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Thesis

A view to obtaining the diploma of

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Option: ***Numerical analysis, PDEs and applications***

Applications of the fixed point technique to problems generated by nonlinear functional differential equations

Presented by:

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Abstract in English

Iterative functional differential equations have a long history-more than a century, during which numerous instances have demonstrated the promising applications of these equations in multiple fields such as medicine, biology, epidemiology, and classical electrodynamics, etc.

The main aim of this dissertation is to contribute to the emerging literature of this type of equations via the study of three classes of second order iterative functional differential equations in *Banach* spaces. We establish some existence and uniqueness results and the continuous dependence on parameters of the unique solution by virtue of a powerful technique that combines the fixed point theory and the *Green*'s function method. More precisely, it is based on the transformation of the considered problem into a fixed point problem by converting it into an equivalent integral equation whose kernel is a *Green*'s function before applying the *Banach* and *Schauder* fixed point theorems.

Keywords: Continuous dependence, existence, fixed point theorem, iterative differential equation, *Green*'s function, uniqueness.

Abstract in French

Les équations différentielles fonctionnelles itératives ont une longue histoire de plus d'un siècle, durant lequel de nombreux exemples ont démontré les applications prometteuses de ces équations dans de multiples domaines tels que la médecine, la biologie, l'épidémiologie et l'électrodynamique classique, etc.

L'objectif principal de cette thèse est de contribuer à la littérature émergente de ce type d'équations via l'étude de trois classes d'équations différentielles fonctionnelles itératives du second ordre dans des espaces de *Banach*. Nous établissons certain résultats d'existence et d'unicité ainsi que la dépendance continue de la solution unique par rapport aux paramètres en vertu d'une technique puissante qui combine la théorie du point fixe et la méthode des fonctions de *Green*. Plus précisément, elle repose sur la transformation du problème considéré en un problème de point fixe en le convertissant en une équation intégrale équivalente dont le noyau est une fonction de *Green* avant d'appliquer les théorèmes de point fixe de *Banach* ou *Schauder*.

Mots-clés: Dépendance continue, existence, théorème du point fixe, équation différentielle itérative, fonction de Green, unicité.

Abstract in Arabic

المعادلات التفاضلية الدالية التكرارية لها تاريخ طويل يمتد لأكثر من قرن، أين أثبتت أمثلة عديدة تطبيقاتها الواعدة في مجالات متنوعة، مثل الطب، علم الأحياء، علم الأوبئة والديناميكا الكهربائية ، إلخ.

الهدف الرئيسي من هذه الأطروحة هو المساهمة في الأدبيات الناشئة لهذا النوع من المعادلات من خلال دراسة ثلاث فئات من المعادلات التفاضلية الدالية التكرارية من الدرجة الثانية في فضاءات بناخ.

نقوم بإثبات بعض نتائج الوجود والوحدانية بالإضافة إلى الارتباط المستمر للحل الوحيد على المعاملات باستعمال تقنية فعالة تمزج بين نظري النقطة الثابتة و طريقة دوال غرين، بتعبير أدق، ترتكز هذه التقنية على تحويل المسألة المطروحة إلى مسألة نقطة ثابتة وذلك بتحويلها إلى معادلة تكاملية مكافئة نواتها دالة غرين قبل تطبيق نظريات النقطة الثابتة لبناخ أو شودر.

الكلمات المفتاحية. الارتباط المستمر، وجود، نظرية النقطة الصامدة، المعادلات التفاضلية التكرارية، دالة غرين، وحدانية.

Dedication

I dedicate this work

- To every researcher who loves science, he illuminated the mind of others with his knowledge.

- To those who are unmatched by anybody in the universe and to whom God has instructed us to show homage, who have lavished me with supplications, and happiness, to whom I owe a great debt of gratitude to my dear mother and father.

- To my dear sister, owner of the words that helped me achieve success.

- To those who supported me in my difficult times, my brothers and my brother-in-law.

- To all my friends and family.

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I cannot find words to express all my gratitude to the jury for their reading of the manuscript and also for their precious time and efforts dedicated to evaluating my work. I sincerely thank the president for the honor that he gave me by chairing the jury. I also owe a special thanks to the members of the committee who honored me to be examiners of my work.

My heartfelt thanks to my family for their moral support and caring they provided, with special thanks to professors, staff and fellows at the department of mathematics of 20 August 1955, university of *Skikda* who have rendered their aid during the preparation of this thesis, and all those who have helped me throughout the years of my studies.

Acronyms

Abbreviation	**Meaning**
FDE	Functional differential equation
DDE	Delayed differential equation
ADE	Advanced differential equation
BVP	Boundary value problem
TPBVP	Two-point boundary value problem
a.e.	Almost everywhere

List of symbols

Sets and numbers

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers

$\mathbb{R}^n$: n-dimensional real Euclidean space

$[a,b]$: the interval of numbers between a and b, including a and b

(a,b) : an open interval

$[a,+\infty)$: left-closed and right–unbounded interval

$\mathcal{C}(\mathbb{X}) := \mathcal{C}(\mathbb{X},\mathbb{X})$ is the space of continuous functions from $\mathbb{X}$ into itself

$\mathcal{C}^1(\mathbb{R}^{n+1},\mathbb{R})$: space of continuously differentiable functions from $\mathbb{R}^{n+1}$ into $\mathbb{R}$

$\mathcal{C}^2([-b,b],\mathbb{R})$: space of twice-continuously differentiable functions from $[-b,b]$ into $\mathbb{R}$

T : a period

P_T : the *Banach* space of all continuous and $T-$periodic functions

$P_T(L,M) := \{x \in P_T : \|x\| \le L,\ |x(t_2) - x(t_1)| \le M|t_2 - t_1|,\ \forall t_1,t_2 \in [0,T]\}$

$P_T(L) = \{x \in P_T : \|x\| \le L\}$

$P_T\left(L_{C_{\ell,m}}\right) = \left\{x \in P_T : \|x\| \le L_{C_{\ell,m}}\right\}$

$\mathcal{CB}_{Int} := \left\{x \in \mathcal{C}([0,b],\mathbb{R}) : x(0) = 0,\ \nu\int_0^\eta x(s)\,ds = x(b),\ \nu \in \mathbb{R}^*,\ \eta \in (0,b)\right\}$

$$\mathcal{CB}_{Int}(L_0,M_0) := \{x \in \mathcal{CB}_{Int} : 0 \le x(t) \le L_0,$$
$$|x(t_2) - x(t_1)| \le M_0|t_2 - t_1|,\ \forall t_1,t_2 \in [0,b]\}$$

$\mathcal{CB}(\alpha,\beta) := \{x \in \mathcal{X} : -\alpha \le x(t) \le \alpha,\ |x(t_2) - x(t_1)| \le \beta|t_2 - t_1|,\ \forall t_1,t_2 \in [-b,b]\}$

Functions

List of Symbols

$|\cdot|$: absolute value

$\|.\|_E$: a norm on E

$\|f\|_\infty$: the uniform norm defined by $\sup|f(x)|$

$x^{[n]}(t)$: the composition of the function $x(t)$ with itself n times or the n^{th} iterate of the function $x(t)$

$\sum\limits_{m=1}^{n}$: the summation from index $m = 1$ to $m = n$

$\lim\limits_{x \longrightarrow x_0}$: limit as x approaches x_0

$\approx$: approximately equal to

$x'(t) := \dfrac{dx(t)}{dt}$: the first derivative of the function $x(t)$ with respect to t

$x''(t) := \dfrac{d^2x(t)}{dt^2}$: the second derivative of the function $x(t)$ with respect to t

sup : the supremum

max : the maximum

min : the minimum

$\exp x$: the exponential function of x

$G(t,x)$: the *Green's* function

Other notations will be explained when they first appear.

CHAPTER 1

General introduction

1.1 Introduction

Functional differential equations (FDEs) which somehow include deviating arguments whether delayed (DDEs) or advanced (ADEs) ones have a long history of playing a crucial role in describing a broad range of phenomena that appear almost in every facet of our lives. They were and still are a very popular subject and an integral part of the vocabulary of several scholars working in various fields, which in turn has led to the appearance of lot of monographs and research papers on this topic, especially within the last few decades.

State-dependent delay differential equations where the delays depend on the state variable itself can date back to *Poisson* [54] in 1806. They have gained a great prominence because they can give more useful and realistic results when using them to model many phenomena appearing in different scientific disciplines such as electrodynamics, biological sciences, epidemiology and medicine, etc. One of the pioneering works is the two-body problem in classical electrodynamics but unfortunately this kind of equations has remained a mathematical terra incognita until the appearance of the *Driver*'s works [24, 25, 26, 27, 28] in the second half of the last century in which he shed light on the importance of considering delays that depend on the state variable in gaining a more realistic modeling. In the following century, these equations have received considerable attention of several researchers at that time which in turn led to a fairly good amount of works in this direction.

An important type of functional differential equations that fascinates many authors as they appear widely in describing various practical phenomena is the iterative functional differential equation which involves derivatives and iterates of the unknown function. Equations of such kind have found many useful applications in a variety of vital phenomena including biological, epidemiological and ecological phenomena. They can manifest themselves in many ways. The base scenario is that they can be seen as a special type of functional differential equations with deviating arguments depending on both the time and the state variable.

Unfortunately, the study of iterative functional differential equations constitutes a challenging task since equations of this type have distinctive characteristics, making them hard to investigate. Indeed, they are quite different from the usual differential equations which hinders the application of the standard existence and uniqueness theorems.

Several approaches have been used to deal with this type of equations such as the *Picard* successive approximation method, the technique of nonexpansive operators and the fixed point theory (see [7, 29, 64, 65, 66] and references therein). The latter approach has proven extremely effective and continue to be a versatile, powerful and indispensable tool for studying nonlinear equations including functional differential equations.

The red thread of this dissertation is to use this theory combined with the *Green*'s functions method to deal with three classes of second order differential equations with iterative terms and boundary conditions. One of the most elegant ways to reach our targets is to transform the proposed problem into a fixed point problem. So, to establish the existence, uniqueness, and stability of solutions, we reduce this problem to finding fixed points of an integral operator defined from an integral equation with a *Green*'s kernel equivalent to the considered problem. To be more precise, by the aid of some functional analysis tools, we use certain useful properties of the obtained kernel together with *Schauder*'s or *Banach*'s fixed point theorems to derive our main results.

1.2 Brief historical background

To the best of the author's knowledge, the first apparition of iterative equations dates back some two hundred years ago (see *Babage* [5], *Schröder* [58] and *Abel* [1]). It is widely regarded that *Babage* was an important pioneer in the study of iterative equations. To the best of our knowledge, one of the most famous examples of this kind of equations is the *Babage* functional equation, which has been appeared in 1815 in his monograph [5] "An essay towards the calculus of function" where he studied the problem of finding a function equalling its n^{th} iterate $x^{[n]}(t)$. While the first serious attempts on the iterative differential equations which laid the foundation in this direction, is said to have begun, actually in 1921 [31]. Later, in 1968, *Andrzej* [4] applied the *Picard* successive approximation method for dealing with a first order differential equation involving the second iterate of the state where 0 is the left end point of the domain. Sixteen years later, *Eder* [29] used the contraction principle mapping to show that every solution either vanishes identically or is strictly monotonic for a special case of the equation that was studied by *Andrzej* but with $x(t_0) = t_0$ $(t_0 \in [-1, 1])$ where $x(t)$ is the state variable.

In 1990 years, *Wang* [62] investigated the same equation of *Andrzej* with $x(a) = a$, and a is an endpoint of the well-defined interval, by the aid of the *Schauder* fixed point theorem while *Fečkan* [30] dealt with it with the initial value $x(0) = 0$ by virtue of the contraction principle mapping in 1993. *Ge* and *Mo* [33] also studied it in 1997 with $x(t_0) = x_0$ on a given compact interval without the restriction that the nonlinearity is monotone. In 1995, *Staněk* [60] focused on the global properties of the solutions for another first order functional differential equation that included a second iterate of the state variable. In 1997, *Si* et al. [56] were interested in finding sufficient conditions for the existence of analytic solutions of a first order functional differential equation that included an m^{th} iterate of the state variable. In 1998, *Si* and *Wang* [57] investigated a class of nonhomogeneous iterative functional differential equations with variable coefficients. They use the *Banach* and *Schauder* fixed point theorems to prove the existence, uniqueness, and stability of smooth solutions. In the following century, due to their promising applications in many areas, this type of equations attracted the attention of several researchers at that time for studying them by different techniques. For example, the first iterative differential equations caught the attention of *Berinde* [7] who studied the existence and approximation of solutions by applying the technique of non-expansive operators together with the use of several convergence theorems from the theory of iterative approximation of fixed points of non-expansive mappings, in his published paper in 2010. More recently, certainly in the last decade, there has been an increasing activity in this area and hence several papers have appeared. For first order iterative differential equations and their applications, we can mention, for instance, [9, 15, 11, 39, 40, 41, 46, 48, 52, 53, 51, 63, 64, 65] and references therein. For recent real applications of such kind of equations. A technique based on fixed point theorem and the *Green*'s functions method was a powerful and reliable tool to discuss some new results on the existence, uniqueness, and stability for some hematopoiesis models with iterative terms [9, 39, 40], iterative *Nicholson*'s blowflies models [15, 41] and other recruitment models with iterative terms arising in biology and population dynamics [52, 53, 51].

Unfortunately, this is not always the case where the situation is deemed to be somewhat different for second and higher order iterative differential equations. They have been studied by only a very few scholars (see [13, 8, 18, 19, 38, 42, 44]). Actually, the presence of the iterates increases the difficulty of the study whenever the order of the iterative differential equation to be investigated is of order two or more.

1.3 Problem statement

The study of iterative functional differential equations which have some distinctive characteristics is not an easy task as it may seem at first blush, especially when dealing with higher-order ones. This is due to many factors including, their iterative terms that engender many insurmountable obstacles while investigating, their emerging theory which is not yet well developed and also the fact that the majority of usual methods often cannot be applied or may lead in many cases to a dead end.

The current thesis aims at filling this gap where the following fundamental research questions must be answered:

(i) How to control the iterative terms?

(ii) Does the problem have at least one solution?

(iii) Does the problem have a unique solution?

(iv) Does the unique solution, if it exists, depends on model parameters?

1.4 Motivation

The driving forces behind our investigation in this direction are multiple and perhaps the most crucial ones are on one hand their ability to mimic many realistic phenomena and on the other hand, the lack of studies that have been dedicated to this theme. Indeed, as we have said before, one of the salient features of iterative differential equations is that they create more realistic models in describing many real phenomena in many areas especially in life sciences which makes them worth the effort to investigate. Nonetheless, even though there are some researchers that have sought to study them, the literature of this kind of equations is somewhat rare and the investigation on them is still a challenging problem. The complexity in their study lies in the iterative terms that can create some hitches and hence disrupt the good progress of the study and can increase the difficulty in proving the desired results.

So, taking inspiration from all the aforementioned works on this topic, we have dealt with three classes of second order differential equations with iterative terms and different boundary conditions.

1.5 Research objectives

The main goal of this study is the application of the fixed point technique for establishing some existence, uniqueness and stability results for three classes of second order differential equations with iterative terms and boundary conditions. More precisely, our work is interested in the following issues: the existence, uniqueness, positivity, boundedness, periodicity and continuous dependence on parameters of solutions for these second-order iterative boundary value problems. The first problem is a second order iterative differential equation with integral boundary conditions, the second one, is a second order two-point value problem while the third one is a periodic second order boundary value problem.

1.6 Research methodology

Our hybrid approach highlights the efficiency and effectiveness of combining the fixed point theory with other techniques and methods such as the *Green*'s functions method and some functional analysis tools to deal with three different iterative boundary value problems.

