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Mémoire

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Résolution numérique des quelques problèmes pour des équations différentielles d'ordre fractionnaire

Option : ***Analyse Numérique des équations aux dérivées partielles***

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Dedicate

To the one who supported me from my first day of school to the present day,to the one who advised me and guided me for the better,to the only person who cried during all my successes “my father”.

To the one that always reminded me of the importance and usefulness of the study,to which I was supported in all my difficult times

“my mother”.

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Contents

In this memory, we study two fractional problems, where the first depends on the Caputo derivative for $n-1<\alpha<n, \quad n\in\mathbb{N}^*$ and the second involving both left Riemann-Liouville and right Caputo-type fractional derivatives.

Firstly, we study the existence and uniqueness of the solution in a Banach space based on the fixed point theorems..

Secondly, we propose an approximate solution of our problems and we show that the proposed numerical solution is converged to the exact solution for each problem

Keywords: Riemann-Liouville fractional derivative, Caputo fractional derivative , Fractional differential equation, Existence and uniqueness of solution, fixed point theorems, the exact solution, the approximate solution.

Dans cette mémoire, nous étudions deux problèmes fractionnaires, où le premier dépend de la dérivée de Caputo pour $n-1 < \alpha < n, \quad n \in \mathbb{N}^*$ et le deuxième dépendent des dérivées fractionnaires à gauche de Riemann-Liouville et à droite de Caputo .

Premièrement, nous étudions l'existence et l'unicité de la solution dans un éspace de Banach basé sur les théorèmes de point fixe.

Deuxièmement, nous proposons une solution approchée de nos problèmes et nous montrons que la solution numérique proposée est convergée vers la solution exacte pour chaque problème

Mots-clés: dérivée fractionnaire au sens de Riemann-Liouville, dérivée fractionnaire au sens de Caputo, equation differentielle fractionnaire, existence et unicité de la solution, théorèmes du point fixe, la solution exacte, la solution approchée.

ملخص

في هذه المذكرة درسنا مسألتين ذات رتب كسرية، حيث أعتمدنا في المسألة الأولى على مشتق كابوتو العادي من أجل $n-1<\alpha<n$، بينما تعتمد الثانية على المشتق الكسري من اليمين و اليسار لكابوتو.

أولا، ندرس وجود ووحدانية الحل في فضاء بناخ بإستعمال نظرية النقطة الثابتة.

ثانيا، إقترحنا حلا تقريبيا للمسألتين وبيننا أن الحل العددي المقترح يتقارب نحو الحل الدقيق لكل مسألة.

الكلمات المفتاحية: المشتق الجزئي لكابوتو، المشتق الكسري لكابوتو، المعادلة التفاضلية الكسرية، وجود ووحدانية الحل، نظرية النقطة الثابتة، الحل الدقيق، الحل التقريبي.

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibniz's letter to l'Hospital in 1695, where the idea of semiderivative was suggested.

During time fractional calculus was built on formal foundations by many famous mathematicians, Liouville, Grnwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. The fractional integral may be used for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model . Analogously the fractional derivative is sometimes used for describing damping. Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc [14, 22].

Ricardo et al. in [2] are proposed a numerical approach to solve a problem of the fractional differential equation with dependence on the Caputo-Katugampola derivative. Inspiring by works [2],[3] and [13] and , we have suggested a novel numerical approach to the fractional problem and generalized proportional fractional problem. This approach is based on obtaining a decomposition formula for the generalized proportional Caputo derivative. Existence and uniqueness theorems for the given problem are deduced

In the first chapters we remind some techniques and special functions which are necessary for the understanding of the fractional calculus's rules, We provide some basic knowledge about fractional integrals and derivatives, such Riemann-

Liouville fractional integral derivative, Caputo fractional derivative. We give a some fixed point theorems: Ascoli-Arzela , Banach, Scheifer, Laray Schauder.

In the second chapter we study the existence and numerical solutions of the following initial value problem

$$\begin{cases} {}^{C}D^{\beta}_{a^{+}}y(t) = f(t, y(t)), \ \ \beta \in \left]n-1, n\right[\ \ 2 \leq n \ \ t \in [a, b] \\ \\ y^{k}(a) = k! y_{k} \ , \ k = 0, 1, .., n-1. \end{cases}$$

Where ${}^{C}D^{\beta}_{a^{+}}$ is the fractional derivative of Caputo type , $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By Banach fixed point theorem, we prove the existence and uniqueness of solution of this problem, and by Scheifer fixed point theorem we can prove only that exist at least one solution and we proposed an approximate solution of our problem and we prove the convergence of this solution towards the exact solution.

In th third chapter, we study the existence, uniqueness and numerical solution of the following mixed fractional boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}_{b^{-}}\left({}^{C}D^{\beta}_{a^{+}}\right) y(t) = f(t, y(t)), \ \ t \in [a, b] \\ {}^{C}D^{\beta}_{0^{+}}y(b) = 0, \ \ y(a) = y_{0} \end{cases}$$

with$0 < \alpha, \beta \leq 1, \alpha + \beta > 1$, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function.

Chapter 1

Preliminaries

We introduced some useful definitions, properties and lemmas that will be used in the remainder of this memory, for more details see [1, 5, 6, 9, 10, 11, 12, 15, 16, 23, 24]

1.1 Functional spaces

1.1.1 Space for continuous and absolutely continuous functions

Definition 1.1.1. Let $\Omega = [a;b]$, $(-\infty < a < b < \infty)$, a finite interval, we denote by $AC([a,b])$the space of primitive functions of integrable functions, that is to say

$$f \in AC([a,b]) = \left\{ f : \exists \varphi \in L^1([a,b]) : f(x) = c + \int_a^x \varphi(t)dt \right\}.$$

and we call, $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$.

Definition 1.1.2. for $n \in \mathbb{N}$ we denote by $AC^n([a,b])$ the space of functions f having derivatives up to order $(n-1)$ absolutely continuous on $[a,b]$ such as

$$AC^n([a,b]) = \left\{ f : [a,b] \to \mathbb{C} \text{ such that } f^{(n-1)} \in AC([a,b]) \right\}$$

In particular $AC^1([a,b]) = AC([a,b])$.

1.1.2 C^n space

Definition 1.1.3. Let $\Omega = [a,b]$, $(-\infty \leq a < b \leq \infty)$, and $n \in \mathbb{N}$, we designate by $C^n(\Omega)$ the space of functions f which have their derivatives of a lower order or equal to n continue on Ω, provided with the standard:

$$\|f\|_{C^n} = \sum_{i=0}^{n} \left\| f^{(i)} \right\|_C = \sum_{i=0}^{n} \max_{x\in\Omega} \left| f^{(i)}(x) \right|.$$

In particular
if $n = 0$, $C^0(\Omega) = C(\Omega)$ the space of continuous functions $f$on provided with the standard: $\|f\|_c = \max_{x\in\Omega} |f(x)|$

1.2 Special functions

The more important functions used in fractional calculus is the Gamma function of Euler. Its interpretation is simply the generalization of the factorial $n!$ and it allows n to take non-integer values.

1.2.1 The Gamma function

Definition 1.2.1. For every $x > 0$,the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

which converges for $x > 0$.

