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Research Article

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Existence and uniqueness of the weak solution for Keller-Segel model coupled with Boussinesq equations

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Abstract: Keller-Segel chemotaxis model is described by a system of nonlinear partial differential equations: a convection diffusion equation for the cell density coupled with a reaction-diffusion equation for chemoattractant concentration. In this work, we study the phenomenon of Keller-Segel model coupled with Boussinesq equations. The main objective of this work is to study the global existence and uniqueness and boundedness of the weak solution for the problem, which is carried out by the Galerkin method.

Keywords: chemotaxis, Keller-Segel, global existence, boundedness, Boussinesq equations, Galerkin method

MSC 2020: 92C17, 35K58, 82C22

1 Introduction

The phenomenon of chemotaxis is one of the most important phenomena that have aroused the interest of many researchers, as chemotaxis is a biological process where cells (bacteria) move towards a more appropriate chemical direction. The Keller and Segel model, which describes chemotaxis, is considered one of the best studied models in mathematical biology; on the other hand, nature cells often live in a viscous fluid and transport cells and chemical substrates with the fluid, and in the meantime the fluid movement is subject to affecting factors. Chemotaxis model is regarded as one of the most important phenomenon studies, with Keller-Segel [1] [1970] being the first to discover it. Chemotaxis is represented by a model to explain the chemotaxis phenomenon, a chemical attraction that exists between organisms. Among the most important works accomplished for this phenomenon, we mention [2–18]. In [18], the authors studied nonlinear stochastic chemotaxis model with Dirichlet boundary conditions with fractional derivative and disturbed by multiplicative noise and prove the existence and uniqueness of mild solution to time and space-fractional. In [15], the authors studied global existence solution for parabolic-eliptic Keller-Segel model, and in [19], the authors studied regularity and asymptotic behavior of the Keller-Segel system of degenerate type with critical nonlinearity and they also studied the global solutions to the coupled

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chemotaxis-fluid equations. In general, the fluid motion is determined by the incompressible Navier-Stokes equations or the Stokes equations, as this type of reaction in the cellular fluid becomes more complex because it includes not only chemotaxis and diffusion but also the transport and dynamics of viscous fluids. Several researchers have also studied many types of such associated models, which we mention in [20], where the authors proposed the following model:

$$\begin{cases} n_t + u\nabla n = \delta\Delta n^m - \nabla(\chi(c)n\nabla c), \\ c_t + u\nabla c = \mu\Delta c - k(c)n, \\ u_t + \nabla p = \upsilon\Delta u - n\nabla\Phi, \\ \nabla u = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \end{cases}$$

and they studied a model arising from biology, consisting of chemotaxis equations coupled with viscous incompressible fluid equations through transport and external forcing and proved global existence of large data and solutions to the Cauchy problem investigated under certain conditions for the chemotaxis-Navier-Stokes system in two space dimensions, and we obtain precisely global existence of weak solutions for the chemotaxis-Stokes system with nonlinear diffusion for the cell density in three space dimensions. In [7], Duan et al. formed a model

$$\begin{cases} n_t + u\nabla n = \delta\Delta n - \nabla(\chi(c)n\nabla c), \\ c_t + u\nabla c = \mu\Delta c - k(c)n, \\ u_t + \nabla p = \upsilon\Delta u - n\nabla\Phi, \\ \nabla u = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+. \end{cases}$$

In [14], Lorz studied a system consisting of the elliptic-parabolic Keller-Segel equations coupled with Stokes equations by transport and gravitational forcing. We show global-in-time existence of solutions for small initial mass in 2D. In 3D, we establish global existence assuming that the initial $L^{3/2}$-norm is small. Moreover, we give numerical evidence for this extension of the Keller-Segel system in 2D for the model defined as

$$\begin{cases} u\nabla c = -\Delta c + n - a_1 c, \\ n_t + u\nabla n = \Delta n - \nabla(\chi n\nabla c), \\ a_2 u_t + \nabla p - n\Delta u + n\nabla\Phi = 0, \\ \nabla u = 0. \end{cases}$$

Now we consider the Keller-Segel model coupled with Boussinesq equations and we study the global existence and uniqueness of solution for the problem defines as

$$\begin{cases} n_t + u\nabla n - \Delta n - \nabla(n\nabla c) - \nabla(n\nabla\theta) = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ c_t + u\nabla c - \Delta c - \tau c - \rho u - b\theta = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta_t + u\nabla\theta - k\Delta\theta - n\theta = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u_t + u\nabla u - \nu\Delta u - \nabla p - (\theta + n)f = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{1}$$

where $n = n(x,t)$ denotes the density of the cells in position $x \in \mathbb{R}^d$ at time t, $c = c(x,t)$ is the concentration of chemical attractant, $\theta = \theta(x,t)$ denotes temperature of the fluid, $u = u(x,t)$ is the field denoting the velocity, τ, ρ, b are positive constants, $k \geq 0$ is the thermal diffusivity, $\nu \geq 0$ is the kinematic viscosity, $f(x,t)$ is the extermal potential force where $\nabla f = 0$, $p(x,t)$ is the pressure. The main objective of this work is to study the problem Keller-Segel coupled with Boussinesq equations, and we demonstrate the global existence and uniqueness of a weak solution for KSB problem with the Dirichlet conditions and initial conditions defined as:

$$\begin{cases} P1\begin{cases} n_t + u\nabla n - \Delta n - \nabla(n\nabla c) - \nabla(n\nabla\theta) = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ n = 0, & x \in \Gamma, \\ n(0,x) = n_0, & x \in \Omega, \end{cases} \\ P2\begin{cases} c_t + u\nabla c - \Delta c - \tau c - \rho u - b\theta = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ c = 0, & x \in \Gamma, \\ c(0,x) = c_0, & x \in \Omega, \end{cases} \\ P3\begin{cases} \theta_t + u\nabla\theta - k\Delta\theta - n\theta = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta = 0, & x \in \Gamma, \\ \theta(0,x) = \theta_0, & x \in \Omega, \end{cases} \\ P4\begin{cases} u_t + u\nabla u - \nu\Delta u - \nabla p - (\theta + n)f = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = 0, & x \in \Gamma, \\ u(0,x) = u_0, & x \in \Omega. \end{cases} \end{cases} \tag{2}$$