The key insight is to transform the proposed equation with the boundary conditions into a fixed point problem. One way to accomplish this is to construct the cornerstones of the technique, which are an appropriate *Banach* space and a suitable subset of it in order to pave the way for the application of the chosen fixed point theorems and also to control the iterative terms and hence to overcome any possible unexpected obstacle in our study. The second step consists in converting our nonlinear problem into an equivalent integral equation whose kernel is a *Green*'s function. Next, by virtue of the obtained integral equation, *Arzelà-Ascoli* theorem and some useful properties of the obtained *Green*'s kernel, we define a continuous operator from a compact set into itself. Finally, the application of the *Schauder* fixed point theorem can put the finishing touches to the proof. Moreover, since generally a well-posed problem should have a unique solution that depends continuously upon its parameters, we added some extra criteria under which the *Banach* fixed point theorem can be applied to derive such a results.

1.7 Contributions

The intriguing features of our contributions can be summarised as follows:

- Several researches and documents have revealed that deviating arguments in many real phenomena actually depend on the time and the state variables. The studied functional differential equations involve iterative terms resulting from many deviating arguments of this kind.

- Up to now, there are few works that focus on the study of second order iterative differential equations. So, our results will enrich and complement some earlier works to some extent (see [3, 10, 35, 43]).

- The approach used here has proven to be an efficacious tool to deal with other iterative problems, particularly when they are of order higher than two or those that model real phenomena arising in life sciences.

1.8 Layout of the thesis

This thesis has been divided into five chapters where the first two chapters are introductory, while the last three ones present some results published in three certified international journals. The first chapter is a general introduction that provides a concise overview of the theme, a brief historical background, the research objectives and methodology, the motivation and contributions of the work, and also shows the outlines of the thesis that includes five chapters, and ends with a general conclusion, some perspectives for the future, a list of our published papers and a bibliography.

The second chapter renders some preliminary notions and tools from functional analysis that will be used as references in the remainder of this manuscript. The *Schauder* and *Banach* fixed point theorems are the basic tools to be used throughout the thesis in showing the existence or uniqueness results.

The third chapter exposes results published in [20]. It is concerned with the existence, uniqueness, and continuous dependence on parameters of positive bounded solutions for the following class of second order iterative differential equations with integral boundary conditions:

$$x''(t) + f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), ..., x^{[n]}(t)\right) = 0, \ 0 < t < b,$$
$$x(0) = 0, \ \nu \int_0^\eta x(s)\, ds = x(b) \ \text{ with } \eta \in (0, b), \ \nu \in \mathbb{R}^*.$$

Here $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(t) = x^{[n-1]}(x(t))$ denote the iterate of the function $x(t)$ and $f : [0, b] \times \mathbb{R}^n \longrightarrow [0, +\infty)$ is a continuous function with respect to its arguments.

For reaching our targets, we establish the equivalence of our problem with a certain integral equation in the first place, then, we apply the well-known *Schauder*'s fixed point theorem to prove the existence of at least one positive solution and finally we add an additional condition under which the *Banach* fixed point theorem can be applied easily and hence ensures the existence of a unique positive bounded solution that depends continuously on the nonlinearity f. An example is given to support the theory.

The fourth chapter seeks to present some results published in [21] where we concentrate on the existence, uniqueness and stability of bounded solutions for the following second order iterative two-point boundary value problem:

$$x''(t) = h\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \quad -b \leq t \leq b,$$
$$x(-b) = \eta_1, \; x(b) = \eta_2, \; \eta_1, \eta_2 \in [-b, b],$$

where $x^{[n]}(t)$ is the $n-th$ iterate of $x(t)$ and h is a continuous function from $[-b, b] \times \mathbb{R}^n$ to $\mathbb{R}$.

The basic tool we use here to tackle this boundary value problem is an hybrid technique that combines the *Schauder* and *Banach* fixed point theorems with the *Green*'s function method. The approach is based on defining an equivalent integral equation whose kernel is a *Green*'s function from which an integral operator is easily constructed and where emphasis is placed on the use of some properties of the obtained kernel to establish our desired results. Furthermore, we will give an example to illustrate the validity of our main theoretical results.

The fifth chapter that contains results published in [45] deals with the existence of at least one periodic solutions for the following class of second order iterative differential equations with periodic parameters:

$$\begin{aligned}\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = &\sum_{m=1}^{n}\sum_{\ell=1}^{\infty} C_{\ell,m}(t)\left(x^{[m]}(t)\right)^{\ell} \\ &+ \frac{d}{dt}g\left(t, x^{[1]}(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + w(t),\end{aligned}$$

where $x^{[2]}(t) = x(x(t))$, $x^{[m]}(t) = x^{[m-1]}(x(t))$ are the iterates of the state $x(t)$, $p, q, C_{\ell,m}, w \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $m = \overline{1, n}$, $\ell = \overline{1, \infty}$, and $g \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R})$.

The main objective is to find some suitable criteria under which this iterative differential equation admits positive periodic solutions. By reducing this iterative differential equation to another integral equation, the existence theorem is established.

Each of the three last chapters, namely chapters 3 to 5, commences with a brief introduction and ends with a short summary.

CHAPTER 2

Primary concepts

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The aim of this chapter is to provide the necessary background for ready reference and certain concepts that will be used through the thesis, including necessary functional analysis tools and notions, two fixed point theorems, the notion of *Green*'s function, and so on.

This chapter is mainly going to focus on providing some background definitions and the basic knowledge necessary for a better understanding of the chapters which follow without concentrating too much on the proofs of some results. We strive to introduce the necessary notions, some well-known results and tools from functional analysis that will be crucial in our arguments. Next, we introduce two fixed point theorems that will subsequently be used in our proofs. Finally, we will end this chapter by presenting the concept of Green function.

2.1 Notation and preliminaries

In this section we collect the useful definitions and preliminary notions and tools.

2.1.1 Convex subset in a vector space

Definition 2.1 [22] Let $\mathbb{X}$ be a vector space over $\mathbb{F}$. A convex subset of $\mathbb{X}$ is a subset $\mathbb{M} \subseteq \mathbb{X}$ such that for every pair of points $x, y \in \mathbb{M}$ and for every $\alpha \in [0,1]$ we have that

$$\alpha x + (1-\alpha)\, y \in \mathbb{M}.$$

Elements of the form $\alpha x + (1-\alpha)\, y$ are called *Convex Combinations* of x and y.

2.1.2 Bounded, closed and compact subset in a normed vector space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$.

Definition 2.2 [55] A subset $\mathbb{M}$ of $\mathbb{X}$ is said to be bounded if there exists $C > 0$ such that

$$\|x\|_{\mathbb{X}} \leq C, \forall x \in \mathbb{M}.$$

Theorem 2.1 *[55] A set $\mathbb{M} \subseteq \mathbb{X}$ is closed if and only if, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{M}$ which converges to an element $x \in \mathbb{X}$, then $x \in \mathbb{M}$.*

Definition 2.3 [55]A set $\mathbb{M} \subseteq \mathbb{X}$ is called compact if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{M}$.

Definition 2.4 [55] The closure of a set $\mathbb{M} \subseteq \mathbb{X}$ (denoted by $\overline{\mathbb{M}}$) is the smallest closed set that contains $\mathbb{M}$.

Definition 2.5 [55] A set $\mathbb{M} \subseteq \mathbb{X}$ is called relatively compact if its closure $\overline{\mathbb{M}}$ is compact.

Corollary 2.1 *[55] A set $\mathbb{M} \subseteq \mathbb{X}$ is relatively compact if and only if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{X}$.*

2.1.3 Continuous, Lipschitz continuous and compact operators

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two normed vector spaces over the same field $\mathbb{F}$.

Definition 2.6 [23] An operator $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be continuous at a point $x_0 \in \mathbb{X}$ if

$$\lim_{x \longrightarrow x_0} \mathcal{S}x = \mathcal{S}x_0.$$

The continuity at $x_0 \in \mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{X}, \ \ (\|x - x_0\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}x_0\|_{\mathbb{Y}} < \varepsilon) .$$

If $\mathcal{S}$ is continuous at every point of $\mathbb{X}$, then $\mathcal{S}$ is said to be continuous on $\mathbb{X}$. The continuity on $\mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \exists \delta > 0, \forall y \in \mathbb{X}, (\|x - y\|_{\mathbb{X}} < \delta) \Longrightarrow (\|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} < \varepsilon) .$$

Definition 2.7 [59] A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is called *Lipschitz continuous* if there is a positive constant C such that

$$\forall x, y \in \mathbb{X} : \|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} \leq C \|x - y\|_{\mathbb{X}} .$$

If $C \in [0, 1[$, $\mathcal{S}$ is called a contraction mapping.

Remark 2.1 If $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ then

$$\mathcal{S} \text{ is a contraction} \Longrightarrow \mathcal{S} \text{ is Lipschitz continuous} \Longrightarrow \mathcal{S} \text{ is continuous on } \mathbb{X}.$$

Theorem 2.2 *[55] A continuous function on a closed bounded interval is bounded and attains its bounds.*

Remark 2.2 The above theorem is hidden in the proof of many theorems and lemmas in the rest of this thesis where we integrate a continuous function over a compact interval.

Definition 2.8 A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be compact if and only if $\mathcal{S}$ maps bounded sets into relatively compact sets, i.e.,

$$[\mathcal{S} \text{ compact}] \Longleftrightarrow \Big[\forall M \subset E, \ (M \text{ bounded}) \Longrightarrow \Big(\overline{\mathcal{S}(M)} \text{ compact}\Big)\Big].$$

Equivalently, $\mathcal{S}$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

2.1.4 Arzelà-Ascoli theorem

Let $\mathbb{X}$ be a compact subset of a normed vector space over $\mathbb{F}$ and let $\mathcal{C}(\mathbb{X})$ be the normed vector space of real valued continuous functions on $\mathbb{X}$ with the *sup*-norm

$$\|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)|.$$

Let $\mathcal{F}$ be a collection of functions in $\mathcal{C}(\mathbb{X})$.

Definition 2.9 [17] The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and $x, y \in \mathbb{X}$ with $\|x - y\|_{\mathbb{X}} < \delta$ we have $|f(x) - f(y)| < \varepsilon$, i.e.,

$$\forall \varepsilon > 0, \ \forall x \in \mathbb{X}, \exists \delta > 0, \ \forall y \in \mathbb{X}, [\|x - y\|_{\mathbb{X}} < \delta] \Longrightarrow [\forall f \in \mathcal{F}, \ \ |f(x) - f(y)| < \varepsilon].$$

Definition 2.10 [17] The collection $\mathcal{F}$ is said to be uniformly bounded if there is an $M \geq 0$ so that $\|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)| \leq M$ for all $f \in \mathcal{F}$, i.e.,

$$\exists M \geq 0 : \|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)| \leq M, \ \forall f \in \mathcal{F}.$$

Theorem 2.3 *[17] If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $\mathcal{C}(\mathbb{X})$, then $\mathcal{F}$ is relatively compact in $\mathcal{C}(\mathbb{X})$.*

2.1.5 Iterations

Definition 2.11 The composition $x \circ y$ of the function x with the function y is

$$(x \circ y)(t) = x(y(t)),$$

The domain of $x \circ y$ is the set of all t in the domain of y such that $y(t)$ is in the domain of x.

Definition 2.12 For $x : E \longrightarrow E$, the n^{th} iterate of function x, denoted by $x^{[n]}$ for some nonnegative integer n, is defined recursively by

$$x^{[0]} = Id_E,$$

and

$$x^{[n+1]} = x \circ x^{[n]},$$

where Id_E is the identity map on E.

For $T > 0$, Let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t)\},$$

the *Banach* space of all continuous and $T-$periodic functions endowed with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|,$$

and for $L, M \geq 0$, let

$$P_T(L, M) = \{x \in P_T : \|x\| \leq L,\ \ |x(t_2) - x(t_1)| \leq M |t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\},$$

be a closed bounded convex subset of P_T.

Now, we present a useful estimate that will help us later.

Lemma 2.1 *[65] If $x, y \in P_T(L, M)$, then*

$$\left\|x^{[k]} - y^{[k]}\right\| \leq \sum_{j=0}^{k-1} M^j \|x - y\|,\ \ k = 1, 2, \ldots$$

where $x^{[k]} = x \circ x \circ \ldots \circ x$ k times.

Proof. We will prove this inequality by induction. So, the proof will now proceed in two steps:

The basis step: For $k=1$, we have

$$\|x-y\| \leq \|x-y\| .$$

then, the inequality holds for $k=1$

The inductive step: Now, we assume that the inequality holds for a given $k=m$ and we want to show that it also holds for $k=m+1$. Suppose that

$$\left\|x^{[m]}-y^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^{j}\|x-y\| ,$$

then

$$\begin{aligned} \left|x^{[m+1]}(t)-y^{[m+1]}(t)\right| &\leq \left|x\left(x^{[m]}(t)\right)-x\left(y^{[m]}(t)\right)\right|+\left|x\left(y^{[m]}(t)\right)-y\left(y^{[m]}(t)\right)\right| \\ &\leq M\left|x^{[m]}(t)-y^{[m]}(t)\right|+\left|x\left(y^{[m]}(t)\right)-y\left(y^{[m]}(t)\right)\right| , \end{aligned}$$

so

$$\begin{aligned} \left\|x^{[m+1]}-y^{[m+1]}\right\| &\leq M\left\|x^{[m]}-y^{[m]}\right\|+\|x-y\| \\ &\leq M \sum_{j=0}^{m-1} M^{j}\|x-y\|+\|x-y\| \\ &\leq \left(\sum_{j=0}^{m-1} M^{j+1}+1\right)\|x-y\| \\ &\leq \sum_{j=0}^{m} M^{j}\|x-y\| . \end{aligned}$$

By induction we deduce that

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^{j}\|\varphi-\psi\| \qquad \forall m \in \mathbb{N} ,$$

which finishes the proof. ■

Remark 2.3 The above inequality remains true if we replace the subset $P_T(L,M)$ with one of the following sets:

$$\begin{aligned} \mathcal{CB}_{Int}\left(L_0, M_0\right) = \{ & x \in \mathcal{CB}_{Int} : \ 0 \leq x(t) \leq L_0, \\ & \left|x\left(t_2\right)-x\left(t_1\right)\right| \leq M_0\left|t_2-t_1\right| , \ \forall t_1, t_2 \in [0, b]\} , \end{aligned}$$

or

$$\mathcal{CB}(\alpha,\beta) = \{x\in\mathcal{X},\ -\alpha\le x(t)\le\alpha,\\ |x(t_2)-x(t_1)|\le\beta|t_2-t_1|,\ \forall t_1,t_2\in[-b,b]\},$$

where

$$\mathcal{CB}_{Int} = \left\{x\in\mathcal{C}([0,b],\mathbb{R}) : x(0)=0,\ \nu\int_0^{\eta}x(s)\,ds = x(b),\ \nu\in\mathbb{R}^*,\ \eta\in(0,b)\right\},$$

$M_0,\beta\ge 0$, $0\le\alpha\le b$, $L_0\le b$ and $\mathcal{X}=(\mathcal{C}([-b,b],\mathbb{R}))$. Here $\mathcal{CB}_{Int}$ and $\mathcal{X}$ are equipped with the supremum norm.

2.2 Fixed point theorems

Fixed point theory is designed to be used in studying different types of equations arising in various fields.

Definition 2.13 [59] Let $(\mathbb{X},\|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$. A fixed point of a mapping $\mathcal{S}:\mathbb{X}\to\mathbb{X}$ of $\mathbb{X}$ into itself is an $x\in\mathbb{X}$ which is mapped onto itself, that is

$$\mathcal{S}(x)=x.$$

2.2.1 Schauder's fixed point theorem

The *Schauder* fixed point theorem relies on the compactness of the operator in *Banach* spaces.