Lemma 1.2.1. *For every $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$ we have:*
1 .$\Gamma(x+1) = x \times \Gamma(x)$
2 .$\Gamma(n+1) = n!$
3 .*The Euler's reflextion:* $\forall x \notin \mathbb{Z}$, $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{sin(\pi x)}$

4 . $\sum_{i=0}^{n-1} \frac{\Gamma(i-\alpha)}{i!} = -\frac{\Gamma(n-\alpha)}{\alpha\,\Gamma(n)}$

For examples: $\Gamma(2) = 1 \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

1.2.2 The Beta function

Definition 1.2.2. For every $x, y \in \mathbb{R}_+^*$,the beta function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Lemma 1.2.2. *For every* $x, y \in \mathbb{R}_+^*$ *we have:*

1 .$B(x,y) = B(y,x)$

2 .$B(x,y+1) = \frac{y}{x} B(x+1,y)$

3 .$B(x+1,y) = \frac{x}{x+y} B(x,y)$

An interesting formula relating the gamma and beta functions is

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

1.2.3 Mittag-Leffler functions

Definition 1.2.3. The one-parametric Mittag-Leffler function (M-L for short) $E_\alpha(z)$ is a special function of $z \in \mathbb{C}$ which depends on the complex parameter α and is defined by the power series

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \tag{1.2.1}$$

One can see that the series 1.2.1 converges in the whole complex plane for all $\Re(\alpha) > 0$.

A first generalization of $E_\alpha(z)$ is the two-parametric M-L function of $z \in \mathbb{C}$ defined by the series

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + \beta)} \qquad \alpha, \beta \in \mathbb{C} \text{ with } \Re(\alpha) > 0. \tag{1.2.2}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function in (1.2.1)
$E_{\alpha,1}(z) = E_\alpha(z)$.
When $\alpha = \beta = 1 : E_{1,1}(z) = E_1(z) = e^z$.

Lemma 1.2.3. *(Integration of the Mittag-Leffler function)*
Let $\Omega = [a;b], (-\infty < a < b < \infty)$, a finite interval, integrating (1.2.2) term-by-term, we obtain:

$$\int_a^x E_{\alpha,\beta}(\lambda s^\alpha)\, s^{\beta-1} ds = (x-a)^\beta E_{\alpha,\beta+1}\left(\lambda (x-a)^\alpha\right) \quad , \qquad (\alpha > 0\ ,\ \beta > 0) \tag{1.2.3}$$

Relationship (1.2.3) is a particular case of the following more general relationship obtained by the fractional order term-by-term integration of the series (1.2.2), for $(\alpha > 0, \beta > 0, \gamma > 0)$:

$$\frac{1}{\Gamma(\gamma)} \int_a^x (x-s)^{\gamma-1} E_{\alpha,\beta}(\lambda s^\alpha)\, s^{\beta-1} ds = (x-a)^{\beta+\gamma-1} E_{\alpha,\beta+\gamma}\left(\lambda (x-a)^\alpha\right)$$

When $\alpha = \beta = 1$ *:*

$$\frac{1}{\Gamma(\gamma)} \int_a^x (x-s)^{\gamma-1} e^{\lambda s} ds = (x-a)^{\gamma} E_{1,\gamma+1}(\lambda\ (x-a)) \quad , \quad (\gamma > 0) \tag{1.2.4}$$

1.3 Fractional integrals and Fractional derivatives

1.3.1 Fractional integrals

Let $\Omega = [a;b], (-\infty < a < b < \infty)$, a finite interval on the real axis $\mathbb{R}$ and $f \in L^1([a,b])$

Definition 1.3.1. The fractional (arbitrary) order integral of a function f of order $\alpha \in \mathbb{R}_+$ is defined by:

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\, ds \qquad (x > a) \tag{1.3.5}$$

When $\alpha = n \in \mathbb{N}$ the definition (1.3.5) coincide with the n th integrals for the forme

$$\begin{aligned} \left(I_{a^+}^{n} f\right)(x) &= \int_a^x ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} f(s_n)\, ds_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s)\, ds \end{aligned}$$

Exemple 1.3.1. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(x > a)$
$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} (s-a)^{\beta} ds$

By using the following change of variable $s = a + (x-a)y$ we find

$$\begin{aligned}
\left(I_{a^+}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-(x-a)\,y-a)^{\alpha-1}((x-a)\,y)^{\beta}\,(x-a)\,dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\alpha-1}(1-y)^{\alpha-1}(x-a)^{\beta+1}y^{\beta}dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\alpha+\beta}(1-y)^{\alpha-1}y^{\beta}dy \\
&= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-y)^{\alpha-1}y^{\beta}dy \\
&= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} B(\beta+1,\alpha)
\end{aligned}$$

And by applying Lemma 1.2.2 we find

$$I_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta} \tag{1.3.6}$$

Lemma 1.3.1. *Let* $f \in L^1([a,b])$ *and* $\alpha, \beta \in \mathbb{R}_+^* : \left(I_{a^+}^{\alpha} \circ I_{a^+}^{\beta} f\right)(x) = \left(I_{a^+}^{\alpha+\beta} f\right)(x)$

1.3.2 Fractional derivative

Definition 1.3.2. The Riemann Liouville fractional derivative $D_{a^+}^{\alpha}$ of a function f of order $\alpha \in \mathbb{R}_+$ is defined by

$$\begin{aligned}\left(D_{a^+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-s)^{n-\alpha-1} f(s)\, ds \qquad (n = [\alpha]+1 \ , \quad x > a)\end{aligned}$$

where $[\alpha]$ mean the right portion for α
In particular, when $\alpha = m \in \mathbb{N}$, $\left(D_{a^+}^{m} f\right)(x) = f^{(m)}(x)$ where $f^{(m)}(x)$ is the usuel derivative of $f(x)$ of order m

Exemple 1.3.2. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(x > a)$, we have $\left(D_{a^+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha}(x-a)^{\beta}$. According to equality (1.3.6) we find

$$\begin{aligned}\left(D_{a^+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^n \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}(x-a)^{n-\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)(n-\alpha+\beta)(n-1-\alpha+\beta)\ldots(n-(n-1)-\alpha+\beta)}{\Gamma(n-\alpha+\beta+1)} \\ &\quad \times (x-a)^{n-n-\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)}{(\beta-\alpha)!}(x-a)^{\beta-\alpha}\end{aligned}$$

that's mean

$$D_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} \tag{1.3.7}$$

In particular , if $\beta = 0$ then the Riemann-Liouville fractional derivative of a constant is, in general, not equal to zero:

$$D^{\alpha}_{a^+}1 = \frac{(x-a)^{\beta-\alpha}}{\Gamma(1-\alpha)}$$

Lemma 1.3.2. *Let $\alpha > 0$, and $n = [\Re(\alpha)] + 1$ the equality $D^{\alpha}_{a^+}f(x) = 0$ is valid, if and only if*

$$f(x) = \sum_{j=1}^{n} c_j (x-a)^{\alpha-j},$$

where $c_j \in \mathbb{R}$, $j = 1, 2, ..., n$ is arbitrary constant.