2 Existence and uniqueness of weak solution of the problem

To simplify the weak solution of the problem (2), a decomposition into four subproblems ($P1$) and ($P2$) and ($P3$) and ($P4$) is adopted. We use the Galerkin method to demonstrate the existence and uniqueness of a weak solution of subproblems ($P1$) and ($P2$) and ($P3$) and ($P4$); therefore, we obtain the existence and uniqueness of a weak solution of the problem (2). The following initial-boundary conditions assumption is used to prove the proposed solution of (2):

$$n_0 \in L^2(\Omega), \tag{3}$$

$$c_0 \in L^2(\Omega), \tag{4}$$

$$\theta_0 \in L^2(\Omega), \tag{5}$$

$$u_0 \in L^2(\Omega). \tag{6}$$

2.1 Existence and uniqueness of weak solution of the problem (*P*1)

In this subsection, we prove the existence and uniqueness of weak solution result of the problem ($P1$) .

Definition 2.1. We say $n \in L^2(0, T; H_0^1(\Omega))$ with $n_t \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of the problem ($P1$) if and only if

$$\langle n_t, \Phi\rangle + B(n, \Phi, t) = 0, \tag{7}$$

where

$$B(n, \Phi, t) = \int\limits_{\Omega} [(\nabla n \nabla\Phi) + n\nabla(u\Phi) + (n\nabla c + n\nabla\theta)\nabla\Phi]\mathrm{d}x, \tag{8}$$

for all $\Phi \in H_0^1(\Omega)$, $0 \le t \le T$, and

$$n(0, x) = n_0 \in L^2(\Omega). \tag{9}$$

Remark 2.1. Note that $n \in C([0, T]; L^2(\Omega))$ as $n \in L^2(0, T; H_0^1(\Omega))$ and $n_t \in L^2(0, T; H^{-1}(\Omega))$. Then equality (9) makes sense.

Before proving the existence and uniqueness of weak solution of the problem ($P1$), we need the following lemma:

Lemma 2.1.

(*i*) *For all* $\Phi \in H_0^1(\Omega)$, *the* $B(n, \Phi, t)$ *is continuous in* $H_0^1(\Omega) \times H_0^1(\Omega)$, *there exists a constant positive* C *such that*

$$|B(n, \Phi, t)| \leq C\|n\|_{H_0^1(\Omega)}\|\Phi\|_{H_0^1(\Omega)}. \tag{10}$$

(*ii*) *For any* $n \in H_0^1(\Omega)$, *there exists a constant positive* β *such that*

$$\beta\|n\|_{H_0^1(\Omega)} \leq B(n, n, t), \quad \forall n \in H_0^1(\Omega). \tag{11}$$

Proof.

(i) We use the Cauchy-Schwarz inequality on (8) and obtain

$$\begin{aligned}|B(n, \Phi, t)| \leq\ & \|\nabla n\|_{L^2(\Omega)}\|\nabla\Phi\|_{L^2(\Omega)} + \|n\|_{L^2(\Omega)}\|\nabla(u\Phi)\|_{L^2(\Omega)} + \|n\|_{L^2(\Omega)}\|\nabla c\|_{L^4(\Omega)}\|\nabla\Phi\|_{L^4(\Omega)}\\ &+ \|n\|_{L^2(\Omega)}\|\nabla\theta\|_{L^4(\Omega)}\|\nabla\Phi\|_{L^4(\Omega)},\end{aligned}$$

and we have

$$|B(n, \Phi, t)| \leq C\|n\|_{H^1(\Omega)}\|\Phi\|_{H^1(\Omega)}.$$

(ii) The expression of $B(n, n, t)$ is obtained

$$B(n, n, t) = \int_\Omega ((\nabla n)^2 + n\nabla(un) + n\nabla c\nabla n + n\nabla\theta\nabla n)\mathrm{d}x,$$

and we have

$$B(n, n, t) \geq \int_\Omega (\nabla n)^2\mathrm{d}x = \|\nabla n\|^2_{L^2(\Omega)},$$

finally, Poincaré inequality, gives $B(n, n, t) \geq \beta\|n\|^2_{H_0^1(\Omega)}$. □

To demonstrate the existence of weak solution of problem ($P1$), we use the Galerkin method and assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega),\\ \forall m;\ w_t \ldots w_m, \ \text{is linearly independent},\\ \text{the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega).\end{cases} \tag{12}$$

We are looking for $n_m = n_m(t)$ solution <<approached>> of the problem in the form:

$$n_m(t) = \sum_{i=1}^m g_{im}(t)w_i, \tag{13}$$

and g_{im} to be determined by the conditions:

$$\begin{cases}\langle n'_m, w_j\rangle + B(n_m, w_j, t) = 0,\\ 1 \leq j \leq m.\end{cases} \tag{14}$$

The nonlinear differential equation system is to be completed by the conditional:

$$n_m(0) = n_{0m}, \quad n_{0m} = \sum_{i=1}^m \alpha_{im}w_j \to u_0 \quad \text{in } H_0^1(\Omega), \quad \text{when } m \to \infty.$$

2.1.1 Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions n_m of the approximation problems (14) and (2.1) converge to a weak solution of $(P1)$. For this, we will need some uniform estimates.

