MATHEMATICS 2

Solved Exercises

MATHEMATICS

2

For first year university students in matter sciences and related disciplines

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WHAT THIS HANDOUT IS FOR?

The more you practice mathematics, the more proficient you become—just like with any other skill. Mastering a mathematics course, however, goes beyond the rote memorization of rules, laws, and theorems. It prioritizes a deep understanding of the underlying concepts through consistent problem-solving, the application of various techniques, and the use of logical reasoning.

This handout presents a collection of solved mathematical exercises intended for first-year undergraduate students in Matter Sciences (SM). Its primary aim is to provide students with a solid foundation while fostering rigorous scientific reasoning, thereby helping them master key concepts and better understand physical and chemical laws, phenomena, and models—especially those that are often challenging to grasp.

The document addresses key topics in analysis and linear algebra as outlined in the Mathematics 2 syllabus. It is structured into four chapters: Chapters 1, 3, and 4 focus on concepts from analysis, while Chapter 2 concentrates on topics from linear algebra. Each chapter begins with a brief overview of the main notions, followed by a set of fully solved exercises designed to reinforce learning through guided practice and problem-solving.

Chapter 1 deals with solving first- and second-order linear ordinary differential equations using different methods such as the integrating factor method and variation of parameters.

Chapter 2 focuses on linear algebra, emphasizing matrices, determinants, diagonalization, and systems of linear equations.

Chapter 3 includes exercises with detailed solutions on Taylor series expansions.

Chapter 4 presents problems involving differential operators, as well as limits and partial derivatives of functions of two variables.

The methodology employed here aims to encourage active learning, strengthen mathematical rigor, and bridge theoretical understanding with practical applications.

The detailed table of contents at the beginning of the document enables students to navigate the material efficiently and locate specific topics as needed.

CHAPTER 1

Exercises Related to the Chapter: Ordinary Differential Equations

Contents

This chapter features a curated selection of solved exercises covering first and second order linear ordinary differential equations (ODEs). These exercises provide a starting point for understanding various physical and chemical phenomena often described using differential equations. They are intended to strengthen students' analytical reasoning, master some usual methods for solving such equations, and equip them to tackle real-world problems involving differential equations.

1.1 Course Recap: Differential Equations

1.1.1 First-Order Linear Differential Equations

Definition. A first-order ordinary differential equation (ODE) is a mathematical equation that involves an unknown function $y(x)$ and its first derivative $\frac{dy}{dx}$. It can be expressed generally as

$$F\left(x, y, \frac{dy}{dx}\right) = 0,$$

or it takes the form

$$\frac{dy}{dx} = f(x, y),$$

where y is the dependent function and "x" is the independent variable.

Definition 1.1 A first-order linear differential equation is an equation that can be expressed in the standard form:

$$\frac{dy}{dx} + P(x)y\,(x) = Q(x),$$

where $P(x)$ and $Q(x)$ are continuous functions on a given open interval or are constants.

Remark 1.1 If $Q(x) = 0$, the equation is said to be homogeneous. In contrast, if $Q \neq 0$, the equation is said to be nonhomogeneous.

An oversimplified Step-by-Step Guide to Solving First Order Linear Differential Equations

For finding the solution of such linear differential equations, we can use the integrating factor method, the integral transforms, or the method of variation of constants.

Integrating Factor Method

1. Rearrange the terms of the given equation in the form
$$\frac{dy}{dx} + P(x)y\,(x) = Q(x).$$
2. Determine a function $\mu(x) = e^{\int P(x)\,dx}$ which is known as the Integrating factor (I.F).
3. Multiply the both sides of the equation by $\mu(x)$ to get
$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x).$$
4. Integrate both sides with respect to x to get
$$\mu(x)y\,(x) = \int \mu(x)Q(x)\,dx + C,$$
where C is some arbitrary constant of integration.
5. Divide both sides by $\mu(x)$ to deduce the general solution as follows:
$$y\,(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right).$$

This method is systematic and can be applied to any linear first-order differential equation where $P(x)$ and $Q(x)$ are known.

Variation of Constants Method

Solving first-order linear differential equations by using the variation of constants method (also known as variation of parameters) requires 4 steps:

1. Rearrange the terms of the given equation in the form
$$\frac{dy}{dx} + P(x)y\,(x) = Q(x).$$
2. Solve the following the associated equation without a second member
$$\frac{dy}{dx} + P(x)y\,(x) = 0,$$

in order to obtain an infinity of solutions of the form

$$y_h = C \ \exp\left(-\int P(x)dx\right).$$

3. Find a particular solution y_p of the given equation (you only need a single particular solution). For example, if you will use a procedure known as the variation of constants method, you need to replace the constant C in the solution y_h with the function $C(x)$. By substituting the particular solution into the original nonhomogeneous equation, you can find $C(x)$ and hence a particular solution.

4. Add the particular solution to the general solution of the associated equation without a second. In other words, the general solution can be written as follows:

$$y = y_p + y_h.$$

1.1.2 Second-Order Differential Equations with Constant Coefficients

Definition 1.2 A second-order linear differential equation with constant coefficients has the standard form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where $a, b,$ and c are real constants, $a \neq 0$, and f is a given function. When $f(x) = 0$, the equation is called homogeneous.

Remark 1.2 It is worth noting that, in Leibniz's notation, the first and second derivatives of the function y are written as $\frac{dy}{dx}$ $\frac{d^2y}{dx^2}$, respectively, whereas in Lagrange's notation, y' and y'' represent the first and second derivatives of the dependent variable y with respect to the independent variable x, respectively.

An oversimplified Step-by-Step Guide to Solving Second Order Linear Differential Equations

Solving Homogeneous Second Order Linear Differential Equations

To solve such an equation, we first find the roots of the following characteristic equation (also called the auxiliary equation):

$$ar^2 + br + c = 0.$$

Solving this quadratic equation for r gives us the roots, and the general solution depends on their nature:

- **Two distinct real roots** r_1 and r_2**:**

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

- **One repeated root** r**:**

$$y(x) = (C_1 + C_2 x)e^{rx}.$$

- **Complex conjugate roots** $r = \alpha \pm i\beta$**:**

$$y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

In each case, C_1 and C_2 are arbitrary constants determined by initial or boundary conditions.

Nonhomogeneous Second-Order Differential Equations

When $f(x)$ is a given nonzero function (called the nonhomogeneous term or forcing function), the equation is said to be nonhomogeneous.

General Solution Structure: The general solution of the nonhomogeneous equation is given by:

$$y(x) = y_h(x) + y_p(p)$$

Where y_h is the general solution of the homogeneous part and y_p is a particular solution to the full nonhomogeneous equation found by:

- Method of undetermined coefficients
- Variation of parameters

Particular Solution Using Variation of Parameters

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation. The method of variation of parameters seeks a particular solution of the form

$$y_p(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

The functions $c_1(x)$ and $c_2(x)$ are obtained by solving:

$$\begin{aligned} c_1'(x) &= \frac{W_1}{W}, \\ c_2'(x) &= \frac{W_2}{W}. \end{aligned}$$

Where

- W is the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

- W_1 and W_2 are defined as

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \text{ and } W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

Once $c_1(x)$ and $c_2(x)$ are computed, integrate them to obtain $c_1(x)$ and $c_2(x)$. Substituting these into the expression for $y_p(x)$ gives the particular solution to the nonhomogeneous equation.

Particular Cases

Particular Case: $f(x) = p(x)e^{kx}$. Let us consider the non-homogeneous linear second-order differential equation

$$y'' + ay' + by = f(x),$$

where the non-homogeneous (forcing) term $f(x)$ is of the form:

$$f(x) = p(x)e^{kx}.$$

with $p(x)$ a polynomial of degree n, and $k \in \mathbb{R}$ a constant.
Strategy for Finding a Particular Solution: In this case, we seek a particular solution $y_p(x)$ of the form:

$$y_p(x) = q(x)e^{kx},$$

where $q(x)$ is a polynomial to be determined. The degree of $q(x)$ depends on the relationship between k and the roots of the characteristic polynomial $r^2 + ar + c = 0$.
Case 1: k is not a root of the characteristic equation. Then the function e^{kx} is not a solution of the homogeneous equation, and

$$\boxed{\deg(q) = \deg(q) = n.}$$

Case 2: k is a simple root of the characteristic equation. Then e^{kx} appears once in the homogeneous solution. To avoid duplication, we multiply by x, so

$$\boxed{\deg(q) = n + 1.}$$

Case 3: k is a double root of the characteristic equation. Then both e^{kx} and xe^{kx} are in the homogeneous solution. To ensure linear independence, we multiply by x^2, giving

$$\boxed{\deg(q) = n + 2.}$$

Particular Case: $f(x) = p(x)\cos(kx)$ or $f(x) = p(x)\sin(kx)$. Let $p(x)$ be a polynomial of degree n. We then seek a particular solution of the form:

$$y_p(x) = u(x)\cos(kx) + v(x)\sin(kx)$$

where $u(x), v(x)$ are polynomials. If $\cos(kx)$ and $\sin(kx)$ are not solutions of the associated homogeneous equation, then:

$$\deg(u) = \deg(v) = n$$

If they are part of the homogeneous solution (e.g., when the characteristic polynomial has complex roots $\alpha \pm i\beta$ with $\beta = k$), we multiply by x or x^2 as needed to obtain linear independence, increasing the degree of $u(x)$ and $v(x)$ accordingly.

1.2 Solved Exercises

Exercise 01

Solve the following first-order differential equation

1) $y'(x) - 4y(x) = 3$, **2)** $y'(x) - y(x) = 2e^x$, **3)** $x\, y'(x) + y(x) = x\sin x$

Solution

1) $y'(x) - 4y(x) = 3$:

Step 1: We first solve the associated homogeneous equation (i.e., **without the second member**)

$$y'(x) = 4y(x).$$

This can be written in differential form

$$\begin{aligned}\frac{dy}{dx} &= 4y(x) \quad \left(\times \frac{dx}{y}\right) \\ &\Longleftrightarrow \frac{1}{y}dy = 4dx.\end{aligned}$$

Integrating both sides

$$\begin{aligned}\int \frac{1}{y}dy &= \int 4dx \Longrightarrow \ln|y| = 4x + c_1 \\ &\Longrightarrow e^{\ln|y|} = e^{4x+c_1}.\end{aligned}$$

Since the absolute value can be absorbed into a constant, we write

$$y = c_2 e^{4x}, \quad \text{where } c_2 = \pm e^{c_1}.$$

Thus, the general solution of the homogeneous equation is

$$\boxed{y_h(x) = c_2 e^{4x}}$$

Step 2: Find a Particular Solution Using Variation of Constants. We assume a particular solution of the form:

$$y_p(x) = \phi(x) e^{4x},$$

where $\phi(x)$ is an unknown function to be determined.

Differentiate $y_p(x)$,

$$y_p'(x) = \phi'(x) e^{4x} + 4\phi(x) e^{4x}.$$

Substitute $y_p(x)$ and $y_p'(x)$ into the original equation

$$\phi'(x) e^{4x} + 4\phi(x) e^{4x} - 4\phi(x) e^{4x} = 3 \Leftrightarrow \phi'(x) e^{4x} = 3.$$

Solve for $\phi'(x)$

$$\phi'(x) = 3e^{-4x}.$$

Integrate

$$\int \phi'(x)\,dx = \int 3e^{-4x}dx \Leftrightarrow \phi(x) = -\frac{3}{4}e^{-4x} + c_3.$$

Thus

$$\boxed{\phi(x) = -\frac{3}{4}e^{-4x} + c_3}$$

So the particular solution becomes

$$\boxed{y_p(x) = \left(-\frac{3}{4}e^{-4x} + c_3\right)e^{4x}.}$$

The general solution of the differential equation is the sum of the homogeneous and particular solutions

$$y(x) = y_h(x) + y_p(x) = c_2e^{4x} + \left(-\frac{3}{4}e^{-4x} + c_3\right)e^{4x}.$$

Thus

$$\boxed{y(x) = Ce^{4x} - \frac{3}{4}, \text{ where } C = c_2 + c_3.}$$

2) $y'(x) - y(x) = 2e^x$:

Step 1: We first solve the associated homogeneous equation (i.e., **without the second member**)

$$y'(x) = y(x).$$

We rewrite this in differential form

$$\begin{aligned}\frac{dy}{dx} &= y(x) \quad \left(\times\frac{dx}{y}\right)\\ &\Longleftrightarrow \frac{1}{y}dy = dx.\end{aligned}$$

Integrating both sides

$$\begin{aligned}\int \frac{1}{y}dy &= \int dx \Longrightarrow \ln|y| = x + c_1\\ &\Longrightarrow e^{\ln|y|} = e^{x+c_1}\\ &\Longrightarrow y = c_2e^x, \quad \text{where } c_2 = \pm e^{c_1}\end{aligned}$$

Thus, the general solution of the homogeneous equation is

$$\boxed{y_h(x) = c_2 e^x}$$

Step 2: Solve the Non-Homogeneous Equation Using Variation of Constants. Assuming a particular solution of the form

$$y_p(x) = \phi(x)\, e^x,$$

Differentiate both sides

$$y_p'(x) = \phi'(x)\, e^x + \phi(x)\, e^{4x}.$$

Substitute $y_p(x)$ and $y_p'(x)$ into the original differential equation:

$$\phi'(x)\, e^x + \phi(x)\, e^x - \phi(x)\, e^{4x} = 2e^x \Leftrightarrow \phi'(x)\, e^x = 2e^x.$$

Solve for $\phi'(x)$

$$\phi'(x) = 2.$$

Integrate

$$\int \phi'(x)\, dx = \int dx \Longleftrightarrow \phi(x) = 2x + c_3.$$

Thus

$$\boxed{\phi(x) = 2x + c_3}$$

Therefore

$$\boxed{y_p(x) = (2x + c_3)\, e^x}$$

The general solution to the differential equation is the sum of the homogeneous and particular solutions

$$y(x) = y_h(x) + y_p(x) = c_2 e^x + (2x + c_3)\, e^x$$

Thus

$$\boxed{y(x) = (2x + C)\, e^{x}, \text{ where } C = c_2 + c_3.}$$

3) $x.y'(x) + y(x) = x \sin x$:

Step 1: Rewrite the equation in standard linear form.

We divide both sides of the equation by x (assuming $x \neq 0$)

$$y'(x) + \frac{1}{x} y(x) = \sin x.$$

This is a first-order linear differential equation of the form

$$\frac{dy}{dx} + P(x) y(x) = Q(x),$$

with

$$P(x) = \frac{1}{x}, \quad Q(x) = \sin x.$$

Step 2: Compute the integrating factor

The integrating factor $\mu(x)$ is given by:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|.$$

Assuming $x > 0$ (unless otherwise specified), we simplify to

$$\boxed{\mu(x) = x.}$$

Step 3: Multiply the equation by the integrating factor

Multiply both sides of the standard form by the integrating factor x, we get:$y'(x) + \frac{1}{x} y(x) = \sin x$

$$x.y'(x) + x \frac{1}{x} y(x) = x \sin x \Longrightarrow x.y'(x) + y(x) = x \sin x.$$

Now, observe that the left-hand side is the derivative of the product $xy(x)$

$$\frac{d}{dx}[xy(x)] = x.y'(x) + y(x).$$

Hence, the equation becomes

$$\frac{d}{dx}\left[xy\left(x\right)\right] = x\sin x$$

Step 4: Integrate both sides

$$\int \frac{d}{dx}\left[xy\left(x\right)\right]dx = \int x\sin x dx$$

The left-hand side simplifies directly

$$xy\left(x\right) = \int x\sin x dx$$

We now compute the integral on the right using integration by parts. Let:

$$\begin{aligned} u &= x \Longrightarrow du = dx, \\ v' &= \sin x \Longrightarrow v = -\cos x. \end{aligned}$$

Then

$$\int x\sin x dx = -x\cos x + \int \cos dx = -x\cos x + \sin x + c.$$

So

$$xy\left(x\right) = -x\cos x + \sin x + c$$

Step 5: Solve for $y\left(x\right)$

Divide both sides by x (again, assuming $x \neq 0$)

$$y\left(x\right) = -\cos x + \frac{\sin x}{x} + \frac{c}{x}$$

Therefore, the general solution is

$$\boxed{y\left(x\right) = -\cos x + \frac{\sin x}{x} + \frac{c}{x}}$$

Exercise 02

Solve the following IVP's (initial-value problems)

$$\textbf{1)} \begin{cases} y' = y\sin x + \sin x \\ y(c) = \delta \end{cases}, \quad \textbf{2)} \begin{cases} y' + 2y = e^{-\frac{x}{3}} \\ y(0) = 2 \end{cases}$$

Solution

1) $y' = y\sin x,\ y(c) = \delta$:

Step 1: We first solve the associated homogeneous equation (i.e., **without the second member**)

$$\begin{cases} y' = y\sin x, \\ y(c) = \delta. \end{cases}$$

We rewrite this in differential form

$$\frac{dy}{dx} = y\sin x \Longrightarrow \frac{1}{y}dy = \sin x dx$$

Integrating both sides from c to x

$$\begin{aligned} \int_c^x \frac{1}{y}dy &= \int_c^x \sin t dt \\ &\Longrightarrow [\ln y(t)]_c^x = [-\cos t]_c^x \\ &\Longrightarrow \ln y(x) - \ln y(c) = -\cos x + \cos c \\ &\Longrightarrow \ln y(x) - \ln \delta = -\cos x + \cos c \\ &\Longrightarrow \ln y(x) = \ln \delta - \cos x + \cos c \\ &\Longrightarrow e^{\ln y(x)} = e^{\ln \delta - \cos x + \cos c} = e^{\ln \delta} \times e^{\cos c - \cos x}. \end{aligned}$$