Lemma 1.3.3. ***(semigroup property)*** *Let $f \in L^1([a,b])$ and $\alpha, \beta \in \mathbb{R}_+$ then, the equation*

$$\left(I^{\alpha}_{a^+} I^{\beta}_{a^+} f\right)(x) = \left(I^{\alpha+\beta}_{a^+} f\right)(x)$$

hold almost everywhere on $[a,b]$.

Lemma 1.3.4. *Let $f \in L^1([a,b])$ and $\alpha \in \mathbb{R}_+$ then, the following equality*

$$\left(D^{\alpha}_{a^+} I^{\alpha}_{a^+} f\right)(x) = f(x)$$

hold almost everywhere on $[a,b]$.
In general case for $\alpha > \beta$ we have $\left(D^{\beta}_{a^+} I^{\alpha}_{a^+} f\right)(x) = \left(I^{\alpha-\beta}_{a^+} f\right)(x)$.

Lemma 1.3.5. *Let $\alpha \in \mathbb{R}_+$, $f \in L^1([a,b])$ and $f_{n-\alpha}(x) \in AC^n[a,b]$ then, the equality*

$$\left(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f\right)(x) = f(x) + \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)} (x-a)^{\alpha-j} \tag{1.3.8}$$

hold almost everywhere on $[a,b]$, where $f_{n-\alpha} = I^{n-\alpha}_{a^+} f$ be the fractional integral of f of order $n-\alpha$.

Definition 1.3.3. For a function $f \in L^1([a,b],\mathbb{R})$, the Caputo fractional derivative of order $\alpha \in \mathbb{R}_+$ of f is defined by

$$\left({}^C D^{\alpha}_{a^+} f\right)(x) = \left(D^{\alpha}_{a^+0} \left[f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(x-a)^k \right] \right), \qquad (n = [\alpha]+1\ ,\ \ x > a)$$

Lemma 1.3.6. *For a function $f \in AC^n([a,b],\mathbb{R})$, then the Caputo fractional derivative $\left({}^C D^{\alpha}_{a^+} f\right)(x)$ exist almost everywhere on $[a,b]$, with*

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+} f\right)(x) &= \left(I^{n-\alpha}_{a^+} f^{(n)} \right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \qquad (n = [\alpha]+1\ ,\ \ x > a)\end{aligned}$$

In particular, when $\alpha = m \in \mathbb{N}$: $\left({}^C D^{m}_{a^+} f\right)(x) = f^{(m)}(x)$

Exemple 1.3.3. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(x > a)$

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+} f\right)(x) &= \left(I^{n-\alpha}_{a^+} f^{(n)} \right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds \\ &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} (s-a)^{\beta-n} ds\end{aligned}$$

By using the following change of variable $s = a + (x-a)\,y$ we obtain

$$\begin{aligned}\left({}^{C}D_{a^+}^{\alpha}f\right)(x) &= \frac{\beta\,(\beta-1)\ldots(\beta-(n-1))}{\Gamma\,(n-\alpha)}\int_0^1 (x-a-(x-a)\,y)^{n-\alpha-1} \\ &\times ((x-a)\,y)^{\beta-n}\,(x-a)\,dy \\ &= \frac{\beta\,(\beta-1)\ldots(\beta-(n-1))}{\Gamma\,(n-\alpha)}\int_0^1 (x-a)^{n-\alpha-1}(1-y)^{n-\alpha-1} \\ &\times (x-a)^{\beta-n+1}y^{\beta-n}dy \\ &= \frac{\beta\,(\beta-1)\ldots(\beta-(n-1))}{\Gamma\,(n-\alpha)}(x-a)^{\beta-\alpha}\int_0^1 y^{\beta-n}(1-y)^{n-\alpha-1}dy \\ &= \frac{\beta\,(\beta-1)\ldots(\beta-(n-1))}{\Gamma\,(n-\alpha)}(x-a)^{\beta-\alpha}B\,(\beta-n+1, n-\alpha)\end{aligned}$$

And by applying Lemma 1.2.2 we find

$$ {}^{C}D_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma\,(\beta+1)}{\Gamma\,(\beta-\alpha+1)}(x-a)^{\beta-\alpha} \tag{1.3.9}$$

in particular, ${}^{C}D_{a^+}^{\alpha}1 = 0$

Lemma 1.3.7. *Let $\alpha > 0$, and $n = [\Re(\alpha)] + 1$ the equality ${}^{C}D_{a^+}^{\alpha}f\,(x) = 0$ is valid, if and only if*

$$f\,(x) = \sum_{j=0}^{n-1} c_j (x-a)^j,$$

where $c_j \in \mathbb{R}$, $\quad j = 1, 2, \ldots, n$ is arbitrary constant.

Lemma 1.3.8. *Let $\alpha \in \mathbb{R}_+$ and $f \in AC^n[a,b]$ or $f \in C^n[a,b]$ then, the equality*

$$\left(I_{a^+}^{\alpha}\ {}^{C}D_{a^+}^{\alpha} f \right)(x) = f(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (x-a)^j \tag{1.3.10}$$

hold almost everywhere on $[a,b]$.

1.4 Fixed point theorems

In this section we present some important fixed point theorems which we use for proving the existence and uniqueness of solutions of our problems

Theorem 1.4.1. ***Ascoli-Arzela theorem***
Let $\mathcal{F} \subseteq C([a,b])$, for certain $a < b$, then $\mathcal{F}$ is relatively compact in $C([a,b])$ if and only if $\mathcal{F}$ is equicotinuous (that's mean for all $\varepsilon > 0$, their exist $\delta > 0$ such that for all $f \in \mathcal{F}$ and x , $x' \in [a,b]$ with $|x - x'| < \delta$ we have $|f(x) - f(x')| < \varepsilon$) , and uniformly bounded (that's mean their exist $C > 0$ such that $\|f\|_\infty \leq C$ for all $f \in \mathcal{F}$).

Theorem 1.4.2. ***Banach fixed point theorem***
Let $\mathcal{N}$ be a contraction on a Banach space E , then $\mathcal{N}$ has a unique fixed point in E.

Theorem 1.4.3. ***Schauder's fixed point theorem***
Let E be a Banach space. U be a closed, convex and nonempty subset of E. Let $\mathcal{N} : U \to U$ be a continuous mapping such that $\mathcal{N}(U)$ is a relatively compact subset of E. Then $\mathcal{N}$ has at least one fixed point in U.

CHAPTER 2

Analysis and numerical solution for generalized initial value problem with Caputo derivative

In this chapter, we study the existence, uniqueness and numerical solution for the following initial value problem.

$$^{C}D_{a^{+}}^{\beta}y\left(t\right)=f\left(t,y\left(t\right)\right),\quad t\in(a,b), \tag{2.0.1}$$

$$y^{(k)}\left(a\right)=k!y_{k},\qquad k=0,1,...,n-1. \tag{2.0.2}$$

Where $n-1<\beta\leq n$ with $n\geq 2$ and $^{C}D_{0^{+}}^{\beta}$ denotes the left Caputo fractional derivative, and $f:[a,b]\times\mathbb{R}\rightarrow\mathbb{R}$ is given function.