Theorem 2.1. (*Energy estimates.*) *There exists a constant C depending only on Ω, T, such that*

$$\max_{0\le t\le T} \|n_m\|_{L^2(\Omega)} + \|n_m\|_{L^2(0,T;\, H_0^1(\Omega))} + \|n_m'\|_{L^2(0,T;\, H^{-1}(\Omega))} \le C\|n_0\|_{L^2(\Omega)}. \tag{15}$$

Proof. In order to prove the estimation (15), we will estimate each terms in the left side of (13) one by one as follows:

1. Multiplying equation (14) by $g_{jm}(t)$, summing for $k = 1,\dots, m$, and then recalling (13) we find

$$\langle n_m', n_m\rangle + B(n_m, n_m, t) = 0, \tag{16}$$

   and we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|n_m\|^2_{L^2(\Omega)}\right] + B(n_m, n_m, t) = 0, \tag{17}$$

   from Lemma (2.1) there exists constant $\beta > 0$ such that

$$\beta\|n_m\|^2_{H_0^1(\Omega)} \le B(n_m, n_m, t), \quad \forall 0 \le t \le T,$$

   and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|n_m\|^2_{L^2(\Omega)}\right) + \beta\|n_m\|^2_{H_0^1(\Omega)} \le 0, \tag{18}$$

   which implies that

$$\|n_m\|^2_{L^2(\Omega)} \le \|n_m(0)\|^2_{L^2(\Omega)} \le \|n_0\|^2_{L^2(\Omega)},$$

   so we have

$$\max_{0\le t\le T} \|n_m\|_{L^2(\Omega)} \le \|n_0\|_{L^2(\Omega)}. \tag{19}$$

2. Integrate inequality (18) from 0 to T and we employ inequality (19) to find

$$\|n_m\|^2_{L^2(0,T;\, H_0^1(\Omega))} = \int_0^T \|n_m\|^2_{H_0^1(\Omega)} \mathrm{d}t.$$

3. Fix any $v \in H_0^1(\Omega)$, with $\|v\|^2_{H_0^1(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$, $(k = 1, \dots, m)$; we use (14) and deduce for all $0 \le t \le T$ that

$$(n_m', v^1) + B(n_m, v^1, t) = 0, \tag{20}$$

   then (20) implies

$$\langle n_m', v\rangle = (n_m', v) = (n_m', v^1) = -B(n_m, v^1, t),$$

   consequently

$$|\langle n_m', v\rangle| \le C\|n_m\|^2_{H_0^1(\Omega)},$$

   since

$$\|v^1\|^2_{H_0^1(\Omega)} \le \|v\|^2_{H_0^1(\Omega)} \le 1,$$

we have

$$\|n_m'\|_{H^{-1}(\Omega)} \leq C\|n_m\|_{H_0^1(\Omega)},$$

and therefore,

$$\|n_m'\|^2_{L^2(0,T;\,H^{-1}(\Omega))} = \int_0^T \|n_m'\|^2_{H^{-1}(\Omega)}\mathrm{d}t \leq C\int_0^T \|n_m\|^2_{H_0^1(\Omega)}\mathrm{d}t \leq C\|n_0\|^2_{L^2(\Omega)}. \qquad \square$$

2.1.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem ($P1$).

Theorem 2.2. (*Existence of weak solution.*) *Under hypothesis* (3), *there exists a weak solution of problem* ($P1$).

Proof. According to the energy estimates (15), we see that the sequence $\{n_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{n_m'\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. Consequently, there exists a subsequence which is also noted by $\{n_m\}_{m=1}^{\infty}$ and a function $n \in L^2(0, T; H_0^1(\Omega))$, with $n' \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} &n_m \to n \ \text{ weakly in } \ L^2(0, T; H_0^1(\Omega)),\\ &n_m' \to n' \ \text{ weakly in } \ L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{21}$$

1. Next fix an integer N and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ having the form:

$$v(t) = \sum_{k=1}^{N} g^{(k)}(t)w_k, \tag{22}$$

where $\{g^{(k)}\}_{k=1}^{N}$ are given smooth functions, we choose $m \geq N$, multiply equation (14) by $g^{(k)}(t)$ $\forall k = 1, \dots, N$, and then integrate with respect to t to find

$$\int_0^t \langle n_m', v\rangle + B(n_m, v, t)\mathrm{d}t = 0, \tag{23}$$

we recall (21) and to find upon passing to weak limits that

$$\int_0^t \langle n', v\rangle + B(n, v, t)\mathrm{d}t = 0, \quad \forall v \in L^2(0, T; H_0^1(\Omega)), \tag{24}$$

as functions of the form (22) are dense in $L^2(0, T; H_0^1(\Omega))$. Hence, in particular

$$\langle n', v\rangle + B(n, v, t) = 0, \quad \forall v \in H_0^1(\Omega) \quad \text{and} \quad \forall t \in [0, T], \tag{25}$$

and from Remark (2.1) we have $n \in C(0, T; L^2(\Omega))$.

2. In order to prove $n(0) = n_0$, we first note from (9) that

$$\int_0^t -\langle n, v'\rangle + B(n, v, t) = (n(0), v(0)), \tag{26}$$

for each $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$. Similarly, from (23) we obtain

$$\int_0^t -\langle n_m, v'\rangle + B(n_m, v, t)\mathrm{d}t = (n_0, v(0)), \tag{27}$$

and we use again (26) and obtain

$$\int_0^t -\langle n, v'\rangle + B(n, v, t)\mathrm{d}t = (n_0, v(0)), \tag{28}$$

since $n_m(0) \to n_0$ in $L^2(\Omega)$. Comparing (26) and (28), we conclude $n(0) = n_0$. □

Theorem 2.3. (*Uniqueness of weak solutions.*) *A weak solution of problem* ($P1$) *is unique.*

