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Mémoire

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On some inequalities for accretive matrices

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Dedication

To all the illuminating stars that had and will always shine leading me to ,to my family to my backbone in this life, to my pure corner, to my sanctuary and dwelling, my husband

. I didecate this work:

To my dear mather to whom I owe who I am ,she was always there for me and who never stopped paying for my happiness.

To my dear father ,for all the advice he gave me, support he showed me and the sacrifices he mad to see me succeed .

To my dear brothers .

To my dear sisters .

To all my big family ,

To all my frinds .

I dedicate this work to people appreciate the meaning ,and the creativity. By the most sincere feelings and the best words emanating from the heart ,I give my thanks and gratitude to those who were the reason for the success of my career and those who stood with me in the most demanding circumstances and motivated me . My dear family and my husband

Expressions of thanks and gratitude from the heart of the respect and appreciation.

Rahma...



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ABSTRACT

In this thesis , we study accretive matrices (with positive real parts) by generalizing the most important concepts and inequalities known associated with matrix monotone function and matrix convex function from positive matrices to matrices with positive real part.

Keywords : Matrices , inequalities, accretive matrices , positive matrices.

المخلص

في هذه المدكرة قمنا بدراسة المصفوفات ذات الجزء الحقيقي الموجب من خلال تعميم أهم المفاهيم و المتباينات المعروفة المرتبطة بمصفوفات الدوال المحدبة ومصفوفات الدوال الرتبية من المصفوفات الموجبة إلى المصفوفات ذات الجزء الحقيقي الموجب.

الكلمات المفتاحية: المصفوفات , المتباينات , المصفوفات ذات الجزء الحقيقي الموجب , المصفوفات الموجبة .

RESUME

Ce mémoire s'attache à l'étude des matrices à parties réelles positives. Pour ce faire, nous procédons à une généralisation des concepts et inégalités lié à fonction monotone matricielle et fonction convexe matricielle les plus marquants établis pour les matrices positives, en les étendant au domaine des matrices à parties réelles positives.

Mots clés : Matrices ,inégalités, matrices positives ,matrices ayant une partie réelle positive .

List of Contents

Dedication	ii
Acknowledgments	iii
Abstract	iv
List of Contents	v
Introduction	1
1 Preliminary	2
1.1 Basic definitions in matrix theory	2
1.2 Unitarily invariant norms	5
1.3 Matrix monotone and matrix convex functions	7
1.4 Positive linear maps	13
1.5 Matrix means of positive matrices	14
1.6 Accretive matrices	17
2 Accretive matrices and matrix monotone function	21
2.1 An alternative formula for $f(A)$	21
2.2 Arbitrary means of accretive matrices	24
2.3 Ando-type inequalities for accretive matrices	28
2.4 Choi-Davis inequalities for accretive matrices	30
2.5 Norm inequalities for accretive matrices	34
3 Accretive matrices and matrix convex function	37
3.1 The class of matrix convex function (\mathfrak{m}')	38
3.2 The class of matrix monotone decreasing function (\mathfrak{m}'')	41

Introduction

The study of accretive matrices represents a significant branch of modern linear algebra due to their extensive applications in various fields such as theoretical physics, signal processing, and control theory. Accretive matrices, defined by their eigenvalues having non-negative real parts, exhibit unique properties that make them a rich subject for mathematical analysis and theoretical exploration. This research aims to delve into the theories and properties associated with accretive matrices, particularly focusing on the generalization of well-known results from positive matrices to this broader class.

Matrix theory, a fundamental aspect of linear algebra, provides the tools necessary to understand complex systems in multiple dimensions. This thesis specifically addresses accretive matrices and their relationship with matrix monotone and matrix convex functions. Matrix monotone functions are those for which $A \leq B$ implies $f(A) \leq f(B)$ for matrices A and B , whereas matrix convex functions satisfy $f(\alpha A + (1 - \alpha)B) \leq \alpha f(A) + (1 - \alpha)f(B)$ for $\alpha \in [0, 1]$.

The primary objective of this thesis is to extend classical results concerning positive matrices to accretive matrices, thereby broadening the scope of existing mathematical frameworks. This includes exploring unitarily invariant norms, positive linear maps, means of positive matrices, and the integral representations of matrix functions. By developing alternative formulas for matrix functions and examining various inequalities such as Ando-type inequalities and Choi-Davis inequalities for accretive matrices, this study seeks to contribute new theoretical insights and practical tools for the field.

In recent years, there has been significant progress in understanding matrix monotone functions beyond positive matrices. This thesis builds on these advancements by

introducing arbitrary matrix means and functional calculus tailored for accretive matrices. This approach enables the generalization of inequalities from the positive definite setting to the accretive setting, facilitating comparisons between different matrix means and inequalities.

The structure of this thesis is as follows:

- **Chapter 1** introduces fundamental concepts and definitions in matrix theory, ensuring that readers are equipped with the necessary background to understand the subsequent discussions.
- **Chapter 2** delves into accretive matrices and matrix monotone functions, discussing recent progress and presenting new results in this area. The chapter also covers alternative formulas for matrix functions and generalizes known inequalities to accretive matrices.
- **Chapter 3** focuses on matrix convex functions for accretive matrices, exploring their properties and extending known results from positive matrices to accretive matrices. Applications of these results to various problems are also discussed.

Through this research, we aim to provide a comprehensive understanding of accretive matrices and their functions, offering new theoretical insights and practical applications. The exploration of accretive matrices is crucial for advancing mathematical theory and its applications in numerous scientific disciplines, from physics to engineering and beyond.

This thesis endeavors to fill the gap in the literature by extending the established results for positive matrices to accretive matrices, thus opening new avenues for future research and development in matrix theory.

CHAPTER 1

Preliminary

This chapter introduces some fundamental concepts and definitions that will be employed extensively throughout this thesis. For a more in-depth discussion of these concepts, please refer to Bhatia (1997) and Simon (1979).

1.1 Basic definitions in matrix theory

In this section, we give some basic definitions of matrices.

Definition 1.1.1. *If $A \in \mathcal{M}_n$, then the matrix $A^* : \mathcal{M}_n \rightarrow \mathcal{M}_n$ that satisfies*

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \text{ for all } x, y \in \mathbb{C}^n$$

is called the adjoint Hermitian conjugate of A .

Definition 1.1.2. *Let $A \in \mathcal{M}_n$.*

- (a) If $A^* = A$, then A is called Hermitian matrix.*
- (b) If $A^*A = AA^* = I$, then A is called a unitary matrix.*
- (c) If $\langle Ax, x \rangle \geq 0$ for all $x \in H$, then A is called a positive semi-definite matrix.*
- (d) If A is positive semi-definite matrix and is invertible, it is called a positive matrix.*
- (e) If $A^*A = AA^*$, then A is called a normal matrix.*

Throughout this thesis, the notation $A \geq 0$ means that the matrix is positive semi-definite. For two Hermitian matrices A, B the notation $B \geq A$ or $A \leq B$ means $B - A \geq 0$. On the other hand, the notation $A > 0$ is used to mean that A is positive. The notation \mathcal{M}_n^+ is used to denote the class of all positive matrices.

Theorem 1.1.3. *Let $A \in \mathcal{M}_n$. If $A \geq 0$, then there exists a unique matrix $B \in \mathcal{M}_n$ such that $B \geq 0$ and $B^2 = A$. Here B is called the positive square root of A and is denoted by $A^{1/2}$ (or \sqrt{A}).*

The following is a basic property.

Theorem 1.1.4. *If $A \in \mathcal{M}_n$, then A^*A and AA^* are positive semi-definite matrices.*

Definition 1.1.5. *If $A \in \mathcal{M}_n$, then the absolute value of A is the positive square root of the positive matrix A^*A , i.e., $|A| = (A^*A)^{1/2}$.*

Definition 1.1.6. *The Cartesian decomposition of matrix $A \in \mathcal{M}_n$, is given by*

$$A = \Re A + i\Im A, \text{ with } \Re A = \frac{A + A^*}{2} \text{ and } \Im A = \frac{A - A^*}{2i},$$

where $\Re A$ is the real part of A and $\Im A$ is the imaginary part of A .

Definition 1.1.7. *The spectrum of a matrix $A \in \mathcal{M}_n$ is the set*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible in } \mathcal{M}_n\},$$

where I is the identity matrix in \mathcal{M}_n .

Theorem 1.1.8. (Spectral Mapping Theorem) *Let $A \in \mathcal{M}_n$ and let p be any polynomial. Then*

$$\sigma(p(A)) = p(\sigma(A)).$$

Remark 1.1.9. *Let $A \in \mathcal{M}_n$. Then*

$$\sigma(A^*) = \{\bar{\lambda} : \lambda \in \sigma(A)\}.$$

where $\bar{\lambda}$ is the complex conjugate of λ .

Moreover, if A is invertible, then

$$\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\}.$$

Definition 1.1.10. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of matrix A . Then the trace of A is given by

$$\text{tr}A = \sum_{i=1}^n \lambda_i.$$

Definition 1.1.11. The singular values of a matrix $A \in \mathcal{M}_n$, denoted by $s_1(A), s_2(A), \dots, s_n(A)$, are the eigenvalues of $|A|$ arranged in decreasing order, and repeated according to multiplicity. That is, the singular values of A are the square roots of the eigenvalues of A^*A .

1.2 Unitarily invariant norms

This section explores several classes of norms defined on \mathcal{M}_n that exhibit the unitary invariance property. These norms have demonstrated significant utility in diverse scientific fields, including geometry, physics, analysis, and applied mathematics.

Definition 1.2.1. A real valued function $N : X \rightarrow \mathbb{R}$, where X is real (or complex) vector space is said to be a norm if for every scalar α , and $x, y \in X$ we have:

(a) $N(x) \geq 0$.

(b) $N(x) = 0$ if and only if $x = 0$.

(c) $N(\alpha x) = |\alpha| N(x)$.

(d) $N(x + y) \leq N(x) + N(y)$.

Definition 1.2.2. A norm $||| \cdot |||$ on \mathcal{M}_n is said to be unitarily invariant if

$$|||UAV||| = |||A|||,$$

for all unitary matrices $U, V \in \mathcal{M}_n$ and all $A \in \mathcal{M}_n$.

