varepsilon|^{p(x)-1}\right)dxdt \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} G_k^\varepsilon(x,t)dxdt+\xi_k^\varepsilon\int_Q |\nabla T_{\delta_2}(u^\varepsilon)|^{p(x)-1}\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} G_k^\varepsilon(x,t)dxdt+\xi_k^\varepsilon\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)(meas(\mathbf{Q})+1)^{\frac{1}{p^-}} \\
&\quad \left(\int_{Q_T} |\nabla T_{\delta_2}(u^\varepsilon)|^{(p(x)-1)p'(x)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_3+C_4\left(\frac{k}{\alpha}\left\|b_\varepsilon^1(u_0^\varepsilon)\right\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} \frac{1}{|v^\varepsilon|}\,|v^\varepsilon f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_4+\int_{\mathbf{A}\cap F_{\delta_2}} \frac{1}{\delta_2}\,|v^\varepsilon f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_4+\frac{1}{\delta_2}\left(\int_{\mathbf{A}\cap F_{\delta_2}} |v^\varepsilon|^2\,dxdt\right)^{\frac{1}{2}}\left(\int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|^2\,dxdt\right)^{\frac{1}{2}} \\
&\to 0 \text{ when } meas(\mathbf{A})\to\mathbf{0}.
\end{aligned}$$

Which shows that $f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ is equi-integrable. By using Vitali's theorem, to get

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\to f_2(x,t,u,v) \text{ strongly in } L^1(Q). \tag{3.146}$$

Using (3.137), (3.142) and the equi-integrability of the sequence $|b_\varepsilon^1(u_0^\varepsilon)|$ in $L^1(\Omega)$ and $|b_\varepsilon^2(v_0^\varepsilon)|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\left(\sup_{\varepsilon}\left(\int_{\{n\le|u^\varepsilon|\le n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt+\int_{\{n\le|v^\varepsilon|\le n+1\}}|\nabla v^\varepsilon|^2dxdt\right)\right)=0. \quad (3.147)$$

- **Step 3: Strong convergence.** The specifie time regularization of $T_k(u)$ (for fixed $k\ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequaence in $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$ such that$\|v_0^\mu\|_{L^\infty(\Omega)}\le k$, $\forall\mu>0$, and $v_0^\mu\to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\|v_0^\mu\|_{L^{p(.)}(\Omega)}\to 0$ as $\mu\to+\infty$.

  For fixed $k\ge 0$ and $\mu>0$, let us consider the unique solution $T_k(u)_\mu\in L^\infty(\Omega)\cap L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right)$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t}+\mu\left(T_k(u)_\mu-T_k(u)\right)=0 \text{ in } \mathcal{D}'(Q), \quad (3.148)$$

$$T_k(u)_\mu(t=0)=v_0^\mu. \quad (3.149)$$

  The behavior of $T_k(u)_\mu$ as $\mu\to+\infty$ is investigated in [18] and we just recall here that (3.148)-(3.149) imply that

$$T_k(u)_\mu\to T_k(u) \text{ strongly in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right) \text{ a.e in } Q, \text{ as } \mu\to+\infty, \quad (3.150)$$

  with $\|T_k(u)_\mu\|_{L^\infty(\Omega)}\le k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t}\in L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$.
  The main estimate is the following

  Lemma 3.4. *Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r)=r$ for $r\le k$, and suppS' is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle\frac{\partial B_S^1(u^\varepsilon)}{\partial t},(T_k(u^\varepsilon)_\mu-T_k(u))\right\rangle dt\ge 0,$$

  where here $\langle.,.\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'(.)}(\Omega)$ and $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$, and where $B_S^1(z)=\int_0^z b_1'(r)S'(r)dr$.

  Proof. See[14], Lemma 1. □

  Now we are to prove that the weak limit η_k and we prove the weak L^1 convergence of the "trunccted" energy $\mathcal{A}(x,t,\nabla T_k(u^\varepsilon))$ as ε tends to 0. In order to show this result we recall the lemma below.

Lemma 3.5. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla T_k(u^\varepsilon)dxdt \leq \int_Q \eta_k \nabla T_k(u)dxdt, \tag{3.151}$$

$$\lim_{\varepsilon\to 0}\int_Q \left[\mathcal{A}\left(x,t,\nabla u^\varepsilon_{\chi_{\{|u^\varepsilon|\leq k\}}}\right) - \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)\right]$$

$$\times\left[\nabla u^\varepsilon_{\chi_{\{|u^\varepsilon|\leq k\}}} - \nabla u_{\chi_{\{||\leq k\}}}\right]dxdt = 0 \tag{3.152}$$

$\eta_k = \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)$ *a.e in Q, for any $k\geq 0$, as ε tends to* 0.

$$\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla T_k(u^\varepsilon) \to \mathcal{A}\left(x,t,\nabla u\right)\nabla T_k(u) \text{ weakly in } L^1\left(Q\right). \tag{3.153}$$

Proof. Let us introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that, for any $n\geq 1$

$$\begin{cases} S_n(r) = r \text{ if } |\mathrm{r}|\leq n, \\ \operatorname{supp}\left(S'_n\right)\subset\left[-(n+1),(n+1)\right], \\ \|S''_n\|_{L^\infty(\mathbb{R})}\leq 1. \end{cases} \tag{3.154}$$

For fixed $k\geq 0$, we consider the test function $S'_n(u^\varepsilon)\left(T_k(u_\varepsilon)-\left(T_k(u)\right)_\mu\right)$ in (3.96), we use the definition (3.154) of S'_n and we definie $W^\varepsilon_\mu = T_k(u_\varepsilon)-\left(T_k(u)\right)_\mu$, to get

$$\int_0^T \left\langle \left(B^1_S(u^\varepsilon)\right)_t, W^\varepsilon_\mu\right\rangle dt + \int_Q S'_n(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla W^\varepsilon_\mu dxdt \tag{3.155}$$

$$+\int_Q S''_n(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon W^\varepsilon_\mu dxdt + \int_Q \gamma(u^\varepsilon)S'_n(u^\varepsilon)W^\varepsilon_\mu dxdt$$

$$= \int_Q f^\varepsilon_1(x,t,u^\varepsilon,v^\varepsilon)S'_n(u^\varepsilon)W^\varepsilon_\mu dxdt.$$

Now we pass to the limit in (3.155) as $\varepsilon\to 0$, $\mu\to+\infty$, $n\to+\infty$ for k real number fixed. In order to perform this task, we prove below the following results for any $k\geq 0$:

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T \left\langle \left(B^1_S(u^\varepsilon)\right)_t, W^\varepsilon_\mu\right\rangle dt \geq 0 \text{ for any } n\geq k, \tag{3.156}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S''_n(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon W^\varepsilon_\mu dxdt = 0, \tag{3.157}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{3.158}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1. \tag{3.159}$$

Proof of (3.156). In view of the definition W_μ^ε, we apply lemma (3.4) with $S = S_n$ for fixed $n \geq k$. As a consequence, (3.156) hold true. □

Proof of (3.157). For any $n \geq 1$ fixed, we have $supp(S_n'') \subset [-(n+1), -n] \cup [n, n+1]$, $\|W_\mu^\varepsilon\|_{L^\infty(Q)} \leq 2k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$, to get

$$\left|\int_Q S_n''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon dxdt\right| \tag{3.160}$$

$$\leq \; 2k \int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt,$$

for any $n \geq 1$, by (3.147) it possible to etablish (3.157) □

Proof of (3.158). For fixed $n \geq 1$ and in view (3.137) . Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon \, dxdt = \int_Q \gamma(u)S_n'(u)(T_k(u) - T_k(u)_\mu)dxdt. \tag{3.161}$$

Appealing now to (3.150) and passing to the limit as $\mu \to +\infty$ in (3.161) allows to conclude that (3.158) holds true. □

Proof of (3.159). By (3.92), (3.142) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\int_Q f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon \, dxdt = \int_Q f_1(x,t,u,v)S_n'(u)(T_k(u) - T_k(u)_\mu)dxdt,$$

using (3.150) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (3.159). □

Now turn back to the proof of Lemma (3.5), due to (3.156)-(3.159), we are in a position to pass to the limit-sup when $\varepsilon \to 0$, then to the limit-sup when $\mu \to +\infty$ and then to the limit as $n \to +\infty$ in (3.155). Using the definition of W_μ^ε, we deduce that for any $k \geq 0$,

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla\left(T_k(u^\varepsilon) - T_k(u)_\mu\right)dxdt \leq 0.$$

Since $\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u^\varepsilon) = \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)$ fo $k \le n$, the above inequality implies that for $k \le n$,

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \tag{3.162}$$

$$\le \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(t,x,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt.$$

Due to (3.136), to have

$$\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon) \to \eta_{n+1}S_n'(u) \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N \text{ as } \varepsilon\to 0,$$

and the strong convergence of $T_k(u)_\mu$ to $T_k(u)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu\to+\infty$, to get

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt \tag{3.163}$$

$$= \int_Q S_n'(u)\eta_{n+1}\nabla T_k(u)dxdt = \int_Q \eta_{n+1}\nabla T_k(u)dxdt,$$

as soon as $k \le n$, since $S_n'(s) = 1$ for $|s| \le n$. Now, for $k \le n$, to have

$$S_n'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon\to 0$, to obtain

$$\eta_{n+1}\chi_{\{|u|\le k\}} = \eta_k\chi_{\{|u|\le k\}} \text{ a.e in } Q - \{|u| = k\} \text{ for } k\le n.$$

Recalling (3.162) and (3.163) allows to conclude that (3.151) holds true. □

Proof of (3.152). Let $k \ge 0$ be fixed. We use the monotone character (3.76) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, to obtain

$$I^\varepsilon = \int_Q \left(\mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}) - \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\right)\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)dxdt \ge 0. \tag{3.164}$$

Inequality (3.164) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$I_1^\varepsilon = \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}dxdt,$$

$$I_2^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u\chi_{\{|u|\le k\}}dxdt,$$

$$I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)dxdt.$$

We pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that we have $u^\varepsilon = T_k(u^\varepsilon)$ and $\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} = \nabla T_k(u^\varepsilon)$ a.e in Q, and we can assume that k is such that $\chi_{\{|u^\varepsilon|\leq k\}}$ almost everywhere converges to $\chi_{\{|u|\leq k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [16]). Using (3.151), to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \\ &\leq \int_Q \eta_k \nabla T_k(u)dxdt.\end{aligned} \tag{3.165}$$

In view of (3.134) and (3.136), to have

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2^\varepsilon &= -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\left(\nabla T_k(u)\right)dxdt \\ &= -\int_Q \eta_k \left(\nabla T_k(u)\right)dxdt.\end{aligned} \tag{3.166}$$

As a consequence of (3.134), we have for all $k > 0$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u \chi_{\{|u|\leq k\}})\left(\nabla T_k(u^\varepsilon) - \nabla T_k(u)\right)dxdt = 0. \tag{3.167}$$

Taking the limit-sup as $\varepsilon \to 0$ in (3.164) and using (3.165), (3.166) and (3.167) show that (3.152) holds true. □

Proof of (3.153). Using (3.152) and the usual Minty argument applies it follows that (3.153) holds true.