Proof. We suppose there exists two weak solutions n_1 and n_2 and we put $N = n_2 - n_1$ then N is also a solution of problem ($P1$) with $N_0 = (n_2 - n_1)(0) \equiv 0$. Setting $v = N$ in identity (17) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|N\|^2_{L^2(\Omega)}\right) + B(N, N, t) = 0,$$

from Lemma (2.1), we have $B(N, N, t) \geq \beta\|N\|^2_{H^1_0(\Omega)} \geq 0$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|N\|^2_{L^2(\Omega)}\right) \leq 0,$$

then integrate with respect to t to find

$$\|N\|^2_{L^2(\Omega)} \leq \|N_0\|^2_{L^2(\Omega)} = 0,$$

thus $N \equiv 0$. □

2.2 Existence and uniqueness of weak solution of problem (*P*2)

In this subsection, we state and prove the existence and uniqueness of weak solution result of the problem ($P2$)

Definition 2.2. We say $c \in L^2(0, T; H^1_0(\Omega))$ with $c_t \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of the problem ($P2$) if and only if

$$\langle c_t, q\rangle + L(c, q, t) = 0, \tag{29}$$

where

$$L(c, q, t) = \int_\Omega [(\nabla c \nabla q) + u\nabla c q + \tau c q + \rho n q + b\theta q]\mathrm{d}x, \tag{30}$$

for all $q \in H^1_0(\Omega)$, $0 \leq t \leq T$, and

$$c(0, x) = c_0 \in L^2(\Omega). \tag{31}$$

Remark 2.2. Note that $c \in C([0, T]; L^2(\Omega))$ as $c \in L^2(0, T; H^1_0(\Omega))$ and $c_t \in L^2(0, T; H^{-1}(\Omega))$. Then equality (31) makes sense.

To demonstrate the existence of weak solution of problem ($P2$) we use the Galerkin method, and we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H^1_0(\Omega), \\ \forall m;\ w_t \ldots w_m \text{ is linearly independent}, \\ \text{the finite linear combinations of } w_i \text{ are dense in } H^1_0(\Omega). \end{cases}$$

We are looking for $c_m = c_m(t)$ solution <<approached>> of the problem in the form

$$c_m(t) = \sum_{i=1}^{m} d_{im}(t) w_i, \tag{32}$$

and d_{im} to be determined by the conditions:

$$\begin{cases} \langle c_m', w_j \rangle + L(c_m, w_j, t) = 0, \\ 1 \le j \le m. \end{cases} \tag{33}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$c_m(0) = c_{0m}, \quad c_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to c_0 \quad \text{in } H_0^1(\Omega), \quad \text{when } m \to \infty. \tag{34}$$

We now propose to send m to infinity and to show a subsequence of our solutions c_m approximation problems (33) and (34) converges towards a weak solution of the problem $(P2)$. For this, we need uniform estimates.

2.2.1 Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions c_m of the approximation problems (33) and (34) converge to a weak solution of problem $(P2)$. For this, we will need some uniform estimates.

Theorem 2.4. (*Energy estimates.*) *There exists a constant C depending only on Ω, T such that*

$$\max_{0 \le t \le T} \|c_m\|_{L^2(\Omega)} + \|c_m\|_{L^2(0,T;\, H_0^1(\Omega))} + \|c_m'\|_{L^2(0,T;\, H^{-1}(\Omega))} \le C \|c_0\|_{L^2(\Omega)}. \tag{35}$$

Proof. In order to prove the estimation (35), we will estimate each term in the left side of (33) one by one as follows:

1. Multiplying equation (33) by $d_{jm}(t)$ and summing for j we find

$$\langle c_m', w_j \rangle + L(c_m, c_m, t) = 0, \tag{36}$$

and we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|c_m\|^2_{L^2(\Omega)}\right] + L(c_m(t), c_m(t)) = 0,$$

and we put $\|v\| = \sqrt{L(v, v)}$ (= is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|c_m\|^2_{L^2(\Omega)}\right) + \|c_m\|^2_{H_0^1(\Omega)} = 0, \tag{37}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|c_m\|^2_{L^2(\Omega)}\right) \le 0$$

and

$$\|c_m\|^2_{L^2(\Omega)} \le \|c_m(0)\|^2_{L^2(\Omega)} \le \|c_0\|^2_{L^2(\Omega)},$$

and so

$$\max_{0 \le t \le T} \|c_m\|_{L^2(\Omega)} \le \|c_0\|_{L^2(\Omega)}. \tag{38}$$

2. Integrate inequality (37) from 0 to T and we use (38) to find

$$\|c_m\|^2_{L^2(0,T;\ H_0^1(\Omega))} = \int_0^T \|c_m\|^2_{H_0^1(\Omega)} \mathrm{d}t.$$

3. Fix any $v \in H_0^1(\Omega)$, with $\|v\|^2_{H_0^1(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, \dots, m)$. We use (33) from all $0 \le t \le T$ that

$$(c_m', v^1) + L(c_m, v^1, t) = 0. \tag{39}$$

Then (39) implies

$$(c_m', v) = \langle c_m', v\rangle = \langle c_m', v^1\rangle = -L(c_m, v^1, t),$$

consequently

$$|\langle c_m', v\rangle| \le C\|c_m\|^2_{H_0^1(\Omega)},$$

and as

$$\|v^1\|^2_{H_0^1(\Omega)} \le \|v\|^2_{H_0^1(\Omega)} \le 1,$$

thus

$$\|c_m'\|_{H^{-1}(\Omega)} \le C\|c_m\|_{H_0^1(\Omega)},$$

and therefore,

$$\|c_m'\|^2_{L^2(0,T;\ H^{-1}(\Omega))} = \int_0^T \|c_m'\|^2_{H^{-1}(\Omega)} \mathrm{d}t \le C\int_0^T \|c_m\|^2_{H_0^1(\Omega)} \mathrm{d}t \le C\|c_0\|^2_{L^2(\Omega)}.$$ □

2.3 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem ($P2$).