Example 1.2.3. The usual operator (or the spectral) norm, which is denoted by $\| \cdot \|$ is defined for any $A \in \mathcal{M}_n$ by

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle| = s_1(A).$$

When A is Hermitian, this norm is given by

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

Example 1.2.4. The Schatten p -norm of matrix A , denoted by $\|\cdot\|_p$, where $1 \leq p < \infty$ is defined by

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}.$$

It is clear that $\|A\|_p = (\operatorname{tr} |A|^p)^{1/p} = (\operatorname{tr} |A^*|^p)^{1/p}$.

Proposition 1.2.5. Let $A \in \mathcal{M}_n$. Then $\|A\| = \lim_{p \rightarrow \infty} \|A\|_p = s_1(A)$.

Here $\|A\|_2$ is called the Hilbert-Schmidt norm and $\|A\|_1$ is called the trace norm.

Proposition 1.2.6. Let $A \in \mathcal{M}_n$. Then

$$\|A\|_2 = (\operatorname{tr} |A|^2)^{1/2} \quad \text{and} \quad \|A\|_1 = \operatorname{tr} |A|.$$

Example 1.2.7. The Ky Fan k -norm, denoted by $\|\cdot\|_{(k)}$, are defined for any matrix $A \in \mathcal{M}_n$, by

$$\|A\|_{(k)} = \sum_{j=1}^k s_j(A), \quad k = 1, 2, \dots, n.$$

The following is an important property of unitarily invariant norm

Proposition 1.2.8. Let $A \in \mathcal{M}_n$. We can easily prove the following properties for unitarily invariant norms:

(a) $\| \|A\| \| = \| \|A^*\| \|$.

$$(b) \quad |||A||| = \left| \left| (A^*A)^{1/2} \right| \right| = ||| |A| |||.$$

$$(c) \quad |||A^*A||| = |||AA^*|||.$$

$$(d) \quad |||AB^*||| = ||| |A| |B| |||.$$

Theorem 1.2.9. *Let $A \in \mathcal{M}_n$ be invertible . Then*

$$\|A\|^{-1} \leq \|A^{-1}\|. \quad (1.2.1)$$

Theorem 1.2.10. *Let $A, B \in \mathcal{M}_n^+$. Then*

$$\|AB\| \leq \frac{1}{4} \|(A+B)^2\|. \quad (1.2.2)$$

For further results on unitarily invariant norms, we refer to (Bhatia, 1997) and (Simon, 1979).

1.3 Matrix monotone and matrix convex functions

This section delves into matrix monotone functions, a fascinating and important class of functions exhibiting a collection of distinctive properties. These properties share a deep connection with those of matrix convex functions. We will explore both types of functions in detail. To begin, let's refresh our understanding of $f(A)$, when A is any matrix.

Let $f : \mathcal{D} \rightarrow \mathbb{C}$ be an analytic complex function on a complex domain \mathcal{D} . The Cauchy integral formula assures that for $a \in \mathcal{D}$,

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a} dz,$$

where Γ is a simple closed curve in \mathcal{D} that winds once around a . Extending this definition to matrices is possible using the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z\mathcal{I} - A)^{-1} dz, \quad (1.3.1)$$

where Γ is a simple closed curve in the definition of A that winds once around each eigenvalue of A . Of course, Γ must lie in \mathbb{D} .

For example, letting $f : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ be $f(z) = z^\lambda$, $0 < \lambda < 1$, we define

$$A^\lambda = \frac{1}{2\pi i} \int_{\Gamma} z^\lambda (z\mathcal{I} - A)^{-1} dz, \quad (1.3.2)$$

where Γ is any closed curve avoiding $(-\infty, 0]$ in the resolvent of A , so that Γ winds once around each eigenvalue of A .

So, fractional powers are not only defined for positive matrices. They can be defined for any matrix whose eigenvalues are not in $(-\infty, 0]$.

Definition 1.3.1. *Let $f : J \rightarrow \mathbb{R}$ be a function where $J \subseteq \mathbb{R}$ is an interval. Then f is called matrix monotone on J if for any two Hermitian matrices A and B with spectra contained in J , $A \leq B$ implies that $f(A) \leq f(B)$.*

The following function is an example of a function that is not matrix monotone, but it is monotone as a real function.

Example 1.3.2. *Consider the function $f(x) = x^2$, where $x \geq 0$. Let A and B be the 2×2 matrices defined as $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. It is clear that $B - A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ that is $B \geq A$. But $B^2 - A^2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ is not positive semi-definite, since it has a negative*

eigenvalue, so $B^2 \not\leq A^2$.

Definition 1.3.3. Let $f : J \rightarrow \mathbb{R}$ be a function where $J \subseteq \mathbb{R}$ is an interval. Then f is called *matrix convex* if for any Hermitian matrices A and B with spectra contained in J , we have

$$f(\alpha A + (1 - \alpha)B) \leq \alpha f(A) + (1 - \alpha)f(B)$$

for all $\alpha \in [0, 1]$.

The following function is an example of a function that is not matrix convex, but it is convex as a real function.

Example 1.3.4. Consider the function $f(x) = x^3, x \geq 0$. Let A and B be the 2×2 matrices defined as $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. Then A and B are positive and

$$\frac{A^3 + B^3}{2} - \left(\frac{A+B}{2}\right)^3 = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$$

which is not positive since it has a negative eigenvalue. So $\frac{A^3+B^3}{2} \not\leq \left(\frac{A+B}{2}\right)^3$, that is f is not matrix convex, on $[0, \infty)$.

Example 1.3.5. The function $f(x) = ax+b$ where $a, b \in \mathbb{R}$ and $a \geq 0$ is matrix monotone function. It is matrix convex for every $a, b \in \mathbb{R}$.

Definition 1.3.6. Let $f : J \rightarrow \mathbb{R}$ be a function where $J \subseteq \mathbb{R}$ is an interval. Then f is called *matrix concave* if for any Hermitian matrices A and B with spectra contained in J , we have

$$f(\alpha A + (1 - \alpha)B) \geq \alpha f(A) + (1 - \alpha)f(B), \quad (1.3.3)$$

for all $\alpha \in [0, 1]$.

Remark 1.3.7. *f is called matrix concave if and only if $-f$ is matrix convex.*

While there is generally no connection between convex (concave) functions and monotone functions, a compelling relationship emerges when considering matrix convexity (concavity). For a deeper exploration of this connection, refer to (Uchiyama 2010) and (Ando and Hiai 2011).

Proposition 1.3.8. *Let $f : (0, \infty) \rightarrow [0, \infty)$ be continuous. Then*

(i) *f is matrix monotone decreasing if and only if f is matrix convex and $f(\infty) < \infty$.*

(ii) *f is matrix monotone increasing if and only if f is matrix concave.*

In this thesis, positive matrix monotone functions will play a key role. So, we adopt the notation

$$\mathbf{m} = \{f : (0, \infty) \rightarrow (0, \infty); f \text{ is a matrix monotone function with } f(1) = 1\}.$$

The following characterization of $f \in \mathbf{m}$ will be useful for our analysis see (Hansen, 2013).

Lemma 1.3.9. *Let $f \in \mathbf{m}$. Then*

$$f(x) = \int_0^1 (1-t) + tx \, d\nu_f(t), \quad x \in (0, \infty)$$

where ν_f is a probability measure on $[0, 1]$.

The above lemma provides a characterization of $f(x)$ for real $x \in (0, \infty)$. In our discussion, we will need to deal with $f(z)$ where $z \in \mathbb{C}$. To justify the use of $f(z)$, we first recall the following celebrated result of Löwner about matrix monotone functions.

Lemma 1.3.10. *Let $f \in \mathfrak{m}$. Then f has an analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$.*

Notice that when $f \in \mathfrak{m}$, the integral representation in Lemma 1.3.9 applies when $x \in (0, \infty)$. However, Lemma 1.3.10 asserts the well definikness of $f(z), z \in \mathbb{C} \setminus (-\infty, 0]$.

In the following, we show that f still have the integral representation in Lemma 1.3.9.

Proposition 1.3.11. *Let $f \in \mathfrak{m}$. Then, for $z \in \mathbb{C} \setminus (-\infty, 0]$, the integral representation*

$$f(z) = \int_0^1 1!_t z \, d\nu_f(t),$$

holds true, where ν_f is as in Lemma 1.3.9.

Proof : Notice that when $z \notin (-\infty, 0]$, the quantity $1!_t z := (1 - t + tz^{-1})^{-1}$ is well defined. For such z , define $g(z) = \int_0^1 1!_t z \, d\nu_f(t)$. We show that $f = g$.

We show that g is analytic in $\mathbb{C} \setminus (-\infty, 0]$. Indeed, let Γ be any closed circle in $\mathbb{C} \setminus (-\infty, 0]$.

First, we show that

$$\int_{\Gamma} \left(\int_0^1 1!_t z \, d\nu_f(t) \right) dz = \int_0^1 \left(\int_{\Gamma} 1!_t z \, dz \right) d\nu_f(t).$$

Notice that (by letting $z = a + re^{i\theta}$)

$$\int_0^{2\pi} \int_0^1 |1!_t(a + re^{i\theta})| \, d\nu_f(t) r d\theta = r \int_0^1 \int_0^{2\pi} |1!_t(a + re^{i\theta})| \, d\theta d\nu_f(t).$$

But the function $F(t, \theta) = |1!_t(a + re^{i\theta})|$ is continuous on the compact set $[0, 1] \times [0, 2\pi]$.