2.1 Existence of solutions

firstly we solve the following linear problem:

$$^{C}D_{a^{+}}^{\beta}y\left(t\right)=h\left(t\right),\quad t\in(a,b), \tag{2.1.3}$$

$$y^{(k)}\left(a\right)=k!y_{k},\qquad k=0,1,...,n-1. \tag{2.1.4}$$

Lemma 2.1.1. *Assume that $y\in AC^{n}\left([a,b],\mathbb{R}\right)$, then y is a solution to the linear boundary value problem (2.1.3)-(2.1.4) if and only if y satisfies the integral equation*

$$y\left(t\right)=\frac{1}{\Gamma\left(\beta\right)}\int_{a}^{t}\left(t-s\right)^{\beta-1}h(s)ds+\sum_{k=0}^{n-1}y_{k}(t-a)^{k},$$

Define the Banach space $E=C\left([a,b],\mathbb{R}\right)$. We introduce the following hypomemory

(H1) $f:[a,b]\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function

(H2) There exist $p, q \in C([a,b], \mathbb{R}_+)$ such that

$$|f(t,y)| \leq p(t)|y| + q(t), \text{for each } t \in [a,b], \text{ and } y \in \mathbb{R}.$$

(H3) There exists $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f |x-y|, \quad \text{for each } t \in [a,b] \quad \text{and all } x, y \in \mathbb{R}.$$

Note that if (H1) and (H3) are satisfied then (H2) is satisfied.

Theorem 2.1.1. *Assume that* $(H1)$ *and* $(H3)$ *hold, then the problem (2.0.1)-(2.0.2) has a unique fixed point in* E*.*

Proof. We consider the operator $N : E \to E$ defined by

$$N(y(t)) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s))ds + \sum_{k=0}^{n-1} y_k (t-a)^k, \tag{2.1.5}$$

We shall show that N satisfies the assumption of Banach fixed point theorem. Let x, y in E. Then, for every $t \in [a,b]$ we have

$$\begin{aligned}
&|N(x(t)) - N(y(t))| \\
&\quad = \left| \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, x(s))ds - \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s))ds \right| \\
&\quad \leq \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |f(s, y_n(s)) - f(s, y(s))| \, ds \\
&\quad \leq \frac{L_f}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |x(s) - y(s)| \, ds \\
&\quad \leq \frac{(b-a)^{\beta+1}}{\Gamma(\beta+1)} \|f(., y_n(.)) - f(., y(.))\|_\infty
\end{aligned}$$

□

Theorem 2.1.2. *Assume that* $(H1)$ *and* $(H2)$ *hold, then the problem (2.0.1)-(2.0.2) has a fixed point in* B_R*, where*

$$B_R = \{y \in E, \text{ such that } \|y\| \leq R\}.$$

with $1 \geq \frac{(b-a)^{\beta}}{\Gamma(\beta+1)} \|p\|_{\infty}$ *and*

$$R \geq \left(\frac{(b-a)^{\beta}}{\Gamma(\beta+1)} \|q\|_{\infty} + \sum_{k=0}^{n-1} |y_k| (b-a)^k \right) \left(1 - \frac{(b-a)^{\beta}}{\Gamma(\beta+1)} \|p\|_{\infty} \right)^{-1}. \tag{2.1.6}$$

Proof. We consider the operator N as in 2.1.5. We shall show that N satisfies the assumption of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1, We shall show that N is a continuous operator.

Let (y_n) be a sequence converges to y in E. Then, for every $t \in [a, b]$ we have

$$\begin{aligned}
&|N(y_n(t)) - N(y(t))| \\
&\quad = \left| \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y_n(s)) ds - \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s)) ds \right| \\
&\quad \leq \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |f(s, y_n(s)) - f(s, y(s))| \, ds \\
&\quad \leq \|f(., y_n(.)) - f(., y(.))\|_{\infty} \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta} ds \\
&\quad \leq \frac{(b-a)^{\beta+1}}{\Gamma(\beta+1)} \|f(., y_n(.)) - f(., y(.))\|_{\infty}
\end{aligned}$$

It implies that, $|Ny_n - Ny| \to 0$, when $n \to \infty$, since f is a continuous function

Step 2, We shall show that $N(B_R) \subset B_R$.

For any $y \in B_R$ and $t \in [a,b]$ we have

$$\begin{aligned}
|N(y(t))| &= \left| \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s,y(s))ds + \sum_{k=0}^{n-1} y_k (t-a)^k \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |f(s,y(s))| \, ds + \sum_{k=0}^{n-1} |y_k| (t-a)^k \\
&\leq \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left(p(s)|y(s)| + q(s)\right) ds + \sum_{k=0}^{n-1} |y_k| (b-a)^k \\
&\leq \frac{(b-a)^\beta}{\Gamma(\beta+1)} \left(\|p\|_\infty R + \|q\|_\infty\right) + \sum_{k=0}^{n-1} |y_k| (b-a)^k \\
&\leq R
\end{aligned}$$

It implies that $N(B_R)$ is uniformly bounded set to E.

Step 3, We shall show that $N(B_R)$ is equicontinuous sets of E.

Let $t_1, t_2 \in (a,b)$ such that $t_1 < t_2$ and $y \in B_R$, then, we have

$$\begin{aligned}
&|N(y(t_2)) - N(y(t_1))| \\
&= \frac{1}{\Gamma(\beta)} \int_a^{t_1} \left((t_2-s)^{\beta-1} - (t_1-s)^{\beta-1} \right) |f(s,y(s))| \, ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} |f(s,y(s))| \, ds + \sum_{k=1}^{n-1} |y_k| \left[(t_2-a)^k - (t_1-a)^k \right] \\
&\leq \frac{\left(\|p\|_\infty R + \|q\|_\infty\right)}{\Gamma(\beta+1)} \left((t_2-a)^\beta - (t_1-a)^\beta \right) + \sum_{k=1}^{n-1} |y_k| \left[(t_2-a)^k - (t_1-a)^k \right]
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the last above inequality tends to zero. By Arzela-Ascoli Theorem we conclude that $N(B_R)$ is relatively compact. As a consequence of Schauder's fixed point Theorem, we deduce that, N has a fixed point $y \in B_R$ which is a solution of the problem (2.0.1)-(2.0.2). □

2.1.1 Ulam stability

We consider the following inequality

$$\left| {}^{C}D_{a^{+}}^{\beta}x\left(t\right)-f\left(t,x\left(t\right)\right)\right| \leq \epsilon,\ \text{for } t\in\left[a,b\right] \text{ and } \epsilon>0. \tag{2.1.7}$$

Definition 2.1.1. Equation (2.0.1) is $\mathbb{E}_{\beta}$-Ulam-Hyers stable if there exists $c>0$ such that for each $\epsilon>0$, and for each solution $x\in C\left(\left[a,b\right],\mathbb{R}\right)$ of (3.2.13) there exists a solution $y\in C\left(\left[a,b\right],\mathbb{R}\right)$ of equation (2.0.1) with

$$\left|x\left(t\right)-y\left(t\right)\right|\leq c\mathbb{E}_{\beta}\left(L_{f}\left(t-a\right)^{\beta}\right)\epsilon. \tag{2.1.8}$$