Theorem 2.5. (*Existence of weak solution.*) *Under hypothesis* (4), *there exists a weak solution of* ($P2$).

Proof. According to the energy estimates (35), we see that the sequence $\{c_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{c_m'\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, and consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $c \in L^2(0, T; H_0^1(\Omega))$ with $c' \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} c_m &\to c \ \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ c_m' &\to c' \ \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{40}$$

1. Next fix an integer N and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{41}$$

where $\{d^{(k)}\}_{k=1}^{N}$ are given smooth functions. We choose $m \ge N$, multiply equation (33) by $d^{(k)}(t)$ $\forall k = 1, \dots, N$, and then integrate with respect to t to find

$$\int_0^t \langle c_m', v\rangle + L(c_m, v, t)\mathrm{d}t = 0, \tag{42}$$

we recall (40) to find upon passing to weak limits that

$$\int_0^t \langle c', v\rangle + L(c, v, t)\mathrm{d}t = 0, \quad \forall v \in L^2(0, T;\ H_0^1(\Omega)). \tag{43}$$

As functions of the form (32) are dense in $L^2(0, T;\ H_0^1(\Omega))$, hence, in particular

$$\langle c', v\rangle + L(c, v, t) = 0, \quad \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0, T] \tag{44}$$

and from Remark (2.2) we have $c \in C(0, T;\ L^2(\Omega))$.

2. In order to prove $c(0) = c_0$, we first note from (31) that

$$\int_0^t -\langle c, v'\rangle + L(c, v, t)\mathrm{d}t = (c(0), v(0)), \tag{45}$$

for each $v \in C^1(0, T;\ H_0^1(\Omega))$ with $v(T) = 0$. Similarly, from (45) we obtain

$$\int_0^t -\langle c_m, v'\rangle + L(c_m, v, t)\mathrm{d}t = (c_0, v(0)), \tag{46}$$

we use again (45) and obtain

$$\int_0^t -\langle c, v'\rangle + L(c, v, t)\mathrm{d}t = (c_0, v(0)), \tag{47}$$

since $c_m(0) \to c_0$ in $L^2(\Omega)$. Comparing (45) and (47), we conclude $c(0) = c_0$. □

Theorem 2.6. (*Uniqueness of weak solutions.*) *A weak solution of problem* ($P2$) *is unique.*

Proof. We suppose there exists two weak solutions c_1 and c_2 and we put that $C = c_2 - c_1$, then C is also a solution of ($P2$) with $C_0 = (c_2 - c_1)(0) \equiv 0$. Setting $v = C$ in identity (44) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|C\|^2_{L^2(\Omega)}\right) + L(C, C, t) = 0,$$

and as $\|C\| = \sqrt{L(C, C)}$ (= norm in $H_0^1(\Omega)$), there $L(C, C, t) = \|C\|^2_{H_0^1(\Omega)} \geq 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|C\|^2_{L^2(\Omega)}\right) \leq 0,$$

then integrate with respect to t to find

$$\|C\|^2_{L^2(\Omega)} \leq \|C_0\|^2_{L^2(\Omega)} = 0,$$

then $C \equiv 0$. □

2.4 Existence and uniqueness of weak solution of the problem ($P3$)

In this subsection, we state and prove the existence and uniqueness of weak solution result of the problem ($P3$).

Definition 2.3. We say $\theta \in L^2(0, T;\ H_0^1(\Omega))$ with $\theta_t \in L^2(0, T;\ H^{-1}(\Omega))$ is a weak solution of the problem ($P3$) if and only if

$$\langle c_t, q\rangle + A(c, q, t) = 0, \tag{48}$$

where

$$A(c,q,t)=\int\limits_{\Omega}[k(\nabla\theta\nabla q)+u\nabla\theta q+n\theta q]\mathrm{d}x, \tag{49}$$

for all $q \in H_0^1(\Omega)$, $0 \le t \le T$, and

$$\theta(0,x)=\theta_0\in L^2(\Omega). \tag{50}$$

Remark 2.3. Note that $\theta \in C([0,T];L^2(\Omega))$ as $\theta \in L^2(0,T;H_0^1(\Omega))$ and $\theta_t \in L^2(0,T;H^{-1}(\Omega))$. Then equality (50) makes sense.

To demonstrate the existence of weak solution of problem $(P3)$, we use the Galerkin method, and we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m;\ w_t \ldots w_m \text{ is linearly independent}, \\ \text{the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{51}$$

We are looking for $\theta_m = \theta_m(t)$ solution <<approached>> of the problem in the form

$$\theta_m(t)=\sum_{i=1}^{m}d_{im}(t)w_i, \tag{52}$$

and d_{im} to be determined by the conditions:

$$\begin{cases} \langle\theta_m', w_j\rangle + A(\theta_m, w_j, t) = 0, \\ 1 \le j \le m. \end{cases} \tag{53}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$\theta_m(0)=\theta_{0m},\quad \theta_{0m}=\sum_{i=1}^{m}\beta_{im}w_j\to\theta_0 \text{ in } H_0^1(\Omega), \text{ when } m\to\infty. \tag{54}$$

We now propose to send m to infinity and to show a subsequence of our solutions θ_m approximation problems (53) and (54) converges towards a weak solution of the problem $(P3)$. For this, we need uniform estimates.

2.4.1 Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions θ_m of the approximation problems (53) and (54) converges to a weak solution of problem $(P3)$. For this, we will need some uniform estimates.