Therefore, $s := \sup_{(t, \theta)} F(t, \theta) < \infty$, and hence

$$\int_0^{2\pi} \int_0^1 |1!_t(a + re^{i\theta})| \, d\nu_f(t) r d\theta \leq 2\pi s < \infty.$$

This means that

$$\int_0^{2\pi} \int_0^1 (1!_t(a + re^{i\theta})) d\nu_f(t) r d\theta = r \int_0^1 \int_0^{2\pi} (1!_t(a + re^{i\theta})) d\theta d\nu_f(t),$$

which is equivalent to

$$\int_{\Gamma} \left(\int_0^1 1!_t z d\nu_f(t) \right) dz = \int_0^1 \left(\int_{\Gamma} 1!_t z dz \right) d\nu_f(t).$$

Then

$$\begin{aligned} \int_{\Gamma} g(z) dz &= \int_{\Gamma} \left(\int_0^1 1!_t z d\nu_f(t) \right) dz \\ &= \int_0^1 \left(\int_{\Gamma} 1!_t z dz \right) d\nu_f(t) \\ &= 0, \end{aligned}$$

where we have used the fact that $z \mapsto 1!_t z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$, for every $t \in [0, 1]$.

Since $\int_{\Gamma} g(z) dz = 0$ for any circle, in the domain, it follows that g is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

Finally, since f and g are analytic functions having the same values in $(0, \infty)$, it follows that $f = g$. This completes the proof. \square

Lemma 1.3.12. *Let $A \in \mathcal{M}_n$ be Hermitian matrix. Then for $f \in \mathfrak{m}$ and any unit vector x ,*

$$\langle f(A)x, x \rangle \leq f(\langle Ax, x \rangle). \quad (1.3.4)$$

Lemma 1.3.13. *Let $A \in \mathcal{M}_n^+$. If $f \in \mathfrak{m}$, then*

$$f(\|A\|) \leq \|f(A)\|.$$

Theorem 1.3.14. *Let $A, B \in \mathcal{M}_n^+$. Then, for any $f \in \mathfrak{m}$,*

$$\|f(A+B)\| \leq \|f(A) + f(B)\| \quad (1.3.5)$$

Lemma 1.3.15. *[20] Let $A \in \mathcal{M}_n^+$. Then for any matrix convex f and any unit vector x ,*

$$\langle f(A)x, x \rangle \geq f(\langle Ax, x \rangle). \quad (1.3.6)$$

Proposition 1.3.16. *[29] let f be matrix convex function. Then for any $A, B \in \mathcal{M}_n^+$ and $t \in [0, 1]$,*

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A) + f(B)}{2}. \quad (1.3.7)$$

1.4 Positive linear maps

This section lays out the foundational properties of positive linear maps. Unital positive linear maps are defined as follows:

Definition 1.4.1. *A map $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_r$ is linear if it is additive and homogeneous, i.e.*

$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$ for any $\lambda, \mu \in \mathbb{C}$ and for any $A, B \in \mathcal{M}_n$.

A linear map $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_r$ is positive if it preserves the matrix order \geq , i.e. $A \in \mathcal{M}_n^+$ implies $\Phi(A) \in \mathcal{M}_r^+$.

A linear map $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_r$ is called unital if it preserves the identity matrix, i.e. if $\Phi(I) = I$.

Proposition 1.4.2. *Let $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_r$ be a positive linear map. Then*

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B).$$

$$\Phi(A^*) = \Phi^*(A).$$

The following two propositions are given in (Furuta et al, 2005), (Bhatia, 2007), respectively.

Theorem 1.4.3. (Choi's inequality) *Let $A, B \in \mathcal{M}_n^+$ and $f \in \mathfrak{m}$. Then*

$$\Phi(f(A)) \leq f(\Phi(A)). \quad (1.4.1)$$

Lemma 1.4.4. *[20] Let $A \in \mathcal{M}_n^+$ and f be matrix convex. Then for any unital positive linear map Φ ,*

$$\Phi(f(A)) \geq f(\Phi(A)). \quad (1.4.2)$$

Proposition 1.4.5. *Let $A \in \mathcal{M}_n^+$ and Φ be a positive linear map. Then*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \quad (1.4.3)$$

1.5 Matrix means of positive matrices

The concept of matrix means, which refers to the relationship between two positive matrices, was introduced by Kubo and Ando in 1980 (Kubo and Ando, 1980). A matrix mean σ on \mathcal{M}_n^+ is a binary operation $A\sigma B$ satisfying the following requirements:

- $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$; for any $A, B, C, D \in \mathcal{M}_n^+$.
- $C^*(A\sigma B)C = (C^*AC)\sigma(C^*BC)$; for any $A, B \in \mathcal{M}_n^+$ and any invertible $C \in \mathcal{M}_n$.
- $A_k \downarrow_k A$ and $B_k \downarrow_k B$ imply $(A_k\sigma B_k) \downarrow_k (A\sigma B)$; for any $A_k, B_k, A, B \in \mathcal{M}_n^+$.
- $I\sigma I = I$.

Standard examples of matrix means are given by (Pusz and Woronowicz, 1975).

- The weighted arithmetic mean $A\nabla_r B = (1-r)A + rB$,
- The weighted Harmonic mean $A!_r B = ((1-r)A^{-1} + rB^{-1})^{-1}$,
- The weighted geometric mean $A\sharp_r B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r A^{\frac{1}{2}}$,

where $A, B \in \mathcal{M}_n^+$ and $0 \leq r \leq 1$. When $r = \frac{1}{2}$, we drop r from the above notations, and we simply write $\nabla, !$ and \sharp .

For two matrix means σ, τ , we say that $\sigma \leq \tau$ if $A\sigma B \leq A\tau B$ for all $A, B \in \mathcal{M}_n^+$. In particular, we have $!_r \leq \sharp_r \leq \nabla_r$, see (Ando, 1978). That is, if $A, B \in \mathcal{M}_n^+$, then

$$A!_r B \leq A\sharp_r B \leq A\nabla_r B, 0 \leq r \leq 1.$$

Other celebrated relations for these means are

$$A\nabla_r B = B\nabla_{1-r} A, A\sharp_r B = B\sharp_{1-r} A, A!_r B = B!_{1-r} A.$$

The theory of matrix means is strongly related to that of matrix monotone functions, where any mean σ on \mathcal{M}_n^+ is characterized by

$$A\sigma_f B = A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}},$$

for a certain matrix monotone function $f \in \mathfrak{m}$, see (kubo and Ando, 1980).

For exemple, the arithmetic mean ∇ is characterized by $f(x) = \frac{1+x}{2}$ because

$$A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = A^{\frac{1}{2}} \frac{(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}})}{2} A^{\frac{1}{2}} = \frac{A + B}{2}.$$

Recalling Lemma 1.3.9 if $f \in \mathfrak{m}$ and σ_f is the corresponding mean (i.e., $A\sigma_f B = A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$), then we have

$$A\sigma_f B = \int_0^1 A!_t B \, d\nu_f(t), \quad (1.5.1)$$

where ν_f is a probability measure on $[0, 1]$; depending on f .

Proposition 1.5.1. *Let $A, B \in \mathcal{M}_n$ be positive. If $f \in \mathfrak{m}$, then*

$$|||A\sigma_f B||| \leq |||A||| \sigma_f |||B|||. \quad (1.5.2)$$

Proposition 1.5.2. *Let $A, B \in \mathcal{M}_n^+$. If $f \in \mathfrak{m}$, then for any unit vector x ,*

$$\langle (A\sigma_f B)x, x \rangle \leq \langle Ax, x \rangle \sigma_f \langle Bx, x \rangle. \quad (1.5.3)$$

Proposition 1.5.3. (Ando's inequality) *Let $A, B \in \mathcal{M}_n^+$ and Φ be a positive unital linear map. Then*

$$\Phi(A\sigma_f B) \leq \Phi(A)\sigma_f\Phi(B). \quad (1.5.4)$$

Proposition 1.5.4. *Let $A, B \in \mathcal{M}_n^+$ and let $f \in \mathfrak{m}$ be such that $f'(1) = r$ for some $r \in (0, 1)$. Then*

$$A!_r B \leq A\sigma_f B \leq A\nabla_r B.$$

Proposition 1.5.5. *Let $A, B \in \mathcal{M}_n$ be such that $0 < mI \leq A, B \leq MI$, for some positive scalars m, M , and let $f, g \in \mathfrak{m}$. Then for every unital positive linear map Φ ,*

$$\Phi^2(A\sigma_f B) \leq K(h)^2 \Phi^2(A\sigma_g B), \quad (1.5.5)$$

where $h = \frac{M}{m}$ and $k(h) = \frac{(h+1)^2}{4h}$ is the well known Kantorovich constant.

Proposition 1.5.6. *[?] Let $A, B \in \mathcal{M}^+$ and let h be continuous non-negative function on $(0, \infty)$. Then The following conditions are equivalent:*

1. h is matrix monotone decreasing.
2. $h(A\nabla B) \leq h(A)\sigma h(B)$, for all symmetric matrix mean σ .
3. h is matrix log-convex, i.e., $h(A\nabla B) \leq h(A)\#h(B)$.

The theory of matrix means, which explores the relationship between positive matrices, has been extensively studied and documented in the literature. For a comprehensive overview of this field, please refer to the following works: (Ando, 1978), (Kubo and Ando, 1980), (Nishio and Ando, 1976), (Mathias, 1992), (Ando, 1979), (Furuta, 2002), and (Furuta and Yanagide, 1998)..

1.6 Accretive matrices

This section aims to introduce matrix means for a broader class of matrices encompassing positive matrices, known as accretive matrices.

Definition 1.6.1. *A matrix $A \in \mathcal{M}_n$ is said to be accretive if its real part is positive (i.e., $\Re A > 0$.)*

An understanding of sectorial matrices is essential when delving into the properties of accretive matrices.

Definition 1.6.2. For $0 \leq \alpha < \frac{\pi}{2}$, we define the sector

$$S_\alpha = \{z \in \mathbb{C} : \Re(z) > 0, |\Im(z)| \leq \tan(\alpha)\Re(z)\}.$$

A matrix A whose numerical range is a subset of a sector S_α , is called a sectorial matrix.

It is clear that a sectorial matrix is necessarily accretive, in this thesis, it will be implicitly understood that the notions of S_α is defined only when $0 \leq \alpha < \frac{\pi}{2}$.

Remark 1.6.3. The numerical range $W(A)$ of an accretive matrix A satisfies

$$W(A) := \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\} \subset \text{the right half complex plane.}$$

In the following, we list some properties of accretive (sectorial) matrices.