Theorem 2.4 *[59] Let $\mathbb{M}$ be a non-empty bounded closed convex subset of a Banach space $(\mathbb{X},\|.\|)$ and let $\mathcal{S}:\mathbb{M}\longrightarrow\mathbb{M}$ be a compact and continuous mapping. Then $\mathcal{S}$ has a fixed point in $\mathbb{M}$.*

An alternative version of the *Schauder* fixed point theorem can be stated as follows:

Theorem 2.5 *[59] Let $\mathbb{M}$ be a non-empty compact convex subset of a Banach space $(\mathbb{X},\|.\|)$ and let $\mathcal{S}:\mathbb{M}\longrightarrow\mathbb{M}$ be a continuous mapping. Then $\mathcal{S}$ has a fixed point in $\mathbb{M}$.*

2.2.2 Banach's fixed point theorem

One of the very helpful tools which is broadly applicable in proving the existence and uniqueness of solutions, is the well-known *Banach* fixed point theorem (also known as the contraction mapping theorem or contractive mapping theorem).

Theorem 2.6 *[59] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a Banach space and let $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ be a contraction on $\mathbb{X}$. Then $\mathcal{S}$ has a unique fixed point $x \in \mathbb{X}$ such that*

$$\mathcal{S}(x) = x.$$

Theorem 2.7 *[36] If $\mathbb{M}$ is a closed subset of a Banach space $\mathbb{X}$ and $\mathcal{S} : \mathbb{M} \to \mathbb{M}$ is a contraction, then $\mathcal{S}$ has a unique fixed point in $\mathbb{M}$.*

2.3 Green's functions for second order ordinary differential equations

Green's functions can be traced back to the excellent work of the British mathematician *George Green* [34] in 1828, where he introduced this notion when he studied the *Poisson*'s equation for the electric potential. The *Green*'s functions method can be used for solving a large number of families including ordinary differential equations where the inverse of the differential operator is an integral operator whose kernel is a *Green*'s function. So, for solving nonhomogeneous boundary value problems for ordinary differential equations, often it is easier to use the of *Green*'s functions method.

Let a_0, a and b be three continuous and differentiable functions on $[c, d]$. Consider the linear second-order differential equation of the form

$$a_0(t)\, x^{''}(t) + a(t)\, x^{'}(t) + b(t)\, x(t) = r(t), \forall t \in [c, d], \tag{2.1}$$

with the boundary conditions

$$\begin{cases} \alpha_0 x(c) + \alpha_1 x'(c) + \alpha_2 x(d) + \alpha_3 x'(d) = \delta \\ \beta_0 x(c) + \beta_1 x'(c) + \beta_2 x(d) + \beta_3 x'(d) = \gamma. \end{cases} \tag{2.2}$$

When α_2, α_3, β_0 and β_1 are null, these conditions are said to be separated.

Definition 2.14 We call homogeneous problem associated with (2.1)-(2.2), the problem (2.3)-(2.4) defined by

$$a_0(t)\,x''(t) + a(t)\,x'(t) + b(t)\,x(t) = 0. \tag{2.3}$$

with the following boundary conditions:

$$\begin{cases} \alpha_0 x(c) + \alpha_1 x'(c) + \alpha_2 y(d) + \alpha_3 x'(d) = 0 \\ \beta_{01} x(c) + \beta_1 x'(c) + \beta_2 x(d) + \beta_3 x'(d) = 0. \end{cases} \tag{2.4}$$

Theorem 2.8 *[50]*

a) Assume $a_0(t) \neq 0, \forall t \in [c, d]$.

If the homogeneous problem associated with (2.1)-(2.2) has only the trivial solution, then the Green's function for this boundary value problem exists and is unique.

b) If the above conditions are assumed to be fulfilled, then the solution of the nonhomogeneous boundary value problem is unique and given by

$$x(t) = \int_c^d G(t,s)\ r(s)\,ds, \tag{2.5}$$

where the kernel $G(t,s)$ is a Green's function.

Remark 2.4 There are several approaches to finding particular solutions of nonhomogeneous equations such as the method of variation of parameters, integrating factor method, and the *Green*'s function method, to name a few. According to the superposition principle the general solution can be found as the sum of the general solution to the homogeneous equation and a particular solution of the nonhomogeneous equation. So, if the homogeneous problem (2.3)-(2.4) has only the trivial solution and the differential equation (2.1) is linear. Then, the unique solution obeying nonhomogeneous boundary conditions is simply the obtained particular solution.

Definition 2.15 The problem is said to be regular if in addition to the previous conditions, we have

1) a_0 is nonzero (except perhaps at a finite number of isolated points).

2) The homogeneous problem (2.3)-(2.4) has only the trivial solution.

Definition 2.16 [47] For the regular problem (2.1)-(2.2), there exists a unique function $G(t,s)$ called a *Green*'s function which is entirely determined by the following four properties:

1− $G(t,s)$ satisfies the homogeneous differential equation (2.3) for any $t \neq s$.

2− For all fixed s in $]c,d[$, the function $G(t,s)$ satisfies the boundary conditions of the problem.

3− The *Green*'s function $G(t,s)$ must be continuous.

4− For all fixed s in $]c,d[$, $\frac{\partial G}{\partial t}(s^{+},s) - \frac{\partial G}{\partial t}(s^{-},s) = \frac{1}{a_0(s)}$ (jump in derivative or jump discontinuity of $\frac{\partial G}{\partial t}$ at $t = s$).

Remark 2.5 Since G depends on the main part of the differential equation, but not on the source term r, once G is found we can immediately solve the more general problem for any arbitrary source term.

Example 2.1 We seek to find the *Green's* function of the following problem:

$$\frac{d^2x(t)}{dt^2} = r(t), \tag{2.6}$$

for all $t \in]0,1[$ with the two boundary conditions

$$\begin{cases} x(0) = 0 \\ x(1) = 0. \end{cases} \tag{2.7}$$

First and foremost, we must check that the rank of

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

is 2. Indeed, the first and the third columns ensure that the rank equals 2.

Solutions of the homogeneous differential equation

$$\frac{d^2x(t)}{\partial t^2} = 0,$$

are of the form

$$x(t) = A\,t + B,$$

where A and B are constants. The boundary conditions (2.7) give $A=0$ and $B=0$, which guarantees the existence of a unique *Green's function. Therefore*

1) On $]0,s[$,

$$G(t,s)=A(s)\,t+B(s)\,,$$

and on $]s,1[$,

$$G(t,s)=C(s)\,t+D(s)\,.$$

2) We must have $G(0,s)=0$, then $B(s)=0$ and we must also have $G(1,s)=0$ i.e.,

$$C(s)+D(s)=0.$$

3) Since G is continuous, then

$$G\left(s^{-},s\right)=\lim_{t\to s^{-}}G(t,s)=\lim_{t\to s^{+}}G(t,s)=G\left(s^{+},s\right),$$

this implies that

$$A(s)\,s+B(s)=C(s)\,s+D(s)\,.$$

4)

$$\frac{\partial G}{\partial t}\left(s^{+},s\right)=C(s)\,,\ \frac{\partial G}{\partial t}\left(s^{-},s\right)=A(s)\,.$$

Consequently

$$C(s)-A(s)=1.$$

$A(s)\,,\ B(s)\,,\ C(s)$ and $D(s)$ are solution of

$$\begin{cases} B(s)=0\\ C(s)+D(s)=0\\ s\left[A(s)-C(s)\right]+B(s)-D(s)=0\\ A(s)-C(s)+1=0,\end{cases}$$

which gives

$$\begin{cases} A(s)=s-1\\ B(s)=0\\ C(s)=s\\ D(s)=-s.\end{cases}$$

We have the *Green*'s function:

$$G(t,s)=\begin{cases} t(s-1) & \text{if } 0\leqslant t\leqslant s\\ s(t-1) & \text{if } s\leqslant t\leqslant 1.\end{cases}$$

Finally, the solution of the problem (2.6)-(2.7) is

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\,r(s)\,ds \\ &= \int_0^t t(s-1)\,r(s)\,ds+\int_t^1 s(t-1)\,r(s)\,ds. \end{aligned} \tag{2.8}$$

CHAPTER 3

Positive solutions for a class of second-order iterative differential equations with integral boundary conditions

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In this chapter, we are interested in investigating a class of second order iterative differential equations with integral boundary conditions.

3.1 Introduction

Boundary value problems (BVPs for short) for functional differential equations are often encountered in the modelling of many phenomena in various engineering and scientific disciplines like physics, science, technology, and engineering. For instance, they arise in describing blood flow problems, population dynamics, cellular systems, heat transfer, chemical engineering, underground water flow, thermoelasticity, plasma physics, acoustic diffusion and so on and so forth.

Different types of boundary conditions can be specified at the boundaries of the domain. One of them is integral boundary conditions which have aroused great interest among the scientific communities where the mathematical literature of the past several decades, contains abundant theoretical achievements in this direction. The basic tools used to deal with problems of this form are *Leray–Schauder* continuation theorem, the coincidence degree theory, the modified wavelet multigrid method, the variational iteration method, the *Richardson* iteration method and the fixed point theory. For instance, we cite the following works where the fixed point theory has acquired greater significance.

In [37], *Infante* applied the theory of fixed point index to deal with the following equation:

$$\begin{gathered} u''(t) + f(t, u(t)) = 0,\ 0 < t < 1, \\ u'(0) = 0,\ u(1) = \int_0^1 \gamma(s, u(s))\, ds. \end{gathered}$$

In [16], by means of *Krasnoselskii's* fixed point theorem, *Boucherif* focused on the existence of positive solutions of the following problem:

$$y''(t) = f(t, y(t)),\ 0 < t < 1,$$

with

$$y(0) - ay'(0) = \int_0^1 g_0(s)\, y(s)\, ds,$$

and

$$y(1) - by'(1) = \int_0^1 g_1(s)\, y(s)\, ds,$$

where $a, b \geq 0$, $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g_0, g_1 : [0,1] \longrightarrow [0, +\infty)$ are continuous functions.

In [6], *Benchohra* et al. used nonlinear alternative *Leray Schauder* type and *Banach* contraction principle to prove the existence of solutions of the following second-order boundary value problem:

$$-y''(t) = f(t, y(t)),\ \text{a.e. } 0 < t < 1,$$

with

$$y(0)=0,\ y(1)=\int_0^1 g(s)\,y(s)\,ds,$$

where $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function and $g:[0,1]\longrightarrow\mathbb{R}$ is an integrable function.

In [2], *Ahmad* et al. used the nonlinear alternative of *Leray-Schauder* type and some suitable theorems of fixed point theory to investigate the following second-order boundary value problem:

$$-x''(t)\in F(t,x(t)),\ 0<t<1,$$

with

$$x(0)=0,\ x(1)=\omega\int_0^\eta x(s)\,ds \text{ with } \eta\in(0,1),$$

where $\omega\neq 2/\eta^2\in\mathbb{R}$, $F:[0,1]\times\mathbb{R}\longrightarrow D(\mathbb{R})$ is a multivalued map, and D is the family of all subsets of $\mathbb{R}$.

In [32], *Galvis* et al. discussed the existence of positive solutions of the following problem:

$$u''(t)+a(t)\,f(u(t))=0,\ 0<t<\gamma,$$

$$u(0)=0,\ u(\gamma)=\nu\int_0^\eta u(s)\,ds \text{ with } \eta\in(0,\gamma),$$

by virtue of the *Schauder* fixed point theorem.

These works motivate us to investigate the following class of second-order iterative differential equations with integral boundary conditions:

$$x''(t)+f\left(x^{[0]}(t),x^{[1]}(t),x^{[2]}(t),\ldots,x^{[n]}(t)\right)=0,\ 0<t<b, \tag{3.1}$$

$$x(0)=0,\ \nu\int_0^\eta x(s)\,ds=x(b)\ \text{ with } \eta\in(0,b),\ \nu\in\mathbb{R}^*, \tag{3.2}$$

where

$$x^{[0]}(t)=t, x^{[1]}(t)=x(t), x^{[2]}(t)=x(x(t)),\ldots,x^{[n]}(t)=x^{[n-1]}(x(t)),$$

and $f:[0,b]\times\mathbb{R}^n\longrightarrow[0,+\infty)$ is a continuous function with respect to its arguments.

Our main contribution here is to show that the fixed point theory can be applied successfully to iterative problems with integral boundary conditions. So, this chapter aims at establishing a set of sufficient conditions to ensure the existence, uniqueness, and continuous dependence on parameters of positive solutions of problem (3.1)-(3.2) where the proofs of the main results rely on the *Schauder* theorem.

This chapter is organized as follows. In section 2, some preliminary materials are stated. In section 3, the existence of at least one positive solution to problem (3.1)-(3.2) is established. Section 4 is consecrated to discuss the existence and continuous dependence on parameters of the unique positive solution of problem (3.1)-(3.2). An example is given to illustrate the effectiveness of the main outcomes in section 5. Finally, a summary is drawn to end this chapter in section 6.

3.2 Preliminaries

Before we start off this section we need to present some definitions.

Let us first define a non-empty subset $\mathcal{CB}_{Int}$ of $\mathcal{C}\left([0,b],\mathbb{R}\right)$ as follows:

$$\mathcal{CB}_{Int}=\left\{x\in\mathcal{C}\left([0,b],\mathbb{R}\right):x\left(0\right)=0,\ \nu\int_0^{\eta}x\left(s\right)ds=x\left(b\right),\ \nu\in\mathbb{R}^*,\ \eta\in\left(0,b\right)\right\}.$$

It's obvious that $\mathcal{CB}_{Int}$ endowed with the norm

$$\left\|x\right\|=\sup_{t\in[0,b]}\left|x\left(t\right)\right|,$$

is a *Banach* space.

For $0\leq L_0\leq b$ and $M_0\geq 0$, let

$$\mathcal{CB}_{Int}\left(L_0,M_0\right)=\left\{x\in\mathcal{CB}_{Int},\ 0\leq x\left(t\right)\leq L_0,\ \left|x\left(t_2\right)-x\left(t_1\right)\right|\leq M_0\left|t_2-t_1\right|,\ \forall t_1,t_2\in[0,b]\right\},$$

be a closed convex and bounded subset of $\mathcal{CB}_{Int}$.

Throughout this work, we impose the following hypotheses which will be used in the sequel:

The function $f(t,x_1,x_2,...,x_n)$ is globally *Lipschitz* in $x_1,...,x_n$. i.e., there exist n positive constants $c_1,\ c_2,...,c_n$ such that

$$\left|f\left(t,x_1,...,x_n\right)-f\left(t,y_1,...,y_2\right)\right|\leq\sum_{i=1}^{n}c_i\left\|x_i-y_i\right\|, \tag{3.3}$$

and for the sake of simplicity, we will adopt the following notations:

$$\begin{aligned} \rho &= \sup_{s\in[0,b]} \left| f\left(s,0,0,...,0\right)\right|, \\ \zeta &= \rho + L_0 \sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M_0^j. \end{aligned}$$

Lemma 3.1 *[32] Let $2b \neq \nu\eta^2$, then for $y \in \mathcal{C}\left(\left[0,b\right],\left[0,+\infty\right)\right)$, the problem*

$$x''\left(t\right) + y\left(t\right) = 0, \tag{3.4}$$

$$x\left(0\right) = 0,\ \ \nu \int_0^{\eta} x\left(t\right)dt = x\left(b\right),\ \ \eta \in \left(0,b\right),\ \ \nu \neq 0, \tag{3.5}$$

has a unique solution given by

$$\begin{aligned} x\left(t\right) &= \frac{2t}{2T - \nu\eta^2}\int_0^T (b-s)y\left(s\right)ds - \frac{\nu t}{2b-\nu\eta^2}\int_0^{\eta}(\eta - s)^2 y\left(s\right)ds \\ &\quad -\int_0^t (t-s)y\left(s\right)ds. \end{aligned}$$

Lemma 3.2 *[32] Let $0 < \nu < \dfrac{2}{\eta^2}$. If $y \in \mathcal{C}\left(0,b\right)$ and $y(t) \geq 0$ on $\left(0,b\right)$, then the unique solution of the problem (3.4)-(3.5) satisfies $x(t) \geq 0$ for $t \in \left[0,b\right]$.*

3.3 Existence results

As indicated at the beginning of this chapter, the main goal of this section is to prove the existence of positive solutions to problem (3.1)-(3.2) by using *Schauder*'s fixed point theorem. So, to apply it, we need to construct an operator $\mathcal{A} : \mathcal{CB}_{Int}\left(L_0,M_0\right) \longrightarrow \mathcal{CB}_{Int}$ which takes the form:

$$\begin{aligned} \left(\mathcal{A}\varphi\right)\left(t\right) &= \frac{2t}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds \\ &\quad - \frac{\nu t}{2b-\nu\eta^2}\int_0^{\eta} (\eta-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds \\ &\quad - \int_0^t (t-s) f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds. \end{aligned} \tag{3.6}$$

So, fixed points of operator $\mathcal{A}$ are solutions of problem (3.1)-(3.2) and vice versa. Since the compactness of $\mathcal{CB}_{Int}\left(L_0,M_0\right)$ is guaranteed by the *Arzelà-Ascoli* theorem, we just have to prove $\mathcal{A}$ is well defined, continuous and $\mathcal{A}\left(\mathcal{CB}_{Int}\left(L_0,M_0\right)\right) \subset \mathcal{CB}_{Int}\left(L_0,M_0\right)$.