Theorem 2.7. (*Energy estimates.*) *There exists a constant C depending only on Ω, T such that*

$$\max_{0\le t\le T}\|\theta_m\|_{L^2(\Omega)}+\|\theta_m\|_{L^2(0,T;\,H_0^1(\Omega))}+\|\theta_m'\|_{L^2(0,T;\,H^{-1}(\Omega))}\le C\|\theta_0\|_{L^2(\Omega)}. \tag{55}$$

Proof. In order to prove the estimation (55) we will estimate each terms in the left side of (53) one by one as follows:

1. Multiplying equation (53) by $d_{jm}(t)$ and summing for j we find

$$\langle\theta_m', w_j\rangle + A(\theta_m,\theta_m,t)=0, \tag{56}$$

and we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\theta_m\|^2_{L^2(\Omega)}\right] + A(\theta_m(t), \theta_m(t)) = 0,$$

and we put $\|v\| = \sqrt{A(v, v)}$ (= is norm in $H^1_0(\Omega)$), so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_m\|^2_{L^2(\Omega)}\right) + \|\theta_m\|^2_{H^1_0(\Omega)} = 0, \tag{57}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_m\|^2_{L^2(\Omega)}\right) \leq 0$$

and

$$\|\theta_m\|^2_{L^2(\Omega)} \leq \|\theta_m(0)\|^2_{L^2(\Omega)} \leq \|\theta_0\|^2_{L^2(\Omega)},$$

and so we have

$$\max_{0\leq t\leq T} \|\theta_m\|_{L^2(\Omega)} \leq \|\theta_0\|_{L^2(\Omega)}. \tag{58}$$

2. Integrate inequality (57) from 0 to T and we use (58) to find

$$\|\theta_m\|^2_{L^2(0,T;\, H^1_0(\Omega))} = \int_0^T \|\theta_m\|^2_{H^1_0(\Omega)}\mathrm{d}t.$$

3. Fix any $v \in H^1_0(\Omega)$, with $\|v\|^2_{H^1_0(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, \ldots, m)$. We use (53) from all $0 \leq t \leq T$ that

$$(\theta'_m, v^1) + A(\theta_m, v^1, t) = 0. \tag{59}$$

Then (59) implies

$$(\theta'_m, v) = \langle\theta'_m, v\rangle = \langle\theta'_m, v^1\rangle = -A(\theta_m, v^1, t),$$

consequently

$$|\langle\theta'_m, v\rangle| \leq C\|\theta_m\|^2_{H^1_0(\Omega)},$$

and as

$$\|v^1\|^2_{H^1_0(\Omega)} \leq \|v\|^2_{H^1_0(\Omega)} \leq 1,$$

thus

$$\|\theta'_m\|_{H^{-1}(\Omega)} \leq C\|\theta_m\|_{H^1_0(\Omega)},$$

and therefore

$$\|\theta'_m\|^2_{L^2(0,T;\, H^{-1}(\Omega))} = \int_0^T \|\theta'_m\|^2_{H^{-1}(\Omega)}\mathrm{d}t \leq C\int_0^T \|\theta_m\|^2_{H^1_0(\Omega)}\mathrm{d}t \leq C\|\theta_0\|^2_{L^2(\Omega)}.$$

□

2.4.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem ($P3$).

Theorem 2.8. (*Existence of weak solution.*) *Under hypothesis* (59), *there exists a weak solution of* ($P3$).

Proof. According to the energy estimates (55), we see that the sequence $\{\theta_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{\theta_m'\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ and consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $\theta \in L^2(0, T; H_0^1(\Omega))$ with $\theta' \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} \theta_m \to \theta \ \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ \theta_m' \to \theta' \ \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{60}$$

1. Next fix an integer N and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ having the form

   $$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{61}$$

   where $\{d^{(k)}\}_{k=1}^{N}$ are given smooth functions. We choose $m \geq N$, multiply equation (53) by $d^{(k)}(t)$ $\forall k = 1, \ldots, N$, and then integrate with respect to t to find

   $$\int_0^t \langle \theta_m', v \rangle + A(\theta_m, v, t) \mathrm{d}t = 0, \tag{62}$$

   we recall (60) to find upon passing to weak limits that

   $$\int_0^t \langle \theta', v \rangle + A(\theta, v, t) \mathrm{d}t = 0, \quad \forall v \in L^2(0, T; H_0^1(\Omega)). \tag{63}$$

   As functions of the form (61) are dense in $L^2(0, T; H_0^1(\Omega))$. Hence in particular

   $$\langle \theta', v \rangle + A(\theta, v, t) = 0, \quad \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0, T] \tag{64}$$

   and from Remark (2.3) we have $\theta \in C(0, T; L^2(\Omega))$.
2. In order to prove $\theta(0) = \theta_0$, we first note from (50) that

   $$\int_0^t -\langle \theta, v' \rangle + A(\theta, v, t) \mathrm{d}t = (\theta(0), v(0)), \tag{65}$$

   for each $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$. Similarly, from (62) we obtain

   $$\int_0^t -\langle \theta_m, v' \rangle + A(\theta_m, v, t) \mathrm{d}t = (\theta_0, v(0)), \tag{66}$$

   we use again (65) and obtain

   $$\int_0^t -\langle \theta, v' \rangle + A(\theta, v, t) \mathrm{d}t = (\theta_0, v(0)), \tag{67}$$

   since $\theta_m(0) \to \theta_0$ in $L^2(\Omega)$. Comparing (65) and (67), we conclude $(\theta_0) = \theta_0$. □

Theorem 2.9. (*Uniqueness of weak solutions.*) *A weak solution of problem* ($P3$) *is unique.*

Proof. We suppose there exists two weak solutions θ_1 and θ_2 and we put that $\theta = \theta_2 - \theta_1$, then θ is also a solution of ($P3$) with $\theta_0 = (\theta_2 - \theta_1)(0) \equiv 0$. Setting $v = \theta$ in identity (64) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\theta\|_{L^2(\Omega)}^2\right) + A(\theta, \theta, t) = 0,$$

and as $\|\theta\| = \sqrt{A(\theta, \theta)}$ (norm in $H_0^1(\Omega)$), $A(\theta, \theta, t) = \|\theta\|^2_{H_0^1(\Omega)} \geq 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\theta\|^2_{L^2(\Omega)}\right) \leq 0,$$

and integrate with respect to t to find

$$\|\theta\|^2_{L^2(\Omega)} \leq \|\theta_0\|^2_{L^2(\Omega)} = 0,$$

then $\theta \equiv 0$. □

3 Existence and uniqueness of weak solution of problem (*P4*)

We state and prove the existence and uniqueness of weak solution result of problem ($P4$)