Proposition 1.6.4. The class of accretive matrices is

- (1) a convex cone of \mathcal{M}_n .
- (2) invariant under inversion.

It is well known that for any matrix $A \in \mathcal{M}_n$, $|||\Re A||| \leq |||A|||$, for any unitarily invariant norm $|||\cdot|||$ on \mathcal{M}_n . The following lemma presents a reversed version of this inequality for sectorial matrices see (Zhang, 2015).

Proposition 1.6.5. Let $A \in \mathcal{M}_n$ be accretive matrix such that $W(A) \subset S_\alpha$ and let $|||\cdot|||$ be any unitarily invariant norm on \mathcal{M}_n . Then

$$\cos \alpha |||A||| \leq |||\Re(A)||| \leq |||A|||.$$

Other interesting needed properties of accretive matrices are given in (Lin, 2015), (Drury, 2014), (Choi et al, 2019) and (Drury 2015) as follows.

Proposition 1.6.6. *If $A \in \mathcal{M}_n$ is accretive, then*

$$\Re(A^{-1}) \leq (\Re A)^{-1}$$

This inequality can be reversed as follows.

Lemma 1.6.7. *If $A \in \mathcal{M}_n$ is accretive with $W(A) \subset S_\alpha$, then*

$$\sec^2(\alpha) \Re(A^{-1}) \geq (\Re A)^{-1}$$

Theorem 1.6.8. *Let $A, B \in \mathcal{M}_n$ be accretive matrices and $0 < \lambda < 1$, then*

$$\Re(A!_\lambda B) \geq (\Re A)!_\lambda(\Re B)$$

Proof : Consider the matrix convex function $f(A) = (\Re(A^{-1}))^{-1}$, then

$$\begin{aligned} f((1-\lambda)A + \lambda B) &\leq (1-\lambda)f(A) + \lambda f(B) \\ (\Re((1-\lambda)A + \lambda B)^{-1})^{-1} &\leq (1-\lambda)(\Re A^{-1})^{-1} + \lambda(\Re B^{-1})^{-1} \\ (\Re((1-\lambda)A^{-1} + \lambda B^{-1})^{-1})^{-1} &\leq (1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1} \\ \Re((1-\lambda)A^{-1} + \lambda B^{-1})^{-1} &\geq ((1-\lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1})^{-1} \\ \Re(A!_\lambda B) &\geq (\Re A)!_\lambda(\Re B), \end{aligned}$$

this completes the proof. □

Theorem 1.6.9. *Let $A, B \in \mathcal{M}_n$ be accretive matrices such that $W(A), W(B) \subset S_\alpha$.*

Then for $0 < \lambda < 1$

$$\Re(A!_\lambda B) \leq \sec^2(\alpha)(\Re A)!_\lambda(\Re B) \tag{1.6.1}$$

Proof : Let A, B be sectorial matrices, we have

$$\begin{aligned}
\Re(A!_{\lambda}B) &= \Re((1 - \lambda)A^{-1} + \lambda B^{-1})^{-1} \\
&\leq ((1 - \lambda)\Re A^{-1} + \lambda\Re B^{-1})^{-1} \\
&\leq (\cos^2(\alpha)(1 - \lambda)(\Re A)^{-1} + \cos^2(\alpha)\lambda(\Re B)^{-1})^{-1} \\
&\leq \sec^2(\alpha) ((1 - \lambda)(\Re A)^{-1} + \lambda(\Re B)^{-1})^{-1} \\
&= \sec^2(\alpha) (\Re A)!_{\lambda}(\Re B),
\end{aligned}$$

which complete the proof □

Lemma 1.6.10. *Let $A \in \mathcal{M}_n$ is accretive with $W(A) \subset S_{\alpha}$ and $t \in (0, 1)$. Then $W(A^t) \subset S_{t\alpha}$.*

Also note that $W(A^{-t}) \subset S_{t\alpha}$. This follows because $W(A^{-1}) \subset S_{\alpha}$ when $W(A) \subset S_{\alpha}$.

CHAPTER 2

Accretive matrices and matrix monotone
function

This chapter explores recent progress in extending the application of matrix monotone functions from positive definite matrices to the broader class of accretive matrices. We achieve this by introducing the concept of arbitrary matrix means and functional calculus specifically tailored for accretive matrices. This framework allows us to generalize well-known inequalities from the positive definite setting to the accretive setting. Examples include comparisons between arithmetic and harmonic means, Ando's inequality, Choi's inequality, and the Ando-Zhan subadditivity inequality.

2.1 An alternative formula for $f(A)$

Inspired by the positive case, in this section we present an easier formula for $f(A)$, when A is accretive and $f \in \mathbf{m}$. We recall that for such parameters, $f(A)$ is defined by (1.3.1). Our main result in this section reads as follows.

Theorem 2.1.1. *Let $f \in \mathbf{m}$ and $A \in \mathcal{M}_n$ be any matrix with eigenvalues vector $\lambda(A)$. If $\lambda(A) \cap (-\infty, 0] = \phi$, then*

$$f(A) = \int_0^1 I_t A \, d\nu_f(t),$$

where ν_f is a probability measure on $[0, 1]$.

Proof : First, we show the result when A is a diagonalizable matrix with $\lambda(A) \cap (-\infty, 0] = \phi$. So, let A be such matrix and let $f \in \mathbf{m}$. If $A = V^{-1}D[\lambda_i]V$, where $D[\lambda_i]$ is diagonal.

Then immediate calculations show that

$$\begin{aligned} \int_0^1 (I!_t A) d\nu_f(t) &= V^{-1} D \left[\int_0^1 (1!_s \lambda_i) d\nu_f(t) \right] V \\ &= V^{-1} D \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \lambda_i} dz \right] V \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z) (zI - A)^{-1} dz, \end{aligned}$$

where we have used Proposition 1.3.11 to obtain the second identity, noting that $\lambda_i \notin (-\infty, 0]$. This shows the result for diagonalizable matrices with no eigenvalues in $(-\infty, 0]$. For the general case, let $A \in \mathcal{M}_n$ with no eigenvalue in $(-\infty, 0]$. Since diagonalizable matrices are dense in \mathcal{M}_n , with respect to the operator norm, we can find a sequence of diagonalizable matrices (A_m) such that $A_m \rightarrow A$ in the usual operator norm, (Serre, 2010, Corollary 5.1). Further, since $A_m \rightarrow A$ and $\lambda(A) \cap (-\infty, 0]$, we may assume without loss of generality that $\lambda(A_m) \cap (-\infty, 0] = \emptyset$ for all m , due to the continuity of the map that maps each matrix $A \in \mathcal{M}_n$ to its eigenvalue set $\lambda(A)$, (Artin, 2011, Proposition 5.2.2). Since $\lambda(A_m) \cap (-\infty, 0] = \emptyset$, and A_m is diagonalizable, the first part of the proof implies that

$$f(A_m) = \int_0^1 I!_t A_m d\nu_f(t). \quad (2.1.1)$$

Since $f \in \mathfrak{m}$, it is analytically continued to $\mathcal{D} := \mathbb{C} \setminus (-\infty, 0]$, and hence it is $n - 1$ continuously differentiable in \mathcal{D} . Therefore, the mapping $A \rightarrow f(A)$ is a continuous mapping on the set of matrices with spectrum in \mathcal{D} , (Higham, 2008, Theorem 1.19). This continuity implies that $f(A_m) \rightarrow f(A)$, which in turns implies (by (2.1.1))

$$\int_0^1 I!_t A_m d\nu_f(t) \rightarrow \frac{1}{2\pi i} \int_{\Gamma} f(z) (zI - A)^{-1} dz, \quad (2.1.2)$$

where Γ is a simple closed curve in \mathcal{D} that surrounds the spectrum of A . It remains to show that

$$\int_0^1 I!_t A_m d\nu_f(t) \rightarrow \int_0^1 I!_t A d\nu_f(t).$$

We first notice that for each $t \in [0, 1]$, the mapping $X \rightarrow I!_t X$ is continuous on the class of matrices with spectrum in \mathcal{D} , (Higham, 2008, Theorem 1.19). Consequently, for every $t \in [0, 1]$,

$$\|I!_t A_m - I!_t A\| \rightarrow 0. \quad (2.1.3)$$

This means that, for large m ,

$$\begin{aligned} \|I!_t A_m - I!_t A\| &\leq \|I!_t A_m\| + \|I!_t A\| \\ &\leq 2\|I!_t A\| + 1, \end{aligned}$$

where the above inequality follows from (2.1.3). Noting the latter inequality and applying the dominated Lebesgue convergence theorem implies that

$$\int_0^1 \|I!_t A_m - I!_t A\| d\nu_f(t) \rightarrow 0,$$

which shows that

$$\int_0^1 I!_t A_m d\nu_f(t) \rightarrow \int_0^1 I!_t A d\nu_f(t). \quad (2.1.4)$$

This together with (2.1.2) complete the proof of the theorem. \square

2.2 Arbitrary means of accretive matrices

We have introduced matrix means for positive matrices earlier in the section 1.5, and we have seen that if $f \in \mathfrak{m}$, then a probability measure ν_f on $[0, 1]$ exists such that for positive A, B , one has

$$A\sigma_f B = A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} \quad (2.2.1)$$

$$= \int_0^1 A!_t B \, d\nu_f(t). \quad (2.2.2)$$

Our goal in this section is to extend the definition of an arbitrary matrix means to the context of accretive matrices. Our central definition in this section reads as follows.

Definition 2.2.1. *Let $A, B \in \mathcal{M}_n$ be two accretive matrices, $f \in \mathfrak{m}$, and let ν_f be the probability measure characterizing σ_f . We define the matrix means σ_f of A and B by*

$$A\sigma_f B = \int_0^1 A!_t B \, d\nu_f(t). \quad (2.2.3)$$

Remark 2.2.2. *Our first remark is that we adopt the above definition for accretive matrices only. Notice that for $A\sigma_f B$ to be defined, we must have $A!_t B$ defined for all $t \in [0, 1]$. This means that we must have $(1-t)A^{-1} + tB^{-1}$ invertible, for all $t \in [0, 1]$. When A and B are both accretive, this is guaranteed. However, if they are not accretive, we have no control over this. This is the main reason we restrict ourselves to accretive matrices in this definition, and in the following discussion.*

Now we begin our investigation by reciting the following result which extends Lemma ?? to any matrix mean.