Thus, the solution of the homogeneous equation is

$$\boxed{y_h(x) = \delta e^{\cos c - \cos x}}$$

Step 2: Solve the non-homogeneous equation using the method of variation of constants. We look for a solution of the form

$$y = \phi(x)\, e^{\cos c - \cos x},$$

Differentiate both sides

$$y'(x) = \phi'(x)\, e^{\cos c - \cos x} + \phi(x) \sin x e^{\cos c - \cos x}.$$

Substitute $y(x)$ and $y'(x)$ into the original equation

$$\phi'(x)\, e^{\cos c - \cos x} = \sin x.$$

Thus

$$\phi'(x) = \sin x e^{\cos x - \cos c}.$$

Integrating both sides from c to x

$$\begin{aligned} \int_c^x \phi'(t)\, dt &= \int_c^x \sin t e^{\cos t - \cos c} dt \\ &\Longrightarrow [\phi(t)]_c^x = \left[-e^{\cos t - \cos c}\right]_c^x \\ &\Longrightarrow \phi(x) - \phi(c) = -e^{\cos x - \cos c} + e^{\cos c - \cos c}. \end{aligned}$$

Since $y(c) = \delta$, then

$$\phi(c)\, e^{\cos c - \cos c} = \delta \Longrightarrow \phi(c) = \delta.$$

Thus

$$\boxed{\phi(x) = \delta - e^{\cos x - \cos c} + 1}$$

Therefore, the general solution is

$$\boxed{y = -1 + (\delta + 1)\, e^{\cos c - \cos x}.}$$

2) $y' + 2y = e^{-\frac{x}{3}}$, $\phi(0) = 2$:

Step 1: Identify the type of equation

This is a first-order linear differential equation of the form

$$\frac{dy}{dx} + P(x) y(x) = Q(x),$$

with

$$P(x) = 2, \quad Q(x) = e^{-\frac{x}{3}}.$$

Step 2: Compute the integrating factor

The integrating factor $\mu(x)$ is given by:

$$\mu(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$$

Step 3: Multiply both sides by the integrating factor

Multiply the whole equation by $\mu(x) = e^{2x}$

$$e^{2x}y' + 2e^{2x}y = e^{2x}e^{-\frac{x}{3}}.$$

Simplify the right-hand side

$$e^{2x}e^{-\frac{x}{3}} = e^{2x-\frac{x}{3}} = e^{\frac{5}{3}x}.$$

Step 4: Left-hand side is a derivative

$$e^{2x}y' + 2e^{2x} = \frac{d}{dy}\left(e^{2x}y\right) = e^{\frac{5}{3}x}$$

Step 5: Integrate both sides

$$\int \frac{d}{dy}\left(e^{2x}y\right)dx = \int e^{\frac{5}{3}x}dx$$

So

$$e^{2x}y = \frac{3}{5}e^{\frac{5}{3}x} + c$$

Step 6: Solve for $y(x)$:

$$y(x) = e^{-2x}\left(\frac{3}{5}e^{\frac{5}{3}x} + c\right).$$

Step 7: Apply the initial condition

$$y(0) = e^{-2(0)}\left(\frac{3}{5}e^{\frac{5}{3}(0)} + c\right) \Longrightarrow 2 = c + \frac{3}{5} \Longrightarrow c = \frac{7}{5}.$$

The general solution is

$$\boxed{y(x) = e^{-2x}\left(\frac{3}{5}e^{\frac{5}{3}x} + \frac{7}{5}\right).}$$

Exercise 03

I. A pizza is taken out of the oven after it has finished baking, where the oven temperature is 350 $^\circ$**F**. The kitchen temperature is 75 $^\circ$**F**, and five minutes later, the pizza's temperature has decreased to 340 $^\circ$**F**. We wish to wait until the pizza cools to 300°**F** before cutting and serving it. How much additional time must we wait?

The kitchen temperature
is 75 $^\circ$F

the oven temperature
is 350 $^\circ$F

II. A cake is taken out of the oven after it has finished baking, where the oven temperature is 450°**F**. The ambient (room) temperature is 70°**F**, and after 10 minutes, the temperature of the cake has decreased to 430°**F**

a) Formulate the initial-value problem that models this situation.

b) Solve the initial-value problem to determine the temperature $T(t)$ of the cake as a function of time.

Solution

I. According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the difference between the object's temperature and the surrounding (ambient) temperature.

If $T(t)$ denotes the temperature of the object as a function of time t, then $\frac{dT}{dt}$ represents the rate of change of that temperature. Let T_s denotes the temperature of the surroundings, so $T_s = 75\ ^{\circ}\mathbf{F}$. The temperature of the pizza when it is removed from the oven is $350\ ^{\circ}\mathbf{F}$, which represents the initial temperature (or initial value), hence $T_0 = 350\ ^{\circ}\mathbf{F}$. Under these conditions, Newton's Law of Cooling can be written in the following form

$$\frac{dT}{dt} = kT - 75k,$$

with $T(0) = 350$.

For solving the obtained ODE, we use the variation of constants method. So we follow the following steps:

Step 1: Solving the associated equation without a second member

$$\frac{dT}{dt} = kT.$$

First, we rewrite this homogeneous equation as follows:

$$\frac{dT}{T} = k\ dt.$$

Next, we integrate both sides of the above equation

$$\int \frac{dT}{T} = \int k\ dt.$$

This gives

$$T_h(t) = C\ \exp(kt),$$

where C represents the constant of integration.

Step 2: Finding a particular solution T_p by using the the variation of the constant as follows:

$$T_p(t) = C(t)\ \exp(kt).$$

Therefore

$$\frac{dT_p}{dt} = T_p'(t) = C'(t) \ \exp(kt) + kC(t) \ \exp(kt).$$

The substitution in the original nonhomogeneous equation gives

$$C'(t) \ \exp(kt) + kC(t) \ \exp(kt) = k \ C(t) \ \exp(kt) - 75k,$$

and thus

$$C'(t) = -75k \ \exp(-kt).$$

Integrating both sides leads to

$$C(t) = 75 \ \exp(-kt).$$

Consequently

$$T_p(t) = 75.$$

Step 3: Deducing the general solution

We have

$$\begin{aligned} T(t) &= T_h(t) + T_p(t) \\ &= C \ \exp(kt) + 75, \end{aligned}$$

where C is an arbitrary constant.

We will now fix the value of the constant C by imposing the initial condition $T(0) = 350$. Indeed

$$T(0) = C + 75 = 350 \Longrightarrow C = 275.$$

Therefore the solution to the given initial value problem is

$$T(t) = 275 \ \exp(kt) + 75.$$

To determine the value of k, we need to use the fact that after 5 minutes the temperature of the pizza is 340 $^{\circ}$**F**. Therefore

$$T(5) = 275 \exp(5k) + 75 = 340,$$

which implies that

$$k = \frac{1}{5} \ln\left(\frac{53}{55}\right) \approx -0.007408.$$

As a result

$$T(t) = 275 \ \exp(-0.007408 \ t) + 75.$$

When is the temperature equals 300°**F**, we get

$$300 = 275 \ \exp(-0.007408 \ t) + 75,$$

this gives

$$\begin{aligned} t &= \left(\frac{-1}{0.007408}\right) \ln\left(\frac{225}{275}\right) \\ &= \left(\frac{-1}{0.007408}\right) \ln\left(\frac{9}{11}\right) \\ &\approx 27. \end{aligned}$$

Consequently, approximately 27 additional minutes are required after the pizza's temperature reaches 340°F before it can be cut and served.

II. a) **Initial-value problem**

Let $T(t)$ denote the cake's temperature (in Fahrenheit degree °**F**) at time t (in minutes) after removal from the oven. By Newton's Law of Cooling,

$$\frac{dT}{dt} = kT - T_s,$$

where T_s is the ambient temperature and k is the (constant) cooling rate. Here $T_s = 70$ °**F**. The initial temperature at $t = 0$ is the oven temperature:

$$T(0) = 450.$$

Thus the initial-value problem is

$$\begin{cases} \frac{dT}{dt} = kT - 70 \\ T(0) = 450 \end{cases}.$$

b) **Solve the initial-value problem:** This is a linear first-order ODE with constant coefficients. The general solution has the form

$$T(t) = Ce^{kt} + 70.$$

where C is a constant determined from the initial condition. Using $T(0) = 450$ gives

$$450 = C + 70 \Longrightarrow C = 380,$$

so

$$T(t) = 380\, e^{kt} + 70.$$

We determine k from the data $T(10) = 430$. Substituting $t = 10$ in the above equation, we get

$$380\, e^{10k} = 360,$$

and hence

$$e^{10k} = \frac{360}{380} = \frac{18}{19}.$$

We infer that

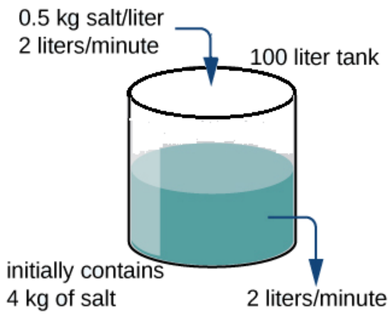
$$k = \frac{1}{10} \ln\left(\frac{18}{19}\right) = -5.4067 \times 10^{-3}.$$

Consequently

$$T(t) = 380e^{\left(-5.4067 \times 10^{-3}\right)t} + 70.$$

Exercise 04

A tank initially contains 100 L. of brine in which 4 kg of salt are dissolved. At time $t = 0$ a second brine solution begins to flow into the tank at a rate of 2 L./min.; this incoming solution has salt concentration 0.5 kg/L. Simultaneously, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate 2 L./min., so the liquid volume in the tank remains constant. Determine the amount of salt in the tank as a function of time, and compute the limiting amount of salt present in the tank as time approaches infinity, assuming that the solution remains perfectly mixed at all times.



Solution

Let $S(t)$ denote the amount of salt (in kg) in the tank at time t (minutes). The rate of change of salt equals the salt entering per minute minus the salt leaving per minute. So

$$\frac{dS}{dt} = \text{Inflow rate} - \text{Outflow rate}.$$

Since the solution flows into the tank at a rate of 2 L./min., and each liter contains 0.5 kg of salt, it follows that 1 kilogram of salt enters the tank every minute. Thus, the inflow rate is equal to 1 kg/min.

To determine the rate at which salt leaves the tank, we must first find the salt concentration in the tank at any given time. Because the amount of salt changes with time, the concentration also varies. However, the total volume of the solution remains constant at 100 liters. Hence, the concentration of salt in the tank is $\frac{S(t)}{100}$ kg/L., and since the mixture flows out at 2

L./min., the amount of salt leaving the tank each minute is $2\left(\frac{S(t)}{100}\right)$ kg/min. Consequently, the outflow rate is given by $\frac{S(t)}{50}$.

Therefore, the governing differential equation can be expressed as follows:

$$\frac{dS(t)}{dt} = 1 - \frac{S(t)}{50},$$

with the initial condition $S(0) = 4$. Thus, the process is modeled by the above initial value problem (IVP) for a first-order linear differential equation (a Cauchy problem).

The aim here is to determine $S(t)$, the amount of salt in the tank at any time t, and to find the limiting amount of salt in the tank as $t \longrightarrow \infty$, assuming the mixture remains perfectly uniform at all times. By virtue of the variation of constants method, we can solve the obtained IVP. So we pursue the following steps:

Step 1: Writing the ODE in linear standard form:

$$\frac{dS(t)}{dt} + \frac{S(t)}{50} = 1$$

Step 2: Solving the associated homogeneous equation

$$\frac{dS_h(t)}{dt} + \frac{S_h(t)}{50} = 0.$$

Integrating leads to

$$S_h(t) = C \ \exp\left(\frac{-t}{50}\right)$$

where C is an arbitrary constant.

Step 3: Seeking a particular solution by letting the constant vary as follows:

$$S_p(t) = C(t) \ \exp\left(\frac{-t}{50}\right).$$

The differentiation of $S_p(t)$ gives

$$S_p(t) = C'(t) \ \exp\left(\frac{-t}{50}\right) - \frac{1}{50} C(t) \ \exp\left(\frac{-t}{50}\right).$$

Now, let's substitute it in the original nonhomogeneous equation

$$C'(t)\ \exp\left(\frac{-t}{50}\right) - \frac{1}{50}C(t)\ \exp\left(\frac{-t}{50}\right) + \frac{C(t)}{50}\ \exp\left(\frac{-t}{50}\right) = 1.$$

Thus

$$C'(t) = \exp\left(\frac{t}{50}\right),$$

and hence

$$C(t) = 50\exp\left(\frac{t}{50}\right) + K$$

where K is a constant of integration. So we infer that

$$S_p(t) = 50 + K\exp\left(\frac{-t}{50}\right).$$

Step 4: Deducing the general solution

$$S(t) = S_h(t) + S_p(t) = A\ \exp\left(\frac{-t}{50}\right) + 50$$

where $A = C + K$ is an arbitrary constant. Moreover, the initial condition $S(0) = 4$ implies that

$$A = -46,$$

and hence

$$S(t) = 50 - 46\exp\left(\frac{-t}{50}\right).$$

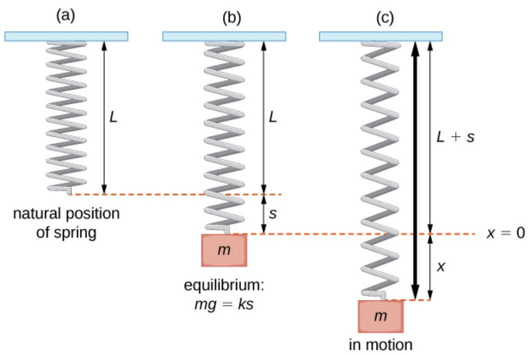
Now, to find the limiting amount of salt in the tank, we take the limit as $t \longrightarrow \infty$

$$\lim_{t\longrightarrow\infty} S(t) = \lim_{t\longrightarrow\infty}\left[50 - 46\exp\left(\frac{-t}{50}\right)\right] = 50.$$

So the amount of salt approaches 50 kg as t approaches infinity.

Exercise 05

Consider a mass hanging from a spring that is fixed to a rigid support (this setup is known as a spring-mass system.) The force of gravity pulls the mass downward, while the spring's restoring force acts upward. As illustrated in the figure below, when these two forces balance each other, the mass is said to be in equilibrium. If the mass is moved away from this equilibrium position, it begins to oscillate vertically, moving up and down around that point.



a) Formulate the mathematical equation that models this behavior.

b) Give the general solution.

Solution

a) Let m represent the mass (in slugs or kilograms), k indicate the spring constant (in lb./ft. or N/M.), g be the acceleration due to gravity (in ft./s^2 or m/s^2), and $x(t)$ stand for the displacement of the mass from its equilibrium position. In spring–mass systems of this type, it is customary to adopt the convention that downward motion is positive. Therefore, a positive displacement means the mass lies below the equilibrium position, while a negative displacement indicates it is above equilibrium. The mathematical model describing the motion of a spring–mass system is derived from Newton's Second Law:

$$\text{Sum of forces} = \text{gravity} + \text{the restoring force} = \text{mass} \times \text{acceleration}.$$

Step 1: Forces acting on the mass

1. Gravitational force (downward): $F_g = mg$

2. Spring restoring force (upward): By virtue of Hooke's law, the restoring force of the spring is proportional to the displacement and acts in the opposite direction from the displacement, so the restoring force is given by $F_s = -k\left(s + x\left(t\right)\right).$

Step 2: Motion about equilibrium

Applying Newton's second law, we get

$$m\ x''\left(t\right) = mg - k\left(s + x\left(t\right)\right) = mg - kx\left(t\right) - ks.$$

However, according to the equilibrium position ($mg = ks$), the differential equation becomes

$$x''\left(t\right) + \left(\frac{k}{m}\right)x\left(t\right) = 0,$$

which is the simple harmonic motion equation, describing the vertical oscillation of the mass about its equilibrium position where the angular frequency is $\omega = \sqrt{\frac{k}{m}}$.

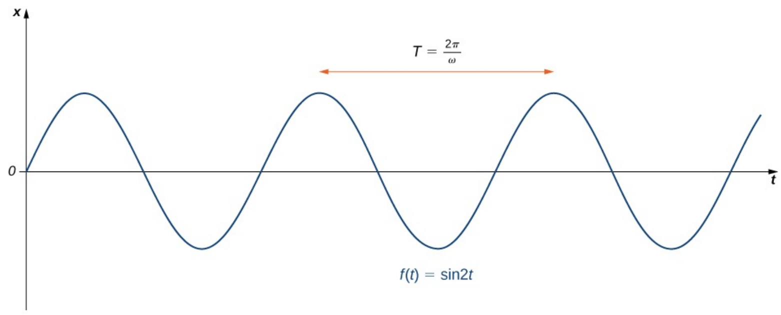
b) The characteristic equation is

$$r^2 + \omega^2 = 0,$$

which has complex conjugate roots. Thus

$$x\left(t\right) = C_1 \cos \omega t + C_2 \sin \omega t.$$

which gives the position of the mass at any point in time. The motion of the mass is called simple harmonic motion. The period of this motion (the time it takes to complete one oscillation) is $T = \frac{2\pi}{\omega}$ and the frequency is $f = \frac{1}{T} = \frac{\omega}{2\pi}$. The below graph denotes the vertical displacement versus time for simple harmonic motion.