Definition 3.1. We say $u \in L^2(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of problem ($P4$) if and only if

$$\langle u_t, q\rangle + L(u, q, t) = 0,$$

where

$$L(c, q, t) = \int_\Omega [(v\nabla u\nabla q) + u\nabla uq + p\nabla q + (\theta + n)fq]\mathrm{d}x, \tag{68}$$

for all $q \in H_0^1(\Omega)$, $0 \leq t \leq T$, and

$$u(0, x) = u_0 \in L^2(\Omega). \tag{69}$$

Remark 3.1. Note that $u \in C([0, T]; L^2(\Omega))$ as $u \in L^2(0, T; H_0^1(\Omega))$ and $u_t \in L^2(0, T; H^{-1}(\Omega))$. Then equality (69) makes sense.

To demonstrate the existence of weak solution of problem (P1) we use the Galerkin method, and we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m;\ w_t \ldots w_m \text{ is linearly independent}, \\ \text{the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{70}$$

We are looking for $u_m = u_m(t)$ solution <<approached>> of the problem in the form

$$u_m(t) = \sum_{i=1}^{m} d_{im}(t)w_i, \tag{71}$$

where d_{im} to be determined by the conditions:

$$\begin{cases} \langle c_m', w_j\rangle + L(c_m, w_j, t) = 0, \\ 1 \leq j \leq m. \end{cases} \tag{72}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$u_m(0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to u_0 \text{ in } H_0^1(\Omega), \quad \text{when } m \to \infty. \tag{73}$$

We now propose to send m to infinity and to show a subsequence of our solutions u_m approximation problems (72) and (73) converges toward a weak solution of problem ($P4$), for this we need uniform estimates.

3.1 Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions u_m of the approximation problems (72) and (73) converges to a weak solution of problem $(P4)$. For this, we will need some uniform estimates.

Theorem 3.1. (*Energy estimates.*) *There exists a constant C depending only on Ω, T such that*

$$\max_{0\leq t\leq T}\|u_m\|_{L^2(\Omega)} + \|u_m\|_{L^2(0,T;\,H_0^1(\Omega))} + \|u'_m\|_{L^2(0,T;\,H^{-1}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{74}$$

Proof. In order to prove the estimation (74) we will estimate each terms in the left side of (72) one by one as follows:

1. Multiplying equation (72) by $d_{jm}(t)$ and summing for j we find

$$\langle u'_m, w_j\rangle + L(u_m, u_m, t) = 0, \tag{75}$$

and we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|u_m\|^2_{L^2(\Omega)}\Big] + L(u_m(t), u_m(t)) = 0, \tag{76}$$

and we put $\|u\| = \sqrt{L(u,u)}$ (= is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_m\|^2_{L^2(\Omega)}\Big) + \|u_m\|^2_{H_0^1(\Omega)} = 0, \tag{77}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_m\|^2_{L^2(\Omega)}\Big) \leq 0$$

and

$$\|u_m\|^2_{L^2(\Omega)} \leq \|u_m(0)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)}, \tag{78}$$

and so

$$\max_{0\leq t\leq T}\|u_m\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}. \tag{79}$$

2. Integrate inequality (77) from 0 to T and we use (79) to find

$$\|u_m\|^2_{L^2(0,T;\,H_0^1(\Omega))} = \int_0^T \|u_m\|^2_{H_0^1(\Omega)}\mathrm{d}t. \tag{80}$$

3. Fix any $v \in H_0^1(\Omega)$, with $\|v\|^2_{H_0^1(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, \ldots, m)$. We use (72) from all $0 \leq t \leq T$ that

$$(u'_m, v^1) + L(u_m, v^1, t) = 0. \tag{81}$$

Then (81) implies

$$(u'_m, v) = \langle u'_m, v\rangle = \langle u'_m, v^1\rangle = -L(u_m, v^1, t),$$

consequently

$$|\langle u'_m, v\rangle| \leq C\|u_m\|^2_{H_0^1(\Omega)},$$

and as

$$\|v^1\|^2_{H_0^1(\Omega)} \leq \|v\|^2_{H_0^1(\Omega)} \leq 1,$$

thus

$$\|u_m'\|_{H^{-1}(\Omega)} \le C\|u_m\|_{H_0^1(\Omega)},$$

and therefore,

$$\|u_m'\|^2_{L^2(0,T;\, H^{-1}(\Omega))} = \int_0^T \|u_m'\|^2_{H^{-1}(\Omega)}\mathrm{d}t \le C\int_0^T \|u_m\|^2_{H_0^1(\Omega)}\mathrm{d}t \le C\|u_0\|^2_{L^2(\Omega)}. \qquad \square$$

3.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P4)$.

Theorem 3.2. (*Existence of weak solution.*) *Under hypothesis* (73), *there exists a weak solution of* $(P4)$.