Proposition 2.2.3. *Let $A, B \in \mathcal{M}_n$ be accretive matrices and let $f \in \mathfrak{m}$. Then*

$$\Re(A\sigma_f B) \geq (\Re A) \sigma_f (\Re B). \quad (2.2.4)$$

As a consequence, if A and B are accretive, then so is $A\sigma_f B$.

Proof : Let $A, B \in \mathcal{M}_n$ be accretive. Then

$$\begin{aligned} \Re(A\sigma_f B) &= \int_0^1 \Re(A!_t B) \, d\nu_f(t) \\ &\geq \int_0^1 \Re(A)!_t \Re(B) \, d\nu_f(t) \quad (\text{by Theorem 1.6.8}) \\ &= (\Re A) \sigma_f (\Re B). \end{aligned}$$

This completes the proof. □

When A and B are sectorial, we have the following reverse of Proposition 2.2.3.

Proposition 2.2.4. *Let $A, B \in \mathcal{S}_\alpha$. If $f \in \mathfrak{m}$, then*

$$\Re(A\sigma_f B) \leq \sec^2 \alpha (\Re A) \sigma_f (\Re B). \quad (2.2.5)$$

Proof : By Definition 2.2.1, we have

$$\begin{aligned} \Re(A\sigma_f B) &= \int_0^1 \Re(A!_t B) \, d\nu_f(t) \\ &\leq \sec^2 \alpha \int_0^1 (\Re(A)!_t \Re(B)) \, d\nu_f(t) \quad (\text{by Theorem 1.6.9}) \\ &= \sec^2 \alpha (\Re A) \sigma_f (\Re B). \end{aligned}$$

This completes the proof. □

Now, we present a generalization of Proposition 1.5.4 from the setting of positive matrices to sectorial ones.

Theorem 2.2.5. *Let $A, B \in \mathcal{S}_\alpha$. If $f \in \mathfrak{m}$ is such that $f'(1) = r$ for some $r \in (0, 1)$, then*

$$\cos^2 \alpha \mathfrak{R}(A!_r B) \leq \mathfrak{R}(A\sigma_f B) \leq \sec^2 \alpha \mathfrak{R}(A\nabla_r B). \quad (2.2.6)$$

Proof : First,

$$\begin{aligned} \mathfrak{R}(A!_r B) &\leq \sec^2 \alpha ((\mathfrak{R}A)!_r(\mathfrak{R}B)) && \text{(by Theorem 1.6.9)} \\ &\leq \sec^2 \alpha ((\mathfrak{R}A)\sigma_f(\mathfrak{R}B)) && \text{(by Proposition 1.5.4)} \\ &\leq \sec^2 \alpha \mathfrak{R}(A\sigma_f B). && \text{(by (2.2.4))} \end{aligned}$$

Thus, we have shown the first inequality. To show the second inequality, we have

$$\begin{aligned} \mathfrak{R}(A\sigma_f B) &\leq \sec^2 \alpha (\mathfrak{R}A) \sigma_f (\mathfrak{R}B) && \text{(by (2.2.5))} \\ &\leq \sec^2 \alpha (\mathfrak{R}A)\nabla_r(\mathfrak{R}B) && \text{(by Proposition 1.5.4)} \\ &= \sec^2 \alpha \mathfrak{R}(A\nabla_r B). \end{aligned}$$

This shows the second desired inequality, and the proof is complete. \square

We notice that when A, B are positive, then α can be taken as $\alpha = 0$, which then retrieves Proposition 1.5.4 as a special case of Theorem 2.2.5.

In studying matrix means, it is customary to compare between different means that arise from different matrix monotone functions. In the next result, we present such comparison for sectorial matrices.

Theorem 2.2.6. *Let $A, B \in \mathcal{S}_\alpha$ such that $0 < mI \leq \Re A, \Re B \leq MI$. If $f, g \in \mathfrak{m}$. Then for every unital positive linear map Φ ,*

$$\| \Phi(\Re(A\sigma_f B))\Phi^{-1}(\Re(A\sigma_g B)) \| \leq \sec^6 \alpha K(h), \quad (2.2.7)$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ and $\|\cdot\|$ denote the usual operator norm.

Proof : Since $0 < mI \leq \Re A \leq MI$,

we have

$$(M - \Re A)(m - \Re A)(\Re A)^{-1} \leq 0,$$

which is equivalent to

$$\Re A + Mm(\Re A)^{-1} \leq (M + m)I,$$

since $(\Re A)^{-1} \geq \Re A^{-1}$, we have

$$\frac{1}{2}\Re A + \frac{1}{2}Mm\Re A^{-1} \leq \frac{1}{2}(M + m)I. \quad (2.2.8)$$

Similarly

$$\frac{1}{2}\Re B + \frac{1}{2}Mm\Re B^{-1} \leq \frac{1}{2}(M + m)I. \quad (2.2.9)$$

Adding (2.2.8) and (2.2.9), we get

$$\Re(A\nabla B) + Mm\Re(A!B)^{-1} \leq (M + m)I. \quad (2.2.10)$$

Then, we have

$$\begin{aligned}
& \| \sec^2 \alpha Mm\Phi(\Re(A\sigma_f B))\Phi^{-1}(\Re(A\sigma_g B)) \| \\
\leq & \frac{1}{4} \| \sec^2 \alpha \Phi(\Re(A\sigma_f B)) + Mm\Phi^{-1}(\Re(A\sigma_g B)) \|^2 && \text{(by Theorem 1.2.10)} \\
\leq & \frac{1}{4} \| \sec^2 \alpha \Phi(\Re(A\sigma_f B)) + Mm\Phi((\Re(A\sigma_g B))^{-1}) \|^2 && \text{(by (1.4.3))} \\
\leq & \frac{1}{4} \| \sec^4 \alpha \Phi(\Re(A\nabla B)) + \sec^2(\alpha)Mm\Phi(\Re(A!B))^{-1} \|^2 && \text{(by (2.2.6))} \\
\leq & \frac{1}{4} \| \sec^4 \alpha \Phi(\Re(A\nabla B)) + \sec^4 \alpha Mm\Phi(\Re(A!B))^{-1} \|^2 \\
\leq & \frac{1}{4} \| \sec^4 \alpha \Phi(\Re(A\nabla B)) + Mm\Re(A!B)^{-1} \|^2 \\
\leq & \frac{1}{4} \sec^8 \alpha (M + m)^2. && \text{(by(2.2.10))}
\end{aligned}$$

That is

$$\| \Phi(\Re(A\sigma_f B))\Phi^{-1}(\Re(A\sigma_g B)) \| \leq \sec^6 \alpha K(h),$$

which completes the proof. □

2.3 Ando-type inequalities for accretive matrices

In this section, we present versions of Ando's inequality (1.5.4). We begin by stating the following needed lemma which concerns the Ando-type inequality for the harmonic matrix mean. For this purpose, we notice that if Φ is a unital positive linear map and A is any matrix, then

$$\Re\Phi(A) = \Phi(\Re A). \tag{2.3.1}$$

Lemma 2.3.1. *Let $A, B \in \mathcal{M}_n$ be accretive and let Φ be a unital positive linear map.*

Then

$$\Phi(\Re A!_t \Re B) \leq \Re(\Phi(A)!_t \Phi(B)). \quad (2.3.2)$$

Proof : Noting (1.5.4), (2.3.1) and Theorem 1.6.8, we have

$$\Phi(\Re A!_t \Re B) \leq \Phi(\Re A)!_t \Phi(\Re B) = \Re \Phi(A)!_t \Re \Phi(B) \leq \Re(\Phi(A)!_t \Phi(B)). \quad (2.3.3)$$

□

Now we are in the position to state the sectorial version of (1.5.4), valid for any matrix mean.

Theorem 2.3.2. *Let $A, B \in \mathcal{S}_\alpha$ and let Φ be a unital positive linear map. If $f \in \mathfrak{m}$, then*

$$\Re \Phi(A\sigma_f B) \leq \sec^2 \alpha \Re(\Phi(A)\sigma_f \Phi(B)). \quad (2.3.4)$$

Proof : Using the definition of $A\sigma_f B$, we get

$$\begin{aligned} \cos^2 \alpha \Re \Phi(A\sigma_f B) &= \Phi(\cos^2 \alpha \Re(A\sigma_f B)) \quad (\text{by (2.3.1)}) \\ &\leq \Phi(\Re A\sigma_f \Re B) \quad (\text{by Proposition 2.2.4}) \\ &\leq \Phi(\Re A)\sigma_f \Phi(\Re B) \quad (\text{by (1.5.4)}) \\ &= \Re \Phi(A)\sigma_f \Re \Phi(B) \quad (\text{by (2.3.1)}) \\ &\leq \Re(\Phi(A)\sigma_f \Phi(B)) \quad (\text{by Proposition 2.2.3}), \end{aligned}$$

which completes the proof. □

As an application of Theorem 2.3.2, we present the following accretive version of Proposition 1.5.2.

Corollary 2.3.3. *Let $A, B \in \mathcal{S}_\alpha$. If $f \in \mathfrak{m}$, then for any unit vector x , we have*

$$\Re \langle (A\sigma_f B)x, x \rangle \leq \sec^2 \alpha \Re \langle \langle Ax, x \rangle \sigma_f \langle Bx, x \rangle \rangle. \quad (2.3.5)$$

Proof : Letting $\Phi(A) = \langle Ax, x \rangle$ in Theorem 2.3.2. Then Φ is a unital positive linear map and

$$\cos^2 \alpha \Re \Phi(A\sigma_f B) \leq \Re (\Phi(A)\sigma_f \Phi(B)) \Rightarrow \cos^2 \alpha \Re \langle (A\sigma_f B)x, x \rangle \leq \Re \langle \langle Ax, x \rangle \sigma_f \langle Bx, x \rangle \rangle,$$

which completes the proof □

2.4 Choi-Davis inequalities for accretive matrices

The main application of Theorem 2.1.1 is showing the following representation, which presents the natural extension of the well known characterization for positive matrices. For completeness of the proof, it is important to recall that a function $f \in \mathfrak{m}$ can be analytically continued to $\mathbb{C} \setminus (-\infty, 0]$. This means that $f(A)$ can be defined similarly for any A whose spectrum is disjoint from $(-\infty, 0]$.