Remark 2.1.1. Let x be a solution of the inequality (3.2.13) then x is a solution of the following integral inequality

$$\left|x\left(t\right)-\left(\frac{1}{\Gamma\left(\beta\right)}\int_{a}^{t}\left(t-s\right)^{\beta-1}f(s,x(s))ds+\sum_{k=0}^{n-1}x_{k}(t-a)^{k}\right)\right|\leq\frac{\epsilon(b-a)^{\beta}}{\Gamma\left(\beta+1\right)}.$$

Theorem 2.1.3. *Assume that $(H1)$ and $(H3)$ hold. Then the equation (2.0.1) is $\mathbb{E}_{\beta}$-Ulam-Hyers stable.*

Proof. Let x be a solution of inequality (3.2.13) and y the unique solution of the following Cauchy problem

$$\begin{aligned}&{}^{C}D_{a^{+}}^{\beta}y\left(t\right)=f\left(t,y\left(t\right)\right), \quad t\in(a,b),\\ &y^{(k)}\left(a\right)=k!x_{k}, \qquad k=0,1,...,n-1.\end{aligned}$$

Therefore,

$$y(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s)) ds + \sum_{k=0}^{n-1} x_k (t-a)^k. \tag{2.1.9}$$

By using (H3), Remark 2.1.1 and **Corollary 2** in **[?]** we obtain

$$\begin{aligned}
|x(t) - y(t)| &\leq \left| x(t) - \left(\frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s)) ds + \sum_{k=0}^{n-1} x_k (t-a)^k \right) \right| \\
&\leq \left| x(t) - \left(\frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, x(s)) ds + \sum_{k=0}^{n-1} x_k (t-a)^k \right) \right| \\
&\quad + \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left| f(s, x(s)) - f(s, y(s)) \right| ds \\
&\leq \frac{\epsilon (b-a)^\beta}{\Gamma(\beta+1)} + \frac{L_f}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left| x(s) - y(s) \right| ds \\
&\leq \frac{\epsilon (b-a)^\beta}{\Gamma(\beta+1)} \mathbb{E}_\beta \left(L_f (t-a)^\beta \right).
\end{aligned}$$

Thus, $|x(t) - y(t)| \leq c \mathbb{E}_\beta \left(L_f (t-a)^\beta \right) \epsilon$, for all $t \in [a, b]$, where $c = \dfrac{(b-a)^\beta}{\Gamma(\beta+1)}$.
The proof is complete. □

2.2 Numerical approach

Theorem 2.2.1. *([2]* ***Theorem 8*** *for $\rho = 1$) Let $N \geq 1$ an integer and $y : [0, 1] \to \mathbb{R}$, be a function of class AC^2. Let consider*

$$\mathcal{A}_N = \frac{1}{\Gamma(2-\beta)} \sum_{k=0}^{N} \frac{\Gamma(k-1+\beta)}{\Gamma(\beta-1)\, k!},$$

$$\mathcal{B}_{N,k} = \frac{\Gamma\left(k-1+\beta\right)}{\Gamma\left(2-\beta\right)\Gamma\left(\beta-1\right)\left(k-1\right)!},\ k=1,2,...,N,$$

and functions $\mathcal{V}_k : [a,b] \to \mathbb{R}$ *by*

$$\mathcal{V}_k\left(t\right) = \int_a^t \left(s-a\right)^{k-1} y'\left(s\right) ds,$$

Then

$${}^{C}D^{\beta}_{a^+}y\left(t\right) = \mathcal{A}_N\left(t-a\right)^{1-\beta} y'\left(t\right) - \sum_{k=1}^{N} \mathcal{B}_{N,k}(t-a)^{1-\beta-k}\mathcal{V}_k(t) + \mathcal{E}_N\left(t\right)$$

with

$$\lim_{N\to\infty} \mathcal{E}_N\left(t\right) = 0,\ \ \forall t \in [a,b],$$

The problem (2.1.3)-(2.1.4) can be rewritten as

$$y\left(t\right) =^{C} D^{n-\beta}_{a^+} I^{1}_{a^+}\left(I^{n-1}_{a^+} f(t,y(t))\right) + \sum_{k=0}^{n-1} y_k(t-a)^k$$

Using the decomposition formula given in Theorem 3.2.1, we find

$$\begin{aligned} y\left(t\right) &= \sum_{k=0}^{n-1} y_k(t-a)^k + \tilde{\mathcal{A}}_N\left(t-a\right)^{1-n+\beta}\left(I^{n-1}_{a^+}f(t,y(t))\right) \\ &\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k}(t-a)^{1-n+\beta-k}\tilde{\mathcal{V}}_k(t) + \tilde{\mathcal{E}}_N\left(t\right) \\ &= \sum_{k=0}^{n-1} y_k(t-a)^k + \tilde{\mathcal{A}}_N\left(t-a\right)^{1-n+\beta}\left(\frac{1}{(n-2)!}\int_a^t (t-s)^{n-2} f(s,y(s))ds\right) \\ &\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k}(t-a)^{1-n+\beta-k}\tilde{\mathcal{V}}_k(t) + \tilde{\mathcal{E}}_N\left(t\right), \end{aligned} \tag{2.2.10}$$

where

$$\tilde{\mathcal{A}}_N = \frac{1}{\Gamma(2-n+\beta)} \sum_{k=0}^{N} \frac{\Gamma(k+n-1-\beta)}{\Gamma(n-1-\beta)\,k!},$$

$$\tilde{\mathcal{B}}_{N,k} = \frac{\Gamma(k+n-1-\beta)}{\Gamma(2-n+\beta)\,\Gamma(n-1-\beta)\,(k-1)!},\ k = 1, 2, ..., N,$$

$$\begin{aligned}
\tilde{\mathcal{V}}_k(s) &= \int_a^t (s-a)^{k-1} \left(I_{a^+}^{n-1} f(s, y(s))\right) ds \\
&= \frac{1}{(n-2)!} \int_a^t (s-a)^{k-1} \left(\int_a^s (s-\tau)^{n-2} f(\tau, y(\tau)) d\tau \right) ds
\end{aligned}$$

and the error $\tilde{\mathcal{E}}_N$ satisfies

$$\left|\tilde{\mathcal{E}}_N(t)\right| \leq M(t) \frac{\exp\left((1-n+\beta)^2 + 1 - n + \beta\right)}{N^{1-n+\beta}(1-n+\beta)\Gamma(2-n+\beta)} (t-a)^{2-n+\beta},$$

where $M(t) = \sup_{s\in[a,t]} \left| I_{a^+}^{n-1} f(t, y(t)) \right|$.
We take the formula of approximate solution, y_N by

$$\begin{aligned}
y(t) &= \sum_{k=0}^{n-1} y_k (t-a)^k + \tilde{\mathcal{A}}_N (t-a)^{1-n+\beta} \left(I_{a^+}^{n-1} f(t, y_N(t))\right) \\
&\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k} (t-a)^{1-n+\beta-k} \tilde{\mathcal{V}}_{k,N}(t) + \tilde{\mathcal{E}}_N(t) \\
&= \sum_{k=0}^{n-1} y_k (t-a)^k + \tilde{\mathcal{A}}_N (t-a)^{1-n+\beta} \left(\frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} f(s, y_N(s)) ds \right) \\
&\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k} (t-a)^{1-n+\beta-k} \tilde{\mathcal{V}}_{k,N}(t) + \tilde{\mathcal{E}}_N(t), \qquad (2.2.11)
\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{V}}_{k,N}(t) &= \int_a^t (s-a)^{k-1} \left(I_{a^+}^{n-1} f(s, y_N(s))\right) ds \\ &= \frac{1}{(n-2)!}\int_a^t (s-a)^{k-1} \left(\int_a^s (s-\tau)^{n-2} f(\tau, y_N(\tau)) d\tau\right) ds\end{aligned}$$

for $k = 1, 2, ..., N$.