Exercise 06

Solve the following differential equations using two different methods

1) $y'' - 4y' + 4y = 2(x-2)e^x$, **2)** $y'' + 5y' + 6y = -3x^2 + x - 2$,

3) $y'' - y = x^3 + x^2$, **4)** $y'' - 2y' + y = x^2 + x - 3$

Solution

1) $y'' - 4y' + 4y = 2(x-2)e^x$:

First Method: Variation of Parameters

Step 1: Solve the Associated Homogeneous Equation

Consider the associated homogeneous differential equation:

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

The discriminant is

$$\Delta = (-4)^2 - 4(4) = 0$$

Since the discriminant is zero, the equation has a repeated real root

$$r = \frac{4}{2} = 2$$

Since the characteristic equation has a repeated root r, the general solution is given by

$$y_h(x) = c_1 y_1(x) + c_2 x y_1(x)$$

Here, $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ are two linearly independent solutions. So

$$\boxed{y_h(x) = c_1 e^{2x} + c_2 x e^{2x}}$$

Step 2: Apply Variation of Parameters

We now apply the method of variation of parameters to find a particular solution to the original non-homogeneous equation.

We look for a particular solution of the form

$$y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x),$$

where the functions $c_1(x)$ and $c_2(x)$ satisfy the system

$$\begin{cases} c_1'(x) y_1(x) + c_2'(x) y_2(x) = 0 \\ c_1'(x) y_1'(x) + c_2'(x) y_2'(x) = f(x), \end{cases}$$

with

$$f(x) = 2(x-2)e^x$$

This system yields the standard formulas

$$c_1'(x) = \frac{W_1}{W}, \ c_2'(x) = \frac{W_2}{W},$$

Compute W, W_1 and W_2

Since

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}, \ W_1 = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix}, \ W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix}.$$

Then

$$\begin{aligned} W &= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x}, \\ W_1 &= \begin{vmatrix} 0 & xe^{2x} \\ 2(x-2)e^{x} & e^{2x}+2xe^{2x} \end{vmatrix} = -2xe^{3x}(x-2), \\ W_2 &= \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2(x-2)e^{x} \end{vmatrix} = 2e^{3x}(x-2). \end{aligned}$$

Now compute the derivatives of the parameters $c_1'(x)$ and $c_2'(x)$

$$\begin{aligned} c_1'(x) &= \frac{W_1}{e^{4x}} = \frac{-2xe^{3x}(x-2)}{e^{4x}} = -2x(x-2)e^{-x}, \\ c_2'(x) &= \frac{W_2}{e^{4x}} = \frac{2e^{3x}(x-2)}{e^{4x}} = 2(x-2)e^{-x}. \end{aligned}$$

Step 3: Integrate to find $c_1(x)$ and $c_2(x)$

Compute $c_1(x)$

$$c_1(x) = \int -2x(x-2)e^{-x}dx$$

We propose the following form for the integral

$$\int -2x(x-2)e^{-x}dx = \left(ax^2+bx+c\right)e^{-x}$$

By differentiating and comparing coefficients, we find

$$a = 2,\ b = 0,\ c = 0$$

So

$$\boxed{c_1(x) = 2x^2e^{-x} + c_3}$$

Compute $c_2(x)$

$$c_2(x) = \int 2(x-2)e^{-x}dx$$

Assume

$$\int 2(x-2)e^{-x}dx = (ax+b)e^{-x}$$

Identifying terms, we find

$$a = -2,\ b = 2$$

So

$$\boxed{c_2(x) = (-2x+2)\,e^{-x} + c_4}$$

Step 4: Construct the Particular Solution

Using

$$y_1(x) = e^{2x} \text{ and } y_2(x) = xe^{2x}$$

We now construct

$$y_p(x) = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$

Substituting the expressions

$$y_p(x) = \left(2x^2e^{-x} + c_3\right)e^{2x} + \left((-2x+2)\,e^{-x} + c_4\right)xe^{2x}$$

Simplifying

$$\boxed{y_p(x) = y_p(x) = c_3e^{2x} + 2xe^{x} + c_4xe^{2x}}$$

Step 5: General Solution

Adding the homogeneous and particular solutions

$$y(x) = y_h(x) + y_p(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{2x} + 2xe^{x} + c_4xe^{2x}$$

Grouping constants

$$\boxed{y(x) = k_1e^{2x} + k_2xe^{2x} + 2xe^{x}}$$

Second Method: Method of Undetermined Coefficients

This method applies when the non-homogeneous term is of the form $f(x) = p(x)\,e^{kx}$, where $p(x)$ is a polynomial.

Here, the non-homogeneous term is

$$p(x) = f(x) = 2(x-2)\,e^{x},\ \ k = 1$$

Since $k = 1$ is not a root of the characteristic equation $r^2 - 4r + 4 = 0$, we propose a particular solution of the form

$$y_p(x) = (ax + b)\, e^x$$

Step 1: Compute Derivatives

We compute the first and second derivatives

$$\begin{aligned} y_p'(x) &= ((ax + b)\, e^x)' = e^x (a + b + ax), \\ y_p''(x) &= (e^x (a + b + ax))' = e^x (2a + b + ax). \end{aligned}$$

Step 2: Substitute into the Equation

$$y'' - 4y' + 4y = e^x (2a + b + ax) - 4e^x (a + b + ax) + 4(ax + b)\, e^x$$

Simplifying

$$y'' - 4y' + 4y = e^x (b - 2a + ax).$$

We want this to match the non-homogeneous term

$$2(x - 2)\, e^x = (2x - 4)\, e^x$$

Matching coefficients

$$a = 2, \;\; b = 0$$

Thus, the particular solution is

$$\boxed{y_p(x) = 2xe^x}$$

Step 3: General Solution

Adding to the homogeneous solution

$$y(x) = y_h(x) + y_p(x) = c_1 e^{2x} + c_2 x e^{2x} + 2xe^x$$

The general solution is

$$\boxed{y(x) = c_1 e^{2x} + c_2 x e^{2x} + 2x e^{x}}$$

2) $y'' + 5y' + 6y = -3x^2 + x - 2$:

First Method: Variation of Parameters

Step 1: Solve the Associated Homogeneous Equation. We first consider the corresponding homogeneous equation

$$y'' + 5y' + 6y = 0.$$

This is a linear differential equation with constant coefficients. We solve it by finding the roots of the characteristic equation

$$r^2 + 5r + 6 = 0.$$

Compute the discriminant

$$\Delta = 5^2 - 4(6)(1) = 25 - 24 = 1 > 0$$

Since the discriminant is positive, the characteristic equation has two distinct real roots

$$r_1 = \frac{-5 + \sqrt{1}}{2} = -2, \quad r_2 = \frac{-5 - \sqrt{1}}{2} = -3$$

Therefore, the general solution of the homogeneous equation is

$$\boxed{y_h(x) = c_1 e^{-2x} + c_2 e^{-3x}}$$

Step 2: Solve the Nonhomogeneous Equation Using Variation of Parameters

We now find a particular solution $y_p(x)$ using the method of variation of parameters. Let

$$y_p(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x),$$

where

$$y_1(x) = e^{r_1 x} = e^{-2x} \text{ and } y_2(x) = e^{r_2 x} = e^{-3x},$$

and $c_1(x)$, $c_2(x)$ are functions to be determined. To find $c_1(x)$ and $c_2(x)$, we use the following formulas

$$c_1'(x) = \frac{W_1}{W},\ c_2'(x) = \frac{W_2}{W},$$

where

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix},\ W_1 = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix},\ W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix},$$

with

$$f(x) = -3x^2 + x - 2$$

Compute the Wronskian

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x}.$$

Compute W_1 and W_2

$$\begin{aligned} W_1 &= \begin{vmatrix} 0 & e^{-3x} \\ -3x^2 + x - 2 & -3e^{-3x} \end{vmatrix} = \left(3x^2 - x + 2\right) e^{-3x}, \\ W_2 &= \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & -3x^2 + x - 2 \end{vmatrix} = -\left(3x^2 - x + 2\right) e^{-2x}. \end{aligned}$$

Integrate to find $c_1(x)$ and $c_2(x)$

$$\begin{aligned} c_1'(x) &= \frac{W_1}{W} = \frac{\left(3x^2 - x + 2\right) e^{-3x}}{-e^{-5x}} = -\left(3x^2 - x + 2\right) e^{2x} \\ &\Longrightarrow c_1(x) = \int -\left(3x^2 - x + 2\right) e^{2x} dx. \end{aligned}$$

So

$$\boxed{c_1(x) = -\tfrac{1}{2}\left(3x^2 - 4x + 4\right) e^{2x} + c_3}$$

$$\begin{aligned} c_2 4(x) &= \frac{W_2}{W} = \frac{(3x^2 - x + 2)\, e^{-2x}}{e^{-5x}} = \left(3x^2 - x + 2\right) e^{3x} \\ &\Longrightarrow\ c_2(x) = \int \left(3x^2 - x + 2\right) e^{3x}. \end{aligned}$$

So

$$\boxed{c_2(x) = (x^2 - x + 1)\, e^{3x} + c_4}$$

Construct the particular solution

$$\begin{aligned} y_p(x) &= c_1(x)\, y_1(x) + c_2(x)\, y_2(x) \\ &= \left(-\frac{1}{2}\left(3x^2 - 4x + 4\right) e^{2x} + c_3\right) e^{-2x} \\ &\quad + \left(\left(x^2 - x + 1\right) e^{3x} + c_4\right) e^{-3x}. \end{aligned}$$

So

$$\boxed{y_p(x) = -\frac{1}{2}x^2 + x - 1 + c_3 e^{-2x} + c_4 e^{-3x}}$$

Step 3: General Solution

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^{-2x} + c_2 e^{-3x} - \frac{1}{2}x^2 + x - 1 + c_3 e^{-2x} + c_4 e^{-3x}. \end{aligned}$$

So

$$\boxed{y(x) = -\frac{1}{2}x^2 + x - 1 + c_5 e^{-2x} + c_6 e^{-3x}}$$

Second Method: Method of Undetermined Coefficients

We observe that the right-hand side of the equation is a polynomial

$$f(x) = -3x^2 + x - 2$$

Since the nonhomogeneous term is a polynomial of degree 2 and does not involve an exponential factor (i.e., e^{kx} with $k = 0$), and $k = 0$ is not a root of the characteristic equation, we try a particular solution of the same degree

$$y_p(x) = ax^2 + bx + c.$$

Compute derivatives

$$\begin{aligned} y_p'(x) &= \left(ax^2 + bx + c\right)' = 2ax + b \\ y_p''(x) &= (2ax + b)' = 2a. \end{aligned}$$

Substitute into the original equation

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= 2a + 5(2ax + b) + 6\left(ax^2 + bx + c\right) \\ &= 6ax^2 + (10a + 6b)x + 2a + 5b + 6c \end{aligned}$$

Match with the right-hand side

$$6ax^2 + (10a + 6b)x + 2a + 5b + 6c = -3x^2 + x - 2.$$

By comparing coefficients

$$\begin{cases} 6a = -3 \\ 10a + 6b = 1 \\ 2a + 5b + 6c = -2 \end{cases},$$

From the first equation $a = -\frac{1}{2}$.

Substitute into the second

$$10\left(-\frac{1}{2}\right) + 6b = 1 \Longrightarrow b = 1$$

Substitute a, b into the third

$$2\left(-\frac{1}{2}\right) + 5(1) + 6c = -2 \Longrightarrow c = -1$$

Hence, the particular solution is

$$\boxed{y_p(x) = -\tfrac{1}{2}x^2 + x - 1}$$

The general solution is

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-2x} + c_2 x e^{-3x} - \frac{1}{2}x^2 + x - 1$$

So

$$\boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-3x} - \tfrac{1}{2}x^2 + x - 1}.$$

3) $y'' - y = x^3 + x^2$:

First Method: Variation of Parameters

Step 1: Solve the Associated Homogeneous Equation

We begin with the homogeneous part of the equation

$$y'' - y = 0$$

The characteristic equation is

$$r^2 - 1 = 0 \Longrightarrow r_1 = 1,\ r_2 = -1$$

Thus, the general solution to the homogeneous equation is

$$\boxed{y_h(x) = c_1 e^{x} + c_2 e^{-x}}$$

Step 2: Solve the Nonhomogeneous Equation

We apply the method of variation of parameters. The particular solution is sought in the form

$$y_p(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

To find the function $c_1(x)$ and $c_2(x)$, we use

$$c_1'(x) = \frac{W_1}{W},\ c_2'(x) = \frac{W_2}{W},$$

where

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix},\ W_1 = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix},\ W_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix},$$

with

$$\begin{aligned} y_1(x) &= e^x, \; y_2(x) = e^{-x} \\ f(x) &= x^3 + x^2. \end{aligned}$$

Compute W, W_1, and W_2

$$\begin{aligned} W &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2e^x e^{-x}, \\ W_1 &= \begin{vmatrix} 0 & e^{-x} \\ x^3 + x^2 & -e^{-x} \end{vmatrix} = -x^2 e^{-x} - x^3 e^{-x}, \\ W_2 &= \begin{vmatrix} e^x & 0 \\ e^x & x^3 + x^2 \end{vmatrix} = x^2 e^x + x^3 e^x. \end{aligned}$$

Expressions for $c_1'(x)$ and $c_2'(x)$

$$\begin{aligned} c_1'(x) &= \frac{W_1}{W} = \frac{-x^2 e^{-x} - x^3 e^{-x}}{-2e^x e^{-x}} = \frac{1}{2} x^2 e^{-x}(x+1), \\ c_2'(x) &= \frac{W_2}{W} = \frac{x^2 e^x + x^3 e^x}{-2e^x e^{-x}} = -\frac{1}{2} x^2 e^x (x+1). \end{aligned}$$

Step 3: Integration

Compute $c_1(x)$

$$c_1(x) = \int \frac{1}{2} x^2 e^{-x}(x+1)\, dx$$

Let us find a function of the form

$$\left(ax^3 + bx^2 + cx + d\right) e^{-x}$$

whose derivative matches the integrand. After identification, we find

$$a = -\frac{1}{2}, b = -2, c = -4, d = -4$$

Thus

$$\boxed{c_1(x) = -\left(\tfrac{1}{2}x^3 + 2x^2 + 4x + 4\right) e^{-x} + c_3}$$

Compute $c_2(x)$

$$c_2(x) = \int -\frac{1}{2}x^2 e^x (x+1)\, dx$$

We seek a function of the form

$$\left(ax^3 + bx^2 + cx + d\right) e^x$$

whose derivative matches the integrand. After identification, we obtain

$$a = -\frac{1}{2}, b = 1, c = -2, d = 2$$

Thus

$$\boxed{c_2(x) = \left(-\tfrac{1}{2}x^3 + x^2 - 2x + 2\right) e^x + c_4}$$

Step 4: Construct the Particular Solution

$$\begin{aligned} y_p(x) &= c_1(x) y_1(x) + c_2(x) y_2(x) \\ &= \left(-\left(\frac{1}{2}x^3 + 2x^2 + 4x + 4\right) e^{-x} + c_3\right) e^x + \left(\left(-\frac{1}{2}x^3 + x^2 - 2x + 2\right) e^x + c_4\right) e^{-x} \\ &= -\left(\frac{1}{2}x^3 + 2x^2 + 4x + 4\right) + c_3 e^x - \frac{1}{2}x^3 + x^2 - 2x + 2 + c_4 e^{-x} \end{aligned}$$

So

$$\boxed{y_p(x) = -x^3 - x^2 - 6x - 2 + c_3 e^x + c_4 e^{-x}}$$

Step 5: General Solution

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 e^x + c_2 e^{-x} - x^3 - x^2 - 6x - 2 + c_3 e^x + c_4 e^{-x} \end{aligned}$$

Renaming constants

$$\boxed{y(x) = -x^3 - x^2 - 6x - 2 + c_5 e^x + c_6 e^{-x}}$$

Second Method: Method of Undetermined Coefficients

Step 1: Homogeneous Solution

As established

$$\boxed{y_h(x) = c_1 e^x + c_2 e^{-x}}$$

Step 2: Particular Solution

Since the right-hand side is a polynomial

$$f(x) = x^3 + x^2,$$

and does not involve any exponential factor, we propose a particular solution of the form

$$y_p(x) = ax^3 + bx^2 + cx + d$$

Compute derivatives

$$\begin{aligned} y_p'(x) &= \left(ax^3 + bx^2 + cx + d\right)' = 3ax^2 + 2bx + c, \\ y_p''(x) &= \left(3ax^2 + 2bx + c\right)' = 2b + 6ax. \end{aligned}$$

Substitute into the original equation

$$y_p'' - y_p = -ax^3 - bx^2 + (6a - c)x + 2b - d$$

Match with

$$x^3 + x^2$$

By identification

$$\begin{cases} a = -1 \\ b = -1 \\ 6a - c = 0 \\ 2b - d = 0 \end{cases}$$

Thus

$$\boxed{y_p(x) = -x^3 - x^2 - 6x - 2}$$

Final General Solution

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-x} - x^3 - x^2 - 6x - 2.$$

So

$$\boxed{y_G(x) = c_1 e^x + c_2 e^{-x} - x^3 - x^2 - 6x - 2}$$

4) $y'' - 2y' + y = x^2 + x - 3$:

First Method: Method of Undetermined Coefficients

Step 1: Solve the Associated Homogeneous Equation

We begin by considering the corresponding homogeneous equation:

$$y'' - 2y' + y = 0.$$

The associated characteristic equation is

$$r^2 - 2r + 1 = 0.$$

We compute the discriminant

$$\triangle = (-2)^2 - 4 = 0$$

Since the discriminant is zero, the characteristic equation has a repeated real root

$$r = 1$$

Therefore, the general solution to the homogeneous equation is

$$\boxed{y_h(x) = c_1 e^x + c_2 x e^x}.$$

Step 2: Solve the Nonhomogeneous Equation

We now turn to solving the full equation

$$y'' - 2y' + y = x^2 + x - 3$$

We observe that the nonhomogeneous term is a polynomial of degree 2, and it can be written as

$$f(x) = x^2 + x - 3$$

Since $e^{0.x} = 1$ and the exponential factor $k = 0$ is not a root of the characteristic equation, we look for a particular solution of the form

$$y_p(x) = ax^2 + bx + c.$$

Compute the derivatives

$$\begin{aligned} y_p'(x) &= \left(ax^2 + bx + c\right)' = 2ax + b, \\ y_p''(x) &= (2ax + b)' = 2a. \end{aligned}$$

Substitute into the left-hand side of the differential equation

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2a - 2(2ax + b) + ax^2 + bx + c \\ &= ax^2 + (b - 2a)x + 2a - 2b + c \end{aligned}$$

Now equate with the right-hand side of the original equation

$$ax^2 + (b - 2a)x + 2a - 2b + c = x^2 + x - 3$$

Step 3: Match Coefficients

By comparing the coefficients of like powers of x, we obtain the following system

$$\begin{cases} a = 1 \\ b - 4a = 1 \\ 2a - 2b + c = -3 \end{cases}$$

Step 4: Solve the System

From the first equation

$$a = 1$$

Substitute into the second

$$b - 4(1) = 1 \Longrightarrow b = 5$$

Substitute $a = 1$ and $b = 5$ into the third

$$2(1) - 2(5) + c = -3 \Longrightarrow c = 5$$

Hence, the particular solution is

$$\boxed{y_p(x) = x^2 + 5x + 5}$$

Step 5: General Solution

The general solution to the original nonhomogeneous differential equation is the sum of the homogeneous and particular solutions

$$\boxed{y(x) = c_1 e^{rx} + c_2 x e^{rx} + x^2 + 5x + 5}.$$

Second Method: Method of Variation of Parameters

Step 1: General Solution of the Homogeneous Equation

We already found that the homogeneous equation

$$\boxed{y_h(x) = c_1 e^x + c_2 x e^x}.$$

Step 2: Setup for Variation of Parameters

Let

$$y_p(x) = c_1(x) e^x + c_2(x) x e^x = c_1(x) e^x + c_2(x) x e^x,$$

where

$$y_1(x) = e^x, \quad y_2(x) = x e^x$$

By applying the same method as in the previous equations, we obtain

$$\begin{aligned}
W(x) &= e^{2x},\ W_1 = -x\left(x^2 + x - 3\right) e^x,\ W_2 = \left(x^2 + x - 3\right) e^x, \\
c_1'(x) &= -x\left(x^2 + x - 3\right) e^{-x},\ c_2'(x) = \left(x^2 + x - 3\right) e^{-x}, \\
c_1(x) &= \left(x^3 + 4x^2 + 5x + 5\right) e^{-x},\ c_2(x) = -x(x + 3) e^{-x},
\end{aligned}$$

$$\boxed{y_p(x) = x^2 + 5x + 5},$$

and

$$\boxed{y(x) = c_1 e^{rx} + c_2 x e^{rx} + x^2 + 5x + 5}.$$

Exercise 07

Solve the following initial value problems:

1.

$$\begin{cases} y''(x) + 2y'(x) + 4y(x) = x\ e^x \\ y(0) = 1 \text{ and } y(1) = 0. \end{cases}$$

2.

$$\begin{cases} y''(x) - 2y'(x) + (1+m^2)\, y(x) = (1+4m^2)\cos(mx) \\ y(0) = 1 \text{ and } y'(0) = 0. \end{cases}$$

Consider and discuss both cases: when $m = 0$ and when $m \neq 0$.

Solution

1. We start by solving the homogeneous equation. Its characteristic equation is $r^2 + 2r + 4 = 0$, whose discriminant is -12 and the roots are $r_1 = -1 - \sqrt{3}i$ and $r_2 = -1 + \sqrt{3}i$.

The solutions of the homogeneous equation are therefore functions of the form

$$\boxed{y_h(x) = C_1 \left(\cos\sqrt{3}x\right) e^{-x} - C_2 \left(\sin\sqrt{3}x\right) e^{-x},}$$

where $C_1, C_2 \in \mathbb{R}$.

Let us now find a solution of the equation with the right-hand side. We can seek for a solution of the form

$$y_p(x) = (ax+b)\ e^x.$$

Thus

$$\begin{aligned} y''(x) + 2y'(x) + 4y(x) &= 7ax\ e^x + 7\ b\ e^x + 4a\ e^x \\ &= (7ax + 7\ b + 4a)\ e^x. \end{aligned}$$

After identifying the coefficients, we find a particular solution for $a = \frac{1}{7}$ and $b = \frac{-4}{49}$. The solutions of the differential equation are therefore functions of the form

$$\boxed{y(x) = C_1 \left(\cos\sqrt{3}x\right) e^{-x} - C_2 \left(\sin\sqrt{3}x\right) e^{-x} + \tfrac{x\ e^x}{7} - \tfrac{4\ e^x}{49},}$$

with $C_1, C_2 \in \mathbb{R}$. Now, if we seek for a solution verifying $y(0) = 1$ and $y(1) = 0$, we must have

$$C_1 = \frac{53}{49} \text{ and } C_2 = \frac{-53\cos\sqrt{3} - 3e^2}{49\sin\sqrt{3}}.$$

Finally

$$\boxed{y(x) = \tfrac{53}{49}\left(\cos\sqrt{3}x\right) e^{-x} - \left(\tfrac{-53\cos\sqrt{3} - 3e^2}{49\sin\sqrt{3}}\right)\left(\sin\sqrt{3}x\right) e^{-x} + \tfrac{x\ e^x}{7} - \tfrac{4\ e^x}{49}.}$$

2. We begin by considering the case $m = 0$, where the differential equation becomes

$$y''(x) - 2y'(x) + y(x) = 1.$$

Its characteristic equation is

$$r^2 - 2r + 1 = 0,$$

which has a double root $r = 1$. But, we know from the course that there is a unique solution to the initial value problem which is

$$\boxed{y(x) = 1.}$$

Let's now consider the case $m \neq 0$. Its characteristic equation has a discriminant equals to $-4m^2$ whose roots are $\pm 2im$. Solutions of the characteristic equation are then $r_1 = 1 - im$ and $r_2 = 1 + im$ and the solutions of the homogeneous equation are therefore functions of the form

$$\boxed{y_h(x) = C_3(\cos mx)\, e^x + C_4(\sin mx)\, e^x,}$$

where $C_3, C_4 \in \mathbb{R}$. Let us now find a particular solution in the form

$$\boxed{y_p(x) = a_1(\cos mx) + b_1(\sin mx).}$$

It follows that

$$\begin{aligned} y_p''(x) - 2y_p'(x) + \left(1+m^2\right) y_p(x) &= -a_1 m^2 (\cos mx) - b_1 m^2 (\sin mx) \\ &\quad -2b_1 m (\cos mx) + 2a_1 m (\sin mx) \\ &\quad + \left(1+m^2\right) a_1 (\cos mx) + \left(1+m^2\right) b_1 (\sin mx), \end{aligned}$$

which leads

$$\begin{cases} -a_1 m^2 - 2b_1 m + (1+m^2)\, a_1 = 1 + 4m^2 \\ -b_1 m^2 + 2a_1 m + (1+m^2)\, b_1 = 0 \end{cases}.$$

Solving this system gives

$$a_1 = 1 \text{ and } b_1 = -2m.$$

The solutions to the equation are therefore functions of the form

$$\boxed{y(x) = C_3(\cos mx)\, e^x + C_4(\sin mx)\, e^x + (\cos mx) - 2m(\sin mx).}$$

The condition $y(0) = 1$ gives $C_3 = 1$ and while the condition $y'(0) = 0$ gives $C_4 = 2m$.

The unique solution to the initial value problem is therefore

$$\boxed{y(x) = (\cos mx)\, e^x + 2m(\sin mx)\, e^x + (\cos mx) - 2m(\sin mx).}$$

CHAPTER 2

Exercises related to the chapter: Matrices, Determinants, Diagonalization, and Linear Systems

Contents

This chapter presents a set of exercises with solutions covering matrices, determinants, and inverses. It also focuses on linear transformations, eigenvalues, eigenvectors, diagonalization, as well as some methods for solving systems of linear equations including Gaussian elimination, Cramer's Rule, and matrix inversion. Detailed solutions emphasize both correct procedures and key concepts, supporting a strong foundational understanding.

2.1 Course Recap: Matrices, Determinants, Diagonalization, and Linear Systems

2.1.1 Definition of a Matrix

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix with m rows and n columns is said to be of order $m \times n$. Elements of a matrix are usually denoted by a_{ij}, where i is the row index and j is the column index.

2.1.2 Common Types of Matrices

- **Zero Matrix**: A matrix where all elements are zero.
- **Identity Matrix**: A square matrix where all diagonal entries are 1 and all off-diagonal entries are 0. Denoted by I_n.
- **Diagonal Matrix**: A square matrix where all non-diagonal entries are zero.
- **Upper Triangular Matrix**: A square matrix where all entries below the main diagonal are zero.
- **Lower Triangular Matrix**: A square matrix where all entries above the main diagonal are zero.
- **Symmetric Matrix**: A matrix A such that $A^T = A$.
- **Skew-symmetric Matrix**: A matrix A such that $A^T = -A$.
- **Orthogonal Matrix**: A square matrix Q such that $Q^T Q = I$.

- **Unitary Matrix**: A complex matrix U such that $U^*U = I$, where U^* is the conjugate transpose.

2.1.3 Matrix Operations

Let A and B be two matrices of the same size, and $c \in \mathbb{R}$.

- **Addition**: $(A + B)_{ij} = A_{ij} + B_{ij}$
- **Scalar multiplication**: $(cA)_{ij} = c \cdot A_{ij}$
- **Matrix multiplication**: If A is of size $m \times n$ and B is of size $n \times p$, then AB is defined and has size $m \times p$.
- **Transpose**: $(A^T)_{ij} = A_{ji}$
- **Conjugate transpose**: For complex matrices, denoted by A^*

2.1.4 Matrix Associated to a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a unique matrix $A \in M_{m\times n}(\mathbb{R})$ such that for all $x \in \mathbb{R}^n$,

$$T(x) = Ax.$$

This matrix is called the **matrix representation** of T in the standard bases.

2.1.5 Rank of a Matrix

The **rank** of a matrix A, denoted $\text{rank}(A)$, is the maximum number of linearly independent rows or columns of A.

Theorem (Rank Theorem):

$$\text{rank}(A) \leq \min(m, n).$$

2.1.6 Inverse of a Matrix

A square matrix A is **invertible** if there exists a matrix B such that:

$$AB = BA = I.$$

The matrix B is called the **inverse** of A, denoted A^{-1}.

Theorem: A matrix is invertible if and only if its determinant is non-zero and its rank is full.

2.1.7 Determinants

Properties:

- $\det(I_n) = 1$
- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(A^T) = \det(A)$
- Row swap changes the sign of determinant.

2.1.8 Inverse Using Determinants

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

2.1.9 Rank Using Determinants

If $\det(A) \neq 0$, then $\text{rank}(A) = n$.

Eigenvalues and Eigenvectors

λ is an **eigenvalue** of A if $Av = \lambda v$ for some non-zero vector v.

Find λ by solving $\det(A - \lambda I) = 0$.

Finding Eigenvalues

$$\det(A - \lambda I) = 0$$

Finding Eigenvectors

$$(A - \lambda I)v = 0$$

Diagonalizable Matrices

A is **diagonalizable** if $A = PDP^{-1}$ for some invertible P and diagonal D.

Theorem: A is diagonalizable iff it has n linearly independent eigenvectors.

2.2 Solved Exercises

Exercise 01

Let

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^3,$$

and

$$w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2.$$

1. Show that the vectors v_1, v_2, v_3 form a basis $\mathcal{B}$ of $\mathbb{R}^3$, and that the vectors w_1, w_2 form a basis $\mathcal{B}'$ of $\mathbb{R}^2$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + z \\ 2y + 3z \end{pmatrix}$$

a. Determine the matrix associated with f relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

b. Write the matrix associated with f relative to the $\mathcal{B}$ and $\mathcal{B}'$.

3. Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$g(v_1) = w_1, \quad g(v_2) = 2w_2, \quad g(v_3) = w_1 + 3w_2.$$

a. Write the matrix of g relative to the bases $\mathcal{B}$ of $\mathbb{R}^3$ and $\mathcal{B}'$ of $\mathbb{R}^2$.

b. Write the matrix of g relative to standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

Solution

1. We show that the vectors v_1, v_2, v_3 form a basis $\mathcal{B}$ of $\mathbb{R}^3$, and that the vectors w_1, w_2 form a basis $\mathcal{B}'$ of $\mathbb{R}^2$:

- **Showing that** $\{v_1, v_2, v_3\}$ **form a basis of** $\mathbb{R}^3$: To do this, we will verify two properties:

- The vectors v_1, v_2, v_3 are linearly independent,
- and they span $\mathbb{R}^3$.

The linear independence: We want to determine whether the only solution to the vector equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3},$$

is the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Substitute the vectors

$$\alpha_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Adding the vectors component-wise, we obtain the system

$$\begin{cases} -\alpha_1 + \alpha_2 + \alpha_3 = 0 \ldots (1) \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \ldots (2) \\ \alpha_1 + \alpha_2 - \alpha_3 = 0 \ldots (3) \end{cases}$$

From the first equation (1)

$$-\alpha_1 + \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_2 + \alpha_3 \ldots (4)$$

Substitute equation (4) into equations (2) and (3)

$$\begin{cases} (\alpha_2 + \alpha_3) - \alpha_2 + \alpha_3 = 0 \\ (\alpha_2 + \alpha_3) + \alpha_2 - \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} 2\alpha_3 = 0 \\ 2\alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = 0 \\ \alpha_2 = 0 \end{cases}$$

Now substitute $\alpha_2 = 0$ and $\alpha_3 = 0$ back into equation (4)

$$\alpha_1 = 0 + 0 = 0.$$

Hence, the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which confirms that the vectors v_1, v_2, v_3 are linearly independent.

The Span: Since we have three linearly independent vectors in $\mathbb{R}^3$, they automatically span $\mathbb{R}^3$(because the dimension of $\mathbb{R}^3$ is 3). Therefore, the set $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$.

- **Showing that $\{w_1, w_2\}$ form a basis of $\mathbb{R}^2$:**

The linear independence: We examine whether

$$\beta_1 w_1 + \beta_2 v_2 = 0_{\mathbb{R}^2},$$

only has the trivial solution $\beta_1 = \beta_2 = 0$. Substitute the vectors

$$\beta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Adding the vectors component-wise, we obtain the system

$$\begin{cases} \beta_2 = 0 \\ \beta_1 + \beta_2 = 0 \end{cases} \Longrightarrow \beta_2 = 0, \ \beta_1 = 0.$$

Thus, w_1 and w_2 are linearly independent.

The Span: There are 2 linearly independent vectors in $\mathbb{R}^2$, and the dimension of $\mathbb{R}^2$ is 2. Therefore, they span $\mathbb{R}^2$. Hence, the set $\{w_1, w_2\}$ forms a basis of $\mathbb{R}^2$.

1. a. The matrix associated with f relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$: To determine the matrix associated with the linear transformation f relative to the standard bases of $\mathbb{R}^3$ (the domain) and $\mathbb{R}^2$ (the codomain), we begin by considering the general structure of such a matrix. Since the codomain of the transformation is $\mathbb{R}^2$, which has dimension 2, the associated matrix must have 2 rows. Similarly, because the domain of the transformation is $\mathbb{R}^3$, which has dimension 3, the associated matrix must have 3 columns.

Therefore, the associated matrix $[f]_{\text{std}}$ of the transformation f, with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, has the general form

$$\boxed{[f]_{\text{std}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.}$$

Let the standard basis vectors of $\mathbb{R}^3$ be

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now compute

$$
\begin{aligned}
f(e_1) &= f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+0\\2(0)+3(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\\
f(e_2) &= f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0+0\\2(1)+3(0)\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}\\
f(e_3) &= f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0+1\\2(0)+3(1)\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}
\end{aligned}
$$

We construct the matrix $[f]_{\text{std}}$ by placing the results of $f(e_1)$, $f(e_2)$, and $f(e_3)$ as columns. Therefore, the matrix associated with the linear transformation f, relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, is

$$
\boxed{[f]_{\text{std}} = \begin{pmatrix}1 & 0 & 1\\0 & 2 & 3\end{pmatrix}.}
$$

2. a. The matrix of f relative to bases of $\mathcal{B}$ in $\mathbb{R}^3$ and $\mathcal{B}'$ in $\mathbb{R}^2$: We compute $f(v_1), f(v_2), f(v_3)$, then express each result as a linear combination of w_1, w_2, and use the coefficients to form the columns of the matrix.