Now when A and B are accretive, (Drury, 2015) showed that the spectrum of the matrix $A^{-1/2}BA^{-1/2}$ is disjoint from $(-\infty, 0]$; justifying the use of $f(A^{-1/2}BA^{-1/2})$ in the following result, with the aid of Theorem 2.1.1.

Theorem 2.4.1. *Let $A, B \in \mathcal{M}_n$ be accretive matrices and let $f \in \mathfrak{m}$. Then*

$$A\sigma_f B = A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

Proof : By definition,

$$\begin{aligned}
A\sigma_f B &= \int_0^1 A!_t B \, d\nu_f(t) \\
&= \int_0^1 ((1-t)A^{-1} + tB^{-1})^{-1} \, d\nu_f(t) \\
&= A^{\frac{1}{2}} \int_0^1 \left((1-t)I + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} \right)^{-1} \, d\nu_f(t) A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}},
\end{aligned}$$

where we have used Theorem 2.1.1 to obtain the last identity. This completes the proof. \square

We refer the reader to (Sano and Sugawara, 2020), where operator means of accretive operators on Hilbert spaces were treated. Our definition coincides with the main definition in (Sano and Sugawara, 2020) when we restrict ourselves to \mathcal{M}_n .

Our first result in this direction is the following relation between $f(\Re A)$ and $\Re(f(A))$.

Proposition 2.4.2. *Let $f \in \mathfrak{m}$ and $A \in \mathcal{M}_n$ be accretive. Then*

$$\Re(f(A)) \geq f(\Re A). \quad (2.4.1)$$

Consequently, if A is accretive, then so is $f(A)$.

Proof : We easily notice that

$$\Re f(A) = \int_0^1 \Re(I!_t A) \, d\nu_f(t) \geq \int_0^1 (I!_t \Re A) \, d\nu_f(t) = f(\Re A), \quad (2.4.2)$$

where we have used Theorem 1.6.8 to obtain the first inequality. \square

On the other hand, a reversed version of Proposition 2.4.2 can be found for sectorial

matrices, as follows.

Proposition 2.4.3. *Let $f \in \mathfrak{m}$ and $A \in \mathcal{S}_\alpha$. Then*

$$\Re(f(A)) \leq \sec^2 \alpha f(\Re A). \quad (2.4.3)$$

Proof : Using Theorem 1.6.9, we easily obtain

$$\Re f(A) = \int_0^1 \Re(I_t A) d\nu_f(t) \leq \sec^2 \alpha \int_0^1 (I_t \Re A) d\nu_f(t) \leq \sec^2 \alpha f(\Re A),$$

which completes the proof. □

Now we are ready to present the first Choi-Davis inequality for accretive matrices extending (1.4.1).

Theorem 2.4.4. *Let $f \in \mathfrak{m}$, Φ be a unital positive linear map and $A \in \mathcal{M}_n$ be accretive.*

Then the following version of the Choi-Davis inequality hold:

$$\Re f(\Phi(A)) \geq \cos^2 \alpha \Re \Phi(f(A)).$$

Proof : Let A be accretive. notice that

$$\begin{aligned}
\Re f(\Phi(A)) &= \Re \int_0^1 I_t \Phi(A) d\nu_f(t) \quad (\text{by Theorem 2.1.1}) \\
&\geq \int_0^1 I_t \Re(\Phi(A)) d\nu_f(t) \quad (\text{by Theorem 1.6.8}) \\
&= \int_0^1 I_t \Phi(\Re A) d\nu_f(t) \quad (\text{by (2.3.1)}) \\
&= f(\Phi(\Re A)) \quad (\text{by Theorem 2.1.1}) \\
&\geq \Phi(f(\Re A)) \quad (\text{by (1.4.1)}) \\
&\geq \cos^2 \alpha \Re \Phi(f(A)), \quad (\text{by Proposition 2.4.3})
\end{aligned}$$

which completes the proof. □

As an application of Theorem 2.4.4, we present the following accretive version of Lemma 1.3.12.

Corollary 2.4.5. *Let $A \in \mathcal{S}_\alpha$ and $f \in \mathfrak{m}$. Then for any unit vector $x \in \mathbb{C}^n$,*

$$\Re \langle f(A)x, x \rangle \leq \sec^2 \alpha \Re f(\langle Ax, x \rangle). \quad (2.4.4)$$

Proof : Letting $\Phi(A) = \langle Ax, x \rangle$ in Theorem 2.4.4, Φ is a unital positive linear map.

Then we have

$$\cos^2 \alpha \Re \Phi(f(A)) \leq \Re f(\Phi(A)) \Rightarrow \Re \langle f(A)x, x \rangle \leq \sec^2 \alpha \Re f(\langle Ax, x \rangle),$$

which completes the proof. □

2.5 Norm inequalities for accretive matrices

In this section, we present norm inequalities for accretive matrices. We begin with the following accretive version of Lemma 1.3.13.

Proposition 2.5.1. *Let $A \in \mathcal{M}_n$ be accretive. Then for $f \in \mathfrak{m}$,*

$$f(\|\Re A\|) \leq \|\Re f(A)\|. \quad (2.5.1)$$

Proof : For accretive A , we have

$$\begin{aligned} \|\Re f(A)\| &\geq \|f(\Re A)\| && \text{(by Proposition 2.4.2)} \\ &\geq f(\|\Re A\|) && \text{(by Lemma 1.3.13)} \end{aligned}$$

completing the proof. □

A reversed version can be stated as follows, when sectorial matrices interfere.

Corollary 2.5.2. *Let $A \in \mathcal{S}_\alpha$. Then for $f \in \mathfrak{m}$,*

$$f(\|\Re A\|) \leq \|\Re f(A)\| \leq \sec^2 \alpha f(\|\Re A\|).$$

Proof : The first inequality follows from Proposition 2.5.1. For the second inequality, Proposition 2.4.3 and Lemma 1.3.12 imply

$$\Re \langle f(A)x, x \rangle = \langle \Re f(A)x, x \rangle \leq \sec^2(\alpha) \langle f(\Re A)x, x \rangle \leq \sec^2 \alpha f(\Re \langle Ax, x \rangle).$$

Notice that since A is accretive matrix, $f(A)$ is accretive by Proposition 2.4.2. Taking

the supremum over $\|x\| = 1$ of the latter inequality implies

$$\begin{aligned}
\|\Re f(A)\| &= \sup_{\|x\|=1} \langle \Re f(A)x, x \rangle \quad (\text{since } f(A) \text{ is accretive}) \\
&\leq \sec^2 \alpha \sup_{\|x\|=1} f(\Re \langle Ax, x \rangle) \quad (\text{by Proposition 2.4.2}) \\
&= \sec^2 \alpha f \left(\sup_{\|x\|=1} \langle \Re Ax, x \rangle \right) \quad (\text{since } f \text{ is increasing}) \\
&= \sec^2 \alpha f(\|\Re A\|) \quad (\text{since } \Re A > 0).
\end{aligned}$$

This completes the proof. □

Now we are ready to present the accretive version of Theorem 1.3.14.

Theorem 2.5.3. *Let $A, B \in \mathcal{S}_\alpha$. Then, for any $f \in \mathfrak{m}$,*

$$|||f(A + B)||| \leq \sec^3 \alpha |||f(A) + f(B)|||.$$

Proof : We have

$$\begin{aligned}
\cos \alpha |||f(A + B)||| &\leq |||\Re f(A + B)||| \quad (\text{by Proposition 1.6.5}) \\
&\leq \sec^2 \alpha |||f(\Re A + \Re B)||| \quad (\text{by Proposition 2.4.3}) \\
&\leq \sec^2 \alpha |||f(\Re A) + f(\Re B)||| \quad (\text{by Theorem 1.3.14}) \\
&\leq \sec^2 \alpha |||\Re f(A) + \Re f(B)||| \quad (\text{by Proposition 2.4.2}) \\
&= \sec^2 \alpha |||\Re(f(A) + f(B))||| \\
&\leq \sec^2 \alpha |||f(A) + f(B)||| \quad (\text{by Proposition 1.6.5}).
\end{aligned}$$

Consequently

$$\| \|f(A + B)\| \| \leq \sec^3 \alpha \| \|f(A) + f(B)\| \|,$$

which completes the proof. \square

Next, we present the accretive version of Proposition 1.5.1.

Theorem 2.5.4. *Let $A, B \in \mathcal{S}_\alpha$. Then for any $f \in \mathfrak{m}$,*

$$\| \|A\sigma_f B\| \| \leq \sec^3 \alpha (\| \|A\| \sigma_f \| \|B\| \|). \quad (2.5.2)$$

Proof : Noting (2.2.5), Proposition 1.6.5 and Proposition 1.5.1, we have

$$\begin{aligned} \| \|A\sigma_f B\| \| &\leq \sec \alpha \| \| \Re(A\sigma_f B) \| \| \leq \sec^3 \alpha \| \| \Re A \sigma_f \Re B \| \| \\ &\leq \sec^3 \alpha (\| \| \Re A \| \sigma_f \| \| \Re B \| \|) \leq \sec^3 \alpha (\| \|A\| \sigma_f \| \|B\| \|), \end{aligned}$$

which completes the proof. \square

CHAPTER 3

Accretive matrices and matrix convex function

In this part, we present some inequalities for accretive matrices when treated via matrix convex functions and matrix monotone decreasing. and we will adopt the notations

$$\mathbf{m}' = \{f : [0, \infty) \rightarrow [0, \infty) \text{ is matrix convex, } f(0) = 0, f(1) = 1\}.$$

$$\mathbf{m}'' = \{h : (0, \infty) \rightarrow (0, \infty), h \text{ is matrix monotone decreasing, } h(1) = 1\}.$$

We notice that, for example, if $f(t) = t^r, 1 \leq r \leq 2$ then $f \in \mathbf{m}'$, while if $h(t) = \frac{1}{2^r}(1+t)^r, -1 \leq r < 0$, then $h \in \mathbf{m}''$. We also notice that $h \in \mathbf{m}''$ if and only if h is matrix convex and $f(\infty) < \infty$. Therefore, discussion of \mathbf{m}'' is equivalent to discussion of matrix convex functions that are finite at ∞ .