Lemma 3.3 *Let $2b \neq \nu\eta^2$, then operator $\mathcal{A} : \mathcal{CB}_{Int}(L_0, M_0) \longrightarrow \mathcal{CB}_{Int}$ given by (3.6) is well defined.*

Proof. To show that $\mathcal{A}$ is well defined it suffices to show that $\mathcal{A}\varphi \in \mathcal{C}([0,b],\mathbb{R})$, $(\mathcal{A}\varphi)(0) = 0$ and $\nu \int_0^\eta (\mathcal{A}\varphi)\, dt = (\mathcal{A}\varphi)(b)$ for all $\varphi \in \mathcal{CB}_{Int}(L_0, M_0)$. It is not difficult to check that $\mathcal{A}\varphi \in \mathcal{C}([0,b],\mathbb{R})$ for $\varphi \in \mathcal{CB}_{Int}(L_0, M_0)$ and that $(\mathcal{A}\varphi)(0) = 0$. Let $\varphi \in \mathcal{CB}_{Int}(L_0, M_0)$, we have

$$\begin{aligned}
(\mathcal{A}\varphi)(b) &= \frac{2b}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\frac{\nu b}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&= \frac{\nu\eta^2}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\frac{\nu b}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds,
\end{aligned}$$

and

$$\begin{aligned}
&\nu\int_0^\eta (\mathcal{A}\varphi)(t)\, dt \\
&= \nu\int_0^\eta \left(\frac{2t}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds\right) dt \\
&\quad +\nu\int_0^\eta \left(-\frac{\nu t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds\right) dt \\
&\quad +\nu\int_0^\eta \left(-\int_0^t (t-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds\right) dt, \\
&= \frac{\nu\eta^2}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\frac{1}{2}\frac{\nu^2\eta^2}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\frac{1}{2}\nu\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds
\end{aligned}$$

which gives

$$\begin{aligned}
&\nu\int_0^\eta (\mathcal{A}\varphi)(t)\, dt \\
&= \frac{\nu\eta^2}{2b-\nu\eta^2}\int_0^b (b-s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \\
&\quad -\frac{\nu b}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds.
\end{aligned}$$

So $\nu\int_0^\eta (\mathcal{A}\varphi)(t)\,dt = (\mathcal{A}\varphi)(b)$. Consequently $\mathcal{A}$ is well defined. ■

Lemma 3.4 *If condition (3.3) is satisfied, then the operator $\mathcal{A}$ defined by (3.6) is continuous.*

Proof. For $\varphi,\psi \in \mathcal{CB}_{Int}(L_0, M_0)$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq\ & \frac{2t}{|2b-\nu\eta^2|}\int_0^b (b-s)\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right. \\ & \left. -f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right| ds \\ & +\frac{|\nu|\,t}{|2b-\nu\eta^2|}\int_0^\eta (\eta-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right. \\ & \left. -f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right| ds \\ & +\int_0^t |(t-s)|\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right. \\ & \left. -f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right| ds. \end{aligned}$$

Taking into account the condition (3.3), we get

$$\begin{aligned} |(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq\ & \frac{2t}{|2b-\nu\eta^2|}\left(\int_0^b (b-s)ds\right)\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\ & +\frac{|\nu|\,t}{|2b-\nu\eta^2|}\left(\int_0^\eta (\eta-s)^2 ds\right)\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\ & +\left(\int_0^t |(t-s)|\,ds\right)\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right), \end{aligned}$$

then

$$\begin{aligned} |(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq\ & \frac{tb^2}{|2b-\nu\eta^2|}\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\ & +\frac{1}{3}\frac{t\eta^3|\nu|}{|2b-\nu\eta^2|}\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\ & +\frac{1}{2}t^2\left(\sum_{i=1}^n c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right). \end{aligned}$$

Consequently

$$\begin{aligned}\left|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)\right| &\leq \frac{b^3}{\left|2b-\nu\eta^2\right|}\left(\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right)\\ &+\frac{1}{3}\frac{b\eta^3\left|\nu\right|}{\left|2b-\nu\eta^2\right|}\left(\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right)\\ &+\frac{1}{2}b^2\left(\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right)\\ &=\left(\frac{b^3}{\left|2b-\nu\eta^2\right|}+\frac{1}{3}\frac{b\eta^3\left|\nu\right|}{\left|2b-\nu\eta^2\right|}+\frac{1}{2}b^2\right)\left(\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right)\\ &=\left(\frac{1}{3}\frac{b\left(3b^2+\left|\nu\right|\eta^3\right)}{\left|2b-\nu\eta^2\right|}+\frac{1}{2}b^2\right)\left(\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\right).\end{aligned}$$

Thanks to Lemma 2.1, we infer that

$$\left|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)\right|\leq\left(b\left(\frac{3b^2+\left|\nu\right|\eta^3}{3\left|2b-\nu\eta^2\right|}+\frac{1}{2}b\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M_0^j\right)\left\|\varphi-\psi\right\|,$$

which establishes the continuity of operator $\mathcal{A}$. ■

Lemma 3.5 *Assume that condition (3.3) holds. If*

$$b\zeta\left(\frac{3b^2+\left|\nu\right|\eta^3}{3\left|2b-\nu\eta^2\right|}+\frac{1}{2}b\right)\leq L_0, \tag{3.7}$$

and

$$\zeta\left(\frac{3b^2+\left|\nu\right|\eta^3}{3\left|2b-\nu\eta^2\right|}+b\right)\leq M_0, \tag{3.8}$$

then $\mathcal{A}\left(\mathcal{CB}_{Int}\left(L_0,M_0\right)\right)\subset\mathcal{CB}_{Int}\left(L_0,M_0\right).$

Proof. It follows from Lemma 3.2, that $(\mathcal{A}\varphi)(t)\geq 0$ for $t\in[0,b]$. So, it remains to prove that $(\mathcal{A}\varphi)(t)\leq L_0$ and $\left|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)\right|\leq M_0\left|t_2-t_1\right|$, for all $\varphi\in\mathcal{CB}_{Int}\left(L_0,M_0\right)$ and $t_1,t_2\in[0,b]$.

For φ in $\mathcal{CB}_{Int}\left(L_0,M_0\right)$, we have

$$\begin{aligned}\left|(\mathcal{A}\varphi)(t)\right| &\leq \frac{2t}{\left|2b-\nu\eta^2\right|}\int_0^b(b-s)\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\ &+\frac{\left|\nu\right|t}{\left|2b-\nu\eta^2\right|}\int_0^{\eta}(\eta-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\ &+\int_0^t\left|(t-s)\right|\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds.\end{aligned}$$

Since

$$\begin{aligned}
&\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| \\
&\leq \left|f\left(\varphi^{[0]}(s),\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)-f\left(\varphi^{[0]}(s),0,0,...,0\right)\right| \\
&+\left|f\left(\varphi^{[0]}(s),0,0,...,0\right)\right| \\
&\leq \rho+\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M_0^j\left\|\varphi\right\| \\
&\leq \rho+L_0\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M_0^j=\zeta,
\end{aligned}$$

then

$$\begin{aligned}
\left|\left(\mathcal{A}\varphi\right)(t)\right| &\leq \frac{2t\zeta}{\left|2b-\nu\eta^2\right|}\int_0^b(b-s)\,ds \\
&+\frac{\left|\nu\right|t\zeta}{\left|2b-\nu\eta^2\right|}\int_0^\eta(\eta-s)^2ds \\
&+\zeta\int_0^t\left|(t-s)\right|ds \\
&= b\zeta\left(\frac{3b^2+\left|\nu\right|\eta^3}{3\left|2b-\nu\eta^2\right|}+\frac{1}{2}b\right).
\end{aligned}$$

In view of assumption (3.7), we get

$$\left(\mathcal{A}\varphi\right)(t)\leq\left|\left(\mathcal{A}\varphi\right)(t)\right|\leq L_0.$$

Let $t_1,t_2\in[0,b]$ with $t_1<t_2$, we have

$$\begin{aligned}
&\left|\left(\mathcal{A}\varphi\right)(t_2)-\left(\mathcal{A}\varphi\right)(t_1)\right| \\
\leq\ &\frac{\left|2t_2-2t_1\right|}{\left|2b-\nu\eta^2\right|}\int_0^b(b-s)\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds \\
&+\frac{\left|\nu t_2-\nu t_1\right|}{\left|2b-\nu\eta^2\right|}\int_0^\eta(\eta-s)^2f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds \\
&+\left|\int_0^{t_1}(t_2-s)f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds\right. \\
&+\int_{t_1}^{t_2}(t_2-s)f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds \\
&\left.-\int_0^{t_1}(t_1-s)f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)ds\right| \\
\leq\ &\frac{\zeta b^2\left|t_2-t_1\right|}{\left|2b-\nu\eta^2\right|}+\frac{1}{3}\frac{\zeta\eta^3\left|\nu\right|\left|t_2-t_1\right|}{\left|2b-\nu\eta^2\right|}+\zeta b\left|t_2-t_1\right| \\
=\ &\zeta\left(\frac{3b^2+\left|\nu\right|\eta^3}{3\left|2b-\nu\eta^2\right|}+b\right)\left|t_2-t_1\right|.
\end{aligned}$$

According to (3.8), we find

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq M_0.$$

Thus, $(\mathcal{A}\varphi)(t) \in \mathcal{CB}_{Int}(L_0, M_0)$ for all $\varphi \in \mathcal{CB}_{Int}(L_0, M_0)$ shows that $\mathcal{A}$ is a self-mapping from $\mathcal{CB}_{Int}(L_0, M_0)$ to $\mathcal{CB}_{Int}(L_0, M_0)$. ■

Our first main result is the following existence theorem:

Theorem 3.1 *Assume that conditions (3.3), (3.7) and (3.8) are satisfied. Then the problem (3.1)-(3.2) has at least one positive solution x in $\mathcal{CB}_{Int}(L_0, M_0)$.*

Proof. We know that problem (3.1)-(3.2) admits a solution x on $\mathcal{CB}_{Int}(L_0, M_0)$ if and only if the operator $\mathcal{A}$ defined by (3.6) admits a fixed point.
Based on Lemmas 3.3, 3.4 and 3.5, we conclude that all requirements of *Schauder*'s fixed point theorem are fulfilled. Thus, $\mathcal{A}$ possesses at least one fixed point on $\mathcal{CB}_{Int}(L_0, M_0)$ which means that problem (3.1)-(3.2) has at least one positive solution. ■

3.4 Existence, uniqueness and continuous dependence results

The main purpose of this section is the investigation of the existence of a unique positive solution that depends continuously on the nonlinearity f.

Theorem 3.2 *Under the assumptions of Theorem 3.1, if*

$$b\left(\frac{3b^2 + |\nu|\,\eta^3}{3\,|2b - \nu\eta^2|} + \frac{1}{2}b\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M_0^j < 1, \tag{3.9}$$

holds, then problem (3.1)-(3.2) has a unique positive solution in $\mathcal{CB}_{Int}(L_0, M_0)$.

Proof. To prove the uniqueness of the solution, we assume the contrary and argue for a contradiction. Let φ, ψ be two distinct solutions of problem (3.1)-(3.2). We notice that by using the same technique as that in the proof of Lemma of 3.4 we can establish that

$$\begin{aligned} |\varphi(t) - \psi(t)| &= |(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\ &\leq \left(b\left(\frac{3b^2 + |\nu|\,\eta^3}{3\,|2b - \nu\eta^2|} + \frac{1}{2}b\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M_0^j\right)\|\varphi - \psi\|. \end{aligned}$$

According to (3.9), we get

$$\|\varphi-\psi\| < \|\varphi-\psi\|,$$

which is a contradiction and implies that the solution must be unique. ■

Now, we show that the unique solution proved in the last theorem depends continuously on the function f. Fortunately, it can be proved without any additional assumptions.

Definition 3.1 The solution of the integral equation

$$\begin{aligned} x(t) &= \frac{2t}{2b-\nu\eta^2}\int_0^b (b-s) f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad -\frac{\nu t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad -\int_0^t (t-s) f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \end{aligned}$$

depends continuously on the function f if $\forall\varepsilon>0$, $\exists\delta(\varepsilon)>0$ such that

$$\begin{aligned} &\Big|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) \\ &-f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\Big| \le \delta(\varepsilon) \Longrightarrow \|\widetilde{x}-x\| \le \varepsilon, \end{aligned}$$

where $\widetilde{x}$ is the unique solution of the following integral equation:

$$\begin{aligned} \widetilde{x}(t) &= \frac{2t}{2b-\nu\eta^2}\int_0^b (b-s) \widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds \\ &\quad -\frac{\nu t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 \widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds \\ &\quad -\int_0^t (t-s) \widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds. \end{aligned}$$

Theorem 3.3 *Assume that all conditions of Theorem 3.2 are satisfied. The unique solution of (3.1)-(3.2) depends continuously on the function f.*

Proof. Let $x \in \mathcal{CB}_{Int}(L_0, M_0)$ be the unique solution of (3.1)-(3.2), so x satisfies

$$\begin{aligned} x(t) &= \frac{2t}{2b-\nu\eta^2}\int_0^b (b-s) f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad -\frac{\nu t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2 f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad -\int_0^t (t-s) f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \end{aligned}$$

Let $\widetilde{x} \in \mathcal{CB}_{Int}(L_0, M_0)$ be a perturbed solution of problem (3.1)-(3.2) with small perturbation in the function f satisfies all conditions of Theorem 3.2. So there exists a continuous function with respect to their arguments $\widetilde{f} : [0, b] \times \mathbb{R}^n \longrightarrow [0, +\infty)$ such that $\widetilde{x}$ satisfies the following integral equation:

$$\begin{aligned}\widetilde{x}(t) &= \frac{2t}{2b-\nu\eta^2}\int_0^b (b-s)\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds \\ &\quad -\frac{\nu t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds \\ &\quad -\int_0^t (t-s)\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds.\end{aligned}$$

Estimating the difference between $x(t)$ and $\widetilde{x}(t)$, we get

$$\begin{aligned}& \left|\widetilde{x}(t) - x(t)\right| \\ \leq\ & \frac{2t}{\left|2b-\nu\eta^2\right|}\int_0^b (b-s)\left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right)\right. \\ & \left. - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| ds \\ & + \frac{|\nu|\, t}{2b-\nu\eta^2}\int_0^\eta (\eta-s)^2\left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right)\right. \\ & \left. - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| ds \\ & + \int_0^t \left|(t-s)\right|\left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right)\right. \\ & \left. - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| ds.\end{aligned}$$

But

$$\begin{aligned}& \left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| \\ =\ & \left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right)\right. \\ & \left. + f\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right|.\end{aligned}$$