Lemma 3.6. *$\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$.*

Proof. Denote $V_\mu^\varepsilon = T_k(v_\varepsilon) - (T_k(v))_\mu$ and choose $S_n'(v^\varepsilon)\Big(T_k(v_\varepsilon) - (T_k(v))_\mu\Big)$ the test function in (3.97). One can get that

$$\begin{aligned}&\int_0^T \left\langle \left(B_S^2(v^\varepsilon)\right)_t, V_\mu^\varepsilon\right\rangle dt + \int_Q S_n'(v^\varepsilon)\nabla v^\varepsilon \nabla V_\mu^\varepsilon dxdt \\ &+\int_Q S_n''(v^\varepsilon)|\nabla v^\varepsilon|^2 V_\mu^\varepsilon dxdt = \int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S_n'(v^\varepsilon)V_\mu^\varepsilon dxdt.\end{aligned} \tag{3.168}$$

By a similar discussion, one has

$$\lim_{\mu\to+\infty}\inf\lim_{\varepsilon\to 0}\int_0^T \left\langle \left(B_S^2(v^\varepsilon)\right)_t, V_\mu^\varepsilon\right\rangle dt \geq 0 \text{ for any } n \geq k, \tag{3.169}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S''_n(v^\varepsilon)|\nabla v^\varepsilon|^2 V^\varepsilon_\mu dxdt = 0, \tag{3.170}$$

and

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'_n(v^\varepsilon)V^\varepsilon_\mu dxdt = 0, \text{ for any } n\geq 1. \tag{3.171}$$

Hence

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S'_n(v^\varepsilon)\nabla v^\varepsilon\nabla V^\varepsilon_\mu dxdt \leq 0. \tag{3.172}$$

□

Similarly, one gets that $\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$. □

- **Step 4:** In this step we prove that (u,v) satisfies (3.84), (3.85)-(3.88) . For any fixed $n\geq 0$ one has

$$\begin{aligned}&\int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\\ =&\int_Q\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(u^\varepsilon)dxdt-\int_Q\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(u^\varepsilon)dxdt.\end{aligned}$$

According to (3.136) and (3.153) one is at liberty to pass to the limit as ε tends to 0 for fixed $n\geq 1$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\\ =&\int_Q\mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(u)dxdt-\int_Q\mathcal{A}(x,t,\nabla u)\nabla T_n(u)dxdt\\ =&\int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u)\nabla u dxdt.\end{aligned} \tag{3.173}$$

Letting n tends to $+\infty$ in (3.173), it follows from estimate (3.147), that

$$\lim\lim_{\varepsilon\to 0}\int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt = 0.$$

Similarly, one can prove

$$\lim\lim_{\varepsilon\to 0}\int_{\{n\leq|v^\varepsilon|\leq n+1\}}|\nabla v^\varepsilon|^2 dxdt = 0.$$

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\text{supp}(S') \subset [-k,k]$. Pontwise multiplication of that approximate equation (3.96) by $(S'(u^\varepsilon), S'(v^\varepsilon))$ leads to

$$\begin{aligned}&\left(B_S^1(u^\varepsilon)\right)_t - div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) \\ &+S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon) + \gamma(u^\varepsilon)S'(u^\varepsilon) = f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q),\end{aligned} \tag{3.174}$$

and

$$\begin{aligned}&\left(B_S^2(v^\varepsilon)\right)_t - div(S'(v^\varepsilon)\nabla v^\varepsilon) \\ &+S''(v^\varepsilon)|\nabla(v^\varepsilon)|^2 = f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{3.175}$$

In what follows to pass to the limit as ε tends to 0 in each term of (3.174). Since S is bounded, and $(S(u^\varepsilon), S(v^\varepsilon))$ converges to $(S(u), S(v))$ a.e in Q and in $(L^\infty(Q))^2$ *-weak, then
$((B_S^1(u^\varepsilon))_t, (B_S^2(v^\varepsilon))_t)$ converges to $((B_S^1(u))_t, (B_S^1(v))_t)$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\text{supp}(S') \subset [-k,k]$,

$$S'(u^\varepsilon)\mathcal{A}(t,x,\nabla u^\varepsilon) = S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|u^\varepsilon|\le k\}} \text{ a.e in } Q.$$

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (3.153) of Lemma(3.5) imply that $S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as ε tends to 0, because $S'(u) = 0$ for $|u| \ge k$ a.e in Q and $S'(v^\varepsilon)\nabla v^\varepsilon$ converges to $S'(v)\nabla v$ weakly in $L^2(Q)$ as ε tends to 0. The pointwise convergence of u^ε to u, the bounded character of S', S'' and (3.153) of Lemma (3.5) allow to conclude that

$$S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) \to S''(u)\mathcal{A}(x,t,\nabla u)\nabla T_k(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \to 0$, and lemma (3.3) shows that

$$S''(v^\varepsilon)\nabla^\varepsilon v\nabla T_k(v^\varepsilon) \to S''(v)\nabla v\nabla T_k(v) \text{ weakly in } L^1(Q)$$

The use of (3.137) to obtain that $\gamma(u^\varepsilon)S'(u^\varepsilon)$ converges to $\gamma(u)S'(u)$ in $L^1(Q)$, and we use (3.92), (3.134) and we obtain that

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ converges to } f_1(x,t,u,v)S'(u) \text{ in } L^1(Q)$$

and

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ converges to } f_2(x,t,u,v)S'(v) \text{ in } L^1(Q).$$

As a consequence of the above convergence result, the position to pass to the limit as ε tends to 0 in equation (3.174) and (3.175), we conclude that (u,v) satisfies (3.85) and (3.86).
It remains to show that $S(u)$ satisfies the initial condition (3.87) and $S(v)$ satisfies the initial condition (3.88). To this end, firstly remark that, S being bounded, $(S(u^\varepsilon), S(v^\varepsilon))$

is bounded in $(L^\infty(Q))^2$, $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ is bounded in $L^\infty(Q) \times L^\infty(Q)$. Secondly, (3.174) and (3.175), the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^1(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p-)'}(]0,T[; W^{-1,p'(.)}(\Omega))$ and $\frac{\partial B_S^2(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^2(]0,T[; H_0^1(\Omega))$. As a consequence, an Aubin's type lemma ([39], Corollary 4) implies that $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ lies in a compact set of $(C(]0,T[; L^1(\Omega)))^2$. It follows that, on the one hand, $B_S^1(u^\varepsilon)(t=0)$ converges to $B_S^1(u)(t=0)$ strongly in $L^1(\Omega)$ and $B_S^2(v^\varepsilon)(t=0)$ converges to $B_S^2(v)(t=0)$ strongly in $L^1(\Omega)$. Due to (3.91), to conclude that (3.87) and (3.88) holds true. As a conclusion of **Step 2** and **Step 4**, the proof of Theorem (3.2) is complete.

□

Chapter 4

The exitence of Renormalized solution for quasilinear parabolic problems with variable exponents and mesure data

4.1 Introduction

The purpose of this chapter is to study the existence of renormalized solutions u to the quasi-linear parabolic problem involving the $p(x)$-Laplacian type operator

$$\begin{cases} u_t - \text{div}\mathcal{A}(x,t,\nabla u) + \mathcal{B}(u) = \mu & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ u(.,0) = u_0(.) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where Ω be a bounded-connected domain of $\mathbb{R}^N$, $(N \geq 2)$ with lipshitz boundary $\partial\Omega$ and $Q = \Omega \times]0,T[$ for any fixed $T > 0$. Let $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function, let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p^- \leq p^+ < N$. The operator $-\text{div}\mathcal{A}(x,t,\nabla u) = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary lions operator (see assumption (4.2)-(4.4)), and $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with $\mathcal{B}(u) = \lambda\,|u|^{p(x)-2}\,u$ is as continuous increasing function for $\lambda > 0$ and $\mathcal{B}(0) = 0$.

Variable-exponent Lebesgue and Sobolev spaces are the natural extensions of the classical constant exponent Lp-spaces. This kind of theory finds many applications, for example in nonlinear elastic mechanics, electrorheological fluids dynamics, and image restoration etc. We refer the readers to[29] .

In the classical case ($p(.) = 2$ or $p(.) = p$ (a constant)), we recall that the notion of renormalized solutions was introduced by Di Perna and Lions [22] in their study of the Boltzmann equation. It has been studied by many authors under various conditions on the data the existence and uniqueness of the renormalized solution for parabolic equations with measure data in the classical Sobolev spaces (see [11], [3],[43] and[37]). In Sobolev space with variable exponents, the authors in [7] have proved the existence of entropy and renormalized solutions for strongly

nonlinear elliptic equations in the framework of Sobolev spaces with variable exponents and in 2014, Chao zhang[45] provides the existence and uniqueness of entrpy solution for $p(x)$-Laplace equations with a Radon measure which is absolutely continuous with respect to the relative $p(x)$-capacity. The corresponding parabolic case equations in [42] have proved the existence of renormalized solutions for a class of nonlinear parabolic systems with variable exponents and, for the corresponding parabolic equations with L^1 data, the authors in [15] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponents and L^1 data. Chao Zhang and Shulin Zhou in [46] proved the existence and uniqueness results renormalized solutions and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data.
The existence of renormalized solution for quasilinear parabolic problem with variable exponents and measure data of (4.1), prove in clasical Sobolev space [37] in case where $u = b(u)$, $b(u_0) \in L^1(\Omega)$ with $b : \mathbb{R} \to \mathbb{R}$ is a increasing C^1-function and $b(0) = 0$, for every μ is diffuse measure and also the purpose of this paper is to extend the results in [6] to the case of parabolic equations.
In the following section 4.2, is devoted to set the main assumption, to the definition of renormalized solutions of (4.1). Section 4.3 is prove that the formulation of renormalized solution does not depend on the decomposition of μ. Finally, to prove the main result of this paper (Theorem (4.1)), on the existence of a renormalized solution.

4.2 Main Assumptions And Definition of Renormalized Solution

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times]0, T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$

$$\mathcal{A}(x, t, \xi).\xi \geq \alpha \, |\xi|^{p(x)} , \tag{4.2}$$

$$|\mathcal{A}(x, t, \xi)| \leq \beta \left[b(x, t) + |\xi|^{p(x)-1} \right], \tag{4.3}$$

$$(\mathcal{A}(x, t, \xi) - \mathcal{A}(x, t, \eta)).(\xi - \eta) > 0, \tag{4.4}$$

$$\mu \in \mathcal{M}_0(Q), \tag{4.5}$$

$$u_0 \text{ is a measurable function in } \Omega, \text{ such that } u_0 \in L^1(\Omega). \tag{4.6}$$

Where $1 < p- \leq p+ < +\infty$, α, β are positives constants and b is a nonnegative function in $L^{p'(.)}(Q)$. And $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\mathcal{B}(0) = 0$.
The definition of a renormalized solution for Problem (4.1) can be stated as follows.