We have

$$\begin{aligned}
f(v_1) &= f\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ 2(1)+3(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\
f(v_1) &= f\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 2(-1)+3(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
f(v_1) &= f\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+(-1) \\ 2(1)+3(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{aligned}$$

$$\begin{cases} f(v_1) = a_{11}w_1 + a_{21}w_2 \\ f(v_2) = a_{12}w_1 + a_{22}w_2 \\ f(v_1) = a_{13}w_1 + a_{23}w_2 \end{cases}$$

Adding the vectors component-wise, we obtain

$$\begin{aligned}
\begin{pmatrix} 0 \\ 5 \end{pmatrix} &= a_{11}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_{21}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{21} \\ a_{11}+a_{21} \end{pmatrix} \\
\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= a_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_{22}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{22} \\ a_{12}+a_{22} \end{pmatrix} \\
\begin{pmatrix} 0 \\ -1 \end{pmatrix} &= a_{13}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_{23}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{23} \\ a_{13}+a_{23} \end{pmatrix}
\end{aligned}$$

which gives the following systems

$$\begin{cases} a_{21} = 0 \\ a_{11} + a_{21} = 5 \end{cases} \implies a_{11} = 5, a_{21} = 0,$$

$$\begin{cases} a_{22} = 2 \\ a_{12} + a_{22} = 1 \end{cases} \implies a_{12} = -1, a_{22} = 2,$$

$$\begin{cases} a_{23} = 0 \\ a_{13} + a_{23} = -1 \end{cases} \Longrightarrow a_{13} = -1, a_{23} = 0,$$

Thus

$$\boxed{[f]_{\mathcal{B}' \longleftarrow \mathcal{B}} = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix}.}$$

3.a. The Matrix of g relative to the bases of $\mathcal{B}$ and $\mathcal{B}'$: We have

$$f(v_1) = w_1, \quad f(v_2) = 2w_2, \quad f(v_3) = w_1 + 3w_2.$$

We write

$$\begin{cases} f(v_1) = 1.w_1 + 0.w_2 \\ f(v_2) = 0.w_1 + 2.w \\ f(v_3) = 1.w_1 + 3.w_2 \end{cases}$$

So

$$\boxed{[g]_{\mathcal{B}' \longleftarrow \mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}.}$$

3.b. The Matrix of g relative to standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$: Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. We want to express X in terms of the basis $\mathcal{B} = \{v_1, v_2, v_3\}$. That is, we want to find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$X = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v$$

Adding the vectors component-wise leads to the system

$$\begin{cases} -\alpha_1 + \alpha_2 + \alpha_3 = x \ldots (1) \\ \alpha_1 - \alpha_2 + \alpha_3 = y \ldots (2) \\ \alpha_1 + \alpha_2 - \alpha_3 = z \ldots (3) \end{cases}$$

First, we find the analytic expression of the linear transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.
So, we need to express $g\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of the coordinates $x, y, z \in \mathbb{R}$. Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. We express this arbitrary vector in terms of the basis $\mathcal{B} = \{v_1, v_2, v_3\}$. We want scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$X = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v$$

Adding the vectors component-wise leads to the system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_2 - \alpha_1 + \alpha_3 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 - \alpha_3 \end{pmatrix},$$

which gives the equations

$$\begin{cases} -\alpha_1 + \alpha_2 + \alpha_3 = x \ldots (1) \\ \alpha_1 - \alpha_2 + \alpha_3 = y \ldots (2) \\ \alpha_1 + \alpha_2 - \alpha_3 = z \ldots (3) \end{cases}$$

From equation (1), solve for α_1

$$\alpha_1 = \alpha_2 + \alpha_3 - x \ldots (4)$$

Substitute equation (4) into equations (2) and (3), we obtain

$$\begin{cases} (\alpha_2 + \alpha_3 - x) - \alpha_2 + \alpha_3 = y \\ (\alpha_2 + \alpha_3 - x) + \alpha_2 - \alpha_3 = z \end{cases} \Leftrightarrow \begin{cases} 2\alpha_3 - x = y \\ 2\alpha_2 - x = z \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = \frac{1}{2}x + \frac{1}{2}y \\ \alpha_2 = \frac{1}{2}x + \frac{1}{2}z \end{cases}$$

Now substitute $\alpha_2 = \frac{1}{2}x + \frac{1}{2}z$ and $\alpha_3 = \frac{1}{2}x + \frac{1}{2}y$ back into equation (4), we get

$$\alpha_1 = \frac{1}{2}x + \frac{1}{2}z + \frac{1}{2}x + \frac{1}{2}y - x = \frac{1}{2}y + \frac{1}{2}z.$$

Since g is linear, then

$$g(X) = g(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v) = \alpha_1 g(v_1) + \alpha_2 g(v_2) + \alpha_3 g(v)$$

where

$$\begin{aligned} g(v_1) &= w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ g(v_2) &= 2w_2 = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ g(v_3) &= w_1 + 3w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned}$$

Now compute $g(X)$

$$g(X) = \left(\frac{1}{2}y + \frac{1}{2}z\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(\frac{1}{2}z + \frac{1}{2}x\right)\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \left(\frac{1}{2}y + \frac{1}{2}x\right)\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

So the analytic expression is

$$\boxed{g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2}x + \frac{3}{2}y + z \\ 3x + \frac{5}{2}y + \frac{3}{2}z \end{pmatrix}.}$$

Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be the standard bases of $\mathbb{R}^3$. We compute

$$\begin{aligned} g(e_1) &= g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}(1) + \frac{3}{2}(0) + (0) \\ 3(1) + \frac{5}{2}(0) + \frac{3}{2}(0) \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \\ g(e_2) &= f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}(0) + \frac{3}{2}(1) + (0) \\ 3(0) + \frac{5}{2}(1) + \frac{3}{2}(0) \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \end{pmatrix} \\ g(e_3) &= f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}(0) + \frac{3}{2}(0) + 1 \\ 3(0) + \frac{5}{2}(0) + \frac{3}{2}(1) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \end{aligned}$$

We construct the matrix $[g]_{\text{std}}$ by placing the results of $g(e_1)$, $g(e_2)$, and $g(e_3)$ as columns. Therefore, the matrix associated with the linear transformation f, relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, is

$$\boxed{[g]_{\text{std}} = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} & 1 \\ 3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}.}$$

Exercise 2

I. Solve the following system

$$\begin{cases} -X+Y-Z=\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \dots (1) \\ X-2Y+Z=\begin{pmatrix} 1 & -2 & 5 \\ -3 & -2 & -3 \\ 1 & -4 & 4 \end{pmatrix} \dots (2) \\ X+4Y-2Z=\begin{pmatrix} 5 & 2 & 4 \\ 2 & 2 & 1 \\ 3 & -3 & 1 \end{pmatrix} \dots (3) \end{cases}$$

Let the matrices

$$A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 2 & 1 & -3 \\ 5 & 4 & -2 \\ 0 & 3 & -4 \end{pmatrix}$$

1. Compute the matrix product $C = A.B$, if the multiplication is defined.

2. Consider the matrix equation $X.C = D$, where

$$D=\begin{pmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix},$$

and X is an unknown matrix.

Determine the values of the unknown matrix X.

Solution

I. Add equations (1) and (2), we get

$$Y=-\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}-\begin{pmatrix} 1 & -2 & 5 \\ -3 & -2 & -3 \\ 1 & -4 & 4 \end{pmatrix}=\begin{pmatrix} 0 & 1 & -6 \\ 2 & 3 & 2 \\ -2 & 3 & -3 \end{pmatrix}$$

From equation (2)

$$X = -Z + \begin{pmatrix} 1 & -2 & 5 \\ -3 & -2 & -3 \\ 1 & -4 & 4 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 & -6 \\ 2 & 3 & 2 \\ -2 & 3 & -3 \end{pmatrix} = -Z + \begin{pmatrix} 1 & 0 & -7 \\ 1 & 4 & 1 \\ -3 & 2 & -2 \end{pmatrix}$$

Now substitute the known values of X and Y into (3)

$$-3Z = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 2 & 1 \\ 3 & -3 & 1 \end{pmatrix} - 4\begin{pmatrix} 0 & 1 & -6 \\ 2 & 3 & 2 \\ -2 & 3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -7 \\ 1 & 4 & 1 \\ -3 & 2 & -2 \end{pmatrix}$$

so

$$Z = \begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & -\frac{35}{3} \\ \frac{7}{3} & \frac{14}{3} & \frac{8}{3} \\ -\frac{14}{3} & \frac{17}{3} & -5 \end{pmatrix}$$

Back to X

$$X = -\begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & -\frac{35}{3} \\ \frac{7}{3} & \frac{14}{3} & \frac{8}{3} \\ -\frac{14}{3} & \frac{17}{3} & -5 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -7 \\ 1 & 4 & 1 \\ -3 & 2 & -2 \end{pmatrix} + \begin{pmatrix} \frac{7}{3} & -\frac{2}{3} & \frac{14}{3} \\ -\frac{4}{3} & -\frac{2}{3} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{11}{3} & 3 \end{pmatrix}$$

Finally

$$X = \begin{pmatrix} \frac{7}{3} & -\frac{2}{3} & \frac{14}{3} \\ -\frac{4}{3} & -\frac{2}{3} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{11}{3} & 3 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & -6 \\ 2 & 3 & 2 \\ -2 & 3 & -3 \end{pmatrix}, Z = \begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & -\frac{35}{3} \\ \frac{7}{3} & \frac{14}{3} & \frac{8}{3} \\ -\frac{14}{3} & \frac{17}{3} & -5 \end{pmatrix}.$$

II.1. $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$: Since the number of columns in matrix A equals the number of rows in matrix B, the multiplication $A \cdot B$ defined. The resulting matrix $C = A \cdot B$ will have dimensions 3×3.

$$C = A \cdot B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & -3 \\ 5 & 4 & -2 \\ 0 & 3 & -4 \end{pmatrix}.$$

So

$$\boxed{C = A \cdot B = \begin{pmatrix} 2 & 7 & -11 \\ 13 & 14 & -7 \\ -6 & -6 & -2 \end{pmatrix}.}$$

II.2. The value of X: We first determine the dimensions of X. Since the size of matrix C is 3×3, and the size of D is 3×3, X must be of size 3×3 in order for the matrix multiplication $X \cdot C$ to be defined.

Let

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now compute the product $X.C$

$$\begin{aligned} X.C &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 2 & 7 & -11 \\ 13 & 14 & -7 \\ -6 & -6 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2a_{11}+13a_{12}-6a_{13} & 7a_{11}+14a_{12}-6a_{13} & -11a_{11}-7a_{12}-2a_{13} \\ 2a_{21}+13a_{22}-6a_{23} & 7a_{21}+14a_{22}-6a_{23} & -11a_{21}-7a_{22}-2a_{23} \\ 2a_{31}+13a_{32}-6a_{33} & 7a_{31}+14a_{32}-6a_{33} & -11a_{31}-7a_{32}-2a_{33} \end{pmatrix} \end{aligned}$$

Since $X.C = D$

$$\begin{pmatrix} 2a_{11}+13a_{12}-6a_{13} & 7a_{11}+14a_{12}-6a_{13} & -11a_{11}-7a_{12}-2a_{13} \\ 2a_{21}+13a_{22}-6a_{23} & 7a_{21}+14a_{22}-6a_{23} & -11a_{21}-7a_{22}-2a_{23} \\ 2a_{31}+13a_{32}-6a_{33} & 7a_{31}+14a_{32}-6a_{33} & -11a_{31}-7a_{32}-2a_{33} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix}$$

we get the following systems of equations

$$(S_1) \begin{cases} 2a_{11}+13a_{12}-6a_{13} = -1 \\ 7a_{11}+14a_{12}-6a_{13} = 1 \\ -11a_{11}-7a_{12}-2a_{13} = 1 \end{cases}$$

$$(S_2)\begin{cases} 2a_{21} + 13a_{22} - 6a_{23} = 2 \\ 7a_{21} + 14a_{22} - 6a_{23} = -2 \\ -11a_{21} - 7a_{22} - 2a_{23} = 2 \end{cases}$$

$$(S_3)\begin{cases} 2a_{31} + 13a_{32} - 6a_{33} = 3 \\ 7a_{31} + 14a_{32} - 6a_{33} = 3 \\ -11a_{31} - 7a_{32} - 2a_{33} = -3 \end{cases}$$

Solving these systems yields the general solutions

$$\begin{aligned} a_{11} &= \frac{8}{15}, a_{12} = -\frac{2}{3}, a_{13} = -\frac{11}{10}, \\ a_{21} &= -\frac{44}{45}, a_{22} = \frac{8}{9}, a_{23} = \frac{19}{15}, \\ a_{31} &= -\frac{4}{45}, a_{32} = \frac{4}{9}, a_{33} = \frac{13}{30} \end{aligned}$$

Therefore, the general solution for matrix X is:

$$\boxed{X = \begin{pmatrix} \frac{8}{15} & -\frac{2}{3} & -\frac{11}{10} \\ -\frac{44}{45} & \frac{8}{9} & \frac{19}{15} \\ -\frac{4}{45} & \frac{4}{9} & \frac{13}{30} \end{pmatrix}}.$$

Exercise 03

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

1. Calculate A^2.

2. Calculate A^3.

3. Compute the matrix expression $A^3 - A^2 - A + I_3$.

4. Deduce that the matrix A is invertible and find its inverse A^{-1}.

Solution

1. Calculate $\mathbf{A^2 = A.A}$:

$$\begin{aligned} A^2 &= A.A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0\cdot 0 + 1\cdot(-1) + 0\cdot 1 & 0\cdot 1 + 1\cdot 2 + 0\cdot 0 & 0\cdot 0 + 1\cdot 0 + 0\cdot(-1) \\ -1\cdot 0 + 2\cdot(-1) + 0\cdot 1 & -1\cdot 1 + 2\cdot 2 + 0\cdot 0 & -1\cdot 0 + 2\cdot 0 + 0\cdot(-1) \\ 1\cdot 0 + 0\cdot(-1) + (-1)\cdot 1 & 1\cdot 1 + 0\cdot 2 + (-1)\cdot 0 & 1\cdot 0 + 0\cdot 0 + (-1)\cdot(-1) \end{pmatrix}, \end{aligned}$$

so

$$\boxed{A^2 = \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}}$$

2. Calculate $\mathbf{A^3 = A^2.A}$:

$$\begin{aligned} A^3 &= A^2.A = \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} (-1)\cdot 0 + 2\cdot(-1) + 0\cdot 1 & (-1)\cdot 1 + 2\cdot 2 + 0\cdot 0 & (-1)\cdot 0 + 2\cdot 0 + 0\cdot(-1) \\ -(2)\cdot 0 + 3\cdot(-1) + 0\cdot 1 & -(2)\cdot 1 + 3\cdot 2 + 0\cdot 0 & -(2)\cdot 0 + 3\cdot 0 + 0\cdot(-1) \\ -(1)\cdot 0 + 1\cdot(-1) + 1\cdot 1 & -(1)\cdot 1 + 1\cdot 2 + 1\cdot 0 & -(1)\cdot 0 + 1\cdot 0 + 1\cdot(-1) \end{pmatrix}, \end{aligned}$$

so

$$\boxed{A^3 = \begin{pmatrix} -2 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix}.}$$

3. Calculate $\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I}_3$:

$$A^3 - A^2 - A + I_3 = \begin{pmatrix} -2 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\boxed{A^3 - A^2 - A + I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{\mathbb{R}^3}.}$$

4. We deduce that A is invertible and calculate A^{-1}: We have

$$\begin{aligned} A^3 - A^2 - A + I_3 &= 0_3 \Leftrightarrow -A^3 + A^2 + A = I_3 \\ &\Leftrightarrow A\left(-A^2 + A + I_3\right) = I_3. \end{aligned}$$

So

$$A^{-1} = A^2 + A + I_3 = -\begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\boxed{A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix}.}$$

Exercise 04

I. Find the rank of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

II. Let the matrix B be given by:

$$B = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{pmatrix}$$

1. Write the transpose of the matrix B, denoted B^T.

2.Find the rank of the transposed matrix B^T.

3. Deduce the rank of the original matrix B.

Solution

I. Finding the Rank of Matrix A : The rank of a matrix is the maximum number of linearly independent rows or columns. Equivalently, it is the number of non-zero rows in the matrix after we perform row reduction (Gaussian elimination or Row Echelon Form).