We introduce our main results in two parts; where we discuss the class \mathbf{m}' first, then the class \mathbf{m}'' . For the results in this section, the following two propositions are essential [14].

Proposition 3.0.5. *Let $f : [0, \infty) \rightarrow [0, \infty)$. The following are equivalent:*

1. f is matrix convex and $f(0) = 0$.
2. $\frac{f(t)}{t}$ is matrix monotone.
3. $\frac{f(t)}{t}$ is matrix concave.

The proof of the following proposition is straightforward.

Proposition 3.0.6. *Let $h : (0, \infty) \rightarrow (0, \infty)$. If h is matrix monotone decreasing then $\frac{1}{h}$ is matrix monotone.*

Consequently

$$f \in \mathbf{m}' \Rightarrow \frac{f(t)}{t} \in \mathbf{m} \text{ and } h \in \mathbf{m}'' \Rightarrow \frac{1}{h} \in \mathbf{m}. \quad (3.0.1)$$

3.1 The class of matrix convex function (\mathfrak{m}')

First, we point out that the class \mathfrak{m}' is too wide, in the following sense. Let $f : [0, \infty) \rightarrow [0, \infty)$ be any matrix convex function such that $f(0) \leq f(t)$, for all $t \in [0, \infty)$.

Then the function

$$g(t) = \frac{f(t) - f(0)}{f(1) - f(0)} \in \mathfrak{m}'.$$

When f is matrix convex and A is accretive, then $f(A)$ is defined via the Dunford integral (1.3.1). Our first result is the following integral representation.

Proposition 3.1.1. *Let $f \in \mathfrak{m}'$. Then*

$$f(t) = \int_0^1 t (1!_{\gamma} t) d\mu_f(\gamma),$$

where μ_f is a probability measure on $[0, 1]$, associated with f .

Proof : For the given f , it is immediate that the function $g(t) = \frac{f(t)}{t} \in \mathfrak{m}$. But then

$$g(t) = \int_0^1 1!_{\gamma} t d\mu_g(\gamma),$$

for some probability measure μ_g on $[0, 1]$. This implies that

$$f(t) = \int_0^1 t (1!_{\gamma} t) d\mu_f(\gamma)$$

where $\mu_f = \mu_g$. □

So, if $f : [0, \infty) \rightarrow [0, \infty)$ is any matrix convex function with $f(t) \geq f(0)$, then

$g = \frac{f(t)-f(0)}{f(1)-f(0)} \in \mathfrak{m}'$, and hence

$$f(t) = f(0) + (f(1) - f(0)) \int_0^1 t(I!_{\gamma}t) d\mu_f(\gamma). \quad (3.1.1)$$

Corollary 3.1.2. *Let $f \in \mathfrak{m}'$ and let A be accretive. Then*

$$f(A) = \int_0^1 A(I!_{\gamma}A) d\mu_f(\gamma),$$

for some probability measure μ_f on $[0, 1]$.

Proof : Since $f \in \mathfrak{m}'$, it follows that $f(t) = tg(t)$ for some $g \in \mathfrak{m}$, by Proposition 3.0.5.

Since $g \in \mathfrak{m}$, it follows by the Theorem 2.1.1 that

$$f(A) = Ag(A) = A \int_0^1 I!_{\gamma}A d\mu_g(\gamma),$$

which completes the proof. □

In [8], it was shown that when $f \in \mathfrak{m}$ and A is accretive, we have

$$\Re(f(A)) \geq f(\Re A). \quad (3.1.2)$$

The next result shows that this inequality is reversed when $f \in \mathfrak{m}'$.

Theorem 3.1.3. *Let $A \in \mathcal{M}_n$ be accretive. Then, for any $f \in \mathfrak{m}'$*

$$\Re(f(A)) \leq f(\Re A). \quad (3.1.3)$$

Proof : To show the above inequality, we use the fact that for $0 < \gamma < 1$,

$$A(I!_{\gamma}A) = \frac{A}{1-\gamma} - \frac{\gamma}{1-\gamma}(I!_{\gamma}A).$$

Applying Corollary 3.1.2 and the above identity, we obtain

$$\begin{aligned} \Re(f(A)) &= \Re\left(\int_0^1 A(I!_{\gamma}A)d\mu_f(\gamma)\right) \\ &= \int_0^1 \Re\left(\frac{A}{1-\gamma} - \frac{\gamma}{1-\gamma}(I!_{\gamma}A)\right) d\mu_f(\gamma) \\ &\leq \int_0^1 \left(\frac{\Re A}{1-\gamma} - \frac{\gamma}{1-\gamma}(I!_{\gamma}\Re A)\right) d\mu_f(\gamma) && \text{(by Theorem 1.6.8)} \\ &= \int_0^1 \Re A(I!_{\gamma}\Re A)d\mu_f(\gamma) \\ &= f(\Re A). \end{aligned}$$

This completes the proof □

Corollary 3.1.4. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be any matrix convex function such that $f(t) \geq f(0)$ for all $t > 0$. Then*

$$\Re(f(A)) \leq f(\Re A).$$

In particular, if f is matrix convex increasing, then $\Re(f(A)) \leq f(\Re A)$.

Proof : Since

$$g(t) = \frac{f(t) - f(0)}{f(1) - f(0)} \in \mathbf{m}',$$

Theorem 3.1.3 applied to the function g implies $\Re(g(A)) \leq g(\Re A)$. Simplifying this last inequality implies $\Re(f(A)) \leq f(\Re A)$, as desired. □

3.2 The class of matrix monotone decreasing function (\mathfrak{m}'')

In this section we look into the class \mathfrak{m}'' , to complement our study. First, an integral representation of elements in \mathfrak{m}'' .

Proposition 3.2.1. *Let $h \in \mathfrak{m}''$. Then*

$$h(t) = \left(\int_0^1 1!_{\gamma} t \, d\mu_h(\gamma) \right)^{-1},$$

where μ_h is a probability measure on $[0, 1]$, associated with h .

Proof : Notice that $g(t) = \frac{1}{h(t)} \in \mathfrak{m}$. Then

$$g(t) = \int_0^1 1!_{\gamma} t \, d\mu_g(\gamma),$$

for some probability measure μ_g on $[0, 1]$. This implies that

$$h(t) = \left(\int_0^1 1!_{\gamma} t \, d\mu_h(\gamma) \right)^{-1}$$

where $\mu_h = \mu_g$. □

Corollary 3.2.2. *Let $h \in \mathfrak{m}''$ and let A be accretive. Then*

$$h(A) = \left(\int_0^1 (I!_{\gamma} A) d\mu_h(\gamma) \right)^{-1},$$

for some probability measure μ_h on $[0, 1]$.

Theorem 3.2.3. *Let $A \in \mathcal{M}_n$ be accretive. Then, for any $f \in \mathfrak{m}''$*

$$h(\Re A) \geq \Re(h(A)). \quad (3.2.1)$$

Proof : We have

$$\begin{aligned} \Re(h(A)) &= \Re \left(\left(\int_0^1 (I!_{\gamma} A) d\mu_h(\gamma) \right)^{-1} \right) \\ &\leq \left(\int_0^1 \Re(I!_{\gamma} A) d\mu_h(\gamma) \right)^{-1} && \text{(by Proposition 1.6.6)} \\ &\leq \left(\int_0^1 I!_{\gamma} \Re A d\mu_h(\gamma) \right)^{-1} && \text{(by Theorem 1.6.8)} \\ &= h(\Re A), \end{aligned}$$

which completes the proof. □

Now, we present the reversed version of Theorem 3.2.3 via sectorial matrices.

Theorem 3.2.4. *Let $A \in \mathcal{M}_n$ is accretive with $W(A) \subset S_{\alpha}$. Then for any $h \in \mathfrak{m}''$*

$$h(\Re A) \leq \sec^4 \alpha \Re(h(A)). \quad (3.2.2)$$

Proof : We have

$$\begin{aligned} \Re(h(A)) &= \Re \left(\left(\int_0^1 (I!_{\gamma} A) d\mu_h(\gamma) \right)^{-1} \right) \\ &\geq \cos^2 \alpha \left(\int_0^1 \Re(I!_{\gamma} A) d\mu_h(\gamma) \right)^{-1} && \text{(by Lemma 1.6.7)} \\ &\geq \cos^4 \alpha \left(\int_0^1 I!_{\gamma} \Re A d\mu_h(\gamma) \right)^{-1} && \text{(by Lemma 1.6.9)} \\ &= \cos^4 \alpha h(\Re A), \end{aligned}$$

which completes the proof. \square

Theorem 3.2.4 implies the following main difference between the classes \mathfrak{m}' and \mathfrak{m}'' .

Now we are ready to present the sectorial version of Lemma 1.4.4, valid for any matrix convex h .

Theorem 3.2.5. *Let $A \in \mathcal{S}_\alpha$ and $h \in \mathfrak{m}''$. Then*

$$\Re h(\Phi(A)) \leq \sec^4 \alpha \Re \Phi(h(A)) \quad (3.2.3)$$

for any unital positive linear map Φ .

Proof : We have

$$\begin{aligned} \Re h(\Phi(A)) &\leq h(\Re \Phi(A)) && \text{(by Theorem 3.2.3)} \\ &\leq \Phi(h(\Re A)) && \text{(by Lemma 1.4.4)} \\ &\leq \sec^4 \alpha \Phi(\Re h(A)) && \text{(by Theorem 3.2.4)} \\ &= \sec^4 \alpha \Re \Phi(h(A)). \end{aligned}$$

This completes the proof. \square

As an application of Theorem 3.2.5, we present the following accretive version of Lemma 1.3.15.