Using (3.3) and Lemma 2.1, we obtain

$$\begin{aligned}& \left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| \\ \leq\ & \left|\widetilde{f}\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right)\right| \\ & + \left|f\left(\widetilde{x}^{[0]}(s), \widetilde{x}^{[1]}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right| \\ \leq\ & \left\|\widetilde{f} - f\right\| + \sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M_0^j \left\|\widetilde{x} - x\right\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|\widetilde{x}-x\| &\leq b\left(\frac{3b^2+|\nu|\,\eta^3}{3\,|2b-\nu\eta^2|}+\frac{1}{2}b\right)\left\|\widetilde{f}-f\right\| \\ &\quad +b\left(\frac{3b^2+|\nu|\,\eta^3}{3\,|2b-\nu\eta^2|}+\frac{1}{2}b\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M_0^j\,\|\widetilde{x}-x\|\,.\end{aligned}$$

Therefore

$$\|\widetilde{x}-x\| \leq \frac{b\left(\dfrac{3b^2+|\nu|\,\eta^3}{3\,|2b-\nu\eta^2|}+\frac{1}{2}b\right)}{1-b\left(\dfrac{3b^2+|\nu|\,\eta^3}{3\,|2b-\nu\eta^2|}+\dfrac{1}{2}b\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M_0^j}\left\|\widetilde{f}-f\right\|.$$

This completes the proof. ■

3.5 Example

Below, we are going to provide an illustrating example. Considering the following boundary-value problem:

$$x''(t)+\sin^2 t+\frac{1}{10}\left(\cos^2 t\right)x^{[1]}(t)+\frac{1}{25}\left(\sin^2 t\right)x^{[2]}(t)=0, \tag{3.10}$$

$$x(0)=0,\ \nu\int_0^{\eta}x(s)\,ds=x(b)\ \text{ with } \eta\in(0,b)\,, \tag{3.11}$$

where

$$f(t,x,y)=\sin^2 t+\frac{1}{10}x\cos^2 t+\frac{1}{25}y\sin^2 t.$$

We have

$$|f(t,y_1,y_2)-f(t,z_1,z_2)|\leq\frac{1}{10}|y_1-z_1|+\frac{1}{25}|y_2-z_2|\,,$$

which means that

$$|f(t,y_1,y_2)-f(t,z_1,z_2)|\leq\sum_{i=1}^{2}c_i\,\|y_i-z_i\|\,,$$

with $c_1=\dfrac{1}{10}$ and $c_2=\dfrac{1}{25}$. In addition, if $b=\dfrac{\pi}{2}$ and $L_0=7,\ M_0=6$ in the definition of $\mathcal{CB}_{Int}(L_0,M_0)$, we have $f>0,\ \rho=\sup_{s\in[0,b]}|f(s,0,0)|=1$ and $\zeta=\dfrac{7}{4}$.

For $\nu = \frac{1}{2}$ and $\eta = \frac{1}{3}$, we have $2b = \pi \neq \nu\eta^2 = \frac{1}{18}$, $\nu = \frac{1}{2} < \frac{2}{\eta^2} = 18$ and

$$b\zeta\left(\frac{3b^2 + |\nu|\,\eta^3}{3\,|2b - \nu\eta^2|} + \frac{1}{2}b\right) \approx 4.362\,3 \leq L_0 = 7,$$

$$\zeta\left(\frac{3b^2 + |\nu|\,\eta^3}{3\,|2b - \nu\eta^2|} + b\right) \approx 4.1516 \leq M_0 = 6,$$

$$\left(b\left(\frac{3b^2 + |\nu|\,\eta^3}{3\,|2b - \nu\eta^2|} + \frac{1}{2}b\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M_0^j\right) \approx 0.947\,25 < 1.$$

It is not hard to check that all the assumptions of Theorems 3.2 and 3.3 are satisfied. Consequently, problem (3.10)-(3.11) possesses a unique positive solution that depends continuously on the function f.

3.6 Conclusion

In this chapter we are concerned with the existence, uniqueness and stability of positive solutions for a second order iterative functional differential equation with integral boundary conditions.

Applying the *Schauder* fixed point theorem, we have established some new sufficient conditions to ensure the existence of at least one positive solution for the proposed boundary value problem. In addition, under the same conditions of the existence theorem and an additional criterion, we have been able to prove that the problem admits a unique positive solution that depends continuously upon its nonlinearity which means that a slight change in the nonlinearity f should not induce a large change in the unique solution. In the end of this chapter, an example is given to show the accuracy of the conditions of our findings.

CHAPTER 4

Bounded solutions for a two-point boundary value problem for a class of second-order iterative differential equations

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In this chapter, we establish the existence, uniqueness and stability results for a second order iterative two-point boundary value problem.

4.1 Introduction

Two-point boundary value problems (TPBVPs) for functional differential equations on a finite interval is a functional differential equation where two conditions are specified at two different points, usually at the endpoints of the interval of interest. This terminology also indicates other cases when some conditions are required at endpoints and others at interior points. We find problems of such kind in various engineering models and physical phenomena, especially in astrodynamics, optimal control, beam deflections, heat flow, and various dynamical systems.

Despite the fact that these problems have highly significant real applications and, although there is an abundant literature on TPBVPs, intensive works still carrying out in this area. They have been widely studied by an overwhelming number of researchers by means of many approaches such as relaxation methods (finite difference methods), shooting methods, linearization methods, the *Hamilton–Jacobi* theory, the method of lower and upper solutions, degree theory, variational methods, the fixed point theory, to name a few.

Alas, the publications on iterative second order problems are very scarce because the introduction of the iterative terms makes these problems more difficult to deal with. For second-order iterative problems with two-point boundary conditions, we cite the following works:

By virtue of the *Schauder* fixed point theorem, *Kaufmann* [38] investigated the following problem:

$$x''(t) = h\left(t, x(t), x^{[2]}(t)\right), \; a \leq t \leq b,$$

where

$$x(a) = a, \quad x(b) = b,$$

or

$$x(a) = b, \quad x(b) = a.$$

Turab et al. [61] used the *Banach* fixed point theorem to establish the existence, uniqueness as well as the *Hyers-Ulam* and *Hyers-Ulam-Rassias* type stability of the solutions of the same equation of *Kaufmann* associated with $x(a) = a$ and $x(b) = b$.

So, inspired and motivated by the aforementioned works, this chapter focuses on the existence, uniqueness and stability of bounded solutions for the following class of second order iterative differential equations with two-point boundary conditions:

$$x''(t) = h\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \quad -b \leq t \leq b, \tag{4.1}$$

$$x(-b) = \eta_1, \; x(b) = \eta_2, \; \eta_1, \eta_2 \in [-b, b], \tag{4.2}$$

where $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(x) = x^{[n-1]}(x(t))$ and h is a continuous function from $[-b, b] \times \mathbb{R}^n$ to $\mathbb{R}$.

In this chapter, by utilizing the *Schauder* fixed point theorem combined with the *Green*'s functions method, new criteria are derived to discuss the existence of at least one bounded solution to problem (4.1) − (4.2). Moreover, under the previous conditions and another proper assumption, the *Banach* fixed point theorem with the *Green*'s functions method guarantee the existence of a unique bounded solution that depends continuously upon the source term h. Since the main idea here is also based on converting the proposed problem into an equivalent integral equation, then, the proofs follow the same strategy which is used in chapter 3. The only difference here is that we use the fixed point theorems together with some properties of the obtained *Green*'s kernel.

The plan of this chapter is as follows: In section 2, we state some necessary basic knowledge, we transform problem (4.1) − (4.2) into an equivalent integral equation and we also prove two useful properties of the obtained *Green*'s kernel. In section 3, the existence of at least one bounded solution of problem (4.1) − (4.2) is analyzed. In section 4, the existence and dependence on parameters of the unique bounded solution to problem (4.1) − (4.2) is discussed. In section 5, an illustrative example is exhibited to support the main theoretical achievements. Finally, we conclude the chapter by a brief conclusion in section 6.

4.2 Preliminaries

In this section, we would like to recall some basic preliminaries and present some definitions and lemmas which are used in what follows.

Let $\mathcal{X} = (\mathcal{C}([-b, b], \mathbb{R}), \|.\|)$ where $b > 0$ be a *Banach* space equipped with

$$\|x\| = \sup_{t \in [-b, b]} |x(t)|.$$

For $\alpha \in [0,b]$ and $\beta \geq 0$, we also define a convex bounded and closed subset of $\mathcal{X}$ as follows:

$$\mathcal{CB}(\alpha,\beta) = \left\{ x \in \mathcal{X}, \ -\alpha \leq x(t) \leq \alpha, \ |x(t_2) - x(t_1)| \leq \beta |t_2 - t_1|, \ \forall t_1, t_2 \in [-b,b] \right\}.$$

Remark 4.1 It results from Arzelà-Ascoli theorem that $\mathcal{CB}(\alpha,\beta) \subset \mathcal{X}$ is compact.

The nonlinear source term h has to comply the following condition:

$$|h(t,x_1,...,x_n) - h(t,z_1,...,z_n)| \leq \sum_{i=1}^{n} a_i \|x_i - z_i\|, \tag{4.3}$$

where $a_1, a_2, ..., a_n$ are positive constants.

Before applying *Schauder*'s fixed point theorem, we will rewrite our problem as an equivalent integral equation. So, the next lemma which guarantees such conversion, will be crucial in defining an integral operator which in turn will help us to reach our desired results.

Lemma 4.1 *If $b \neq 0$, then $x \in \mathcal{CB}(\alpha,\beta) \cap \mathcal{C}^2([-b,b],\mathbb{R})$ is a solution of (4.1) − (4.2) if and only if $x \in \mathcal{CB}(\alpha,\beta)$ is a solution of the following integral equation:*

$$x(t) = \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) + \int_{-b}^{b} G(t,s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds, \tag{4.4}$$

where $G(t,s)$ can be expressed as

$$G(t,s) = \frac{1}{2b} \begin{cases} (t-b)(s+b), & -b \leq s \leq t \leq b, \\ (t+b)(s-b), & -b \leq t \leq s \leq b. \end{cases} \tag{4.5}$$

Proof. Let $x \in \mathcal{CB}(\alpha,\beta) \cap \mathcal{C}^2([-b,b],\mathbb{R})$ is a solution of (4.1) − (4.2). An integration of equation (4.1) from $-b$ to t leads to

$$\int_{-b}^{t} x''(s)\, ds = \int_{-b}^{t} h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

Hence

$$x'(t) = x'(-b) + \int_{-b}^{t} h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

Integrating again from $-b$ to t, we get

$$\int_{-b}^{t} x'(s)\, ds = \int_{-b}^{t} x'(-b)\, ds + \int_{-b}^{t} \left[\int_{-b}^{x} h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \right] dx,$$

which implies that

$$x(t) = x(-b) + x'(-b)(t+b) + \int_{-b}^{t} (t-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

By (4.2), we obtain

$$x(t) = \eta_1 + x'(-b)(t+b) + \int_{-b}^{t} (t-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds, \tag{4.6}$$

and

$$x(b) = \eta_2 = \eta_1 + x'(-b)\, 2b + \int_{-b}^{b} (b-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \tag{4.7}$$

Thus

$$x'(-b) = \frac{\eta_2 - \eta_1}{2b} - \frac{1}{2b} \int_{-b}^{b} (b-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \tag{4.8}$$

The substitution of (4.8) in (4.6) gives

$$\begin{aligned} x(t) &= \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) - \frac{1}{2b} \int_{-b}^{b} (t+b)(b-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad + \int_{-b}^{t} (t-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &= \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) - \frac{1}{2b} \int_{-b}^{t} (b-t)(b+s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad - \frac{1}{2b} \int_{t}^{b} (t+b)(b-s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds, \end{aligned}$$

which is equivalent to

$$x(t) = \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) + \int_{-b}^{b} G(t,s)\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

Conversely, if we suppose that $x \in \mathcal{CB}(\alpha, \beta)$ and we differentiate both sides of the equation (4.4) with respect to t, we get equation (4.1) together with the given boundary conditions (4.2). ■

Lemma 4.2 *Green's function (4.5) satisfies*

$$\int_{-b}^{b} |G(t,s)|\, ds \leq \frac{1}{2} b^2. \tag{4.9}$$

Proof. We have

$$\int_{-b}^{b} |G(t,s)|\, ds \leq \frac{1}{2b}\int_{-b}^{t} |(t-b)(s+b)|\, ds + \frac{1}{2b}\int_{t}^{b} |(t+b)(s-b)|\, ds.$$

Since $(t-b)(s+b) \leq 0$ for $-b \leq s \leq t$ and $(t+b)(s-b) \leq 0$ for $t \leq s \leq b$, then

$$|(t-b)(s+b)| = (b-t)(b+s) \text{ and } |(t+b)(s-b)| = (b-s)(b+t),$$

which gives

$$\begin{aligned}\int_{-b}^{b} |G(t,s)|\, ds &= \frac{1}{2b}\int_{-b}^{t} (b-t)(b+s)\, ds + \frac{1}{2b}\int_{t}^{b} (b-s)(b+t)\, ds \\ &= \frac{1}{2}b^2 - \frac{1}{2}t^2 \\ &\leq \frac{1}{2}b^2.\end{aligned}$$

Thereby, the proof is finished. ■

Lemma 4.3 *Green's function* (4.3) *satisfies*

$$\int_{-b}^{b} |G(t_2,s) - G(t_1,s)| \leq 5b\,|t_2 - t_1|.$$

Proof. Let $t_1, t_2 \in [-b,b]$ with $t_1 < t_2$ and let

$$\begin{aligned}G_1(t,s) &= \frac{1}{2b}(t-b)(s+b), \\ G_2(t,s) &= \frac{1}{2b}(t+b)(s-b).\end{aligned}$$

We have

$$\begin{aligned}\int_{-b}^{b} |G(t_2,s) - G(t_1,s)|\, ds &= \int_{-b}^{t_1} |G_1(t_2,s) - G_1(t_1,s)|\, ds \\ &+ \int_{t_1}^{t_2} |G_1(t_2,s) - G_2(t_1,s)|\, ds \\ &+ \int_{t_2}^{b} |G_2(t_2,s) - G_2(t_1,s)|\, ds.\end{aligned}$$

Since

$$\begin{aligned}\int_{-b}^{t_1} |G_1(t_2,s) - G_1(t_1,s)|\, ds &= \int_{-b}^{t_1} \left|\frac{1}{2b}(t_2-b)(s+b) - \frac{1}{2b}(t_1-b)(s+b)\right| ds \\ &\leq \frac{1}{2b}|t_2 - t_1| \int_{-b}^{t_1} (b+s)\, ds \\ &= \frac{1}{4b}|t_2 - t_1|(b+t_1)^2 \\ &\leq b\,|t_2 - t_1|,\end{aligned}$$

$$\begin{aligned}\int_{t_1}^{t_2} \left|G_1(t_2,s)-G_2(t_1,s)\right| ds &= \int_{t_1}^{t_2} \left|\frac{1}{2b}(t_2-b)(s+b)-\frac{1}{2b}(t_1+b)(s-b)\right| ds \\ &\leq \frac{1}{2b}\int_{t_1}^{t_2}(b-t_2)(s+b)\,ds+\frac{1}{2b}\int_{t_1}^{t_2}(t_1+b)(b-s)\,ds \\ &\leq 3b\left|t_2-t_1\right|,\end{aligned}$$

and

$$\begin{aligned}\int_{t_2}^{b} \left|G_2(t_2,s)-G_2(t_1,s)\right| ds &= \int_{t_2}^{b} \left|\frac{1}{2b}(t_2+b)(s-b)-\frac{1}{2b}(t_1+b)(s-b)\right| ds \\ &\leq \frac{1}{2b}\left|t_2-t_1\right|\int_{t_2}^{b}(b-s)\,ds \\ &= \frac{1}{4b}\left|t_2-t_1\right|(b-t_2)^2 \\ &\leq b\left|t_2-t_1\right|.\end{aligned}$$

Then

$$\int_{-b}^{b}\left|G(t_2,s)-G(t_1,s)\right| ds \leq 5b\left|t_2-t_1\right|.$$

The proof is now completed. ■

4.3 Existence results

The main objective of this section is to establish some suitable conditions for guaranteeing the existence of at least one bounded solution of problem (4.1) − (4.2).