We begin by applying elementary row operations to reduce matrix A to Row Echelon Form (REF). We will use row 1 (R_1) as the pivot row. Perform the following row operations

$$\begin{aligned} R_2 &= R_2 - R_1, \\ R_3 &= R_3 - 2R_1, \\ R_4 &= R_4 - 3R_1, \end{aligned}$$

so

$$A \sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{pmatrix}$$

Now, use row 2 (R_2) as the new pivot to eliminate entries below it

$$\begin{aligned} R_3 &= R_3 + 3R_2, \\ R_4 &= R_4 + R_2, \end{aligned}$$

so

$$\boxed{A \sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.}$$

We now observe that there are only two non-zero rows, so

$$\boxed{\text{Rank}\,(A) = 2.}$$

II. 1. Transpose of the Matrix B: The transpose of a matrix is obtained by interchanging its rows and columns. Thus, the transpose of B, denoted by B^T, is

$$\boxed{B^T = \begin{pmatrix} 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & 4 \\ -3 & 0 & 3 & -2 \end{pmatrix}.}$$

2. Finding the Rank of B^T: We now perform row reduction to bring B^T to row echelon form. Use row 1 as the pivot row. Perform the following row operations

$$\begin{aligned} R_2 &= R_2 - 2R_1, \\ R_3 &= R_3 + 3R_1. \end{aligned}$$

So, the matrix becomes

$$B^T \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 6 & -3 & -5 \end{pmatrix}.$$

Normalize row 2 by dividing the entire row by -3

$$R_2\left(\frac{1}{-3}\right)R_2,$$

so

$$B^T \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 6 & -3 & -5 \end{pmatrix}.$$

Now, eliminate entry below the pivot in column 2

$$R_3 = R_3 - 6R_2,$$

the matrix becomes

$$B^T \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 3 & 7 \end{pmatrix}.$$

Normalize row 3

$$R_3 = \frac{1}{3}R_3,$$

so

$$B^T \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 1 & \frac{7}{3} \end{pmatrix}.$$

The final row-echelon form is

$$\boxed{B^T \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 1 & \frac{7}{3} \end{pmatrix}.}$$

There are 3 non-zero rows, so

$$\boxed{\text{Rank}\left(B^T\right) = 3.}$$

3. Deduce the rank of the Matrix B: It is a fundamental property of matrices that

$$\text{Rank}\,(B) = \text{Rank}\left(B^T\right).$$

Therefore, we conclude that

$$\boxed{\text{Rank}\,(B) = 3.}$$

Exercise 05

Compute the following determinants

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 5 & 14 & -3 & 2 \\ 15 & 18 & -9 & 5 \\ -10 & -10 & 6 & -4 \\ 20 & 8 & -12 & 7 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{vmatrix}$$

$$\Delta_4 = \begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix}, \quad \Delta_5 = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix}$$

Solution

Computing Δ_1 : To simplify the determinant, we perform a row operation. Specifically, we subtract row 2 (R_2) from row 3 (R_3)

$$R_3 = R_3 - R_2$$

After applying this operation, the matrix becomes

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

Now, we perform a cofactor expansion along the first row, since the first entry in the first row is 0, which simplifies the calculation

$$\Delta_1 = -\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (1)(0) - (1)(1) = -1$$

Thus

$$\boxed{\Delta_1 = -1.}$$

Computing Δ_2 : In the first column (C_1) we factor out 5, and in the third column (C_3) we factor out -3. So, the determinant becomes

$$\Delta_2 = (5)(-3)\begin{vmatrix} 1 & 14 & 1 & 2 \\ 3 & 18 & 3 & 5 \\ -2 & -10 & -2 & -4 \\ 4 & 8 & 4 & 7 \end{vmatrix}.$$

We observe that column 1 and Column 3 are identical. Therefore, the determinant is zero, because a determinant with two equal columns is always zero

$$\boxed{\Delta_2 = (5)(-3)(0) = 0.}$$

Computing Δ_3 : We apply the following elementary row operations

$$\begin{aligned} R_2 &= R_2 - R_1, \\ R_3 &= R_3 - 4R_2, \\ R_4 &= R_4 - R_3. \end{aligned}$$

The transformed matrix becomes

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since the matrix has three identical rows, we immediately conclude

$$\boxed{\Delta_3 = 0.}$$

Computing Δ_4 : We perform the elementary row operation

$$R_1 = R_1 + R_2 + R_3.$$

This transforms the matrix into

$$\begin{pmatrix} 3a+b & 3a+a & 3a+b \\ a & a+b & a \\ a & a & a+b \end{pmatrix}.$$

We factor out $(3a+b)$ from the first row since all entries in the first row are equal to $(3a+b)$

$$\Delta_4 = (3a+b)\begin{vmatrix} 1 & 1 & 1 \\ a & a+b & a \\ a & a & a+b \end{vmatrix}.$$

Now apply the following elementary column operations

$$\begin{aligned} C_2 &= C_2 - C_1, \\ C_3 &= C_3 - C_1. \end{aligned}$$

This gives

$$\Delta_4 = (3a+b)\begin{vmatrix} 1 & 0 & 0 \\ a & a & 0 \\ a & 0 & a \end{vmatrix}.$$

Now the matrix is in upper triangular form. Its determinant is the product of the diagonal entries

$$\boxed{\Delta_4 = a^2\,(3a+b)\,.}$$

Computing Δ_5 : We perform the following operations on the columns

$$\begin{aligned} C_1 &= xC_1, \\ C_2 &= yC_2, \\ C_3 &= xC_3. \end{aligned}$$

This transforms the matrix into

$$\begin{pmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{pmatrix},$$

so

$$\Delta_5 = \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Since each entry of the third row is (xyz), we factor (xyz) out

$$\Delta_5 = \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}.$$

We interchange (R_2) and (R_3)

$$\Delta_5 = -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}.$$

Then we interchange (R_1) and (R_2) again

$$\Delta_5 = (-)(-)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}.$$

So

$$\begin{aligned}\Delta_5 &= \begin{vmatrix} y & z \\ y^2 & z^2 \end{vmatrix} - \begin{vmatrix} x & z \\ x^2 & z^2 \end{vmatrix} + \begin{vmatrix} x & y \\ x^2 & y^2 \end{vmatrix} \\ &= yz^2 - y^2z - \left(xz^2 - x^2z\right) + xy^2 - x^2y.\end{aligned}$$

Thus

$$\boxed{\Delta_5 = -x^2y + x^2z + xy^2 - xz^2 - y^2z + yz^2.}$$

Exercise 06

Using determinants, discuss the rank of the following matrix A according to the values os of of the parameter α

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}.$$

Solution We start by computing the determinant of A

$$\det A = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}.$$

Using cofactor expansion along the first row

$$\det A = \alpha \begin{vmatrix} \alpha & 1 \\ 1 & \alpha \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & \alpha \end{vmatrix} + \begin{vmatrix} 1 & \\ 1 & 1 \end{vmatrix}$$

Now calculate the 2×2 determinants

$$\det A = \alpha\left(\alpha^2 - 1\right) - (\alpha - 1) + (1 - \alpha) = \alpha^3 - 3\alpha + 2.$$

We attempt to factor this cubic polynomial. Try rational root theorem. Test $\alpha = 1$

$$(1)^3 - 3(1) + 2 = 0,$$

so $\alpha = 1$ is root.

Now factor out $(\alpha - 1)$

$$\begin{aligned}
\alpha\left(\alpha^2 - 1\right) - (\alpha - 1) + (1 - \alpha) &= \alpha(\alpha - 1)(\alpha + 1) - (\alpha - 1) - (\alpha - 1) \\
&= \alpha(\alpha - 1)(\alpha + 1) - 2(\alpha - 1) \\
&= (\alpha - 1)(\alpha(\alpha + 1) - 2) \\
&= (\alpha - 1)\left(\alpha^2 + \alpha - 2\right)
\end{aligned}$$

Factor further

$$\det A = (\alpha - 1)(\alpha - 1)(\alpha + 2).$$

Thus,

$$\boxed{\det A = (\alpha - 1)^2(\alpha + 2).}$$

Now, we determine the rank based on the determinant

Case 1: $\alpha \neq 1$ **and** $\alpha \neq -2$. In this case $\det A \neq 0$, so the matrix A is of full rank. Since A is 3×3 matrix, we conclude

$$\boxed{\text{Rank}(A) = 3.}$$

Case 1: $\alpha = 1$**.** Substitute into the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

All rows are identical. Therefore, the matrix has only one linearly independent row, so

$$\boxed{\text{Rank}(A) = 1.}$$

Case 1: $\alpha = 0$**.** Substitute into the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Check the determinant

$$\det A = (\alpha - 1)^2 (\alpha + 2) = ((-2) - 1)^2 ((-2) + 2) = 0.$$

Now check the rank by evaluating the determinant of a 2×2 minor. Take the leading principal minor

$$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-2)(-2) - (1)(1) = 3 \neq 0$$

So at least one 2×2 minor is non-zero, Thus

$$\boxed{\text{Rank}(A) = 2.}$$

Finally

$$\boxed{\text{Rank}(A) = \begin{cases} 3, \text{ if } \alpha \neq 1 \textbf{ and } \alpha \neq -2 \\ 2, \text{ if } \alpha = -2 \\ 1, \text{ if } \alpha = 1 \end{cases}.}$$

Exercise 07

Let

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & -2 & -2 \end{pmatrix}$$

1. Show that the matrix A is invertible.

2. Calculate the inverse matrix A^{-1}.

Solution

1. Check if A is invertible: To check whether A is invertible, we compute its determinant using cofactor expansion along the first row

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} \\ &= 0 + 2(4-3) + (4-3) = 2 + 1 = 3 \neq 0.\end{aligned}$$

Since $\det(A) = 3 \neq 0$, the matrix A is invertible.

2. The inverse matrix A^{-1}: We use the formula

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

We now compute the adjugate matrix adj(A), which is the transpose of the cofactor matrix com(A).

Compute Cofactors

$$\begin{aligned}\det A_{11} &= \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} = 0, \det A_{12} = \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} = 1, \det A_{13} = \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} = 1 \\ \det A_{21} &= \begin{vmatrix} -2 & 1 \\ -2 & -2 \end{vmatrix} = 6, \det A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5, \det A_{23} = \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} = 4 \\ \det A_{31} &= \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3, \det A_{32} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3, \det A_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3\end{aligned}$$

The cofactor matrix is

$$\begin{aligned}\operatorname{com}(A) &= \begin{pmatrix} \det A_{11} & -\det A_{12} & \det A_{13} \\ -\det A_{21} & \det A_{22} & -\det A_{23} \\ \det A_{31} & -\det A_{32} & \det A_{33} \end{pmatrix} \\ &= \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \end{pmatrix}\end{aligned}$$

So

$$\operatorname{com}(A) = \begin{pmatrix} 0 & -1 & 1 \\ -6 & -5 & -4 \\ -3 & -3 & -3 \end{pmatrix}.$$

The transpose of the cofactor matrix (i.e. the adjugate $\operatorname{adj}(A)$) is

$$\operatorname{adj}(A) = (\operatorname{com}(A))^T = \begin{pmatrix} 0 & -6 & -3 \\ -1 & -5 & -3 \\ 1 & -4 & -3 \end{pmatrix}.$$

Now

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{3}\begin{pmatrix} 0 & -6 & -3 \\ -1 & -5 & -3 \\ 1 & -4 & -3 \end{pmatrix}.$$

Thus

$$\boxed{A^{-1} = \begin{pmatrix} 0 & -2 & -1 \\ -\frac{1}{3} & -\frac{5}{3} & -1 \\ \frac{1}{3} & -\frac{4}{3} & -1 \end{pmatrix}.}$$

Exercise 08

Solve the following system of equations using three different methods

$$\begin{cases} x - y - 2z = 1 \\ -3x + y - z = 5 \\ -4x + 2y + 3z = 6. \end{cases}$$

1. Using Gaussian elimination (pivot method)

2. By inverting the coefficient matrix

3. Using Cramer's rule

Solution

1. Gaussian Elimination (Pivot Method): We write the augmented matrix

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ -3 & 1 & -1 & 5 \\ -4 & 2 & 3 & 6 \end{array}\right)$$

We perform the following row operations to simplify the matrix

$$\begin{aligned} R_2 &= R_2 - (-3)\, R_1, \\ R_3 &= R_3 - (-3)\, R_1 \end{aligned}$$

The resulting matrix is

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -2 & -7 & 8 \\ 0 & -2 & -5 & 10 \end{array}\right)$$

Next, perform the following operation on row 3

$$R_3 = R_3 - R_2$$

The resulting matrix is

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -2 & -7 & 8 \\ 0 & 0 & 2 & 2 \end{array}\right).$$

From row 3

$$2z = 2 \Longrightarrow z = 1.$$

From row 2

$$-2y - 7(1) = 8 \Longrightarrow y = -\frac{15}{2}$$

From row 1

$$x - \left(-\frac{15}{2}\right) - 2(1) = 1 \Longrightarrow x = -\frac{9}{2}.$$

Thus, the solution of the system is

$$\boxed{x = \frac{-9}{2}, y = \frac{-15}{2}, z = 1.}$$

2. Matrix Inversion Method: We begin by expressing the system in matrix form

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -3 & 1 & -1 \\ -4 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}.$$

Now, we compute the determinant. Using Gaussian Elimination

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -2 & -7 & 8 \\ 0 & 0 & 2 & 2 \end{array}\right).$$

from Method 1

$$\det(A) = (1)(2)(-2) = -4 \neq 0 \text{ (Upper Triangular Matrix)}$$

Since $\det(A) = 3 \neq 0$, the matrix A is invertible.

To calculate A^{-1}**:** We use the formula

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

We now compute the adjugate matrix adj(A), which is the transpose of the cofactor matrix com(A).

The cofactor matrix is

$$\operatorname{com}(A) = \begin{pmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & -1 \\ -4 & 3 \end{vmatrix} & \begin{vmatrix} -3 & 1 \\ -4 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -4 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} \end{pmatrix}$$

So

$$\operatorname{com}(A) = \begin{pmatrix} 5 & 13 & -2 \\ -1 & -5 & 2 \\ 3 & 7 & -2 \end{pmatrix}.$$

The transpose of the cofactor matrix (i.e. the adjugate adj(A)) is

$$\operatorname{adj}(A) = (\operatorname{com}(A))^T = \begin{pmatrix} 5 & -1 & 3 \\ 13 & -5 & 7 \\ -2 & 2 & -2 \end{pmatrix}.$$

Now

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{4} \begin{pmatrix} 5 & -1 & 3 \\ 13 & -5 & 7 \\ -2 & 2 & -2 \end{pmatrix}.$$

Now,

$$AX = B \Longrightarrow X = A^{-1}B = -\frac{1}{4}\begin{pmatrix} 5 & -1 & 3 \\ 13 & -5 & 7 \\ -2 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{15}{2} \\ 1 \end{pmatrix}.$$

Thus, the solution of the system is

$$\boxed{x = \frac{-9}{2}, y = \frac{-15}{2}, z = 1.}$$

Cramer's Rule: We observe that $\det(A) \neq 0$. So, we can apply Cramer's Rule. The solution is given by:

$$x = \frac{\Delta_x}{\det(A)}, \quad y = \frac{\Delta_y}{\det(A)}, \quad z = \frac{\Delta_z}{\det(A)},$$

where

$$\Delta_x = \begin{vmatrix} 1 & -1 & -2 \\ 5 & 1 & -1 \\ 6 & 2 & 3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} 1 & 1 & -2 \\ -3 & 5 & -1 \\ -4 & 6 & 3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} 1 & -1 & 1 \\ -3 & 1 & 5 \\ -4 & 2 & 6 \end{vmatrix}.$$

Computing Δ_x:

$$\begin{aligned} \Delta_x &= \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & -1 \\ 6 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ 6 & 2 \end{vmatrix} \\ &= 5 + 21 - 8 = 18. \end{aligned}$$

So

$$\boxed{\Delta_x = 18.}$$

Computing Δ_y:

$$\begin{aligned} \Delta_y &= \begin{vmatrix} 5 & -1 \\ 6 & 3 \end{vmatrix} - \begin{vmatrix} -3 & -1 \\ -4 & 3 \end{vmatrix} - 2\begin{vmatrix} -3 & 5 \\ -4 & 6 \end{vmatrix} \\ &= 21 + 13 - 4 = 30. \end{aligned}$$

So

$$\boxed{\Delta_y = 30.}$$

Computing Δ_z:

$$\begin{aligned}\Delta_z &= \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 5 \\ -4 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 1 \\ -4 & 2 \end{vmatrix} \\ &= -4 + 2 - 2 = -4.\end{aligned}$$

So

$$\boxed{\Delta_z = -4.}$$

Substituting into the formula

$$\begin{aligned} x &= \frac{18}{-4} = -\frac{9}{2}, \\ y &= -\frac{15}{2}, \\ z &= \frac{-4}{-4} = 1. \end{aligned}$$

Hence, the system has a unique solution

$$\boxed{x = \frac{-9}{2}, y = \frac{-15}{2}, z = 1.}$$

Exercise 09

Consider the following system of linear equations, which depends on the parameter m :

$$(S)\begin{cases} 2x + y - z = 1 \ldots (1) \\ x + m\ y + z = 1 \ldots (2) \\ 3x + y - m\ z = 2 \ldots (3) \end{cases}$$

1. Solve the system using the Gaussian elimination method.

2. Solve the system using Cramer's rule.

Solution

1. Solution by the Gauss Method: The given system of equations can be represented in matrix form as

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & m & 1 \\ 3 & 1 & -m \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Thus, the augmented matrix becomes

$$\overline{A} = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & m & 1 & 1 \\ 3 & 1 & -m & 2 \end{array}\right)$$

We perform the following row operations to simplify the matrix

$$\begin{aligned} R_2 &\longrightarrow R_2 - \frac{1}{2}R_1, \\ R_3 &\longrightarrow R_3 - \frac{1}{2}R_1. \end{aligned}$$

The resulting matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & m - \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} - m & \frac{1}{2} \end{array}\right)$$

We now interchange the second and third rows to simplify further

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} - m & \frac{1}{2} \\ 0 & m - \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right).$$

Next, perform the following operation on row 3

$$R_3 \longrightarrow R_3 - \frac{\left(m - \frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} R_2 = R_3 + (2m - 1)\, R_2.$$

The resulting matrix becomes

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2}-m & \frac{1}{2} \\ 0 & 0 & 2m\,(2-m) & m \end{array}\right).$$

Thus, the final form of the matrix after applying Gaussian elimination is

$$\boxed{\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2}-m & \frac{1}{2} \\ 0 & 0 & 2m\,(2-m) & m \end{array}\right).}$$

We now analyze the solution according to different values of the parameter m.