Corollary 3.2.6. *Let $A \in \mathcal{S}_\alpha$ and $h \in \mathfrak{m}''$. Then for any unit vector $x \in \mathbb{C}^n$,*

$$\sec^4 \alpha \Re \langle h(A)x, x \rangle \geq \Re h(\langle Ax, x \rangle). \quad (3.2.4)$$

Proof : Letting $\Phi(A) = \langle Ax, x \rangle$ in Theorem 3.2.5, Φ is a unital positive linear map.

Then we have

$$\sec^4 \alpha \Re \Phi(h(A)) \geq \Re h(\Phi(A)) \Rightarrow \sec^4 \alpha \Re \langle h(A)x, x \rangle \geq \Re h(\langle Ax, x \rangle),$$

which completes the proof. \square

We notice that if $h \in \mathfrak{m}''$, then $h((1-t)A + tB) \leq (1-t)h(A) + th(B)$ for the positive matrices A, B . The next result presents the accretive version of this.

Theorem 3.2.7. *Let $A, B \in \mathcal{S}_\alpha$ and $h \in \mathfrak{m}''$. Then for any $t \in [0, 1]$,*

$$\Re h(tA + (1-t)B) \leq \sec^4 \alpha \Re(t h(A) + (1-t)h(B)). \quad (3.2.5)$$

Proof : We have

$$\begin{aligned} \Re h(tA + (1-t)B) &\leq h(t \Re A + (1-t)\Re B) && \text{(by Theorem 3.2.3)} \\ &\leq t h(\Re A) + (1-t)h(\Re B) && \text{(by convexity of } h) \\ &\leq \sec^4 \alpha \Re(t h(A) + (1-t)h(B)) && \text{(by Theorem 3.2.4)}. \end{aligned}$$

This completes the proof. \square

In what follows, we present a Hermite-Hadamard inequality for accretive matrices.

Theorem 3.2.8. *Let $A, B \in \mathcal{S}_\alpha$ and $h \in \mathfrak{m}''$. Then*

$$\cos^4 \alpha \Re h\left(\frac{A+B}{2}\right) \leq \int_0^1 \Re h((1-t)A + tB) dt \leq \sec^4 \alpha \Re\left(\frac{h(A) + h(B)}{2}\right). \quad (3.2.6)$$

Proof : Let $t \in [0, 1]$. Then

$$\begin{aligned}
\int_0^1 \Re h((1-t)A + tB) dt &\geq \cos^4 \alpha \int_0^1 h((1-t)\Re A + t\Re B) dt && \text{(by Theorem 3.2.4)} \\
&\geq \cos^4 \alpha h\left(\frac{\Re A + \Re B}{2}\right) && \text{(by Proposition 1.3.16)} \\
&\geq \cos^4 \alpha \Re h\left(\frac{A + B}{2}\right) && \text{(by Theorem 3.2.3)}.
\end{aligned}$$

Thus, we have shown the first inequality. To show the second inequality, we have

$$\begin{aligned}
\int_0^1 \Re h((1-t)A + tB) dt &\leq \int_0^1 h((1-t)\Re A + t\Re B) dt && \text{(by Theorem 3.2.3)} \\
&\leq \int_0^1 (1-t)h(\Re A) + th(\Re B) dt \\
&= \frac{h(\Re A) + h(\Re B)}{2} \\
&\leq \sec^4 \alpha \Re\left(\frac{h(A) + h(B)}{2}\right) && \text{(by Theorem 3.2.4)}.
\end{aligned}$$

This completes the proof. □

References

- [1] Abu-omar, A. and Kittaneh, F. (2015), Numerical radius inequalities for $n \times n$ operator matrices, **Linear Algebra Appl.**, 468, 1826.
- [2] Ando, T. (1994), Majorization and inequalities in matrix theory, **Linear Algebra Appl.**, 199, 1767.
- [3] Ando, T. (1979), Concavity of certain maps on positive definite matrices and applications to Hadamard products, **Linear Algebra Appl.**, 26, 203241.
- [4] Ando, T. (1978), **Topics on operator inequalities**, Lecture Note, Sapporo.
- [5] Ando, T. and Hiai, F. (2011), Operator log-convex functions and operator means, **Math. Ann.**, 350, 611-630.
- [6] Ando, T. and Zhan, X. (1999), Norm inequalities related to operator monotone functions, **Math. Ann.**, 315, 771-780.
- [7] Artin, M.(2011), **Algebra**, 2nd edition, Prentice Hall.
- [8] Bedrani, Y. Kittaneh, F. and Sababheh, M. (2020), From positive to accretive matrices, [ArXiv: 2002.11090](https://arxiv.org/abs/2002.11090).
- [9] Bedrani, Y. Kittaneh, F. and Sababheh, M. (2021), Numerical radii of accretive matrices, **Linear Multilinear Algebra**, 69, 957-970.
- [10] Bedrani, Y. Kittaneh, F. and Sababheh, M. (2021), On the weighted geometric mean of accretive matrices, **Ann. Funct. Anal.**, 12, DOI: <https://doi.org/10.1007/s43034-020-00094-6>.

- [11] Bedrani, Y. Kittaneh, F. and Sababheh, M. (2021), **Accretive matrices and matrix monotone function**, Results Math 77,52(2022). DOI: <https://doi.org/10.1007/s00025-021-01590-4>.
- [12] Bhatia, R. (2007), **Positive Definite Matrices**, Princeton University Press, Princeton.
- [13] Bhatia, R. and Kittaneh, F. (2000), Notes on matrix arithmeticgeometric mean inequalities, **Linear Algebra Appl.**, 308, 203211.
- [14] Bhatia, R. (1997), **Matrix Analysis**, Springer-Verlag, New York.
- [15] Drury, S. and Lin, M. (2014), Singular value inequalities for matrices with numerical ranges in a sector, **Oper. Matrices**, 8, 11431148.
- [16] Drury, S. (2015), Principal powers of matrices with positive definite real part, **Linear Multilinear Algebra**, 63, 296-301.
- [17] Furuta, T. Hot, J.M. Pečarić, J. and Seo, Y. (2005), **MondPečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space**, Element, Zagreb.
- [18] Furuta, T. and Yanagide, M. (1998), Generalized means and convexity of inversion for positive operators, **Amer. Math. Monthly**, 105, 258-259.
- [19] Furuta, T. (2002), **Invitation to Linear Operators: Form Matrix to Bounded Linear Operators on a Hilbert Space**, Taylor and Francis.
- [20] Furuta, T. Mičić Hot, J. Pečarić, J. and Seo, Y. **MondPečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space**, Element, Zagreb, 2005.

- [21] Fujii, J.I. and Seo, Y. (2016), Tsallis relative operator entropy with negative parameters. **Adv. Oper. Theory**, 1, 219236.
- [22] Halmos, P.R. (1982), **A Hilbert space problem book**, 2^{ed}, Grad.Texts Math. 19, Springer, New York.
- [23] Hansen, F. (2013), The fast track to Lowners theorem, **Linear Alg. Appl.**, 438, 4557-4571.
- [24] Hiai, F. and Zhan, X. (2002), Inequalities involving unitarily invariant norms and operator monotone functions, **Linear Algebra Appl.**, 341, 151169.
- [25] Higham, N. (2008), Functions of Matrices: Theory and Computation, **SIAM**, <https://doi.org/10.1137/1.9780898717778>.
- [26] Hoa, D.T. Binh, D.T.H. and Toan, H.M. (2013), On some inequalities with matrix means, **RIMS Kokyuroku**, 1893, 67-71.
- [27] Kubo, F. and Ando, T. (1979/80), Means of positive linear operators, **Math. Ann.**, 246, 205-224.
- [28] Kittaneh, F. Moslehian, M.S. and Yamazaki, T. (2015), Cartesian decomposition and numerical radius inequalities, **Linear Algebra Appl.**, 471, 4653.
- [29] Moslehian, M.S. **Matrix HermiteHadamard type inequalities**, Houston J. Math., **39** (2013), 177189.
- [30] Lin, M. (2015), Extension of a result of Hanynsworth and Hartfiel, **Arch. Math.**, 104, 93-100.
- [31] Lin, M. (2016), Some inequalities for sector matrices. **Oper. Matrices**, 10, 915921.

- [32] Lin, M. Sun, F. (2017), A property of the geometric mean of accretive operators, **Linear Multilinear Algebra**, 65, 433-437.
- [33] Tan, F. and Xie, A. (2019), On the Logarithmic Mean of Accretive Matrices, **Filomat**, 33, 4747-4752.
- [34] Mathias, R. (1992), Matrices with positive definite Hermitian part: inequalities and linear systems, **SIAM J. Matrix Anal. Appl.**, 13, 640-654.
- [35] Raïssouli, M. Moslehian, M.S. and Furuichi, S. (2017), Relative entropy and Tsallis entropy of two accretive operators, **C. R. Acad. Sci. Paris Ser.**, 355, 687-693.
- [36] Sano, T. and Sugawara, K. (2020), Operator means for operators with positive definite real part, **Adv. Oper. Theory**, <https://doi.org/10.1007/s43036-019-00017-w>.
- [37] Serre, D. (2010), **Matrices, theory and applications**, Springer-Verlag, New York.
- [38] Tan, F. and Xie, A. (2020), An extension of the AMGMHM inequality, **Bull. Iran. Math. Soc.**, 46, 245-251.
- [39] Tan, F. and Chen, H. (2019), Inequalities for sector matrices and positive linear maps, **Electronic Journal of Linear Algebra**, 34, 418-423.
- [40] Uchiyama, M. (2010), Operator monotone functions, positive definite kernels and majorization, **Proc. Amer. Math. Soc.**, 138, 3985-3996.
- [41] Mao, Y. and Mao, Y. (2020), Inequalities for the Heinz Mean of Sector Matrices, **Bull. Iran. Math. Soc.**, 46, 1767-1774.
- [42] Zhang, F. (2015), A matrix decomposition and its applications, **Linear Multilinear Algebra**, 63, 2033-2042.