Definition 4.1. *Let $\mu \in M_0(Q)$ and $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$, let $u_0 \in L^1(\Omega)$, $(f; F; div(G); g)$ a decomposition of μ. A measurable function u defined on Q is a renormalized solution of problem* (4.1) *if :*

$$T_k(u - g) \in L^{p^-}(]0, T[; W_0^{1,p(.)}(\Omega)) \text{ for any } k > 0 \ , \ \mathcal{B}(u) \in L^1(Q) , \tag{4.7}$$

$$\text{and } v{=}u - g \in L^{\infty}\left(]0,T[;L^{1}\left(\Omega\right)\right) \cap L^{q^{-}}(]0,T[;W_{0}^{1,q(.)}(\Omega)), \tag{4.8}$$

for all continuous functions $q(x)$ on $\overline{\Omega}$ satisfying $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ for all $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int\limits_{\{n\le|u-g|\le n+1\}} |\nabla u|^{p(x)} dxdt = 0, \tag{4.9}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support on $\mathbb{R}$, so

$$\begin{aligned}
&(S(v))_t - div(\mathcal{A}(x,t,\nabla u)S'(v)) + S''(v)\mathcal{A}(x,t,\nabla u)\nabla v\\
&+\mathcal{B}(u)S'(v) = fS'(v) + FS'(v) + GS''(v)\nabla v\\
&-div(GS'(v)) \text{ in } \mathcal{D}'(Q),
\end{aligned} \tag{4.10}$$

$$S(v)(t=0) = S(u_0) \text{ in } \Omega. \tag{4.11}$$

The following are explained as shown below on definition (4.1).

Remark 4.1. *Note that, all terms in (4.10) are well defined. Indeed, let $k > 0$ such that $supp(S') \subset [K, K]$, let $S(u-g) = S(T_k(u-g)) \in L^{p^-}(]0,T[;W_0^{1;p(.)}(\Omega))$ and $\frac{\partial S(u-g)}{\partial t} \in \mathcal{D}'(Q)$. The term $S'(u-g)\mathcal{A}(x,t,\nabla u)$ identifes with $S'(T_k(v)+g)\mathcal{A}(x,t,\nabla(T_k(v)+g))$ a.e. in Q, where $v = u-g$ and $u = T_k(v)+g$ in $\{|u-g| \le k\}$, assumptions (4.3) imply that*

$$\begin{aligned}
&|S'(T_k(u-g))\mathcal{A}(x,t,\nabla u)|\\
&\le \beta \left\|S'\right\|_{L^\infty(\mathbb{R})} \left[b(x,t) + |\nabla(T_k(v)+g)|^{p(x)-1}\right] \ a.e \text{ in } Q.
\end{aligned} \tag{4.12}$$

Using (4.3) and (4.7), it follows that $S'(u-g)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. The term $S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)$ identifes with $S''(u-g)\mathcal{A}(x,t,\nabla(T_k(v)+g))\nabla T_k(u-g)$ and in view of (4.3), (4.7) and (4.12), to obtain $S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g) \in L^1(Q)$ and $S'(u-g)\mathcal{B}(u) \in L^1(Q)$. Finally $f\ S'(u-g)$ and $GS''(u-g)\nabla T_k(u-g)$ belongs to $L^1(Q)$ and $FS' \in (L^{p'(.)}(Q))^N$ and $GS'(u-g) \in (L^{p'(.)}(Q))^N$ in view of (4.7) and because S' is a bounded function on $\mathbb{R}$. Also $\frac{\partial S(u-g)}{\partial t} \in L^{(p^-)'}(]0,T[;W^{-1,p'(.)}(\Omega)) + L^1(Q)$ and $S(u-g) \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, which implies that $S(u-g) \in C(]0,T[;L^1(\Omega))$.

Let it first be proven that the formulation of renormalized solution does not depend on the decomposition of μ. This fact essentially relies on Lemma (1.2)

Proposition 4.1. *Let u be a renormalized solution of (4.1). Then u satisfies (4.7)-(4.11) for every decomposition $(f;F;div(G);g)$ of μ.*

Proof. Assume that u satisfies the conditions of Definition (4.1) for $(f;F;div(G);g)$, and let $(\widetilde{f};\widetilde{F};div(\widetilde{G});\widetilde{g})$ be a different decomposition of μ. Note that since, by Lemma (1.2), $g-\widetilde{g} \in C(]0,T[;L^1(\Omega))$ let u- $\widetilde{g} \in L^\infty(]0,T[;L^1(\Omega))$, hence it is almost everywhere finite. First of all to prove that $T_k(u-\widetilde{g}) \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ for every $k > 0$.

Let us introduce a sequence of increasing $C^{\infty}(\mathbb{R})$-functions S_n such that, for any $n \geq 1$

$$\begin{cases} S_n(r) = r \text{ if } |\mathrm{r}| \leq n, \\ \operatorname{supp}(S_n') \subset [-(n+1),(n+1)], \\ \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \end{cases} \tag{4.13}$$

To choose as test function $T_k(S_n(u-g)+g-\widetilde{g})$ in (4.10) and use Lemma (1.2), to obtain

$$A+B+D+E = F+I+H+M. \tag{4.14}$$

Where

$$\begin{aligned}
A &= \int_0^T \left\langle (S_n(u-g)+g-\widetilde{g})_t \,, T_k(S_n(u-g)+g-\widetilde{g})\right\rangle dt, \\
B &= \int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
D &= -\int_Q S_n''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
E &= \int_Q S_n'(u-g)\mathcal{B}(u)T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
F &= \int_Q \left((S_n'(u-g)-1)\,f+\widetilde{f}\right) T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
I &= \int_Q ((S_n'(u-g)-1)\,F+\widetilde{F})\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
H &= \int_Q \left((S_n'(u-g)-1)\,G+\widetilde{G}\right)\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\
M &= \int_Q S_n''(u-g)G\nabla(u-g)T_k(S_n(u-g)+g-\widetilde{g})dxdt.
\end{aligned}$$

By initial condition (4.11) and Lemma (1.2), to obtain

$$A = \int_\Omega \Theta_k(S_n(u-g))(T)dx - \int_\Omega \Theta_k(S_n(u_0))dx, \tag{4.15}$$

where $\Theta_k(r) = \int_0^r T_k(s)ds$ is a positive Lipschitz continuous function. Using (4.15) and the definition (4.13) of S_n, leads to

$$A \geq -k\int_\Omega |u_0|\,dx, \;\; \forall n \geq 1. \tag{4.16}$$

Let $E_k = \{(x,t) \in Q : |S_n(u-g) + g - \widetilde{g}| \leq k\}$, we have

$$\begin{aligned} B &= \int_{E_k} |S_n'(u-g)|^2 \mathcal{A}(x,t,\nabla u)\nabla u dxdt \qquad (4.17) \\ &- \int_{E_k} |S_n'(u-g)|^2 \mathcal{A}(x,t,\nabla u)\nabla g dxdt \\ &+ \int_{E_k} S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla(g-\widetilde{g})dxdt \\ & \qquad (4.18) \\ &= B_1 + B_2 + B_3, \end{aligned}$$

the properties of S_n, and because $0 < S_n' < 1$ let $(S_n'(s))^{p^-} \leq S_n'(s)$, $(S_n'(s))^{(p^-)'} \leq S_n'(s)$, $S_n'(s) \leq S_n'(s)^2 + \chi_{\{n \leq |s| \leq n+1\}}$, to obtain

$$\begin{aligned} B_1 &\geq \alpha \int_{E_k} |S_n'(u-g)|^{p-} |\nabla u|^{p(x)} dxdt \qquad (4.19) \\ &- \alpha \int_{\{n \leq |v| \leq n+1\}} |\nabla u|^{p(x)} dxdt. \end{aligned}$$

Using (4.2), (4.3), and Young's inequality, to deduce that

$$\begin{aligned} |B_2| + |B_3| &\leq \left[C\left(\frac{1}{\varepsilon}\right) \|c\|_{L^{p'(.)}}^{(p^-)'} + C\left(\frac{1}{\varepsilon}\right) \|\nabla g\|_{L^{p(.)}}^{p^-} \right. \qquad (4.20) \\ &\left. + C\|\nabla(g-\widetilde{g})\|_{L^{p(.)}}^{p^-} \right] + \frac{\alpha}{2}\int_{E_k} |S_n'(u-g)|^{p-} |\nabla u|^{p(x)} \\ &+ \int_{\{n \leq |u-g| \leq n+1\}} |\nabla u|^{p(x)} dxdt, \end{aligned}$$

and,

$$\begin{aligned} |D| + |M| &\leq C\left[\|b\|_{L^{p'(.)}}^{(p^-)'} + \|\nabla g\|_{L^{p(.)}}^{p^-} + \|G\|_{L^{p'(.)}}^{(p^-)'} \right] \qquad (4.21) \\ &+ C\int_{\{n \leq |u-g| \leq n+1\}} |\nabla u|^{p(x)} dxdt. \end{aligned}$$

Using (4.2) and Young's inequality, to obtain

$$|E|+|F|+|H|+|I| \leq C\left[\|\mathcal{B}(u)\|_{L^1}+\|f\|_{L^1}+\left\|\widetilde{f}\right\|_{L^1}\right. \tag{4.22}$$
$$+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}|G|^{p'(x)}\,dxdt+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}\left|\widetilde{G}\right|^{p'(x)}dxdt$$
$$+\int_{E_k}\frac{1}{p(x)(\frac{\alpha}{2}p'^{\frac{p'(x)}{p(x)}}}|\nabla g|^{p(x)}\,dxdt+\int_{E_k}\frac{1}{p(x)(\frac{\alpha}{2}p'^{\frac{p'(x)}{p(x)}}}|\nabla\widetilde{g}|^{p(x)}\,dxdt$$
$$+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}|F|^{p'(x)}\,dxdt+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}\left|\widetilde{F}\right|^{p'(x)}dxdt$$
$$\left.+\|u_0\|_{L^1}\right]+\alpha\int_{E_k}|S'_n(u-g)|^{p^-}\,|\nabla u|^{p(x)}\,dxdt$$
$$+C\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt.$$

Using (4.14) to (4.22), we deduce that

$$\alpha\int_{E_k}|S'_n(u-g)|^{p^-}\,|\nabla u|^{p(x)}\leq \tag{4.23}$$
$$C\left[\|\mathcal{B}(u)\|_{L^1}+\|f\|_{L^1}+\left\|\widetilde{f}\right\|_{L^1}+\|G\|_{L^{p'(.)}}^{(p^-)'}+\left\|\widetilde{G}\right\|_{L^{p'(.)}}^{(p^-)'}\right.$$
$$\left.+\|\nabla g\|_{L^{p(.)}}^{p^-}+\|\nabla\widetilde{g}\|_{L^{p(.)}}^{p^-}+\|F\|_{L^{p'(.)}}^{(p^-)'}+\left\|\widetilde{F}\right\|_{L^{p'(.)}}^{(p^-)'}+\|u_0\|_{L^1}\right]$$
$$+C\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt.$$