Case 1: m = 2

Substituting $m = 2$ into the matrix. The matrix becomes

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2}-2 & \frac{1}{2} \\ 0 & 0 & 0 & 2 \end{array}\right).$$

The third row corresponds to the contradictory equation

$$0 = 2$$

Therefore, The system is inconsistent and has no solution when $m = 2$.

Case 2: m = 0

Substituting $m = 0$ into the matrix. The matrix becomes

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Here, the third row is entirely zero, indicating a dependent system with infinitely many solutions. The system reduces to two equations in three unknowns

$$\begin{cases} 2x + y - z = 1 \\ -\frac{1}{2}y + \frac{3}{2}z = \frac{1}{2} \end{cases} \Longleftrightarrow \begin{cases} 2x + y - z = 1 \ldots (4) \\ -y + 3z = 1 \ldots (5) \end{cases}$$

From equation (5)

$$y = 3z - 1 \ldots (6)$$

Substitute equation (6) into equation (4), we get

$$2x + (3z - 1) - z = 2x + 2z - 1 = 1 \Longrightarrow x = 1 - z.$$

Therefore, when $m = 0$, the system has infinitely many solutions given by:

$$\boxed{x = 1 - z, \quad y = 3z - 1}$$

Case 3: $m \neq 0$ **and** $m \neq 2$

In this case, the third row gives the equation

$$2m\,(2 - m)\,z = m \Longrightarrow z = \frac{1}{2\,(2 - m)} \quad (\text{since } m \neq 0 \text{ and } m \neq 2)$$

Substitute this value into the second row

$$-\frac{1}{2}y + \left(\frac{3}{2} - m\right)\left(\frac{1}{2\,(2 - m)}\right) = \frac{1}{2} \Longrightarrow y = \frac{1}{2m - 4}$$

Now, substitute y and z into the first row to find x

$$2x + \frac{1}{2m - 4} - \frac{1}{2\,(2 - m)} = 1 \Longrightarrow x = \frac{m - 3}{2m - 4}$$

Therefore, the system has the unique solution

$$\boxed{x = \frac{m - 3}{2m - 4}, \quad y = \frac{1}{2m - 4}, \quad z = \frac{1}{2\,(2 - m)}}$$

The solution to the system depends on the parameter m as follows:

$$\begin{cases} \text{No solution} & \text{if } m = 2 \\ \text{Infinitely many solutions: } x = 1 - z,\ y = 3z & \text{if } m = 0 \\ \text{Unique solution: } x = \dfrac{m-3}{2m-4},\ y = \dfrac{1}{2m-4},\quad z = \dfrac{1}{2\,(2-m)} & \text{if } m \neq 0, m \neq 2 \end{cases}$$

2. Solution Using Cramer's Rule: We begin by expressing the system in matrix form

$$AX = B,$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & m & 1 \\ 3 & 1 & -m \end{pmatrix},\quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\quad B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

To determine whether the system is of Cramer type, we compute the determinant $\Delta = \det A$ of A by expanding along the first row

$$\begin{aligned} \Delta &= \det A = (2)\begin{vmatrix} m & 1 \\ 1 & -m \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 3 & -m \end{vmatrix} + (-1)\begin{vmatrix} 1 & m \\ 3 & 1 \end{vmatrix} \\ &= (2)\left(-m^2 - 1\right) - (1)\,(-m-3) + (-1)\,(1-3m) \\ &= -2m^2 - 2 + m + 3 - 1 + 3m \\ &= 4m - 2m^2. \end{aligned}$$

So

$$\boxed{\det A = 4m - 2m^2.}$$

Alternatively, from the Gaussian elimination performed earlier, the reduced matrix was

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} - m & \frac{1}{2} \\ 0 & 0 & 2m\,(2-m) & m \end{array}\right).$$

Only one row interchange was performed, so

$$\det A = (-\mathbf{1})\,(2)\left(-\frac{1}{2}\right)(2m\,(2-m)) = 2m\,(2-m)\,.$$

If $m \neq 0$, **and** $m \neq 2$, we observe that $\det A \neq 0$. So, we can apply Cramer's rule. The solution is given by:

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta},$$

where

$$\Delta_x = \begin{vmatrix} 1 & 1 & -1 \\ 1 & m & 1 \\ 2 & 1 & -m \end{vmatrix},\quad \Delta_y = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -m \end{vmatrix},\quad \Delta_z = \begin{vmatrix} 2 & 1 & 1 \\ 1 & m & 1 \\ 3 & 1 & 2 \end{vmatrix}.$$

Computing Δ_x:

$$\begin{aligned} \Delta_x &= \begin{vmatrix} m & 1 \\ 1 & -m \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & -m \end{vmatrix} - \begin{vmatrix} 1 & m \\ 2 & 1 \end{vmatrix} \\ &= -m^2 - 1 - (-m-2) - (1-2m) = 3m - m^2. \end{aligned}$$

So

$$\boxed{\Delta_x = 3m - m^2.}$$

Computing Δ_y:

$$\begin{aligned} \Delta_y &= 2\begin{vmatrix} 1 & 1 \\ 2 & -m \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & -m \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \\ &= 2\,(-m-2) - (-m-3) - (2-3) = -m. \end{aligned}$$

So

$$\boxed{\Delta_y = -m.}$$

Computing Δ_z:

$$\begin{aligned} \Delta_z &= 2\begin{vmatrix} m & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & m \\ 3 & 1 \end{vmatrix} \\ &= 2\,(2m-1) - (2-3) + (1-3m) = m. \end{aligned}$$

So

$$\boxed{\Delta_z = m.}$$

Substituting into the formula

$$\begin{aligned} x &= \frac{3m - m^2}{4m - 2m^2} = \frac{-m\,(m-3)}{-2m\,(m-2)} = \frac{m-3}{2\,(m-2)}, \\ y &= \frac{-m}{4m - 2m^2} = \frac{-m}{-2m\,(m-2)} = \frac{1}{2\,(m-2)}, \\ z &= \frac{m}{4m - 2m^2} = \frac{m}{-2m\,(m-2)} = -\frac{1}{2\,(m-2)}. \end{aligned}$$

Hence, the system has a unique solution for $m \neq 0$, and $m \neq 2$

$$\boxed{x = \frac{m-3}{2\,(m-2)},\ y = \frac{1}{2\,(m-2)},\ z = -\frac{1}{2\,(m-2)}.}$$

If $m = 0$, **or** $m = 2$, $\det A = 0$. So? we can not apply Cramer's rule. Compute the rank of the matrix

$$\text{Minor} = \Delta' = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \neq 0 \Longrightarrow \text{rank}\,(A) = 2$$

Now analyze the augmented matrix

$$\overline{A} = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & m & 1 & 1 \\ 3 & 1 & -m & 2 \end{array}\right).$$

according to the parameter m.

If $m = 2$, the matrix becomes

$$\overline{A} = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & -2 & 2 \end{array}\right).$$

Since

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \neq 0,$$

So

$$\boxed{\operatorname{rank}\left(\overline{A}\right) = 3.}$$

Since $\operatorname{rank}\left(\overline{A}\right) \neq \operatorname{rank}(A)$, the system is inconsistent and has no solution if $m = 2$.

If $m = 0$, the matrix becomes

$$\overline{A} = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

Swap (R_2) and (R_1)

$$\overline{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

We perform the following row operations to simplify the matrix

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1, \\ R_3 &\rightarrow R_3 - 3R_1. \end{aligned}$$

The resulting matrix is

$$\overline{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{pmatrix}.$$

We perform the following row operation

$$R_3 \rightarrow R_3 - R_2$$

The resulting matrix is

$$\overline{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 2 non-zero rows. So

$$\boxed{\text{rank}\left(\overline{A}\right) = 2.}$$

Since $\text{rank}\left(\overline{A}\right) \neq \text{rank}(A) = 2$, the system has infinitely many solutions. From the second equation of the reduced system

$$(S')\begin{cases} 2x - z = 1 - y \\ x + z = 1 \end{cases}$$

we get

$$x = 1 - z.$$

Substitute the value of y in the first equation

$$2\,(1 - z) - z = 1 - y \Longrightarrow y = 3z - 1$$

So, if $m = 0$, the system has infinitely many solutions given by:

$$\boxed{x = 1 - z, \;\; y = 3z - 1}$$

Exercise 10

Solve the following systems using Cramer's rule:

$$(S_1)\begin{cases} 3x - 2y + 2z + 3u - t = 4 \\ 5x + y - 3z + 2u + 6t = -7 \\ x - 5y + 7z + 4u - 8t = 15 \end{cases}, \quad (S_2)\begin{cases} 6x + 4y + 5z - 3t = 1 \\ 5x - y + 2z + t = 3 \\ 7x + 9y + 8z - 7t = 2 \end{cases},$$

$$(S_3)\begin{cases} 4x + 2y = 5 \\ 3x - y = 7 \\ 2x + 6y = -4 \end{cases}, \quad (S_4)\begin{cases} 2x + 3y = 7 \\ 3x - y = 5 \\ x - 5y = -11 \end{cases}, \quad (S_5)\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 14 \\ x + 4y - z = -2 \end{cases}$$

$$(S_6)\begin{cases} -x + y - z = 0 \\ x + y - z = -1 \\ 2x + y - z = 1 \end{cases}, \quad (S_7)\begin{cases} x + y + z = 3 \\ x + y - z = -1 \\ 2x + 2y - z = 0 \end{cases}$$

Solution

Solving (S_1): This system consists of three equations and five unknowns (x, y, z, u, t). Since the number of equations is less than the number of variables, Cramer's rule cannot be applied directly.

We consider the coefficient matrix A_1 of the system

$$A_1 = \begin{pmatrix} 3 & -2 & 2 & 3 & -1 \\ 5 & 1 & -3 & 2 & 6 \\ 1 & -5 & 7 & 4 & -8 \end{pmatrix}$$

To determine the solvability of the system, we compute the rank of A_1 using Gaussian elimination. We first swap rows to bring a leading 1 to the top

$$\text{Swap } R_1 \longleftrightarrow R_2$$

The matrix becomes

$$A_1 = \begin{pmatrix} 1 & -5 & 7 & 4 & -8 \\ 5 & 1 & -3 & 2 & 6 \\ 3 & -2 & 2 & 3 & -1 \end{pmatrix}.$$

We now eliminate entries below the pivot in the first column

$$\begin{aligned} R_2 &= R_2 - 5R_1, \\ R_3 &= R_3 - 3R_1. \end{aligned}$$

The matrix becomes

$$A_1 = \begin{pmatrix} 1 & -5 & 7 & 4 & -8 \\ 0 & 26 & -38 & -18 & 46 \\ 0 & 13 & -19 & -9 & 23 \end{pmatrix}.$$

Next, eliminate below the pivot in column 2

$$R_3 = R_3 - \frac{13}{26}R_2 = R_3 - \frac{1}{2}R_2$$

The matrix becomes

$$A_1 = \begin{pmatrix} 1 & -5 & 7 & 4 & -8 \\ 0 & 26 & -38 & -18 & 46 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since there are two non-zero rows, we conclude that

$$\boxed{\text{Rank}(A_1) = 2}$$

We now extract a consistent square subsystem by choosing the first two equations and the variables x and y

$$(S_1') \begin{cases} 3x - 2y = 4 - 2z - 3u + t \\ 5x + y = -7 + 3z - 2u - 6t \end{cases}$$

The coefficient matrix of this subsystem is

$$\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}$$

We compute its determinant Δ_1

$$\Delta_1 = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \neq 0,$$

Since $\Delta_1 \neq 0$, we can apply Cramer's rule to solve for x and y in terms of the free variables z, u, and t.

$$\begin{cases} 3x - 2y = 4 - 2z - 3u + t \\ 5x + y = -7 + 3z - 2u - 6t \end{cases}$$

We compute the determinants Δ_x and Δ_y

$$\Delta_x = \begin{vmatrix} 4 - 2z - 3u + t & -2 \\ -7 + 3z - 2u - 6t & 1 \end{vmatrix} = 4z - 7u - 11t - 10,$$

$$\Delta_y = \begin{vmatrix} 3 & 4-2z-3u+t \\ 5 & -7+3z-2u-6t \end{vmatrix} = 9u - 23t + 19z - 41.$$

Using Cramer's Rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\Delta_1} = \frac{4z-7u-11t-10}{13} = \frac{4}{13}z - \frac{7}{13}u - \frac{11}{13}t - \frac{10}{13}, \\ y &= \frac{\Delta_y}{\Delta_1} = \frac{9u-23t+19z-41}{13} = \frac{9}{13}u - \frac{23}{13}t + \frac{19}{13}z - \frac{41}{13}. \end{aligned}$$

Now we substitute the expressions for x and y into the third equation of the original system (S_1)

$$x - 5y + 7z + 4u - 8t = 15.$$

Substituting

$$\left(\frac{4}{13}z - \frac{7}{13}u - \frac{11}{13}t - \frac{10}{13}\right) - 5\left(\frac{9}{13}u - \frac{23}{13}t + \frac{19}{13}z - \frac{41}{13}\right) + 7z + 4u - 8t = 15,$$

so the equation is satisfied. Therefore, the system is consistent and contains free variables, it has infinitely many solutions. The general solution can be expressed in terms of the parameter z, u, and t.

$$\boxed{x = \frac{4}{13}z - \frac{7}{13}u - \frac{11}{13}t - \frac{10}{13}, y = \frac{9}{13}u - \frac{23}{13}t + \frac{19}{13}z - \frac{41}{13}.}$$

Solving (S_2): This is a system of three equations in four unknowns (x, y, z, t). Since the number of variables exceeds the number of equations, , Cramer's rule cannot be applied directly to solve the full system

We define the coefficient matrix A_2 of the system

$$A_2 = \begin{pmatrix} 6 & 4 & 5 & -3 \\ 5 & -1 & 2 & 1 \\ 7 & 9 & 8 & -7 \end{pmatrix}$$

To determine the solvability of the system, we compute the rank of A_2 by evaluating the 3×3 minors. The relevant determinants are:

$$\begin{vmatrix} 6 & 4 & 5 \\ 5 & -1 & 2 \\ 7 & 9 & 8 \end{vmatrix} = 0, \quad \begin{vmatrix} 6 & 4 & -3 \\ 5 & -1 & 1 \\ 7 & 9 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 9 & 8 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 6 & 5 & -3 \\ 5 & 2 & 1 \\ 7 & 8 & -7 \end{vmatrix} = 0$$

Since all 3×3 minors vanish, we conclude

$$\operatorname{Rank}(A_2) \leq 2.$$

Now consider the following 2×2 submatrix

$$A_2' = \begin{pmatrix} 6 & 4 \\ 5 & -1 \end{pmatrix}$$

The determinant of A_2' is

$$\det(A_2') = \begin{vmatrix} 6 & 4 \\ 5 & -1 \end{vmatrix} = -26 \neq 0$$

Since at least one 2×2 minor is non-zero, we conclude:

$$\boxed{\operatorname{Rank}(A_2) = 2.}$$

We now extract a consistent square subsystem by choosing the first two equations and the variables x and y

$$(S_2') \begin{cases} 6x + 4y = 1 - 5z + 3t \\ 5x - y = 3 - 2z - t \end{cases}$$

The coefficient matrix of this subsystem is

$$A_2' = \begin{pmatrix} 6 & 4 \\ 5 & -1 \end{pmatrix} \quad \text{with } \det(A_2') = \begin{vmatrix} 6 & 4 \\ 5 & -1 \end{vmatrix} = -26 \neq 0$$

Since $\det(A_2')$, we can apply Cramer's rule to solve for x and y in terms of the free variables z and t.

$$\begin{cases} 6x + 4y = 1 - 5z + 3t \\ 5x - y = 3 - 2z - t \end{cases}$$

We compute the determinants Δ_x and Δ_y

$$\Delta_x = \begin{vmatrix} 1 - 5z + 3t & 4 \\ 3 - 2z - t & -1 \end{vmatrix} = t + 13z - 13,$$

$$\Delta_y = \begin{vmatrix} 6 & 1 - 5z + 3t \\ 5 & 3 - 2z - t \end{vmatrix} = 13z - 21t + 13.$$

Using Cramer's rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\det(A_2')} = \frac{t + 13z - 13}{-26} = \frac{1}{2} - \frac{1}{2}z - \frac{1}{26}t, \\ y &= \frac{\Delta_y}{\det(A_2')} = \frac{13z - 21t + 13}{-26} = \frac{21}{26}t - \frac{1}{2}z - \frac{1}{2}. \end{aligned}$$

Now we substitute the expressions for x and y into the third equation (S_2)

$$7x + 9y + 8z - 7t = 2.$$

Substituting

$$7\left(\frac{1}{2} - \frac{1}{2}z - \frac{1}{26}t\right) + 9\left(\frac{21}{26}t - \frac{1}{2}z - \frac{1}{2}\right) + 8z - 7t = -1 \neq 2,$$

the expressions for x and y do not satisfy the third equation. Therefore, the system (S_2) is inconsistent and has no solutions.

Solving (S_3): This is an overdetermined system, meaning it contains more equations (3) than unknowns (2). As a result, the system is not square, and Cramer's rule cannot be applied directly to all three equations simultaneously. However, we can proceed by selecting any two equations to form a square system. We solve that reduced system using Cramer's rule, and then verify whether the obtained solution also satisfies the third equation.