Theorem 2.2.2. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies (H_3). For $N \in \mathbb{N}$, let y and y_N as in (3.2.17) and (2.2.11) respectively. If*

$$a < b < a + \left(\frac{\Gamma(2-n+\beta)}{L}\right)^{\frac{1}{\beta}}. \tag{2.2.12}$$

Then, $y_N(t)$ tends to $y(t)$ as N tends to ∞.

Proof. It follows from (3.2.17) and (2.2.11) that

$$\begin{aligned}|y_N(t) - y(t)| \leq\ & \left|\tilde{\mathcal{A}}_N\right| (t-a)^{1-n+\beta} \left| I_{a^+}^{n-1} f(t, y_N(t)) - I_{a^+}^{n-1} f(t, y(t)) \right| \\ & - \sum_{k=1}^{N} \left|\tilde{\mathcal{B}}_{N,k}\right| (t-a)^{1-n+\beta-k} \left|\tilde{\mathcal{V}}_{k,N}(t) - \tilde{\mathcal{V}}_k(t)\right| + \left|\tilde{\mathcal{E}}_N(t)\right|,\end{aligned}$$

for all $t \in [a, b]$. Define

$$\delta_{y_N} = \max_{t\in[a,b]} |y_N(t) - y(t)|.$$

Then, we have

$$|f(t, y_N(t)) - f(t, y(t))| \leq L_f |x_N(t) - x(t)| \leq L\delta_{y_N},$$

$$\begin{aligned}\left|I_{a^+}^{n-1}f(t,y_N(t))-I_{a^+}^{n-1}f(t,y(t))\right| &\leq \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}\left|f(s,y_N(s))-f(s,y(s))\right|ds\\ &\leq \frac{L\delta_{y_N}}{(n-1)!}(t-a)^{n-1}\\ &\leq L\delta_{y_N}(t-a)^{n-1}.\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathcal{V}}_{k,N}(t)-\tilde{\mathcal{V}}_k(t)\right| &\leq \left|\int_a^t (s-a)^{k-1}\left(I_{a^+}^{n-1}f(s,y_N(s))\right)ds-\int_a^t (s-a)^{k-1}\left(I_{a^+}^{n-1}f(s,y(s))\right)ds\right|\\ &\leq \int_a^t (s-a)^{k-1}\left|I_{a^+}^{n-1}f(s,y_N(s))-I_{a^+}^{n-1}f(s,y(s))\right|ds\\ &\leq \frac{L\delta_{y_N}}{k+n-1}(t-a)^{n+k-1}\\ &\leq \frac{L\delta_{y_N}}{k}(t-a)^{n+k-1}.\end{aligned}$$

As a similar estimation in [2], we obtain that

$$\begin{aligned}\left|\tilde{\mathcal{A}}_N\right| &= \frac{1}{\Gamma(2-n+\beta)}\left|\sum_{k=0}^{N}\frac{\Gamma(k+n-1-\beta)}{\Gamma(n-1-\beta)\,k!}\right|\\ &\leq \frac{1}{(1-n+\beta)\pi}\frac{\Gamma(N+n-\beta)}{\Gamma(N+1)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{N}\left|\tilde{\mathcal{B}}_{N,k}\right|(t-a)^{1-n+\beta-k}\left|\tilde{\mathcal{V}}_{k,N}(t)-\tilde{\mathcal{V}}_k(t)\right|\\ &\qquad\leq L\delta_{y_N}(t-a)^{\beta}\left[\frac{1}{(1-n+\beta)\pi}\frac{\Gamma(N+n-\beta)}{\Gamma(N+1)}+\frac{1}{\Gamma(2-n+\beta)}\right].\end{aligned}$$

Thus

$$|y_N(t) - y(t)| \leq L\delta_{y_N}(t-a)^{\beta}\left[\frac{2}{(1-n+\beta)\pi}\frac{\Gamma(N+n-\beta)}{\Gamma(N+1)} + \frac{1}{\Gamma(2-n+\beta)}\right] + \left|\tilde{\mathcal{E}}_N(t)\right|$$

It implies that

$$\delta_{y_N} \leq L\delta_{y_N}(t-a)^{\beta}\left[\frac{2}{(1-n+\beta)\pi}\frac{\Gamma(N+n-\beta)}{\Gamma(N+1)} + \frac{1}{\Gamma(2-n+\beta)}\right] + \max_{t\in[a,b]}\left|\tilde{\mathcal{E}}_N(t)\right| \tag{2.2.13}$$

where $\left|\tilde{\mathcal{E}}_N(t)\right| \to 0$ as $n \to \infty$. By using that fact $\lim_{N\to\infty}\frac{\Gamma(N+n-\beta)}{\Gamma(N+1)} = 0$, (see e.g. [6]) and setting $N \to \infty$ in 2.2.13 we get

$$\lim_{N\to\infty}\delta_{y_N}\left[1 - \frac{L}{\Gamma(2-n+\beta)}(b-a)^{\beta}\right] \leq 0,$$

from the definition of b , we obtain $\delta_{y_N} \to 0$ as $N \to \infty$. □

CHAPTER 3

Analysis and numerical solution for mixed fractional boundary value problem

In this chapter, we study the existence, uniqueness and numerical solution of the following mixed fractional boundary value problem.

$$ {}^{C}D_{b^-}^{\alpha}\left({}^{C}D_{a^+}^{\beta}y\left(t\right)\right)=f\left(t,y\left(t\right)\right),\quad t\in(a,b), \tag{3.0.1}$$

$$ {}^{C}D_{0^+}^{\beta}y\left(b\right)=0,\quad y\left(a\right)=y_0. \tag{3.0.2}$$

Where $0<\alpha,\beta\leq 1$, $\alpha+\beta>1$ and ${}^{C}D_{1^-}^{\alpha},{}^{C}D_{0^+}^{\beta}$ denotes respectively the right and left Caputo fractional derivative, and $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is given function.