Using the properties of S_n and the fact that $\widetilde{g}$ belongs to $L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, the result that will be deduced from preceding inequality that, for all $n\geq 1$,

$$\int_Q \chi_{E_k}\,|\nabla(S_n(u-g))|^{p(x)}\,dxdt\leq C,$$

inequality (1.3) implies that

$$\int_0^T \chi_{E_k} \min\left\{\|\nabla(S_n(u-g))\|^{p-}_{L^{p(x)}(\Omega)}, \|\nabla(S_n(u-g))\|^{p+}_{L^{p(x)}(\Omega)}\right\}$$
$$\leq \int_Q \chi_{E_k} |\nabla(S_n(u-g))|^{p(x)}\, dxdt \leq C,$$

then is $S_n(u-g)$ is bounded in $L^{p-}(]0,T[;W_0^{1,p(.)}(\Omega))$. Since $\nabla(T_k(S_n(u-g)+g-\widetilde{g})) = \chi_{E_k}$ $\nabla(S_n(u-g)+g-\widetilde{g})$ and since g, $\widetilde{g}\in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, this implies that $v_n = T_k(S_n(u-g)+g-\widetilde{g})$ is bounded in $L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ and converges, up to a subsequence converge to v weakly in $L^{p-}(]0,T[;W_0^{1,p(.)}(\Omega))$, thus also in $\mathcal{D}'(Q)$; but $v_n \to T_k(u-\widetilde{g})$ a.e. in Q and is bounded by k, so that $v_n \to T_k(u-\widetilde{g})$ in $\mathcal{D}'(Q)$. Then $T_k(u-\widetilde{g}) = v \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, for all $k>0$. To prove that (4.9) holds true for $\widetilde{g}$. Using the admissible test function $\theta_h(S_n(u-g)+g-\widetilde{g})$ in (4.10) with $S=S_n$; $\theta_h(s) = T_{h+1}(s) - T_h(s)$, the coercive character (4.2) and the use of Young's inequality it possible to obtain that

$$\begin{aligned}
&\alpha\int_{F_n} |S_n'(u-g)|^2 |\nabla u|^{p(x)} dxdt \qquad (4.24)\\
\leq\ & Ck\int_Q \left(|\mathcal{B}(u)| + |f| + \left|\widetilde{f}\right|\right)\theta_h\left(S_n(u-g)+g-\widetilde{g}\right)dxdt\\
&+\int_\Omega \overline{\theta_h}\left(S_n(u_0)\right)dx + \int_{F_n}\left(\frac{p^+-1}{p^+}|F|^{p'(x)} + \frac{p^+-1}{p^+}\left|\widetilde{F}\right|^{p'(x)}\right.\\
&+\frac{p^+-1}{p^+}|G|^{p'(x)} + \frac{p^+-1}{p^+}\left|\widetilde{G}\right|^{p'(x)} + \frac{p^+-1}{p^+}|b|^{p'(x)}\\
&\left.+\frac{C_1}{p(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}|\nabla g|^{p(x)} + \frac{C_1}{p(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}|\nabla \widetilde{g}|^{p(x)}\right)\\
&+C\int_{\{n\leq|u-g|\leq n+1\}} |\nabla u|^{p(x)} dxdt + \omega(n).
\end{aligned}$$

Where $F_n = \{h \leq |S_n(u-g)+g-\widetilde{g}| \leq h+1\}$. Taking the limit as n tends to $+\infty$ in (4.24), using (4.9) and the convergence of χ_{F_n} to $\chi_{\{h\leq|u-g|\leq h+1\}}$ shows that for any $h>0$.

$$\begin{aligned}\alpha \int\limits_{\{h\leq|u-\widetilde{g}|\leq h+1\}} |\nabla u|^{p(x)}dxdt \leq \int\limits_{\{|u_0|>h\}} |u_0|\,dx \\ + \int\limits_{\{|u-\widetilde{g}|\geq h\}} C \left(|\mathcal{B}(u)|+|f|+\left|\widetilde{f}\right|\right) dxdt + \int\limits_{\{h\leq|u-\widetilde{g}|\leq h+1\}} C_2 \Big(|G|^{p'(x)} \\ + \left|\widetilde{G}\right|^{p'(x)} + |F|^{p'(x)} + \left|\widetilde{F}\right|^{p'(x)} + |b|^{p'(x)} \\ + |\nabla g|^{p(x)} + |\nabla \widetilde{g}|^{p(x)}\Big)\, dxdt,\end{aligned} \tag{4.25}$$

which yields, as h tends to infinity (recall that $u-\widetilde{g}$ is almost everywhere finite),

$$\lim_{h\to\infty} \int\limits_{\{h\leq|u-\widetilde{g}|\leq h+1\}} |\nabla u|^{p(x)}dxdt = 0. \tag{4.26}$$

In the following to prove that the renormalized equation (4.10) and the initial condition (4.11) hold with $\widetilde{g}$ as well. For every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support and let $\varphi\in C_c^\infty(Q)$, we chose $S'(S_n(u-g)+g-\widetilde{g})$ as test function in

(4.10) (with $S = S_n$ in (4.10) and to Lemma (1.2), the result is:

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g)+g-\widetilde{g})_t, S'(S_n(u-g)+g-\widetilde{g})\varphi\rangle \, dt \qquad (4.27)\\
&+\int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
&+\int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
&+\int_Q S_n''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&+\int_Q S_n'(u-g)\mathcal{B}(u)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
=\;&\int_Q \left((S_n'(u-g)-1)\,f+\widetilde{f}\right) S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,F+\widetilde{F}\right) \nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,F+\widetilde{F}\right) \nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,G+\widetilde{G}\right) S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,G+\widetilde{G}\right) \nabla\left(S'(S_n(u-g)+g-\widetilde{g})\right)\varphi dxdt\\
&+\int_Q S_n''(u-g)G\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt.
\end{aligned}$$

In what follows to pass to the limit as n tends to ∞ in each term of (4.27). In what follows $\omega(n)$ stands for any quantity that vanishes as n diverges. For the parabolic contribution in

(4.27)

$$\int_0^T \langle (S_n(u-g)+g-\widetilde{g})_t \,, S'\left(S_n(u-g)+g-\widetilde{g}\right)\varphi\rangle \, dt \tag{4.28}$$
$$= \int_0^T \langle (S\left(S_n(u-g)+g-\widetilde{g}\right))_t \,, \varphi\rangle \, dt$$
$$= -\int_Q S\left(S_n(u-g)+g-\widetilde{g}\right)\varphi_t dxdt = \int_0^T \langle (S(u-\widetilde{g}))_t \,, \varphi\rangle \, dt + \omega(n).$$

Recall that, since $\operatorname{supp}(S') \subset [-k,k]$ and, $\operatorname{supp}\left(S_n'(u-g)S'S_n(u-g)+g-\widetilde{g})\right) \subset \{|u-g| \leq n+1, |u-\widetilde{g}| \leq k\}$; then ∇u may be replaced by $\omega = \nabla\left(T_{k+1}\left(u-\widetilde{g}\right)+\widetilde{g}\right) \in \left(L^{p(\cdot)}\left(Q\right)\right)^N$ in all the terms of (4.27). Using the definition of S_n, ($S_n' \to 1$ and is bounded by1), we obtain

$$\int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt$$
$$= \int_Q S_n'(u-g)\mathcal{A}(x,t,\omega)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt$$
$$\underset{n\to+\infty}{\longrightarrow} \int_Q \mathcal{A}(x,t,\omega)\nabla\varphi S'(u-\widetilde{g})dxdt \tag{4.29}$$
$$= \int_Q \mathcal{A}(x,t,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt.$$

And,

$$\int_Q S_n'(u-g)\mathcal{A}(t,x,\nabla u)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt$$
$$= \int_Q S_n'(u-g)\mathcal{A}(x,t,\omega)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt$$
$$\underset{n\to+\infty}{\longrightarrow} \int_Q \mathcal{A}(x,t,\omega)\nabla(S'(u-\widetilde{g}))\varphi dxdt \tag{4.30}$$
$$= \int_Q \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt,$$

and,

$$\int_Q S_n'(u-g)\mathcal{B}(u)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_Q \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt, \tag{4.31}$$

the definition of S_n', ($S_n'' \to 0$) allows to obtain that

$$\begin{aligned} &\int_Q S_n''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \\ = &\int_Q S_n''(u-g)\mathcal{A}(x,t,\omega)\nabla\left(T_{k+1}\left(u-\widetilde{g}\right)+\widetilde{g}-g\right) \\ &S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} 0, \end{aligned} \tag{4.32}$$

repeating the arguments that lead to (4.29), (4.30), (4.31) and (4.32), we obtain

$$\int_Q \left(\left(S_n'(u-g)-1\right)f+\widetilde{f}\right)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_Q \widetilde{f}S'(u-\widetilde{g})\varphi dxdt, \tag{4.33}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)F+\widetilde{F}\right)S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_Q \widetilde{F}S'(u-\widetilde{g})\nabla\varphi dxdt, \tag{4.34}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)F+\widetilde{F}\right)\nabla\left(S'(S_n(u-g)+g-\widetilde{g})\right)\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_Q \widetilde{F}\nabla\left(S'(u-\widetilde{g})\right)\varphi dxdt, \tag{4.35}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)G+\widetilde{G}\right)S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_Q \widetilde{G}S'(u-\widetilde{g})\nabla\varphi dxdt, \tag{4.36}$$

$$\int\limits_Q \left(\left(S_n'(u-g)-1\right) G+\widetilde{G}\right) \nabla \left(S'(S_n(u-g)+g-\widetilde{g})\right) \varphi dxdt$$

$$\underset{n\to+\infty}{\longrightarrow} \int\limits_Q \widetilde{G}\nabla \left(S'(u-\widetilde{g})\right) \varphi dxdt, \tag{4.37}$$

$$\int\limits_Q S_n''(u-g)G\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.38}$$

As a consequence of the above convergence results in a position to pass to the limit as n tends to $+\infty$ in (4.27) and to conclude that u satisfies (4.10) (with g instead of $\widetilde{g}$). It remains to show that $S(u-\widetilde{g})$ satisfies the initial condition (4.11). To this end, for $\psi \in C_0^\infty(\Omega)$ we take $\varphi = (T-t)\psi$ in (4.27), it possible to obtain

$$\lim_{n\to+\infty} \int\limits_0^T \langle (S_n(u-g)+g-\widetilde{g})_t, S'(S_n(u-g)+g-\widetilde{g})\varphi\rangle \, dt \tag{4.39}$$

$$\begin{aligned}
&+\int\limits_Q \mathcal{A}(t,x,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt + \int\limits_Q \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt \\
&+\int\limits_Q \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt = \int\limits_Q fS'(u-\widetilde{g})\varphi dxdt + \int\limits_Q G\nabla\varphi S'(u-\widetilde{g})dxdt \\
&+\int\limits_Q G\varphi\nabla\left(S'(u-\widetilde{g})\right) dxdt + \int\limits_Q F\nabla\varphi S'(u-\widetilde{g})dxdt \\
&+\int\limits_Q F\varphi\nabla\left(S'(u-\widetilde{g})\right) dxdt.
\end{aligned}$$