Now, we will need to construct an appropriate operator $\mathcal{F}$ from $\mathcal{CB}(\alpha,\beta)$ to $\mathcal{X}$. Let us denote the right hand side of the integral equation in Lemma 4.1 as follows:

$$(\mathcal{F}\psi)(t)=\eta_1+\frac{\eta_2-\eta_1}{2b}(t+b)+\int_{-b}^{b}G(t,s)\,h\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)ds. \quad (4.10)$$

From Lemma 4.1, fixed points of $\mathcal{F}$ are solutions of problem (4.1) − (4.2) and vice versa. So, our key aim here is to prove that $\mathcal{F}$ admits at least one fixed point.

Lemma 4.4 *If condition (4.3) holds. Then the operator $\mathcal{F}$ is continuous.*

Proof. It's obvious that $\mathcal{F}$ is well defined and for all $\varphi,\psi\in\mathcal{CB}(\alpha,\beta)$, we have

$$\left|(\mathcal{F}\varphi)(t)-(\mathcal{F}\psi)(t)\right|\leq\int_{-b}^{b}\left|G(t,s)\right|\left|h\left(s,\varphi(s),...,\varphi^{[n]}(s)\right)-h\left(s,\psi(s),...,\psi^{[n]}(s)\right)\right|ds.$$

Taking into account (4.3) and Lemma 2.1, we get

$$\begin{aligned}\left|h\left(s,\varphi\left(s\right),...,\varphi^{[n]}\left(s\right)\right)-h\left(s,\psi\left(s\right),...,\psi^{[n]}\left(s\right)\right)\right| &\leq \sum_{i=1}^{n}a_i\left\|\varphi^{[i]}-\psi^{[i]}\right\| \\ &\leq \sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\left\|\varphi-\psi\right\|. \qquad (4.11)\end{aligned}$$

It follows from (4.9) and (4.11) that

$$\begin{aligned}\left|\left(\digamma\varphi\right)(t)-\left(\digamma\psi\right)(t)\right| &\leq \sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\left\|\varphi-\psi\right\|\int_{-b}^{b}\left|G\left(t,s\right)\right|ds \\ &\leq \frac{b^2}{2}\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\left\|\varphi-\psi\right\|.\end{aligned}$$

This immediately implies that operator $\digamma$ is continuous. ■

Lemma 4.5 *If condition (4.3) is satisfied and if*

$$2\left|\eta_1\right|+\left|\eta_2\right|+\frac{b^2}{2}\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\leq\alpha, \qquad (4.12)$$

where

$$h_0=\max_{s\in[-b,b]}\left|h\left(s,0,...,0\right)\right|,$$

then

$$-\alpha\leq\left(\digamma\psi\right)(t)\leq\alpha,$$

for all $t\in[-b,b]$ and $\psi\in\mathcal{CB}\left(\alpha,\beta\right).$

Proof. If $\psi\in\mathcal{CB}\left(\alpha,\beta\right)$, we have

$$\begin{aligned}\left|\left(\digamma\psi\right)(t)\right| &= \left|\eta_1+\frac{\eta_2-\eta_1}{2b}\left(t+b\right)+\int_{-b}^{b}G\left(t,s\right)h\left(s,\psi\left(s\right),\psi^{[2]}\left(s\right),...,\psi^{[n]}\left(s\right)\right)ds\right| \\ &\leq 2\left|\eta_1\right|+\left|\eta_2\right|+\int_{-b}^{b}\left|G\left(t,s\right)\right|\left|h\left(s,\psi\left(s\right),\psi^{[2]}\left(s\right),...,\psi^{[n]}\left(s\right)\right)\right|ds.\end{aligned}$$

It follows from (4.3) and Lemma 2.1 that

$$\left|h\left(s,\psi^{[1]}\left(s\right),...,\psi^{[n]}\left(s\right)\right)\right|\leq h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j. \qquad (4.13)$$

Furthermore, (4.9), (4.12) and (4.13) lead to

$$\left|\left(\digamma\psi\right)(t)\right|\leq 2\left|\eta_1\right|+\left|\eta_2\right|+\frac{b^2}{2}\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\leq\alpha.$$

Consequently $-\alpha\leq\left(\digamma\psi\right)(t)\leq\alpha$ for all $t\in[-b,b]$ and $\psi\in\mathcal{CB}\left(\alpha,\beta\right).$ ■

Lemma 4.6 *If condition (4.3) holds and if*

$$\frac{|\eta_1-\eta_2|}{2|b|}+5b\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\leq\beta, \tag{4.14}$$

then

$$|(\digamma\psi)(t_2)-(\digamma\psi)(t_1)|\leq\beta|t_2-t_1|,$$

for all $t_1,t_2\in[-b,b]$ *and* $\psi\in\mathcal{CB}(\alpha,\beta)$.

Proof. If $t_1,t_2\in[-b,b]$ and $\psi\in\mathcal{CB}(\alpha,\beta)$ with $t_1<t_2$, we have

$$\begin{aligned}|(\digamma\psi)(t_2)-(\digamma\psi)(t_1)| &\leq\left|\eta_1+\frac{\eta_2-\eta_1}{2b}(t_2+b)-\eta_1-\frac{\eta_2-\eta_1}{2b}(t_1+b)\right|\\ &+\int_{-b}^{b}|G(t_2,s)-G(t_1,s)|\left|h\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)\right|ds.\end{aligned}$$

From Lemma 4.3 and (4.13), we deduce that

$$\begin{aligned}|(\digamma\psi)(t_2)-(\digamma\psi)(t_1)| &\leq\frac{|\eta_1-\eta_2|}{2|b|}|t_2-t_1|+5b\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)|t_2-t_1|\\ &=\left(\frac{|\eta_1-\eta_2|}{2|b|}+5b\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\right)|t_2-t_1|.\end{aligned}$$

Thanks to (4.14), we have

$$|(\digamma\psi)(t_2)-(\digamma\psi)(t_1)|\leq\beta|t_2-t_1|,$$

which completes the proof. ■

Remark 4.2 It results from Lemmas 4.5 and 4.6 that $\digamma$ maps $\mathcal{CB}(\alpha,\beta)$ into itself, i.e., $\digamma(\mathcal{CB}(\alpha,\beta))\subset\mathcal{CB}(\alpha,\beta)$.

Theorem 4.1 *Suppose that conditions (4.3), (4.12) and (4.14) hold. Then problem (4.1) − (4.2) admits at least one bounded solution* x *in* $\mathcal{CB}(\alpha,\beta)$.

Proof. Thanks to Remark 4.1, $\mathcal{CB}(\alpha,\beta)$ is compact. Furthermore, from Lemma 4.4 and Remark 4.2, all requirements of *Schauder*'s fixed point theorem are fulfilled. Then, $\digamma$ has a fixed point x in $\mathcal{CB}(\alpha,\beta)$ such that $\digamma x=x$, which means that x is a solution of problem (4.1) − (4.2). ■

4.4 Existence, uniqueness and continuous dependence results

Now, besides the previous hypotheses, we impose further an additional condition under which problem (4.1) − (4.2) can admit a unique solution that depends continuously on parameters.

Theorem 4.2 *Besides the assumptions of Theorem 4.1, we assume that*

$$\frac{1}{2}b^2\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j<1. \tag{4.15}$$

Then problem (2.1)-(2.2) has a unique bounded solution.

Proof. Let $\varphi,\psi\in\mathcal{CB}(\alpha,\beta)$. Similarly as in the proof of Lemma 4.4, we have

$$\left|(\mathcal{F}\varphi)(t)-(\mathcal{F}\psi)(t)\right|\le\left(\frac{b^2}{2}\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\|\varphi-\psi\|.$$

From condition (4.15), we deduce that $\mathcal{F}$ is a contraction. So, the *Banach* fixed point theorem ensures that $\mathcal{F}$ has a unique fixed point in $\mathcal{CB}(\alpha,\beta)$ which is exactly the unique solution of problem (4.1) − (4.2). ■

Our goal here is to establish the continuous dependence of the unique solution upon the function h.

Definition 4.1 The solution of the integral equation (4.4) depends continuously on the function h if $\forall\varepsilon>0$, $\exists\delta(\varepsilon)>0$ such that

$$\begin{aligned}&\Big|\widetilde{h}\left(\widetilde{x}^{[0]}(s),\widetilde{x}^{[1]}(s),\widetilde{x}^{[2]}(s),...,\widetilde{x}^{[n]}(s)\right)\\&-h\left(x^{[0]}(s),x^{[1]}(s),x^{[2]}(s),...,x^{[n]}(s)\right)\Big|\le\delta(\varepsilon)\Longrightarrow\|\widetilde{x}-x\|\le\varepsilon,\end{aligned}$$

where $\widetilde{x}$ is the unique solution of the integral equation

$$\widetilde{x}(t)=\eta_1+\frac{\eta_2-\eta_1}{2b}(t+b)+\int_{-b}^{b}|G(t,s)|\,\widetilde{h}\left(s,\widetilde{x}(s),\widetilde{x}^{[2]}(s),...,\widetilde{x}^{[n]}(s)\right)ds.$$

Theorem 4.3 *Under the assumption of Theorem 4.2, the unique solution of problem (4.1) − (4.2) depends continuously on function h.*

Proof. Let $x \in \mathcal{CB}(\alpha,\beta)$ be the unique solution of problem (4.1) − (4.2). Therefore, x satisfies

$$x(t) = \eta_1 + \frac{\eta_2-\eta_1}{2b}(t+b) + \int_{-b}^{b} |G(t,s)|\, h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

If $\widetilde{x} \in \mathcal{CB}(\alpha,\beta)$ is a perturbed solution of (4.1) − (4.2) with a small perturbation in function the h and satisfies Theorem 4.2, then there exists a continuous function $\widetilde{h}$ from $[-b,b]\times\mathbb{R}^n$ to $\mathbb{R}$ such that

$$\widetilde{x}(t) = \eta_1 + \frac{\eta_2-\eta_1}{2b}(t+b) + \int_{-b}^{b} |G(t,s)|\, \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) ds.$$

We have

$$\begin{aligned} |\widetilde{x}(t) - x(t)| \leq & \int_{-b}^{b} |G(t,s)| \left| \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) \right. \\ & \left. - h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right| ds, \end{aligned}$$

and taking into account conditions (4.3), (4.9) and Lemma 2.1, we obtain

$$\begin{aligned} & \left| \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right| \\ & \leq \left| \widetilde{h}\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - h\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) \right| \\ & + \left| h\left(s, \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)\right) - h\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right| \\ & \leq \left\| \widetilde{h} - h \right\| + \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j \left\| \widetilde{x} - x \right\|. \end{aligned}$$

So

$$|\widetilde{x}(t) - x(t)| \leq \frac{1}{2}b^2 \left(\left\| \widetilde{h} - h \right\| + \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j \left\| \widetilde{x} - x \right\| \right).$$

Using (4.15) , we obtain

$$\left\| \widetilde{x} - x \right\| \leq \frac{\frac{1}{2}b^2}{1 - \frac{1}{2}b^2 \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j} \left\| \widetilde{h} - h \right\|.$$

This completes the proof. ■

4.5 Example

In this section, we give an example to show the feasibility and effectiveness of our outcomes.

Let us consider the following problem:

$$x''(t) = \frac{1}{11}\sin 15t + \frac{1}{5}x(t) + \frac{1}{7}x^{[2]}(t) + \frac{1}{9}x^{[3]}(t), \quad -\frac{\pi}{30} \le t \le \frac{\pi}{30}, \tag{4.16}$$

$$x\left(-\frac{\pi}{30}\right) = \frac{1}{12\pi}, \; x\left(\frac{\pi}{30}\right) = \frac{1}{8\pi}, \tag{4.17}$$

where

$$h(t, x_1, x_2, x_3) = \frac{1}{11}\sin 15t + \frac{1}{5}x_1 + \frac{1}{7}x_2 + \frac{1}{9}x_3, \; b = \frac{\pi}{30}, \; \eta_1 = \frac{1}{12\pi} \text{ and } \eta_2 = \frac{1}{8\pi}.$$

So

$$a_1 = \frac{1}{5}, \; a_2 = \frac{1}{7}, \; a_3 = \frac{1}{9} \text{ and } h_0 = \frac{1}{11}.$$

Let $\mathcal{CB}(\alpha, \beta) = \mathcal{CB}\left(\frac{\pi}{31}, \frac{\pi}{4}\right)$. We get

$$2|\eta_1| + |\eta_2| + \frac{b^2}{2}\left(h_0 + \alpha \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j\right) \approx 0.093\,74 \le \alpha = \frac{\pi}{31} \approx 0.101\,34,$$

$$\frac{|\eta_1 - \eta_2|}{2|b|} + 5b\left(h_0 + \alpha \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j\right) \approx 0.14924 \le \beta = \frac{\pi}{4} \approx 0.78540,$$

and

$$\frac{1}{2}b^2 \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j \approx 0.0039587 < 1.$$

We can easily check that all requirements of Theorems 4.1, 4.2 and 4.3 are satisfied. Thus, problem (4.16) − (4.17) admits a unique bounded solution in $\mathcal{CB}\left(\frac{\pi}{31}, \frac{\pi}{4}\right)$ that depends continuously on the function h.

4.6 Conclusion

This chapter dealt with a class of iterative second-order differential equations with two point boundary conditions.

The method used here is an attractive hybrid technique that combines the *Schauder* fixed point theorem and the *Green*'s function method. The essence of this technique lies in converting the proposed problem into an equivalent integral equation whose kernel is a *Green*'s function. To be more precise, it consists in finding an integral operator that will help us together with some useful properties of the obtained kernel in applying the *Schauder* fixed point theorem comfortably and hence in proving the existence of at least one bounded solution of the iterative boundary value problem.

Furthermore, for the existence, uniqueness and continuous dependence on parameters of bounded solutions, we added an extra condition under which *Banach* fixed point theorem can be applied and hence the solution remained unique such that small experimental or physical errors in the initial measurements do not lead to a dramatic effect on it. Finally, an example justifying the validity of the derived theoretical results is provided.

CHAPTER 5

Periodic solutions for a class of second-order iterative differential equations with periodic coefficients

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In this chapter, we are concerned with the study of the existence of periodic solutions for a class of second order iterative differential equations with periodic coefficients.

5.1 Introduction

Second-order differential equations with periodic coefficients have taken great interest by many scholars due to their significant role in providing a more accurate and realistic modelling of plenty of real phenomena in different fields, ranging from life sciences to physics, technology, and engineering.

The existence problem for periodic solutions is still an active area of research. It has attracted the attention of many scholars and has yielded many interesting results. These kinds of equations have been investigated by using various methods such as the upper and lower solutions method, the monotone iterative technique, the continuation method of topological degree, fixed point theory, the variational method, and critical point theory, etc.

However, relatively few authors have discussed the existence of periodic solutions for second-order iterative differential equations. Some interesting results about this subject can be found in [12] and [14]. For example:

In [12], the authors used the *Banach* and *Schauder* fixed point theorems to obtain some sufficient conditions that guarantee the existence of periodic solutions of the equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p\left(t\right)\frac{d}{dt}x(t)+q\left(t\right)x(t)\\&\qquad=\frac{d}{dt}g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right),\end{aligned}$$

where p and q are two T-periodic continuous functions, $f(t,x_1,x_2,...,x_n)$ and $g(t,x_1,x_2,...,x_n)$ are supposed periodic in t with a common period T and are globally *Lipschitz* with respect to their arguments.