3.1 Existence of solutions

firstly we solve the following linear problem:

$$ {}^{C}D_{b^-}^{\alpha}\left({}^{C}D_{a^+}^{\beta}y\left(t\right)\right)=h\left(t\right),\quad t\in(a,b), \tag{3.1.3}$$

$$ {}^{C}D_{0^+}^{\beta}y\left(b\right)=0,\quad y\left(a\right)=y_0. \tag{3.1.4}$$

Lemma 3.1.1. *Assume that $y\in AC^{1}\left([0,1],\mathbb{R}\right)$, then y is a solution to the linear boundary value problem (3.1.3)-(3.1.4) if and only if y satisfies the integral equation*

$$ y\left(t\right)=\int_a^b G(t,\tau)h(\tau)d\tau+y_0,$$

where

$$G(t,\tau)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\begin{cases}\displaystyle\int_a^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & a\leq\tau\leq t\leq b,\\ \displaystyle\int_a^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & a\leq t\leq\tau\leq b.\end{cases} \tag{3.1.5}$$

Proof. Firstly, we apply the right-hand side fractional integral $I_{b^-}^{\alpha}$ to equation (2.1.3), and using ${}^{C}D_{0^+}^{\beta}y(b)=0$ we get

$${}^{C}D_{a^+}^{\beta}y(t)=I_{b^-}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}h(s)ds, \tag{3.1.6}$$

and we apply the left-hand side fractional integral $I_{a^+}^{\beta}$ to the last above equality and using $y(a)=y_0$ we obtain,

$$y(t)=I_{0^+}^{\beta}I_{1^-}^{\alpha}h(t)+y_0. \tag{3.1.7}$$

then

$$\begin{aligned}y(t)&=I_{a^+}^{\beta}I_{b^-}^{\alpha}h(t)+y_0\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}\left(\int_s^b(\tau-s)^{\alpha-1}h(\tau)d\tau\right)ds+y_0\end{aligned}$$

It follows from Fubini theorem, that

$$\begin{aligned}y(t)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_a^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\right)h(\tau)d\tau\\&\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^b\left(\int_a^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\right)h(\tau)d\tau+y_0,\end{aligned}$$

the proof is finished. □

It easy to show that

Lemma 3.1.2. *The function G satisfy the following properties:*

1. *The function $G(t,\tau)$ is nonnegative and continuous functions.*
2. $\int_a^b G(t,\tau)\,d\tau \leq \frac{(b-a)^{\alpha+\beta}}{(\alpha+\beta-1)(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}$.

In the rest of paper taking $\mathcal{K} = \frac{(b-a)^{\alpha+\beta}}{(\alpha+\beta-1)(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}$. Define the Banach space $E = C([a,b], \mathbb{R})$. We introduce the following hypomemory
(H1) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
(H2) There exist $p, q \in C([a,b], \mathbb{R}_+)$ such that

$$|f(t,y)| \leq p(t)|y| + q(t), \text{ for each } t \in [a,b], \text{ and } y \in \mathbb{R}.$$

(H3) There exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \text{for each } t \in [a,b] \text{ and all } x, y \in \mathbb{R}.$$

Theorem 3.1.1. *Assume that $(H1)$ and $(H2)$ hold, then the problem $(3.0.1)-(3.0.2)$ has a fixed point in B_R, where*

$$B_R = \{y \in E, \textit{ such that } \|y\| \leq R\}.$$

such that $1 > \mathcal{K}\|p\|_\infty$ with

$$R \geq (\mathcal{K}\|q\|_\infty + |y_0|)(1 - \mathcal{K}\|p\|_\infty)^{-1} \tag{3.1.8}$$

Proof. We consider the operator $N : E \to E$ defined by

$$N\left(y\left(t\right)\right)=\int_{a}^{b}G\left(t,\tau\right)f\left(\tau,y\left(\tau\right)\right)d\tau+y_{0}$$

Etape 1 : we prove that N is a continuous operator.

Let (y_n) be a sequence converges to y in E; then for every $t \in [a,b]$ we have

$$\begin{aligned}
&\left|N\left(y_{n}\left(t\right)\right)-N\left(y\left(t\right)\right)\right| \\
&\quad=\left|\int_{a}^{b}G\left(t,\tau\right)f\left(\tau,y_{n}\left(\tau\right)\right)d\tau-\int_{a}^{b}G\left(t,\tau\right)f\left(\tau,y\left(\tau\right)\right)\right| \\
&\quad\leq\int_{a}^{b}G\left(t,\tau\right)\left|\left(f\left(\tau,y_{n}\left(\tau\right)\right)-f\left(\tau,y\left(\tau\right)\right)\right)\right|d\tau \\
&\quad\leq\mathcal{K}\left\|f\left(.,y_{n}\left(.\right)\right)-f\left(.,y\left(.\right)\right)\right\|_{\infty}
\end{aligned}$$

It implies that, $|Ny_n - Ny| \to 0$, when $n \to \infty$, since f is a continuous function

Etape 2: we prove that $N(B_R) \subset B_R$. For any $y \in B_R$ and $t \in (a,b)$ we have

$$\begin{aligned}
\left|N\left(y\left(t\right)\right)\right| &= \left|\int_{a}^{b}G\left(t,\tau\right)f\left(\tau,y\left(\tau\right)\right)d\tau+y_{0}\right| \\
&\leq \int_{a}^{b}G(t,\tau)\left|f\left(\tau,y\left(\tau\right)\right)\right|d\tau+\left|y_{0}\right| \\
&\leq \mathcal{K}\left(\left\|p\right\|_{\infty}R+\left\|q\right\|_{\infty}\right)+\left|y_{0}\right| \\
&\leq R
\end{aligned}$$

Etape 3: we proof that $N(B_R)$ is equicontinuous sets of E. Let $t_1, t_2 \in (a,b)$ such that $t_1 <$

t_2 and $u \in B_R$, then, we have

$$\begin{aligned}
&|N(y(t_2)) - N(y(t_1))| \\
&= \left| \int_a^b G(t_2,\tau) f(\tau, y(r))\, dr - \int_a^b G(t_1,\tau) f(\tau, y(\tau))\, d\tau \right| \\
&= \left| \int_a^b (G(t_2, r) - G(t_1,\tau)) f(\tau, y(\tau))\, d\tau \right| \\
&\leq (\|p\|_\infty R + \|q\|_\infty) \int_a^b |G(t_2,\tau) - G(t_1,\tau)|\, d\tau
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero, since $G(t,\tau)$ is continuous function. By Arzela-Ascoli Theorem we conclude that $N(B_R)$ relatively compact. Applying Schauder fixed point Theorem, it follows that N has a fixed point $y \in B_R$. □

3.2 Numerical approach

Theorem 3.2.1. *([2] (Theorem 10) for $\rho = 1$) Let $N \geq 1$ an integer and $y : [0,1] \to \mathbb{R}$, be a function of class AC^2. Let consider*

$$\mathcal{A}_N = \frac{1}{\Gamma(2-\beta)} \sum_{k=0}^{N} \frac{\Gamma(k-1+\beta)}{\Gamma(\beta-1)\, k!},$$

$$\mathcal{B}_{N,k} = \frac{\Gamma(k-1+\beta)}{\Gamma(2-\beta)\,\Gamma(\beta-1)\,(k-1)!},\ k = 1, 2, ..., N,$$

and functions $\mathcal{V}_k : [a,b] \to \mathbb{R}$ by

$$\mathcal{V}_k(t) = \int_a^t (s-a)^{k-1} y'(s)\, ds,$$

Then

$$ {}^{C}D_{a^{+}}^{\beta}y\left(t\right) = \mathcal{A}_{N}\left(t-a\right)^{1-\beta}y'\left(t\right)-\sum_{k=1}^{N}\mathcal{B}_{N,k}(t-a)^{1-\beta-k}\mathcal{V}_{k}(t)+\mathcal{E}_{N}\left(t\right) $$

where

$$ \lim_{N\to\infty}\mathcal{E}_{N}\left(t\right)=0,\quad \forall t\in[a,b]. $$

The problem (3.1.3)-(3.1.4) is equivalent to the following Cauchy problem

$$ {}^{C}D_{a^{+}}^{\beta}y\left(t\right)=I_{b^{-}}^{\alpha}f\left(t,y\left(t\right)\right)=\frac{1}{\Gamma(\alpha)}\int_{t}^{b}(s-t)^{\alpha-1}f(s,y(s))ds $$

$$ y\left(a\right)=y_{0}. $$

From the last problem we obtain that

$$ y\left(t\right)=y_{0}+\frac{1}{\Gamma(\alpha)}\left({}^{C}D_{a^{+}}^{1-\beta}\int_{a}^{t}\int_{s}^{b}(\tau-s)^{\alpha-1}f(\tau,y(\tau))d\tau ds\right). $$