As far as the parabolic contribution in (4.39) is concerned, since $\varphi(0)\neq 0$, $S_n(u-g)=S_n(u_0)$ and $(g-\widetilde{g})(0)=0$ and the integration by parts

$$\int\limits_0^T \left\langle (S_n(u-g)+g-\widetilde{g})_t\, , S'\left(S_n(u-g)+g-\widetilde{g}\right)\varphi\right\rangle dt \tag{4.40}$$

$$\begin{aligned}
&= \int\limits_0^T \left\langle (S'S_n(u-g)+g-\widetilde{g})_t\, , \varphi\right\rangle dt = -\int\limits_\Omega S\left(S_n\left(u_0\right)\right)\varphi\left(0\right) dx \\
&- \int\limits_Q S\left(S_n(u-g)+g-\widetilde{g}\right)\varphi_t dxdt = -\int\limits_\Omega S\left(u_0\right)\varphi\left(0\right) dx \\
&- \int\limits_Q S\left(u-\widetilde{g}\right)\varphi_t dxdt + \omega\left(n\right).
\end{aligned}$$

Secondly, we use φ as test function in (4.10) (with $\widetilde{g}$), then leads to

$$\begin{aligned}
&-\int_{\Omega} S(u-\widetilde{g})(0)\,dx \\
&-\int_{Q} S(u-\widetilde{g})\,\varphi_t dxdt + \int_{Q} \mathcal{A}(x,t,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt \\
&+\int_{Q} \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt + \int_{Q} \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt \\
=\ &\int_{Q} fS'(u-\widetilde{g})\varphi dxdt + \int_{Q} G\nabla\varphi S'(u-\widetilde{g})dxdt \\
&+\int_{Q} G\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt \\
&+\int_{Q} F\nabla\varphi S'(u-\widetilde{g})dxdt + \int_{Q} F\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt.
\end{aligned} \tag{4.41}$$

From (4.39), (4.40) and (4.41) the conclusion is that $\int_{\Omega} S(u-\widetilde{g})(0)\,\psi dx = \int_{\Omega} S(u_0)\,\psi dx$; for all $\psi \in C_0^\infty(\Omega)$, then $S(u-\widetilde{g})(0) = S(u_0)$ in Ω.

□

4.3 The Existence of Result

This section is devoted to establish the existence of a renormalized solution .

Theorem 4.1. *Under assumptions* (4.2)-(4.6) *there exists at least a renormalized solution* u *of Problem* (4.1).

Proof. (of Theorem (4.1)) The proof is divided into 6 steps. In **Step 1**, we introduce an approximate problem. **Step 2** is devoted to establish a few a priori estimates. In **Step 3**, the limit u of the approximate solutions u^ε is introduced and $u-g$ is shown to belongs to $L^\infty(]0,T[;L^1(\Omega))$ and to satisfy (4.7)-(4.8). In **Step 4**, the definition of a time regularization of the field $T_k(u)$ and to establish Lemma (4.1), which allows to control the parabolic contribution that arises in the monotonicity method when passing to the limit. **Step 5** is devoted to prove an energy estimate (Lemma (4.2)). At last, **Step 6** is devoted to prove that u satisfies (4.9)-(4.11) of Definition (4.1) .

- **Step 1.** Let us introduce the following regularization of the data: for $\varepsilon > 0$ fixed

$$u_0^\varepsilon \text{ are a sequence of } C_c^\infty(\Omega) \text{ functions such that} \tag{4.42}$$

$$u_0^\varepsilon \to u_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0.$$

In view of Proposition (1.3), we can find

$$\begin{aligned} &\mu^{\varepsilon} \in C_c^{\infty}(\Omega) : \|\mu^{\varepsilon}\|_{L^1(Q)} \leq C \text{ and} \\ &\mu^{\varepsilon} = f^{\varepsilon} + F^{\varepsilon} - div(G^{\varepsilon}) + (g^{\varepsilon})_t \, , \end{aligned} \tag{4.43}$$

and such that

$$f^{\varepsilon} \in C_c^{\infty}(\Omega) : f^{\varepsilon} \to f \text{ in } L^1(Q) \text{ as } \varepsilon \text{ tends to } 0, \tag{4.44}$$

$$F^{\varepsilon} \in C_c^{\infty}(\Omega) : F^{\varepsilon} \to F \text{ in } \Big(L^{p'(.)}(Q) \Big)^N \text{ as } \varepsilon \text{ tends to } 0, \tag{4.45}$$

$$G^{\varepsilon} \in C_c^{\infty}(\Omega) : G^{\varepsilon} \to G \text{ in } \Big(L^{p'(.)}(Q) \Big)^N \text{ as } \varepsilon \text{ tends to } 0, \tag{4.46}$$

$$g^{\varepsilon} \in C_c^{\infty}(\Omega) : g^{\varepsilon} \to g \text{ in } L^{p-}(]0,T[;V) \text{ as } \varepsilon \text{ tends to } 0 \;, \tag{4.47}$$

Let us now consider the following regularized problem

$$\begin{aligned} &(u^{\varepsilon})_t - div\mathcal{A}(x,t,\nabla u^{\varepsilon}) + \mathcal{B}(u^{\varepsilon}) \\ &= \mu^{\varepsilon} = f^{\varepsilon} + F^{\varepsilon} - div(G^{\varepsilon}) + (g^{\varepsilon})_t \text{ in } Q, \end{aligned} \tag{4.48}$$

$$u^{\varepsilon} = 0 \text{ on }]0,T[\times \partial\Omega, \tag{4.49}$$

$$u^{\varepsilon}(t=0) = u_0^{\varepsilon} \text{ in } \Omega. \tag{4.50}$$

As a consequence, proving existence of a weak solution $u^{\varepsilon} \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ of (4.48)-(4.50) is an easy task (see [30]).

- **Step 2** Using $T_k(u^{\varepsilon} - g^{\varepsilon})$ as a test function in (4.48) leads to

$$\begin{aligned} &\int_{\Omega} \overline{T}_k(u^{\varepsilon} - g^{\varepsilon})(t)dx + \int_0^t \int_{\Omega} \mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla T_k(u^{\varepsilon} - g^{\varepsilon})dxds \\ &+ \int_0^t \int_{\Omega} \mathcal{B}(u^{\varepsilon})\, T_k(u^{\varepsilon} - g^{\varepsilon})dxds = \int_0^t \int_{\Omega} f^{\varepsilon} T_k(u^{\varepsilon} - g^{\varepsilon})dxds \\ &+ \int_0^t \int_{\Omega} F^{\varepsilon}\nabla T_k(u^{\varepsilon} - g^{\varepsilon})dxds \int_0^t \int_{\Omega} G^{\varepsilon}\nabla T_k(u^{\varepsilon} - g^{\varepsilon})dxds \\ &+ \int_{\Omega} \overline{T}_k(u_0^{\varepsilon})dx, \end{aligned} \tag{4.51}$$

for almost every t in $(0,T)$, and where $\overline{T}_k(r) = \int_0^r T_k(s)ds$. Using assumptions (4.2)-(4.3) and the definition of $\overline{T}_k(r)$ in (4.51), to obtain

$$\int_{\Omega} \overline{T}_k(u^\varepsilon - g^\varepsilon)(t)dx + \alpha \int_{E_k} |\nabla u^\varepsilon|^{p(x)}\, dxds \tag{4.52}$$
$$\leq k\,\|\mu^\varepsilon\|_{L^1(Q)} + k\,\|\mathcal{B}(u^\varepsilon)\|_{L^1(Q)} + \beta \int_{E_k} b(t,x)\,|\nabla g^\varepsilon|\, dxdt$$
$$+\beta \int_{E_k} |\nabla u^\varepsilon|\,|\nabla g^\varepsilon|\, dxds + k\,\|u_0^\varepsilon\|_{L^1(Q)},$$

where $E_k = \{(x,t) \in Q : |u^\varepsilon - g^\varepsilon| \leq k\}$, using young's inequality, we get

$$\begin{aligned} &\int_{\Omega} \overline{T}_k(u^\varepsilon - g^\varepsilon)(t)dx + \left(\alpha - \beta\frac{p^+ - 1}{p^+}\right) \int_{E_k} |\nabla u^\varepsilon|^{p(x)}\, dxds \\ \leq\ & k\,\|\mu^\varepsilon\|_{L^1(Q)} + k\,\|\mathcal{B}(u^\varepsilon)\|_{L^1(Q)} + \beta\,\|b\|_{L^{p'(\cdot)}(Q)}\,\|\nabla g^\varepsilon\|_{L^{p(\cdot)}(Q)} \\ &+ \frac{\beta}{p^+}\int_{E_k} |\nabla g^\varepsilon|^{p(x)}\, dxdt + k\,\|u_0^\varepsilon\|_{L^1(Q)}. \end{aligned} \tag{4.53}$$

Also, to obtain

$$\begin{aligned} &k \int_{\{(x,t)\in Q: |u^\varepsilon - g^\varepsilon| > k\}} |\mathcal{B}(u^\varepsilon)|\, dxdt \leq k\,\|\mu^\varepsilon\|_{L^1(Q)} \\ &+\beta\,\|b\|_{L^{p'(\cdot)}(Q)}\,\|\nabla g^\varepsilon\|_{L^{p(\cdot)}(Q)} \\ &+\frac{\beta}{p^+}\int_{E_k} |\nabla g^\varepsilon|^{p(x)}\, dxdt + k\,\|u_0^\varepsilon\|_{L^1(Q)}. \end{aligned} \tag{4.54}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(u^\varepsilon - g^\varepsilon)$ as test function in (4.48). Reasoning as above, using that $\nabla T_{k,1}(s) = \nabla s \chi_{\{k \leq |s| \leq k+1\}}$ and appling young's inequality, we obtain

$$\begin{aligned} &\int_{\{k \leq |u^\varepsilon - g^\varepsilon| \leq k+1\}} |\nabla(u^\varepsilon - g^\varepsilon)|^{p(x)}\, dxdt \\ &\leq Ck \int_{|u_0^\varepsilon| > k} |u_0^\varepsilon|\, dx + Ck \int_{|u^\varepsilon - g^\varepsilon| > k} |\mathcal{B}(u^\varepsilon)|\, dxdt \\ &+Ck \int_{|u^\varepsilon - g^\varepsilon| > k} |f^\varepsilon|\, dxdt + C(\int_{|u^\varepsilon - g^\varepsilon| > k} |F^\varepsilon|^{p'(x)}\, dxdt \\ &+ \int_{|u^\varepsilon - g^\varepsilon| > k} |G^\varepsilon|^{p'(x)}\, dxdt) \leq C, \end{aligned}$$

inequality (1.3) implies that

$$\int_0^T \chi_{\{k\leq|u^\varepsilon-g^\varepsilon|\leq k+1\}} \min\left\{\|\nabla(u^\varepsilon-g^\varepsilon)\|^{p-}_{L^{p(x)}(\Omega)}, \|\nabla(u^\varepsilon-g^\varepsilon)\|^{p+}_{L^{p(x)}(\Omega)}\right\}$$
$$\leq \int_{\{k\leq|u^\varepsilon-g^\varepsilon|\leq k+1\}} |\nabla(u^\varepsilon-g^\varepsilon)|^{p(x)}\, dxdt \leq C. \tag{4.55}$$