Proof. According to the energy estimates (74), we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{u_m'\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, consequently there exists a subsequence which is also noted by $\{u_m\}_{m=1}^{\infty}$ and a function $u \in L^2(0, T; H_0^1(\Omega))$ with $u' \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} &u_m \to u \ \text{ weakly in } L^2(0, T; H_0^1(\Omega)),\\ &u_m' \to u' \ \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{82}$$

1. Next fix an integer N and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t)w_k, \tag{83}$$

   where $\{d^{(k)}\}_{k=1}^{N}$ are given smooth functions. We choose $m \ge N$, multiply equation (72) by $d^{(k)}(t)$ $\forall k = 1, \ldots, N$, and then integrate with respect to t to find

$$\int_0^t \langle u_m', v\rangle + L(u_m, v, t)\mathrm{d}t = 0, \tag{84}$$

   we recall (82) to find upon passing to weak limits that

$$\int_0^t \langle u', v\rangle + L(u, v, t)\mathrm{d}t = 0, \quad \forall v \in L^2(0, T; H_0^1(\Omega)). \tag{85}$$

   As functions of the form (83) are dense in $L^2(0, T; H_0^1(\Omega))$. Hence, in particular

$$\langle u', v\rangle + L(u, v, t) = 0, \quad \forall v \in H_0^1(\Omega) \text{ and } \forall t \in [0, T] \tag{86}$$

   and from Remark (3.1) we have $u \in C(0, T; L^2(\Omega))$.
2. In order to prove $u(0) = u_0$, we first note from (69) that

$$\int_0^t -\langle u, v'\rangle + L(u, v, t)\mathrm{d}t = (u(0), v(0)), \tag{87}$$

   for each $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$. Similarly, from (84) we obtain

$$\int_0^t -\langle u_m, v'\rangle + L(u_m, v, t)\mathrm{d}t = (u_0, v(0)), \tag{88}$$

we use again (87) and obtain

$$\int_0^t -\langle u, v'\rangle + L(u, v, t)\mathrm{d}t = (u_0, v(0)), \tag{89}$$

since $u_m(0) \to u_0$ in $L^2(\Omega)$. Comparing (87) and (89), we conclude $u(0) = u_0$. □

Theorem 3.3. (*Uniqueness of weak solutions.*) *A weak solution of problem* ($P4$) *is unique.*

Proof. We suppose there exist two weak solutions u_1 and u_2 and we put that $U = u_2 - u_1$ then U is also a solution of ($P4$) with $U_0 = (u_2 - u_1)(0) \equiv 0$. Setting $v = U$ in identity (77) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|U\|^2_{L^2(\Omega)}\right) + L(U, U, t) = 0,$$

and as $\|U\| = \sqrt{L(U, U)}$ (= norm in $H_0^1(\Omega)$), there $L(U, U, t) = \|U\|^2_{H_0^1(\Omega)} \geq 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|U\|^2_{L^2(\Omega)}\right) \leq 0,$$

then integrate with respect to t to find

$$\|U\|^2_{L^2(\Omega)} \leq \|U_0\|^2_{L^2(\Omega)} = 0,$$

then $U \equiv 0$. □

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References

[1] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399415.

[2] M. D. Betterton and M. P. Brenner, *Collapsing bacterial cylinders*, Phys. Rev. E **64** (2001), 061904.

[3] M. P. Brenner, L. S. Levitov, and E. O. Budrene, *Physical mechanisms for chemotactic pattern formation by bacteria*, Biophys. J. **74** (1998), no. 4, 1677–1693.

[4] E. O. Budrene and H. C. Berg, *Dynamics of formation of symmetrical patterns by chemotactic bacteria*, Nature (London) **376** (1995), 49–53.

[5] E. O. Budrene and H. C. Berg, *Complex patterns formed by motile cells of Escherichia coli*, Nature (London) **349** (1991), 630–633.

[6] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci. **56** (1981), 217–237.

[7] R. Duan, A. Lorz, and P. Markowich, *Global solutions to the coupled Chemotaxis-Fluid equations*, Comm. Partial Differential Equations **35** (2010), no. 9, 1635–1673.

[8] M. A. Herrero, E. Medina, and J. J. L. Velazquez, *Self-similar blow-up for a reactiondiffusion system*, J. Comput. Appl. Math. **97** (1998), 99–119.

[9] M. A. Herrero and J. J. L. Velazquez, *A blow-up mechanism for a chemotaxis model*, Ann. Scoula Norm. Pisa IV **35** (1997), 633–683.

[10] M. A. Herrero, E. Medina, and J. J. L. Velazquez, *Finite-time aggregation into a single point in a reaction-diffusion system*, Nonlinearity **10** (1997), 1739–1754.

[11] T. Hillen and A. Potapov, *The one-dimensional chemotaxis model: global existence and asymptotic profile*, Math. Methods Appl. Sci. **27** (2004), 1783–1801.

[12] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Differential Equations **215** (2005), 52–107.

[13] W. Jager and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819–824.

[14] A. Lorz, *A coupled Keller-Segel model: Global existence for small initial data and blow-up delay*, Commun. Math. Sci. **10** (2012), no. 2, 555–574.

[15] C. Messikh, A. Guesmia, and S. Saadi, *Global existence and uniqueness of the weak solution in Keller-Segel model*, Glob. J. Sci. Front. Res. F Math. Decision Sci. **14** (2014), no. 2, 1–11.

[16] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. **5** (1995), 581–601.

[17] K. Osaki and A. Yagi, *Finite dimensional attractor for one-dimensional Keller-Segel equations*, Funkcial. Ekvac. **44** (2001), 441–469.

[18] A. Slimani, A. Rahai, A. Guesmia, and L. Bouzettouta, *Stochastic chemotaxis model with fractional derivative driven by multiplicative noise*, Int. J. Anal. Appl. **19** (2021), no. 6, 858–889, DOI: https://doi.org/10.28924/2291-8639-19-2021-858.

[19] M. Mizuno and T. Ogawa, *Regularity and asymptotic behavior for the Keller-Segel system of degenerate type with critical nonlinearity*, J. Math. Sci. Univ. Tokyo **20** (2013), 375–433.

[20] J.-G. Liu and A. Lorz, *A coupled chemotaxis-fluid model: Global existence*, Ann. I. H. Poincaré – AN **28** (2011), no. 5, 643–652.