In [14], the *Krasnoselskii* fixed point theorem is applied to show the existence of positive periodic solutions for the following equation:

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p\left(t\right)\frac{d}{dt}x(t)+q\left(t\right)x(t)\\&\qquad=\frac{d}{dt}f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+\sum_{i=1}^{n}c_i\left(t\right)x^{[i]}\left(t\right).\end{aligned}$$

In this chapter, we are interested in the following class of second order iterative differential equations:

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}C_{\ell,m}(t)\left(x^{[m]}(t)\right)^{\ell}$$
$$+\frac{d}{dt}g\left(t,x^{[1]}(t),x^{[2]}(t),....,x^{[n]}(t)\right)+w(t), \quad (5.1)$$

where $x^{[2]}(t)=x(x(t))$, $x^{[m]}(t)=x^{[m-1]}(x(t))$, $p,q,C_{\ell,m},w\in\mathcal{C}(\mathbb{R},\mathbb{R})$, $m=\overline{1,n}$, $\ell=\overline{1,\infty}$, and $g\in\mathcal{C}^1(\mathbb{R}^{n+1},\mathbb{R})$ satisfies some other conditions that will be specified later.

In this chapter, we discuss the existence of periodic solutions of equation (5.1) by using the same technique that was utilized in chapter 4 which will be a proper vehicle for allowing us to achieve our objectives. As we said before, most of the hard tasks in this study revolve around controlling the iterative terms and reaching acceptable results especially the periodicity and boundedness of the sought solutions.

We will follow three main steps: first of all, we define the *Banach* space and a suitable subset of it to get rid of the headache caused by the presence of the iterative terms and also to pave the way to an easy application of the chosen fixed point theorems; secondly, we reformulate equation (5.1) with the periodic boundary conditions as an integral equation with a *Green*'s function kernel and thirdly, by employing *Schauder*'s fixed point theorem as well as some properties of the obtained *Green*'s kernel, we establish a set of sufficient conditions that prove the existence of periodic solutions of equation (5.1).

The plan of the chapter is structured as follows. In section 2, we begin with introducing some necessary definitions, lemmas and corollaries required to derive the main outcomes. In section 2, we state and prove our existence results. Finally, a conclusion is given in the last section.

5.2 Preliminaries

In this Section, we present some notations and preliminary material needed to establish our main findings and we also state and prove some preliminary results.

For $T>0$, let

$$P_T=\{x\in\mathcal{C}(\mathbb{R},\mathbb{R}),\ x(t+T)=x(t)\}.$$

Endowing P_T with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

we obtain a *Banach* space and for every $L,\ M \geq 0$

$$P_T(L,M) = \{x \in P_T : \|x\| \leq L,\ \ |x(t_2) - x(t_1)| \leq M |t_2 - t_1|,\ t_1, t_2 \in \mathbb{R}\},$$

is a closed convex and bounded subset of P_T.

For convenience, we list the following assumptions:

p, q, $C_{\ell,m}$, $m = \overline{1,n}$, $\ell = \overline{1,\infty}$ and w are continuous and T-periodic real-valued functions such that:

$$p(t+T) = p(t),\ q(t+T) = q(t),\ C_{\ell,m}(t+T) = C_{\ell,m}(t),\ w(t+T) = w(t),\ \ (5.2)$$

and

$$\int_0^T p(s)\,ds > 0,\ \int_0^T q(s)\,ds > 0. \qquad (5.3)$$

We will try to accomplish the desired results by making the following assumptions on the function g:

The function $g(t, x_1, x_2, ..., x_n)$ is supposed T-periodic in t and globally *Lipschitz* in $x_1, ..., x_n$. i.e.,

$$g(t+T, x_1, x_2, ..., x_n) = g(t, x_1, x_2, ..., x_n), \qquad (5.4)$$

and there exist n positive constants $\sigma_1, \sigma_2, ..., \sigma_n$ such that

$$|g(t, x_1, x_2, ..., x_n) - g(t, y_1, y_2, ..., y_n)| \leq \sum_{m=1}^{n} \sigma_m \|x_m - y_m\|, \qquad (5.5)$$

and

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \geq 1, \qquad (5.6)$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s) ds \right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Lemma 5.1 *[49] Suppose that (5.2), (5.3), and (5.6) hold. Then, there are two continuous and T-periodic functions a and b such that*

$$b(t) > 0,\ \int_0^T a(u)\,du > 0,\ \ a(t) + b(t) = p(t)\ \text{ and } \frac{d}{dt}b(t) + a(t)b(t) = q(t),$$

for all $t \in \mathbb{R}$.

Lemma 5.2 *[62] Suppose that all conditions of Lemma 5.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t),$$

has a $T-$periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$\begin{aligned} G(t,s) = {} & \frac{\int_t^s \exp\left[\int_t^u b(v)\,dv + \int_u^s a(v)\,dv\right] du}{\left[\exp(\int_0^T a(u)\,du) - 1\right]\left[\exp(\int_0^T b(u)\,du) - 1\right]} \\ & + \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv + \int_u^{s+T} a(v)\,dv\right] du}{\left[\exp(\int_0^T a(u)\,du) - 1\right]\left[\exp(\int_0^T b(u)\,du) - 1\right]}. \end{aligned} \tag{5.7}$$

Corollary 5.1 *[62] Green's function G defined by (5.7) satisfies the following properties:*

$$\begin{gathered} G(t,t+T) = G(t,t), \quad G(t+T,s+T) = G(t,s), \\ \frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp(\int_t^s b(v)\,dv)}{\exp(\int_0^T b(v)\,dv) - 1}. \end{gathered} \tag{5.8}$$

Lemma 5.3 *Suppose (5.2)-(5.4) and (5.6) hold. If $x \in P_T(L,M) \cap \mathcal{C}^2(\mathbb{R},\mathbb{R})$, then x is a solution of (5.1) if and only if $x \in P_T(L,M)$ is a solution of the following integral equation:*

$$\begin{aligned} x(t) = {} & \sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T} C_{\ell,m}(s)\left(x^{[m]}(s)\right)^{\ell} G(t,s)\,ds + \int_t^{t+T} G(t,s)\,w(s)\,ds \\ & + \int_t^{t+T} \left[E(t,s) - a(s)G(t,s)\right] g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds, \end{aligned}$$

where

$$E(t,s) = \frac{\exp(\displaystyle\int_t^s b(v)\,dv)}{\exp(\displaystyle\int_0^T b(v)\,dv) - 1}. \tag{5.9}$$

Proof. Suppose that $x \in P_T(M, L) \cap \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ is a solution of equation (5.1). From Lemma 5.2, we have

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t,s) \left[\sum_{m=1}^{n} \sum_{\ell=1}^{\infty} C_{\ell,m}(s) \left(x^{[m]}(s) \right)^{\ell} \right] ds \\ &+ \int_t^{t+T} G(t,s) \left[\frac{d}{ds} g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right] ds \\ &+ \int_t^{t+T} G(t,s) w(s) \, ds \end{aligned}$$

An integration by parts gives

$$\begin{aligned} &\int_t^{t+T} G(t,s) \left[\frac{d}{ds} g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right] ds \\ &\qquad = G(t,s) \, g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \Big|_t^{t+T} \\ &\qquad - \int_t^{t+T} \left[\frac{d}{ds} G(t,s) \right] g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \, . \end{aligned}$$

It follows from Corollary 5.1 that

$$\left[G(t,s) g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right]_t^{t+T} = 0,$$

and

$$\begin{aligned} &\int_t^{t+T} G(t,s) \left[\frac{d}{ds} g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right] ds \\ &\qquad = \int_t^{t+T} g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \left[E(t,s) - a(s) G(t,s) \right]. \end{aligned}$$

Consequently

$$\begin{aligned} x(t) &= \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} \int_t^{t+T} C_{\ell,m}(s) \left(x^{[m]}(s) \right)^{\ell} G(t,s) \, ds \\ &+ \int_t^{t+T} \left[E(t,s) - a(s) G(t,s) \right] g\left(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \int_t^{t+T} G(t,s) w(s) \, ds. \end{aligned}$$

It is not hard to obtain the converse implication. Indeed, the derivation of the last integral equation completes the proof. ■

Lemma 5.4 *[62] Let* $A=\int_0^T p(u)\,du$ *and* $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2\geq 4B, \tag{5.10}$$

then

$$\min\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\geq \frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=M_1,$$

and

$$\max\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\leq \frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=M_2.$$

Corollary 5.2 *[62] Functions* G *and* E *satisfy*

$$0<\alpha_1\leq G(t,s)\leq\alpha_2 \text{ and } |E(t,s)|\leq\beta, \tag{5.11}$$

where

$$\alpha_1=\frac{T}{\left(e^{M_2}-1\right)^2},\ \alpha_2=\frac{T\exp(\int_0^T p(u)\,du)}{\left(e^{M_1}-1\right)^2} \text{ and } \beta=\frac{\exp(\int_0^T b(v)\,dv))}{\exp(\int_0^T b(v)\,dv)-1}.$$

We will use in the sequel the following notations:

$$\lambda=\max_{t\in[0,T]}|b(t)|,\ \lambda_1=\max_{t\in[0,T]}|a(t)|,\ \gamma=\exp\left(-\int_0^T b(v)\,dv\right),$$

$$\Gamma=T\lambda\beta,\rho_1=\max_{t\in[0,T]}|g(t,0,....,0)|,\ \varsigma=\rho_1+L\sum_{m=1}^{n}\sum_{j=0}^{m-1}\sigma_m M^j,$$

$$\delta=\frac{1}{\left[\exp\left(\int_0^T a(v)\,dv\right)-1\right]\left[\exp\left(\int_0^T b(v)\,dv\right)-1\right]},$$

$$\mu=Te^{2M_2}\delta\left[T\lambda\gamma\left(2e^{2M_2}+1\right)+e^{M_2}+1\right],$$

$$P_T(L)=\{x\in P_T:\|x\|\leq L\}.$$

and we suppose that

$$T\left(\alpha_2 L^{\ell-1}\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\sum_{j=0}^{m-1}\ell L_{C_{\ell,m}}M^j+(\alpha_2\lambda_1+\beta)\sum_{m=1}^{n}\sum_{j=0}^{m-1}\sigma_m M^j\right)<\infty, \tag{5.12}$$

$$\alpha_2 T\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}L_{C_{\ell,m}}L^\ell+(\beta+\alpha_2\lambda_1)T\varsigma+\alpha_2 T L_w\leq L, \tag{5.13}$$

and

$$\left(\lambda_1\varsigma+L_w+\sum_{m=1}^{n}\sum_{l=1}^{\infty}L^\ell L_{C_{\ell,m}}\right)(2\alpha_2+\mu)+\varsigma(2\beta+\Gamma)\leq M. \tag{5.14}$$

Lemma 5.5 *[12] For any $t_1, t_2 \in [0,T]$, we have*

$$\int_{t_1}^{t_1+T} \left|G_2(t_2,s) - G_2(t_1,s)\right| ds \le \mu \left|t_2 - t_1\right|.$$

Lemma 5.6 *For any $t_1,\ t_2 \in [0,T]$, we have*

$$\int_{t_1}^{t_1+T} \left|E(t_2,s) - E(t_1,s)\right| ds \le \Gamma \left|t_2 - t_1\right|.$$

Proof. Let $t_1,\ t_2 \in [0,T]$, we have

$$\begin{aligned}\int_{t_1}^{t_1+T} \left|E(t_2,s) - E(t_1,s)\right| ds &= \int_{t_1}^{t_1+T} \frac{\left|\exp\left(\int_{t_2}^{s} b(v)\,dv\right) - \exp\left(\int_{t_1}^{s} b(v)\,dv\right)\right|}{\left(\exp\left(\int_0^T b(v)\,dv\right) - 1\right)} ds \\ &\le \frac{\int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} b(v)\,dv\right)\left|1 - \exp\left(\int_{t_1}^{t_2} b(v)\,dv\right)\right| ds}{\left(\exp\left(\int_0^T b(v)\,dv\right) - 1\right)}.\end{aligned}$$

We apply the mean value theorem to the function $f(t) = \exp\left(\int_t^{t_2} b(v)\,dv\right)$ over the interval $[t_1, t_2]$, then there exists $\xi \in]t_1, t_2[$ such that

$$\begin{aligned}\left|f(t_2) - f(t_1)\right| &= \left|\exp\left(\int_{t_2}^{t_2} b(v)\,dv\right) - \exp\left(\int_{t_1}^{t_2} b(v)\,dv\right)\right| \\ &= \left|1 - \exp\left(\int_{t_1}^{t_2} b(v)\,dv\right)\right| \\ &\le \left|f'(\xi)\right| \left|t_2 - t_1\right| \\ &= b(\xi)\exp\left(\int_{\xi}^{t_2} b(v)\,dv\right)\left|t_2 - t_1\right|.\end{aligned}$$

So

$$\begin{aligned}&\int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} b(v)\,dv\right)\left|1 - \exp\left(\int_{t_1}^{t_2} b(v)\,dv\right)\right| ds \\ &\le \int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} b(v)\,dv\right) b(\xi)\exp\left(\int_{\xi}^{t_2} b(v)dv\right)\left|t_2 - t_1\right| ds \\ &\le \sup_{s\in[0,T]} \left|b(s)\right| \left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\left(\int_{\xi}^{t_2} b(v)dv\right)\exp\left(\int_{t_2}^{s} b(v)\,dv\right) ds \\ &= \sup_{s\in[0,T]} \left|b(s)\right| \left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\left(\int_{\xi}^{t_2} b(v)dv + \int_{t_2}^{s} b(v)\right) ds \\ &= \sup_{s\in[0,T]} \left|b(s)\right| \left|t_2 - t_1\right| \int_{t_1}^{t_1+T} \exp\left(\int_{\xi}^{s} b(v)dv\right) ds \\ &\le \sup_{s\in[0,T]} \left|b(s)\right| \left|t_2 - t_1\right| \exp\left(\int_0^T b(v)dv\right) \int_{t_1}^{t_1+T} ds \\ &= \lambda T \exp\left(\int_0^T b(v)dv\right)\left|t_2 - t_1\right|.\end{aligned}$$

Consequently

$$\begin{aligned}\int_{t_1}^{t_1+T}\left|E\left(t_2,s\right)-E\left(t_1,s\right)\right|ds &\leq \frac{\int_{t_1}^{t_1+T}\exp\left(\int_{t_2}^{s}b\left(v\right)dv\right)\left|1-\exp\left(\int_{t_1}^{t_2}b\left(v\right)dv\right)\right|ds}{\left(\exp\left(\int_0^T b\left(v\right)dv\right)-1\right)}.\\ &=\lambda T\frac{\exp\left(\int_0^T b(v)dv\right)}{\left(\exp\left(\int_0^T b\left(v\right)dv\right)-1\right)}\left|t_2-t_1\right|\\ &=\lambda T\beta\left|t_2-t_1\right|=\Gamma\left|t_2-t_1\right|,\end{aligned}$$

which completes the proof. ■

5.3 Existence results

Now, we will discuss the existence of at least one periodic solution of equation (5.1). Let's begin our study with defining an operator $\mathcal{H}:P_T\left(L,M\right)\longrightarrow P_T$ as follows:

$$\begin{aligned}\left(\mathcal{H}\varphi\right)\left(t\right)&=\sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T}C_{\ell,m}\left(s\right)\left(\varphi^{[m]}\left(s\right)\right)^{\ell}G\left(t,s\right)ds\\ &+\int_t^{t+T}\left[E\left(t,s\right)-a\left(s\right)G\left(t,s\right)\right]g\left(s,\varphi^{[1]}\left(s\right),\varphi^{[2]}\left(s\right),....,\varphi^{[n]}\left(s\right)\right)ds\\ &+\int_t^{t+T}G\left(t,s\right)w\left(s\right)ds. \end{aligned}\tag{5.15}$$

Clearly, $\left(\mathcal{H}\varphi\right)\left(t+T\right)=\left(\mathcal{H}\varphi\right)\left(t\right)$ which shows that operator $\mathcal{H}$ is well defined. Furthermore, fixed points of the operator $\mathcal{H}$ are solutions of equation (5.1) and vice versa.