From the estimation (4.53), (4.55) and the properites of $\overline{T}_k$ and u_0^ε, we have

$$u^\varepsilon - g^\varepsilon \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \tag{4.56}$$

and

$$u^\varepsilon - g^\varepsilon \text{ is bounded in } L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega)), \tag{4.57}$$

by Lemma 2.1 in [15] and by (4.53), (4.55) and si $2-\dfrac{1}{N+1}<p(.)<N$, we obtain

$$v^\varepsilon = u^\varepsilon - g^\varepsilon \text{ is bounded in } L^{q-}(]0,T[\,;W_0^{1,q(x)}(\Omega)), \tag{4.58}$$

for all continuous variable exponents $q\in C(\overline{\Omega})$ satisfying

$$1\leq q(x) < \frac{N(p(x)-1)+p(x)}{N+1}$$

, for all $x\in\Omega$. And

$$T_k\left(u^\varepsilon-g^\varepsilon\right) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{4.59}$$

and by (4.54), to obtain

$$\mathcal{B}(u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{4.60}$$

independently of ε.
Proceeding as in [12], [13] that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S'\subset[-k,k]$)

$$S\left(u^\varepsilon-g^\varepsilon\right) \text{ is bounded in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{4.61}$$

and

$$\left(S\left(u^\varepsilon-g^\varepsilon\right)\right)_t \text{ is bounded in } L^1(Q)+L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right). \tag{4.62}$$

In fact, as a consequence of (4.59), by Stampacchia's Theorem, we obtain (4.61). To show that (4.62) holds true, we multiply the equation (4.48) by $S'(u^\varepsilon-g^\varepsilon)$ to obtain

$$\begin{aligned}\left(S\left(u^\varepsilon-g^\varepsilon\right)\right)_t &= div(S'\left(u^\varepsilon-g^\varepsilon\right)\mathcal{A}(x,t,\nabla u^\varepsilon))\\ &-\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla\left(S'\left(u^\varepsilon-g^\varepsilon\right)\right)\\ &-\mathcal{B}\left(u^\varepsilon\right)S'\left(u^\varepsilon-g^\varepsilon\right)+f^\varepsilon S'\left(u^\varepsilon-g^\varepsilon\right)\\ &+F^\varepsilon S'\left(u^\varepsilon-g^\varepsilon\right)-div\left(G^\varepsilon S'\left(u^\varepsilon-g^\varepsilon\right)\right)\\ &+G^\varepsilon\nabla\left(S\left(u^\varepsilon-g^\varepsilon\right)\right) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{4.63}$$

Since supp(S') and supp(S'') are both included in $[-k;k]$; u^{ε} may be replaced by $(T_k(v^{\varepsilon})+g^{\varepsilon})$ in $\{|u^{\varepsilon}-g^{\varepsilon}|\leq k\}$, where $v^{\varepsilon}=u^{\varepsilon}-g^{\varepsilon}$. To have

$$\begin{aligned}&|S'(u^{\varepsilon}-g^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})|\\&\leq\beta\|S'\|_{L^{\infty}}\left[b(x,t)+|T_k(v^{\varepsilon})+g^{\varepsilon}|^{p(x)-1}\right].\end{aligned}\tag{4.64}$$

As a consequence, each term in the right hand side of (4.63) is bounded either in $L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and obtain (4.62).

Now an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n\geq 1$, consider the Lipschitz continuous function θ_n defined through $\theta_n(s)=T_{n+1}(s)-T_n(s)$. Remark that $||\theta_n||_{L^{\infty}}\leq 1$ for any $n\geq 1$ and that $\theta_n(s)\to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^{\varepsilon}-g^{\varepsilon})$ in (4.48) leads to

$$\begin{aligned}&\int_{\Omega}\widetilde{\theta_n}(u^{\varepsilon}-g^{\varepsilon})(t)\,dx+\int_{Q}\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla(\theta_n(u^{\varepsilon}-g^{\varepsilon}))\,dxdt\\&+\int_{Q}\mathcal{B}(u^{\varepsilon})\,\theta_n(u^{\varepsilon}-g^{\varepsilon})dxdt\\&+\int_{Q}F^{\varepsilon}\nabla(\theta_n(u^{\varepsilon}-g^{\varepsilon}))\,dxdt\\&+\int_{Q}G^{\varepsilon}\nabla(\theta_n(u^{\varepsilon}-g^{\varepsilon}))\,dxdt\\&+\int_{\Omega}\widetilde{\theta_n}(u_0^{\varepsilon})\,dx,\end{aligned}\tag{4.65}$$

for almost every t in $]0,T[$ and where $\widetilde{\theta_n}(r)=\int_0^r\theta_n(s)ds\geq 0$. Hence, dropping a nonnegative term

$$
\begin{aligned}
&\int\limits_{\{n\leq|u^{\varepsilon}-g^{\varepsilon}|\leq n+1\}} \mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla u^{\varepsilon}dxdt \\
\leq\ & \int\limits_{\{n\leq|u^{\varepsilon}-g^{\varepsilon}|\leq n+1\}} \mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla g^{\varepsilon}dxdt - \int\limits_{Q}\mathcal{B}\left(u^{\varepsilon}\right)\theta_n(u^{\varepsilon}-g^{\varepsilon})dxdt \\
& +\int\limits_{Q} f^{\varepsilon}\theta_n(u^{\varepsilon}-g^{\varepsilon})dxdt + \int\limits_{Q} F^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt \\
& +\int\limits_{Q} G^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt + \int\limits_{\Omega}\widetilde{\theta_n}\left(u_0^{\varepsilon}\right)dx \\
\leq\ & \beta \int\limits_{\{n\leq|u^{\varepsilon}-g^{\varepsilon}|\leq n+1\}} \left[|b(x,t)|\,|\nabla g^{\varepsilon}| + |\nabla v^{\varepsilon}+\nabla g^{\varepsilon}|^{p(x)-1}|\nabla g^{\varepsilon}|\right]dxdt \\
& +\int\limits_{Q}|\mathcal{B}\left(u^{\varepsilon}\right)|\,\theta_n(u^{\varepsilon}-g^{\varepsilon})dxdt + \int\limits_{Q}|f^{\varepsilon}|\,\theta_n(u^{\varepsilon}-g^{\varepsilon})dx \\
& +\int\limits_{Q} F^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt + \int\limits_{Q} G^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt \\
& +\int\limits_{\Omega}\widetilde{\theta_n}\left(u_0^{\varepsilon}\right)dx.
\end{aligned}
\tag{4.66}
$$

Using assumption (4.43), (4.66) and applying Young's inequality, to obtain

$$
\begin{aligned}
&\frac{\alpha}{2}\int\limits_{\{n\leq|u^{\varepsilon}-g^{\varepsilon}|\leq n+1\}} |\nabla u^{\varepsilon}|^{p(x)}dxdt \\
&\leq C\left(\int\limits_{\{|u^{\varepsilon}-g^{\varepsilon}|\geq n\}} \frac{p^{+}-1}{p^{+}}|b|^{p'(x)}\right. \\
&\left. +\frac{C_1}{p(x)(\frac{\alpha}{2}p'(x))^{\frac{p(x)}{p'(x)}}}|\nabla g^{\varepsilon}|^{p(x)} + |G^{\varepsilon}|^{p'(x)} + |F^{\varepsilon}|^{p'(x)}\right)dxdt \\
&+\int\limits_{\{|u^{\varepsilon}-g^{\varepsilon}|\geq n\}}|\mathcal{B}\left(u^{\varepsilon}\right)|\,dxdt + \int\limits_{\{|u^{\varepsilon}-g^{\varepsilon}|\geq n\}}|f^{\varepsilon}|\,dxdt \\
&+\int\limits_{\left\{|u_0^{\varepsilon}|\geq n\right\}}|u_0^{\varepsilon}|\,dx.
\end{aligned}
\tag{4.67}
$$

- **Step 3** Arguing again as in [[12],[13],[14]] estimate (4.61) and (4.62) implies that, for a subsequence still indexed by ε,

$$u^{\varepsilon}-g^{\varepsilon} \text{ converges a.e where to } u-g \text{ in } Q, \tag{4.68}$$

using (4.48), (4.59) and (4.64), we get

$$u^{\varepsilon} \text{ converge almost every where to } u \text{ in } Q, \tag{4.69}$$

$$\begin{aligned} &T_k(u^{\varepsilon}-g^{\varepsilon}) \text{ converge weakly to } T_k(u-g) \\ &\text{in } L^{p-}\left(]0,T[\,,W_0^{1,p(.)}(\Omega)\right), \end{aligned} \tag{4.70}$$

$$\chi_{\{|u^{\varepsilon}-g^{\varepsilon}|\leq k\}}\mathcal{A}(t,x,\nabla u^{\varepsilon}) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N, \tag{4.71}$$

as ε tends to 0 for any $k>0$ and any $n\geq 1$ and where for any $k>0$, η_k belongs to $\left(L^{p'(.)}(Q)\right)^N$. Since $\mathcal{B}(u^{\varepsilon})$ is a continuous incrassing function, from the monotone convergence theorem, by (4.59) and (4.69), to obtain that

$$\mathcal{B}(u^{\varepsilon}) \text{ converge weakly to } \mathcal{B}(u) \text{ in } L^1(Q). \tag{4.72}$$

Now establish that $u-g$ belongs to $L^{\infty}(]0,T[\,;L^1(\Omega))$. Indeed using (4.53) and $\left|\overline{T}_k(s)\right|\geq |s|-1$ leads to

$$\begin{aligned} \int_{\Omega} |u^{\varepsilon}-g^{\varepsilon}|(t)dx \;\leq\; & meas(\Omega)+k\left\|\mu^{\varepsilon}\right\|_{L^1(Q)}+k\left\|\mathcal{B}(u^{\varepsilon})\right\|_{L^1(Q)} \\ & +\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|b\|_{L^{p'(.)}(Q)}\left\|\nabla g^{\varepsilon}\right\|_{L^{p(.)}(Q)} \\ & +\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\left\|\nabla g^{\varepsilon}\right\|_{L^{p(.)}(Q)}\left\|\nabla u^{\varepsilon}\right\|_{L^{p(.)}(Q)} \\ & +k\left\|u_0^{\varepsilon}\right\|_{L^1(\Omega)}. \end{aligned}$$

Using (4.68) and (4.42)-(4.47) , to have $u-g$ belongs to $L^{\infty}(]0,T[\,;L^1(\Omega))$. Now in a position to exploit (4.67). Since $u^{\varepsilon}-g^{\varepsilon}$ is bounded in $L^{\infty}(]0,T[\,;L^1(\Omega))$, we get

$$\lim_{n\to+\infty}\left(\sup_{\varepsilon}(meas\{|u^{\varepsilon}-g^{\varepsilon}|\geq n\})\right)=0, \tag{4.73}$$

using the equi-integrability of the sequences $|\nabla g^{\varepsilon}|^{p(x)}$, $|G^{\varepsilon}|^{p'(x)}$, $|F^{\varepsilon}|^{p'(x)}$, $|f^{\varepsilon}|$, $|\mathcal{B}(u^{\varepsilon})|$ and $|u_0^{\varepsilon}|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\left(\sup_{\varepsilon}\int_{\{n\leq|u^{\varepsilon}-g^{\varepsilon}|\leq n+1\}}|\nabla u^{\varepsilon}|^{p(x)}dxdt\right)=0. \tag{4.74}$$

- **Step 4** The specifie time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequaence in $L^\infty(\Omega) \cap W_0^{1,p(.)}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \leq k, \ \forall \mu > 0$, and $v_0^\mu \to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\|v_0^\mu\|_{L^{p(.)}(\Omega)} \to 0$ as $\mu \to +\infty$.