Using the decomposition formula given in Theorem 3.2.1, we find

$$ \begin{aligned} y\left(t\right) &= y_{0}+\tilde{\mathcal{A}}_{N}\left(t-a\right)^{\beta}\int_{t}^{b}(s-t)^{\alpha-1}f(s,y(s))ds \\ &\quad -\sum_{k=1}^{N}\tilde{\mathcal{B}}_{N,k}(t-a)^{\beta-k}\tilde{\mathcal{V}}_{k}(t)+\tilde{\mathcal{E}}_{N}\left(t\right) \end{aligned} \tag{3.2.9} $$

where the error $\tilde{\mathcal{E}}_{N}(t)\to 0$ as $N\to\infty$ for all $t\in[a,b]$, and

$$ \tilde{\mathcal{A}}_{N}=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(1+\beta\right)}\sum_{k=0}^{N}\frac{\Gamma\left(k-\beta\right)}{\Gamma\left(-\beta\right)k!}, $$

$$ \tilde{\mathcal{B}}_{N,k}=\frac{\Gamma\left(k-\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(1+\beta\right)\Gamma\left(-\beta\right)\left(k-1\right)!},\ k=1,2,...,N, $$

and

$$\tilde{\mathcal{V}}_k(t) = \int_a^t (s-a)^{k-1}\left(\int_s^b (\tau - s)^{\alpha-1} f(\tau, y(\tau))d\tau\right) ds$$

Take the formula of approximate solution, y_N by

$$\begin{aligned} y_N(t) &= y_0 + \tilde{\mathcal{A}}_N (t-a)^{\beta} \int_t^b (s-t)^{\alpha-1} f(s, y_N(s))ds \\ &\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k}(t-a)^{\beta-k} \tilde{\mathcal{V}}_{k,N}(t) \end{aligned} \tag{3.2.10}$$

where

$$\tilde{\mathcal{V}}_{k,N}(t) = \int_a^t + (s-a)^{k-1}\left(\int_s^b (\tau - s)^{\alpha-1} f(\tau, y_N(\tau))d\tau\right) ds$$

Theorem 3.2.2. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies (H_2). For $N \in \mathbb{N}$, let y and y_N as in (3.2.9) and (3.2.10). Suppose that*

$$a < b < a + \left(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{L}\right)^{\dfrac{1}{\alpha+\beta}}. \tag{3.2.11}$$

Therefore, $y_N(t)$ tends to $y(t)$ as N tends to ∞.

Proof. It follows from (3.2.9) and (3.2.10) that

$$\begin{aligned} |y_N(t) - y(t)| &\leq \left|\tilde{\mathcal{A}}_N\right| (t-a)^{\beta} \int_t^b (s-t)^{\alpha-1} \left|f(s, y_N(s)) - f(s, y(s))\right| ds \\ &\quad + \sum_{k=1}^{N} \left|\tilde{\mathcal{B}}_{N,k}\right| (t-a)^{\beta-k} \left|\tilde{\mathcal{V}}_{k,N}(t) - \tilde{\mathcal{V}}_k(t)\right| + \left|\tilde{\mathcal{E}}_N(t)\right|. \end{aligned} \tag{3.2.12}$$

The estimation of $\left|\tilde{\mathcal{A}}_N\right|$ is given by

$$\left|\tilde{\mathcal{A}}_N\right| \leq \frac{1}{\Gamma(\alpha)\beta\pi}\frac{\Gamma(N+1-\beta)}{\Gamma(N+1)}, \tag{3.2.13}$$

for more detail see [2] (Theorem 10) for $\rho = 1$.
Taking $e_{y_N} = \max\limits_{t\in[a,b]} |y_N(t) - y(t)|$, then, by using (H3) we get

$$|f(t, y_N(t)) - f(t, y(t))| \leq L\,|y_N(t) - y(t)| \leq L e_{y_N}, \tag{3.2.14}$$

and

$$\begin{aligned}\left|\tilde{\mathcal{V}}_{k,N}(t) - \tilde{\mathcal{V}}_k(t)\right| &\leq \int_a^t (s-a)^{k-1}\left(\int_s^b (\tau - s)^{\alpha-1}\,|f(\tau, y_N(\tau))d\tau - f(\tau, y(\tau))|\,d\tau\right) ds \\ &\leq \frac{L(b-a)^{\alpha} e_{y_N}}{k\alpha}(t-a)^k. \end{aligned}\tag{3.2.15}$$

It follows from (3.2.13)-(3.2.15) that

$$\begin{aligned}&\sum_{k=1}^{N}\left|\tilde{\mathcal{B}}_{N,k}\right|(t-a)^{\beta-k}\left|\tilde{\mathcal{V}}_{k,N}(t) - \tilde{\mathcal{V}}_k(t)\right| \\ &\qquad\leq \frac{L(b-a)^{\beta+\alpha} e_{y_N}}{\Gamma(\alpha+1)}\left[\frac{1}{\beta\pi}\frac{\Gamma(N+1-\beta)}{\Gamma(N+1)} + \frac{1}{\Gamma(\beta+1)}\right].\end{aligned}\tag{3.2.16}$$

The inequalities (3.2.12)-(3.2.14) and (3.2.16) yields

$$e_{y_N} \leq \frac{L(b-a)^{\beta+\alpha} e_{y_N}}{\Gamma(\alpha+1)}\left[\frac{2}{\beta\pi}\frac{\Gamma(N+1-\beta)}{\Gamma(N+1)} + \frac{1}{\Gamma(\beta+1)}\right] + \max_{t\in[a,b]}\left|\tilde{\mathcal{E}}_N(t)\right| \tag{3.2.17}$$

By using $\lim_{N\to\infty} \frac{\Gamma(N+1-\beta)}{\Gamma(N+1)} = 0$ (see e.g. [6]) and setting $N \to \infty$ in (3.2.17) we get

$$\lim_{N\to\infty} e_{y_N} \left(1 - \frac{L}{\Gamma(\alpha+1)\Gamma(\beta+1)} (b-a)^{\beta+\alpha}\right)$$

and by the definition of b , we must have $x_N(t)$ tends to $x(t)$ as N tends to ∞. □

In this memory, a fractional boundary value problem with Caputo derivative and a mixed fractional boundary value problem with Caputo derivative are provided. A numerical technique based on a decomposition formula for Caputo derivative operator is given. Furthermore, the convergence analysis of the method is achived.

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