Remark 5.1 Thanks to Arzelà-Ascoli theorem, we can prove the compactness of the aforementioned set $P_T\left(L,M\right)$.

Now, we are able to state the following result:

Theorem 5.1 *Let $C_{\ell,m}\in P_T\left(L_{C_{\ell,m}}\right)$. Suppose that conditions (5.2)-(5.6), (5.10) and (5.12)-(5.14) hold. Then equation (5.1) has at least one periodic solution in $P_T\left(L,M\right)$.*

Proof. For establishing the existence result, we use *Schauder*'s fixed point theorem and for sake of clarity, the proof will be done in two steps.

Step 1: We show that $\mathcal{H}$ is a continuous operator on $P_T(L, M)$. Let $\varphi, \psi \in P_T(L, M)$. We have

$$\begin{aligned} |(\mathcal{H}\varphi)(t) - (\mathcal{H}\psi)(t)| \leq & \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} \int_t^{t+T} |C_{\ell,m}(s)|\, G(t,s) \left| \left(\varphi^{[m]}(s)\right)^{\ell} - \left(\psi^{[m]}(s)\right)^{l} \right| ds \\ & + \int_t^{t+T} |E(t,s)| \left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right. \\ & \left. - g\left(s, \psi^{[1]}(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right) \right| ds \\ & + \int_t^{t+T} |a(s)|\, G(t,s) \left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right. \\ & \left. - g\left(s, \psi^{[1]}(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right) \right| ds. \end{aligned}$$

From (5.5) and Lemma 2.1, we obtain

$$\begin{aligned} & \left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - g\left(s, \psi^{[1]}(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right) \right| \\ & \qquad \leq \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sigma_m M^j \left\| \varphi - \psi \right\|. \end{aligned} \tag{5.16}$$

It follows from the mean value theorem, that

$$\left| \left(\varphi^{[m]}(s)\right)^{\ell} - \left(\psi^{[m]}(s)\right)^{\ell} \right| \leq \ell \left|\eta_m(s)\right|^{\ell-1} \left\| \varphi^{[m]} - \psi^{[m]} \right\|, \tag{5.17}$$

for some constant $\eta_m(s)$ such that $\varphi^{[m]}(s) \leq \eta_m(s) \leq \psi^{[m]}(s)$.

Taking into account (5.16) and (5.17) and using Corollary 5.2 we get the following estimate:

$$\begin{aligned} |(\mathcal{H}\varphi)(t) - (\mathcal{H}\psi)(t)| \leq & \ \alpha_2 \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} \sum_{j=0}^{m-1} \ell T L^{\ell-1} L_{C_{\ell,m}} M^j \left\| \varphi - \psi \right\| \\ & + (\alpha_2 \lambda_1 + \beta)\, T \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sigma_m M^j \left\| \varphi - \psi \right\|, \end{aligned}$$

this yields

$$\begin{aligned} \|(\mathcal{H}\varphi - (\mathcal{H}\psi\| = & \sup_{t\in[0,T]} |(\mathcal{H}\varphi)(t) - (\mathcal{H}\psi)(t)| \\ \leq & \ T \left(\alpha_2 L^{\ell-1} \sum_{m=1}^{n} \sum_{\ell=1}^{\infty} \sum_{j=0}^{m-1} \ell L_{C_{\ell,m}} M^j + (\alpha_2 \lambda_1 + \beta) \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sigma_m M^j \right) \left\| \varphi - \psi \right\|. \end{aligned}$$

Using (5.12), we infer that $\mathcal{H}$ is continuous.

Step 2: The main concern now is to prove that $\mathcal{H}$ maps the set $P_T(L, M)$ into itself.

First, we will prove that $\|\mathcal{H}\varphi\| \leq L,\ \forall \varphi \in P_T(L, M)$.

Let $\varphi \in P_T(L, M)$. Using (5.5), we obtain

$$\left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\right| \leq \varsigma. \tag{5.18}$$

In view of Corollary 5.2 and (5.18) we get

$$\begin{aligned}
|(\mathcal{H}\varphi)(t)| = {} & \sum_{m=1}^{n}\sum_{\ell=1}^{\infty}\int_t^{t+T} |C_{\ell,m}(s)| \left|\left(\varphi^{[m]}(s)\right)^{\ell}\right| G(t,s)\, ds \\
& + \int_t^{t+T} |E(t,s)| \left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\right| ds \\
& + \int_t^{t+T} |a(s)|\, G(t,s) \left| g\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\right| ds \\
& + \int_t^{t+T} G(t,s)\, |w(s)|\, ds. \\
& \leq (\beta + \alpha_2\lambda_1) T\varsigma + \alpha_2 T\left(L_w + \sum_{m=1}^{n}\sum_{\ell=1}^{\infty} L_{C_{\ell,m}} L^{\ell}\right).
\end{aligned}$$

Thus, from (5.13) we find that

$$|(\mathcal{H}\varphi)(t)| \leq L,\ \forall \varphi \in P_T(L, M). \tag{5.19}$$

Now, we will show that $|(\mathcal{H}\varphi)(t_2) - (\mathcal{H}\varphi)(t_1)| \leq M|t_2 - t_1|,\ \forall\ \varphi \in P_T(L, M)$ and $\forall$ $t_1,\ t_2 \in \mathbb{R}$.

For $\varphi \in P_T(L, M)$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
& \left|\int_{t_2}^{t_2+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell} G(t_2,s)\, ds - \int_{t_1}^{t_1+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell} G(t_1,s)\, ds\right| \\
& = \left|\int_{t_2}^{t_1} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell} G(t_2,s)\, ds\right. \\
& \quad + \int_{t_1+T}^{t_2+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell} G(t_2,s)\, ds \\
& \quad \left. + \int_{t_1}^{t_1+T} \left[C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{\ell}\right] \left(G(t_2,s) - G(t_1,s)\right) ds\right| \\
& \leq \int_{t_2}^{t_1} |C_{\ell,m}(s)| \left|\left(\varphi^{[m]}(s)\right)^{\ell}\right| G(t_2,s)\, ds \\
& \quad + \int_{t_1+T}^{t_2+T} |C_{\ell,m}(s)| \left|\left(\varphi^{[m]}(s)\right)^{\ell}\right| G(t_2,s)\, ds \\
& \quad + \int_{t_1}^{t_1+T} |C_{\ell,m}(s)| \left|\left(\varphi^{[m]}(s)\right)^{\ell}\right| |G(t_2,s) - G(t_1,s)|\, ds\, .
\end{aligned}$$

Using Corollary 5.2 and Lemma 5.5, we arrive at

$$\begin{aligned}\left|\int_{t_2}^{t_2+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{l} G(t_2,s)\,ds-\int_{t_1}^{t_1+T} C_{\ell,m}(s)\left(\varphi^{[m]}(s)\right)^{l} G(t_1,s)\,ds\right|&\\ \leq 2\alpha_2 L^{\ell} L_{C_{\ell,m}}\left|t_2-t_1\right|+L^{\ell} L_{C_{\ell,m}}\mu\left|t_2-t_1\right|&\\ =L^{\ell} L_{C_{\ell,m}}\left(2\alpha_2+\mu\right)\left|t_2-t_1\right| .&\qquad(5.20)\end{aligned}$$

Arguing as before, we also get

$$\begin{aligned}&\left|\int_{t_2}^{t_2+T} E(t_2,s)\, g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right) ds\right.\\ &\qquad\left.-\int_{t_1}^{t_1+T} E(t_1,s)\, g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds\right|\\ &\qquad\leq\int_{t_2}^{t_1}\left|E(t_2,s)\right|\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\ &\qquad+\int_{t_1+T}^{t_2+T}\left|E(t_2,s)\right|\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\ &\qquad+\int_{t_1}^{t_1+T}\left|E(t_2,s)-E(t_1,s)\right|\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds .\end{aligned}$$

Lemma 5.6, Corollary 5.2 and (5.18) give

$$\begin{aligned}&\left|\int_{t_2}^{t_2+T} E(t_2,s)\, g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right) ds\right.\\ &\qquad\left.-\int_{t_1}^{t_1+T} E(t_1,s)\, g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right) ds\right|\\ &\qquad\leq\varsigma\left(2\beta+\Gamma\right)\left|t_2-t_1\right| . \qquad (5.21)\end{aligned}$$

Furthermore

$$\begin{aligned}&\left|\int_{t_2}^{t_2+T} G(t_2,s)\, w(s)\, ds-\int_{t_1}^{t_1+T} G(t_1,s)\, w(s)\, ds\right|\\ &\qquad\leq\int_{t_2}^{t_1} G(t_2,s)\left|w(s)\right| ds+\int_{t_1+T}^{t_2+T} G(t_2,s)\left|w(s)\right| ds\\ &\qquad+\int_{t_1}^{t_1+T}\left|G(t_2,s)-G(t_1,s)\right|\left|w(s)\right| ds .\end{aligned}$$

Using Corollary 5.2 and Lemma 5.5, we obtain

$$\left|\int_{t_2}^{t_2+T} G(t_2,s)\, w(s)\, ds-\int_{t_1}^{t_1+T} G(t_1,s)\, w(s)\, ds\right|\leq L_w\left(2\alpha_2+\mu\right)\left|t_2-t_1\right| . \qquad (5.22)$$

We have also

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a(s)\,G(t_2,s)\,g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)ds\right.\\
&\left.-\int_{t_1}^{t_1+T} a(s)\,G(t_1,s)\,g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)ds\right|\\
&\leq \int_{t_2}^{t_1} |a(s)|\,G(t_2,s)\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)\right|ds\\
&+\int_{t_1+T}^{t_2+T} |a(s)|\,G(t_2,s)\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)\right|ds\\
&+\int_{t_1}^{t_1+T} |a(s)|\,|G(t_2,s)-G(t_1,s)|\left|g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)\right|ds.
\end{aligned}$$

By means of Corollary 5.2, Lemma 5.5 and (5.18) , we find

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a(s)\,G(t_2,s)\,g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)ds\right.\\
&\qquad\left.-\int_{t_1}^{t_1+T} a(s)\,G(t_1,s)\,g\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),....,\varphi^{[n]}(s)\right)ds\right|\\
&\qquad\leq \lambda_1\varsigma\,(2\alpha_2+\mu)\,|t_2-t_1|\,. \qquad (5.23)
\end{aligned}$$

According to (5.20)-(5.23), we get

$$\begin{aligned}
|(\mathcal{H}\varphi)(t_2)-(\mathcal{H}\varphi)(t_1)| \leq &\left(\sum_{m=1}^{n}\sum_{\ell=1}^{\infty} L^{\ell}L_{C_{\ell,m}}(2\alpha_2+\mu)\,|t_2-t_1|\right)+\varsigma\,(2\beta+\Gamma)\,|t_2-t_1|\\
&+L_w\,(2\alpha_2+\mu)\,|t_2-t_1|+\lambda_1\varsigma\,(2\alpha_2+\mu)\,|t_2-t_1|\,.
\end{aligned}$$

So

$$\begin{aligned}
&|(\mathcal{H}\varphi)(t_2)-(\mathcal{H}\varphi)(t_1)|\\
&\leq \left[\left(\lambda_1\varsigma+L_w+\sum_{m=1}^{n}\sum_{\ell=1}^{\infty} L^{\ell}L_{C_{\ell,m}}\right)(2\alpha_2+\mu)+\varsigma\,(2\beta+\Gamma)\right]|t_2-t_1|\,.
\end{aligned}$$

In light of (5.14) and Lemma 5.6, we get

$$|(\mathcal{H}\varphi)(t_2)-(\mathcal{H}\varphi)(t_1)| \leq M\,|t_2-t_1|\,,\ \forall\varphi\in P_T(L,M)\,,\ \forall t_1,t_2\in\mathbb{R}. \qquad (5.24)$$

From (5.19) and (5.24), we conclude that $\mathcal{H}(P_T(L,M))\subset P_T(L,M)$.

As a consequence of these two steps and the *Schauder* fixed theorem we conclude that operator $\mathcal{H}$ has at least one fixed point in $P_T(L,M)$ which is a solution of equation (5.1). ■

5.4 Conclusion

The existing results in the literature reveal that, despite the great deal of interest being shown in the investigation of iterative differential equations over recent decades, their theory remains an emerging one and has not yet been well-developed. We have seen how the same technique we used to deal with second-order iterative differential equations with boundary conditions continues to work when investigating second-order iterative differential equations with periodic coefficients. So, the present work aimed to contribute to the investigation of the last kind of problems. More precisely we have probed into the existence of periodic solutions for a second order iterative differential equation with periodic coefficients by following the steps below.

Firstly, based on some earlier publications, we introduced a new second order iterative differential problem and we set up an appropriate subset of the *Banach* space of continuous periodic functions. Secondly, we established an equivalence between this problem and a certain integral equation with a *Green*'s function type kernel. Finally, with the aid of *Schauder*'s fixed point theorem, some properties of the obtained kernel and the *Arzelà-Ascoli* theorem, we proved that the corresponding integral operator admits at least one fixed point which is a solution of the original problem.

General conclusion and perspectives

The present dissertation was devoted to study three classes of second order differential equations with iterative terms and boundary conditions. The intent of the first chapter was to give a brief look at the topic and an overview of the content of the thesis. After having presented some concepts, tools and enough preliminary results that allow us to build a better understanding of the remaining chapters, we have investigated these classes of equations by virtue of a forceful technique that combines the fixed point theory and the *Green*'s functions method. So, based on this hybrid technique, we have succeeded in establishing some existence, uniqueness, and stability results that have been the subject of three publications in well-established journals.

Herein, we will strive, without fear of repetition to highlight the key steps of our approach. For reaching our desired targets, we have proceeded as follows: Firstly, we have constructed an appropriate *Banach* space and a suitable subset of it in order to control the iterative terms and to make the model more realistic and the sought results more acceptable and credible. Secondly, we have converted each iterative problem into an equivalent integral equation whose kernel is a *Green*'s function. The main aim in this step was to transform each equation with its boundary conditions into a fixed point problem where the existence of solutions becomes that of fixed points of an integral self-mapping. Thirdly, we have applied *Schauder*'s fixed point theorem alone or together with some useful properties of the obtained *Green*'s kernel to establish the existence results. Finally, under some additional criteria, we have used the *Banach* fixed point theorem to guarantee the existence and continuous dependence on parameters of the unique solution.

We intend to further study other classes of iterative differential equations since the findings presented in this thesis naturally offer many research perspectives. For example,

- There is a potential for adopting the technique used here more widely and in more depth. It may be used as a means to handle various iterative differential problems that appear frequently in modelling real life phenomena such as disease transmission models, model for a two-body problem of classical electrodynamics, *Nicholson*'s blowflies model, hematopoiesis models, neural networks models, models for fisheries management, etc.

- It would also be very useful to be able to extend the study to iterative differential equations of order greater than two or to fractional iterative equations.

- It would seems important also to study the existence of almost-periodic, pseudo-almost-periodic, or anti-periodic solutions for iterative problems.

- It would appear crucial to use numerical methods or software to determine the solution or even give numerical simulations to illustrate the obtained results which occupy an ever more prominent place in our perspectives.

In conclusion, we believe that the findings presented in this thesis which generalize and improve many previous results in the literature, will contribute even a little to enriching the emerging theory of iterative differential problems.

List of publications

1. Chouaf, S., Bouakkaz, A., Khemis, R.: On bounded solutions of a second-order iterative boundary value problem. Turkish J. Math. **46**(2), 453–464 (2022). Indexed by Thomson Reuters

2. Chouaf, S., Khemis, R., Bouakkaz, A.: Some existence results on positive solutions for an iterative second-order boundary-value problem with integral boundary conditions. Bol. Soc. Parana. Mat. (3) **40**, 1–10 (2022). Indexed by Scopus

3. Khemis, R., Bouakkaz, A., Chouaf, S.: On the existence of periodic solutions of a second order iterative differential equation. Acta Math. Univ. Comen. **92**(1), 9-22 (2023). Indexed by Scopus

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