  For fixed $k \geq 0$ and $\mu > 0$, let us consider the unique solution $T_k(u)_\mu \in L^\infty(\Omega) \cap L^{p-}\left(]0,T[; W_0^{1,p(.)}(\Omega)\right)$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t} + \mu\left(T_k(u)_\mu - T_k(u)\right) = 0 \text{ in } \mathcal{D}'(Q), \tag{4.75}$$

$$T_k(u)_\mu(t=0) = v_0^\mu. \tag{4.76}$$

  The behavior of $T_k(u)_\mu$ as $\mu \to +\infty$ is investigated in [18] and we just recall here that (4.75)-(4.76) imply that

$$\begin{aligned} &T_k(u)_\mu \to T_k(u) \text{ strongly in } L^{p-}\left(]0,T[; W_0^{1,p(.)}(\Omega)\right) \\ &\text{a.e in, } Q \text{ as } \mu \to +\infty, \end{aligned} \tag{4.77}$$

  with $\|T_k(u)_\mu\|_{L^\infty(\Omega)} \leq k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t} \in L^{(p-)'}\left(]0,T[; W^{-1,p'(.)}(\Omega)\right)$.
  The main estimate is the following

  Lemma 4.1. *Let $v^\varepsilon = u^\varepsilon - g^\varepsilon$. Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r) = r$ for $r \leq k$, and $suppS'$ is compact. Then*

$$\liminf_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, S'(v^\varepsilon)\left(T_k(v^\varepsilon)_\mu - T_k(v)\right)\right\rangle dt \geq 0,$$

  where here $\langle .,. \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'(.)}(\Omega)$ and $L^\infty(\Omega) \cap V(\Omega)$.

  Proof. See[14], Lemma 1. □

- **Step 5** Here to prove that the weak limit η_k and to prove the weak L^1 convergence of the "truncted" energy $\mathcal{A}(x,t,\nabla T_k(v^\varepsilon))$ as ε tends to 0. In order to show this result recall the Lemma below.

  Lemma 4.2. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon\to 0} \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)dxdt \leq \int_Q \eta_k \nabla T_k(v)dxdt, \tag{4.78}$$

$$\lim_{\varepsilon\to 0} \int_Q \left[\mathcal{A}\left(x,t,\nabla u^\varepsilon_{\chi_{\{|v^\varepsilon|\leq k\}}}\right) - \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|v|\leq k\}}}\right)\right] \tag{4.79}$$

$$\times \left[\nabla u^{\varepsilon}_{\chi_{\{|v^{\varepsilon}|\leq k\}}} - \nabla u_{\chi_{\{|v|\leq k\}}}\right] dxdt = 0$$

$\eta_k = \mathcal{A}\left(x, t, \nabla u_{\chi_{\{|v|\leq k\}}}\right)$ *a.e in Q, for any $k \geq 0$, as ε tends to 0.*

$$\mathcal{A}\left(x,t,\nabla u^{\varepsilon}\right)\nabla T_k(v^{\varepsilon}) \to \mathcal{A}\left(x,t,\nabla u\right)\nabla T_k(v) \quad \text{weakly in } L^1(Q). \tag{4.80}$$

Proof. For $k \geq 0$, to consider the test function $S'_n(v^{\varepsilon})\left(T_k(u_{\varepsilon}) - (T_k(u))_{\mu}\right)$ in (4.48), and use the definition (4.13) of S'_n and we definie $W^{\varepsilon}_{\mu} = T_k(u_{\varepsilon}) - (T_k(u))_{\mu}$, to get

$$\begin{aligned}
&\int_0^T \left\langle (u^{\varepsilon} - g^{\varepsilon})_t, S'_n(v^{\varepsilon})W^{\varepsilon}_{\mu}\right\rangle dt + \int_Q S'_n(v^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla W^{\varepsilon}_{\mu} dxdt \\
&+ \int_Q S''_n(v^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla v^{\varepsilon} W^{\varepsilon}_{\mu} dxdt \\
+ \quad &\int_Q \mathcal{B}(u^{\varepsilon})S'_n(v^{\varepsilon})W^{\varepsilon}_{\mu} dxdt \\
&+ \int_Q F^{\varepsilon} S'_n(v^{\varepsilon})\nabla W^{\varepsilon}_{\mu} dxdt \\
&+ \int_Q F^{\varepsilon} S''_n(v^{\varepsilon})W^{\varepsilon}_{\mu}\nabla v^{\varepsilon} dxdt \\
&+ \int_Q G^{\varepsilon} S'_n(v^{\varepsilon})\nabla W^{\varepsilon}_{\mu} dxdt \\
&+ \int_Q G^{\varepsilon} S''_n(v^{\varepsilon})W^{\varepsilon}_{\mu}\nabla v^{\varepsilon} dxdt.
\end{aligned} \tag{4.81}$$

Now pass to the limit in (4.81) as $\varepsilon \to 0$, $\mu \to +\infty$, $n \to +\infty$ for k fixed real number. In order to perform this task, to prove below the following results for any $k \geq 0$

$$\liminf_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \left\langle (u^{\varepsilon} - g^{\varepsilon})_t, S'_n(v^{\varepsilon})W^{\varepsilon}_{\mu}\right\rangle dt \geq 0 \text{ for any } n \geq k, \tag{4.82}$$

$$\lim_{n\to+\infty} \lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q S''_n(v^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla v^{\varepsilon} W^{\varepsilon}_{\mu} dxdt = 0, \tag{4.83}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q \mathcal{B}(u^{\varepsilon})S'_n(v^{\varepsilon})W^{\varepsilon}_{\mu} dxdt = 0, \text{ for any } n \geq 1, \tag{4.84}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f^\varepsilon S_n'(v^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{4.85}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q F^\varepsilon S_n'(v^\varepsilon)\nabla W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{4.86}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q F^\varepsilon S_n''(v^\varepsilon)W_\mu^\varepsilon \nabla v^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{4.87}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q G^\varepsilon S_n'(v^\varepsilon)\nabla W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{4.88}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q G^\varepsilon S_n''(v^\varepsilon)W_\mu^\varepsilon \nabla v^\varepsilon dxdt = 0, \text{ for any } n \geq 1. \tag{4.89}$$

Proof of (4.82). In view of the definition W_μ^ε, we apply lemma (4.1) with $S = S_n$ for fixed $n \geq k$. As a consequence, (4.82) hold true. □

Proof of (4.83). For $n \geq 1$, we have $supp(S_n'') \subset [-(n+1), -n] \cup [n, n+1]$, $\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q)} \leq 2k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$. Using assumptions (4.3) and applying Young's inequality, we get

$$\begin{aligned}
&\left|\int_Q S_n''(v^\varepsilon)\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla v^\varepsilon W_\mu^\varepsilon dxdt\right| \qquad (4.90)\\
&\leq C_3 \int_{\{n\leq|v^\varepsilon|\leq n+1\}} \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon dxdt\\
&+C_4 \int_{\{n\leq|v^\varepsilon|\}} \left(|b(x,t)|^{p'(x)} + |\nabla g^\varepsilon|^{p(x)}\right) dxdt,
\end{aligned}$$

for $n \geq 1$, using assumptions (4.73)-(4.74) and the equi-integrability of the sequences $|\nabla g^\varepsilon|^{p(x)}$ in $L^1(Q)$, permits to pass to the limit as n tends to $+\infty$ in (4.90) and to etablish (4.83) □

Proof of (4.84). For $n \geq 1$ and in view (4.72). Lebesgue's convergence theorem implies that for any $\mu > 0$ and $n \geq 1$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_Q \mathcal{B}(u^\varepsilon)S_n'(v^\varepsilon)W_\mu^\varepsilon\ dxdt\\
&= \int_Q \mathcal{B}(u)S_n'(v)(T_k(v) - T_k(v)_\mu)dxdt. \qquad (4.91)
\end{aligned}$$

Appealing now to (4.77) and passing to the limit as $\mu \to +\infty$ in (4.91) allows to conclude that (4.84) holds true. □

Proof of (4.85). By (4.44), (4.68) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \int_Q f^\varepsilon S_n'(v^\varepsilon) W_\mu^\varepsilon \, dxdt = \int_Q f S_n'(v)(T_k(v) - T_k(v)_\mu) dxdt,$$

using (4.77) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (4.85). □

Proof of (4.86). By (4.45), we have

$$F^\varepsilon S_n'(v^\varepsilon) \to F S_n'(v) \text{ a.e. in } Q, \text{and}$$
$$|F^\varepsilon S_n'(v^\varepsilon)| \leq (n+1) \|F^\varepsilon\|_{L^{p'(\cdot)}(Q)} \text{ a.e. in } Q.$$

Let us recal the main properrties of W_μ^ε. For $\mu > 0$, W_μ^ε converges to $(T_k(v) - T_k(v)_\mu)$ weakly in $L^{p^-}(]0,T[; W_0^{1,p}(\Omega))$ as $\varepsilon \to 0$. Taking into account that

$$\|W_\mu^\varepsilon\|_{L^\infty(Q)} \leq 2k \text{ for any } \varepsilon > 0, \ \mu > 0, \tag{4.92}$$

to be able to deduce that

$$\begin{aligned} &W_\mu^\varepsilon \to (T_k(v) - T_k(v)_\mu) \qquad \text{a.e. in } Q \text{ and} \\ &L^\infty(Q) \text{ weakly-* as } \varepsilon \to 0, \end{aligned} \tag{4.93}$$

also to deduce that

$$\lim_{\varepsilon \to 0} \int_Q F^\varepsilon S_n'(v^\varepsilon) \nabla W_\mu^\varepsilon dxdt = \int_Q F S_n'(v) \nabla (T_k(v) - T_k(v)_\mu) dxdt, \tag{4.94}$$

and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[; W_0^{1,p}(\Omega))$ as $\mu \to +\infty$, as consequece (4.86) holds true. □

Proof of (4.87). For $n \geq 1$ and from (4.45), (4.70), it follows that

$$\begin{aligned} &\lim_{\mu \to +\infty} \lim_{\varepsilon \to 0} \int_Q F^\varepsilon S_n''(v^\varepsilon) \nabla v^\varepsilon W_\mu^\varepsilon dxdt \\ &= \lim_{\mu \to +\infty} \int_Q F S_n''(v) \nabla v (T_k(v) - T_k(v)_\mu) dxdt = 0. \end{aligned}$$

□

Proof of (4.88). Using (4.46) and (4.70) lead to $G^{\varepsilon}S'_n(v^{\varepsilon})$ tends to $GS'_n(v)$ strongly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as $\varepsilon \to 0$. We deduce that

$$\lim_{\varepsilon\to 0}\int_Q G^{\varepsilon}S'_n(v^{\varepsilon})\nabla W^{\varepsilon}_{\mu}dxdt = \int_Q GS'_n(v)\nabla(T_k(v) - T_k(v)_{\mu})dxdt,$$

for $\mu > 0$, by (4.93) and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$ allows to conclude (4.88). □

Proof of (4.89). From (4.46) and (4.70), it follows that

$$\lim_{\varepsilon\to 0}\int_Q G^{\varepsilon}\nabla S'_n(v^{\varepsilon})W^{\varepsilon}_{\mu}dxdt = \int_Q G\nabla S'_n(v)(T_k(v) - T_k(v)_{\mu})dxdt = 0, \text{ for } n \geq 1.$$

□

Now turn back to the proof of Lemma (4.2), due to (4.82)-(4.89), in a position to pass to the limit-sup when $\varepsilon \to 0$, then to the limit-sup when $\mu \to +\infty$ and to the limit as $n \to +\infty$ in (4.81). Using the definition of W^{ε}_{μ}, we deduce that for $k \geq 0$,

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^{\varepsilon})S'_n(v^{\varepsilon})\nabla\left(T_k(v^{\varepsilon}) - T_k(v)_{\mu}\right)dxdt \leq 0.$$

Since $\mathcal{A}(x,t,\nabla u^{\varepsilon})S'_n(v^{\varepsilon})\nabla T_k(v^{\varepsilon}) = \mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla T_k(v^{\varepsilon})$ for $k \leq n$, the above inequality implies that for $k \leq n$,

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla T_k(v^{\varepsilon})dxdt \leq \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^{\varepsilon})S'_n(v^{\varepsilon})\nabla T_k(v)_{\mu}dxdt. \tag{4.95}$$

Due to (4.71), $\mathcal{A}(x,t,\nabla u^{\varepsilon})S'_n(v^{\varepsilon}) \to \eta_{n+1}S'_n(v)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as $\varepsilon \to 0$ and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$, we get

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^{\varepsilon})S'_n(v^{\varepsilon})\nabla T_k(v)_{\mu}dxdt = \int_Q S'_n(v)\eta_{n+1}\nabla T_k(v)dxdt = \int_Q \eta_{n+1}\nabla T_k(v)dxdt, \tag{4.96}$$

for $k \leq n$, since $S'_n(s) = 1$ for $|s| \leq n$. Now for $k \leq n$, we have

$$S'_n(v^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|v^\varepsilon|\leq k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|v^\varepsilon|\leq k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon \to 0$, to obtain

$$\eta_{n+1}\chi_{\{|v|\leq k\}} = \eta_k\chi_{\{|v|\leq k\}} \text{ a.e in } Q - \{|v| = k\} \text{ for } k \leq n.$$

Recalling (4.95) and (4.96) allows to conclude that (4.78) holds true. □

Proof of (4.79). Let $k \geq 0$ be fixed. We use the monotone character (4.4) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, to obtain

$$I^\varepsilon = \int_Q \big(\mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}) - \mathcal{A}(x,t,\nabla u\chi_{\{|v|\leq k\}})\big) \big(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\big)\, dxdt \geq 0. \tag{4.97}$$

Inequality (4.97) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$\begin{aligned}
I_1^\varepsilon &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}})\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}dxdt,\\
I_2^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}})\nabla u\chi_{\{|v|\leq k\}}dxdt,\\
I_3^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|v|\leq k\}})\big(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\big)\, dxdt.
\end{aligned}$$

To pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that $v^\varepsilon = u^\varepsilon - g^\varepsilon$ and $\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} = \big(\nabla T_k(v^\varepsilon) - g^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}\big)$ a.e in Q, assume that k is such that $\chi_{\{|v^\varepsilon|\leq k\}}$ almost everywhere converges to $\chi_{\{|v|\leq k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [16]). Using (4.78), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)dxdt\\
&+ \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}})\nabla g^\varepsilon dxdt\\
&\leq \int_Q \eta_k\nabla T_k(v)dxdt + \int_Q \eta_k\nabla g\chi_{\{|v|\leq k\}}dxdt.
\end{aligned} \tag{4.98}$$

In view of (4.70) and (4.71),

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2^\varepsilon &= -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}})\,(\nabla T_k(v)+\nabla g)\,dxdt \\ &= -\int_Q \eta_k\,(\nabla T_k(v)+\nabla g)\,dxdt. \end{aligned} \tag{4.99}$$

As a consequence of (4.47) and (4.70), for all $k > 0$

$$\begin{aligned}\lim_{\varepsilon\to 0} I_3^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u \chi_{\{|v|\le k\}}) \\ &\quad \big(\nabla T_k(v^\varepsilon)+\nabla g^\varepsilon \chi_{\{|v^\varepsilon|\le k\}} - \nabla T_k(v)+\nabla g \chi_{\{|v|\le k\}}\big)\,dxdt = 0. \end{aligned} \tag{4.100}$$

Taking the limit-sup as $\varepsilon \to 0$ in (4.97) and using (4.98), (4.99) and (4.100) show that (4.79) holds true. □

Proof of (4.80). Using (4.79) and the usual Minty argument applies it follows that (4.80) holds true.

□

- **Step 6:** In this step to prove that u satisfies (4.9)-(4.11). To this end, remark that $v^\varepsilon = u^\varepsilon - g^\varepsilon$ and for fixed $n \le 0$ one has

$$\begin{aligned}&\int_{\{n\le |u^\varepsilon - g^\varepsilon|\le n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \\ &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(v^\varepsilon)dxdt - \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(v^\varepsilon)dxdt \\ &\quad + \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n+1\}}\nabla g^\varepsilon dxdt \\ &\quad - \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n\}}\nabla g^\varepsilon dxdt. \end{aligned}$$

According to (4.71) and (4.80) one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 1$, to obtain

$$\lim_{\varepsilon\to 0} \int_{\{n\leq |u^\varepsilon - g^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \tag{4.101}$$
$$= \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(v)dxdt - \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_n(v)dxdt$$
$$+\int_Q \mathcal{A}(x,t,\nabla u)\chi_{\{|v|\leq n+1\}}\nabla g dxdt$$
$$-\int_Q \mathcal{A}(x,t,\nabla u)\chi_{\{|v|\leq n\}}\nabla g dxdt$$
$$= \int_{\{n\leq |u^\varepsilon - g^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt.$$

Taking the limit as n tends to $+\infty$ in (4.101) and using the estimate (4.74) , that u satisfies (4.9). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\mathrm{supp}(S') \subset [-k,k]$. Pontwise multiplication of that approximate equation (4.48) by $S'(u^\varepsilon - g^\varepsilon)$ leads to

$$\begin{aligned} &(S(u^\varepsilon - g^\varepsilon))_t - div(S'(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) \\ &+S''(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon - g^\varepsilon) + \mathcal{B}(u^\varepsilon)S'(u^\varepsilon - g^\varepsilon) \\ &= f^\varepsilon S'(u^\varepsilon - g^\varepsilon) + F^\varepsilon S'(u^\varepsilon - g^\varepsilon) - div(S'(u^\varepsilon - g^\varepsilon)G^\varepsilon) \\ &+S''(u^\varepsilon - g^\varepsilon)G^\varepsilon\nabla(u^\varepsilon - g^\varepsilon) \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{4.102}$$

In what follows to pass to the limit as ε tends to 0 in each term of (4.102). Since S is bounded, and $S(u^\varepsilon - g^\varepsilon)$ converges to $S(u-g)$ a.e in Q and in $L^\infty(Q) * -$weak, then $(S(u^\varepsilon - g^\varepsilon))_t$ converges to $(S(u^\varepsilon - g^\varepsilon))_t$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\mathrm{supp}(S') \subset [-k,k]$, we have $S'(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon) = S'(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}}$ a.e in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (4.80) of Lemma(4.2) imply that $S'(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u-g)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as ε tends to 0, because $S'(u-g) = 0$ for $|u-g| \geq k$ a.e in Q. The pointwise convergence of $u^\varepsilon - g^\varepsilon$ to $u-g$, the bounded character of S', S'' and (4.80) of Lemma (4.2) allow to conclude that

$$\begin{aligned} &S''(u^\varepsilon - g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon - g^\varepsilon) \\ &\to S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla T_k(u-g) \text{ weakly in } L^1(Q) \end{aligned}$$

as $\varepsilon \to 0$. The use of (4.72) to obtain that $\mathcal{B}(u^\varepsilon)S'(u^\varepsilon - g^\varepsilon)$ converges to $\mathcal{B}(u)S'(u-g)$ in $L^1(Q)$, and we use (4.44), (4.45), (4.46), (4.47) and (4.70) and we obtain that $f^\varepsilon S'(u^\varepsilon - g^\varepsilon)$ converges to $fS'(u-g)$ in $L^1(Q)$, the term $F^\varepsilon S'(u^\varepsilon - g^\varepsilon)$ converges to $FS'(u-g)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ and the term $G^\varepsilon S'(u^\varepsilon - g^\varepsilon)$ converges to $GS'(u-g)$ strongly in $\left(L^{p'(\cdot)}(Q)\right)^N$

and $S''(u^\varepsilon - g^\varepsilon)G^\varepsilon\nabla(u^\varepsilon - g^\varepsilon)$ converges to $S''(u-g)G\nabla(u-g)$ weakly in $L^1(Q)$. As a consequence of the above convergence result, the position to pass to the limit as ε tends to 0 in equation (4.102) and to conclude that u satisfies (4.10). It remains to show that $S(u-g)$ satisfies the initial condition (4.11). To this end, firstly remark that, S being bounded, $S(u^\varepsilon - g^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (4.102) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u^\varepsilon - g^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p-)'}(]0,T[;W^{-1,p'(.)}(\Omega))$. As a consequence, an Aubin's type Lemma ([**?**], Corollary 4) implies that $S(u^\varepsilon - g^\varepsilon)$ lies in a compact set of $C^0(]0,T[;L^1(\Omega))$. It follows that, on one hand, $S(u^\varepsilon - g^\varepsilon)(t=0)$ converges to $S(u-g)(t=0)$ strongly in $L^1(\Omega)$. Due to (4.42), to conclude that (4.11) holds true. As a conclusion of **Step 3** and **Step 6**, the proof of Theorem (4.1) is complete.

□

Conclusion and Perspectives

In this thesis, we have studied the existence of solutions for problems of the parabolic type with boundary conditions of the Dirichlet type involving not very regular data (L^1 or measured).
For these problems, the need to work in Lebesgue and Sobolev spaces with variable exponents is motivated by the appearance of these spaces in the modeling of electrorheological fluids and in the image restoration [18]. During the thesis I used the notion of renormalized solution which is adapted to the treatment of elliptical and parabolic partial differential equations with L^1 data and measure.
Note that in the literature, very few parabolic problems have been processed in the variable exponent space to include the fractional case and L^1 data. It seems that this difficulty is related to the understanding and definition suitable functional spaces. Consequently, interesting questions open up avenues for research in this section.
Perspectives: In the near future, i want to focus on the numerical part and application of image processing.

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