We choose equations (1) and (2), forming the following reduced system:

$$(S_3') \begin{cases} 4x + 2y = 5 \\ 3x - y = 7 \end{cases}$$

This system is square (2 equations in 2 unknowns), so we can now apply Cramer's rule.

The coefficient matrix of this subsystem is

$$A_3' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}.$$

We first compute the determinant of this matrix

$$\det(A_3') = \begin{vmatrix} 6 & 4 \\ 5 & -1 \end{vmatrix} = -10 \neq 0$$

Since $\det(A_3')$, the system has a unique solution, and we can proceed with Cramer's rule.

We now compute the determinants Δ_x and Δ_y

$$\Delta_x = \begin{vmatrix} 5 & 2 \\ 7 & -1 \end{vmatrix} = -19,$$

$$\Delta_y = \begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix} = 13.$$

Using Cramer's rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\det(A_3')} = \frac{-19}{-10} = \frac{19}{10}, \\ y &= \frac{\Delta_y}{\det(A_3')} = \frac{13}{-10} = -\frac{13}{10}. \end{aligned}$$

We now verify whether this solution satisfies the third equation of the original system

$$2x + 6y = -4$$

Substituting

$$2\left(\frac{19}{10}\right) + 6\left(-\frac{13}{10}\right) = -4$$

Since the left-hand side equals the right-hand side, the solution satisfies the third equation.

The solution obtained from the reduced system also satisfies the third equation. Therefore, the original system (S_3) is consistent, and the unique solution is

$$\boxed{x = \frac{19}{10}, \quad y = -\frac{13}{10}}$$

Solving (S_4): This system is overdetermined, as it contains three equations and only two unknowns. Therefore, it does not form a square system, and Cramer's rule cannot be applied directly to all three equations.

Let us choose equations (1) and (2)

$$(S_4') \begin{cases} 2x + 3y = 7 \\ 3x - y = 5 \end{cases}$$

This forms a square system with two equations in two unknowns, making it suitable for Cramer's rule.

The coefficient matrix of this subsystem is

$$A_4' = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}.$$

We first compute the determinant of this coefficient matrix

$$\det(A_4') = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0$$

Since $\det(A_4')$, the system has a unique solution, and we can proceed with Cramer's rule.

We now compute the determinants Δ_x and Δ_y

$$\Delta_x = \begin{vmatrix} 7 & 3 \\ 5 & -1 \end{vmatrix} = -22,$$

$$\Delta_y = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = -11.$$

Using Cramer's rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\det(A_4')} = \frac{-22}{-11} = 2, \\ y &= \frac{\Delta_y}{\det(A_4')} = \frac{-11}{-11} = 1. \end{aligned}$$

We now verify whether this solution satisfies the third equation of the original system

$$x - 5y = -11$$

Substituting

$$(2) - 5(1) = -3 \neq -11$$

The left-hand side is not equal to the right-hand side. This indicates that the third equation is not satisfied by the solution of the first two equations. The system (S_4) is inconsistent. It has now solution satisfying all three equations.

Solving (S_5): The coefficient matrix is

$$A_5 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 4 & -1 \end{pmatrix}$$

its determinant is

$$\det(A_5) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 4 & -1 \end{vmatrix} = 3 \neq 0$$

Since $\det(A_5) \neq 0$, the system has a unique solution, and we can apply Cramer's rule.

We now compute the determinants Δ_x, Δ_y, and and Δ_z

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 14 & -1 & 3 \\ -2 & 4 & -1 \end{vmatrix} = -4,$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 14 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 4,$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 14 \\ 1 & 4 & -2 \end{vmatrix} = 18.$$

Using Cramer's rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\det(A_5)} = -\frac{4}{3}, \\ y &= \frac{\Delta_y}{\det(A_5)} = \frac{4}{3} = 1, \\ y &= \frac{\Delta_z}{\det(A_5)} = \frac{18}{3} = 6 \end{aligned}$$

Therefore, the system (S_3) is consistent, and the unique solution is

$$\boxed{x = -\frac{4}{3}, \quad y = \frac{4}{3}, z = 6}$$

Solving (S_6): The coefficient matrix is

$$A_6 = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

its determinant is

$$\det(A_6) = \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 0$$

Since $\det(A_7) = 0$, Cramer's rule is not applicable, and we analyze the system using rank.

First we calculate the rank of A_6 using determinant. We observe that

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$$

So

$$\boxed{\text{Rank}(A_6) = 2}$$

The augmented matrix is

$$\overline{A}_6 = \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 2 & 1 & -1 & 1 \end{array}\right).$$

Now, we calculate the rank of $\overline{A}_6$ using determinants. We observe that

$$\begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

So

$$\boxed{\text{Rank}(\overline{A}_6) = 3}$$

Since Rank$(\overline{A}_6)$ $\neq$Rank(A_6), the system (S_6) is inconsistent and has no solution.

Solving (S_7): The coefficient matrix is

$$A_7 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

its determinant is

$$\det(A_7) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{vmatrix} = 0$$

Since $\det(A_7) = 0$, Cramer's rule is not applicable, and we analyze the system using rank..

First we calculate the rank of A_7 using determinant. We observe that

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$$

So

$$\boxed{\text{Rank}(A_7) = 2}$$

The augmented matrix is

$$\overline{A}_7 = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & -1 & -1 \\ 2 & 2 & -1 & 0 \end{array}\right).$$

Now, Checking relevant 3×3 determinants

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 2 & -1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 2 & -1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 0 \end{vmatrix} = 0.$$

All relevant 3×3 submatrices (such as the ones formed by adding the constants column) have determinant 0, so

$$\boxed{\text{Rank}(\overline{A}_7) = 2}$$

Since the rank of the coefficient matrix A_7 equals the rank of the augmented matrix $\overline{A}_7$ but is less than the number of unknowns, the system has infinitely many solutions. We can proceed by selecting any two equations to form a square system. We solve that reduced system using Cramer's rule, and then verify whether the obtained solution also satisfies the third equation.
We choose equations (1) and (2), forming the following reduced system:

$$(S_7') \begin{cases} x + y + z = 3 \\ x + y - z = -1 \end{cases} \Leftrightarrow \begin{cases} x + z = 3 - y \\ x - z = -1 - y \end{cases}$$

The coefficient matrix of this subsystem is

$$A_7' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We first compute the determinant of this coefficient matrix

$$\det(A_7') = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

Since $\det(A_7')$, the system has a unique solution, and we can proceed with Cramer's rule.
We now compute the determinants Δ_x and Δ_y

$$\Delta_x = \begin{vmatrix} 3 - y & 1 \\ -1 - y & -1 \end{vmatrix} = 2y - 2,$$

$$\Delta_y = \begin{vmatrix} 1 & 3 - y \\ 1 & -1 - y \end{vmatrix} = -4.$$

Using Cramer's rule:

$$\begin{aligned} x &= \frac{\Delta_x}{\det(A_7')} = \frac{2y-2}{-2} = 1-y, \\ y &= \frac{\Delta_y}{\det(A_7')} = \frac{-4}{-2} = 2. \end{aligned}$$

We now verify whether this solution satisfies the third equation of the original system

$$2x + 2y - z = 0$$

Substituting

$$2(1-y) + 2y - (2) = 0$$

The left-hand side equal to the right-hand side. This indicates that the third equation is satisfied by the solution of the first two equations.
The system (S_4) has infinitely many solutions given by

$$\boxed{x = 1 - y,\ z = 2}$$

Exercise 11

Let a, b, and c be three real numbers. Determine the conditions under which the following matrix A is diagonalizable:

$$A = \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}$$

Solution Since A is upper triangular, its eigenvalues are on the diagonal:

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$$

So,

the eigenvalue 1 has algebraic multiplicity 2 (appears twice).

The eigenvalue c has algebraic multiplicity 1 (once).

We distinguish cases depending on the value of c

The geometric multiplicity of the eigenvalue $\lambda = 1$. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associated to the eigenvalue $\lambda = 1$. We solve the equation

$$(A - I)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} 1-1 & a & 1 \\ 0 & 1-1 & b \\ 0 & 0 & c-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 0 & a & 1 \\ 0 & 0 & b \\ 0 & 0 & c-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields the system of equations

$$\begin{cases} z + ay = 0 \\ bz = 0 \\ z\,(c-1) = 0 \end{cases}$$

We analyze the system based on values of b and c.

Case A: $\boxed{b \neq 0}$

From $bz = 0$, we get $z = 0$.

From $z + ay = 0$, we get $ay = 0$.

Subcase A1 if $\boxed{a \neq 0}$: Then $y = 0$. So eigenvectors take the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Subcase A2 if $\boxed{a = 0}$: Then y is free, and eigenvectors have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So the eigenspace is

$$\boxed{\begin{cases} 1 - \text{dimensional if } a \neq 0, \\ 2 - \text{dimensional if } a = 0. \end{cases}}$$

Case B $\boxed{b = 0}$: The second equation $bz = 0$ is always satisfied. The system becomes

$$\begin{cases} z + ay = 0 \\ z\,(c - 1) = 0 \end{cases}$$

We analyze further depending on c.

Subcase B.1 If $\boxed{c = 1}$: Then z is free

From $z + ay = 0$, we get

$$\begin{cases} y = -\dfrac{1}{a} z \quad \text{if } a \neq 0, \\ z = 0 \quad \text{if } a = 0. \end{cases}$$

If $a = 0$, then $z = 0$ and y is free. So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

If $a \neq 0$, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -\dfrac{z}{a} \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -\dfrac{1}{a} \\ 1 \end{pmatrix}.$$

So eigenspace is

$$\boxed{\begin{cases} 2-\text{dimensional if } a \neq 0, \\ 2-\text{dimensional if } a = 0. \end{cases}}$$

Subcase B.2 If $c \neq 1$: Then $z = 0 \Longrightarrow ay = 0$.

If $a = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

If $a \neq 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

So the eigenspace is

$$\boxed{\begin{cases} 1-\text{dimensional if } a \neq 0, \\ 2-\text{dimensional if } a = 0. \end{cases}}$$

The geometric multiplicity of eigenvalue $\lambda = c$. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = c$. We solve the equation

$$(A - cI)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} 1-c & a & 1 \\ 0 & 1-c & b \\ 0 & 0 & c-c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 1-c & a & 1 \\ 0 & 1-c & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} z - x\,(c-1) + ay = 0 \\ bz - y\,(c-1) = 0 \end{cases}$$

We now analyze this system based on the value of c.

Case A: If $c = 1$, the system becomes

$$\begin{cases} z + ay = 0 \\ bz = 0 \end{cases}$$

We analyze this system based on the values of a and b.

Subcase A1: $b \neq 0$.

From the second equation

$$bz = 0 \Longrightarrow z = 0$$

From the first equation $ay = 0$.

if $a \neq 0 \Longrightarrow y = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

if $a = 0 \Longrightarrow y = 0$ is free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So eigenspace is

$$\boxed{\begin{cases} 1 - \text{dimensional if } a \neq 0, \\ 2 - \text{dimensional if } a = 0. \end{cases}}$$

Subcase A2: $b = 0$. Then the second equation is always satisfied.

From the first equation

$$z + ay = 0 \Longrightarrow z = \begin{cases} -ay \text{ if } a \neq 0 \\ 0 \text{ if } a = 0 \end{cases}$$

If $a = 0$, then $z = 0$, and y is free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

If $a \neq 0$, then $z = -ay$. So eigenvectors have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -a \end{pmatrix}.$$

So eigenspace is

$$\boxed{\begin{cases} 1 - \text{dimensional if } a \neq 0, \\ 2 - \text{dimensional if } a = 0. \end{cases}}$$

<u>**Case A:**</u> If $c \neq 1$

From the second equation

$$bz - y(c-1) = 0 \Longrightarrow y = \frac{b}{c-1} z.$$

Substitute into the first equation

$$\begin{aligned} z - x(c-1) + a\left(\frac{b}{c-1}z\right) &= 0 \Longrightarrow z - x(c-1) + ab\frac{z}{c-1} = 0 \\ &\Longrightarrow x = \frac{c+ab-1}{(c-1)^2}z \end{aligned}$$

So all variables are expressed in terms of z, which is free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{c+ab-1}{(c-1)^2}z \\ \frac{b}{c-1}z \\ z \end{pmatrix} = z \begin{pmatrix} \frac{c+ab-1}{(c-1)^2} \\ \frac{b}{c-1} \\ 1 \end{pmatrix}.$$

Hence, the eigenspace is generated by a single vector $\Longrightarrow$ 1−dimensional.

The matrix A is diagonalizable if and only if the sum of the geometric multiplicities of all its eigenvalues equals its size (in this case, 3), and each eigenvalue's geometric multiplicity equals its algebraic multiplicity. Therefore, the matrix A is diagonalizable if and only if

$$\boxed{a = 0 \text{ and } c \neq 1.}$$

Exercise 12

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear endomorphism, whose matrix in the standard (canonical) basis is

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix}$$

1. Find the eigenvalues of the matrix A.

2. Show that there exists a basis of eigenvectors in which the matrix of the transformation f is diagonalizable.

3. Determine the change of basis matrix P, from the standard basis to the eigenbasis.

4. Compute $P^{-1}$and evaluate $P^{-1}AP$.

5. Show, by direct computation, that the matrix A satisfies its characteristic polynomial.

Solution

1. Find the eigenvalues of the matrix A: To determine the eigenvalues of A, we compute its characteristic polynomial, defined as

$$P_A(\lambda) = \det(A - \lambda I_3)$$

Given the matrix A, we have

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 3 & -4-\lambda & 12 \\ 1 & -2 & 5-\lambda \end{vmatrix}$$

To compute the determinant, we expand along the first row

$$\begin{aligned} \det(A - \lambda I_3) &= (2-\lambda)\begin{vmatrix} -4-\lambda & 12 \\ -2 & 5-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & -4-\lambda \\ 1 & -2 \end{vmatrix} \\ &= (2-\lambda)\left(\lambda^2 - \lambda + 4\right) + 4(\lambda - 2) \\ &= (2-\lambda)\left(\lambda^2 - \lambda + 4 - 4\right) \\ &= (2-\lambda)\left(\lambda^2 - \lambda\right) \\ &= \lambda(\lambda - 1)(2-\lambda). \end{aligned}$$

Thus

$$\boxed{P_A(\lambda) = \lambda(\lambda - 1)(2-\lambda).}$$

So the eigenvalues are

$$\boxed{\lambda = 0, \quad \lambda = 1, \quad \lambda = 2.}$$

2. Show that A is diagonalizable: Since the characteristic polynomial splits into distinct roots, and we can find one linearly independent eigenvector for each eigenvalue, the matrix is diagonalizable.

3. Determine the change of basis matrix P: First, we compute the eigenvectors

$\underline{\lambda = 0}$: Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = 0$. We solve the equation

$$(A - (0)\, I_3)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} 2-0 & 0 & 4 \\ 3 & -4-0 & 12 \\ 1 & -2 & 5-0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} 2x + 4z = 0 \\ 3x - 4y + 12z = 0 \\ x - 2y + 5z = 0 \end{cases}$$

From the first equation

$$2x + 4z = 0 \Longrightarrow x = -2z.$$

Substitute into the third equation

$$(-2z) - 2y + 5z = 0 \Longrightarrow 3z - 2y \Longrightarrow y = \frac{3}{2}z.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ \frac{3}{2}z \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ \frac{3}{2} \\ 1 \end{pmatrix}.$$

So, the eigenvector for $\lambda = 0$ is

$$v_1 = \boxed{\begin{pmatrix} -2 \\ \frac{3}{2} \\ 1 \end{pmatrix}.}$$

$\underline{\lambda = 1}$: Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = 1$. We solve the equation

$$(A - (1)\, I_3)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} 2-1 & 0 & 4 \\ 3 & -4-1 & 12 \\ 1 & -2 & 5-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 1 & 0 & 4 \\ 3 & -5 & 12 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} x + 4z = 0 \\ 3x - 5y + 12z = 0 \\ x - 2y + 4z = 0 \end{cases}$$

From the first equation

$$x + 4z = 0 \Longrightarrow x = -4z.$$

Substitute into the third equation

$$(-4z) - 2y + 4z = 0 \Longrightarrow -2y = 0 \Longrightarrow y = 0.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4z \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

So, the eigenvector for $\lambda = 1$ is

$$v_2 = \boxed{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.}$$

$\underline{\lambda = 2}$: Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = 2$. We solve the equation

$$(A - 2I_3)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} 2-2 & 0 & 4 \\ 3 & -4-2 & 12 \\ 1 & -2 & 5-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 0 & 0 & 4 \\ 3 & -6 & 12 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} 4z = 0 \\ 3x - 6y + 12z = 0 \\ x - 2y + 3z = 0 \end{cases}$$

From the first equation

$$4z = 0 \Longrightarrow z = 0.$$

Substitute into the third equation

$$x - 2y + 3(0) = 0 \Longrightarrow x - 2y = 0 \Longrightarrow x = 2y.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

So, the eigenvector for $\lambda = 2$ is

$$v_3 = \boxed{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.}$$

So the change of basis matrix P, formed by placing the eigenvectors as columns, is

$$P = \begin{pmatrix} -2 & -4 & 2 \\ \frac{3}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

4. Compute $P^{-1}$and evaluate $P^{-1}AP$:

The inverse of P: To calculate P^{-1}**:** We use the formula

$$A^{-1} = \frac{1}{\det(P)} \text{adj}(P).$$

We compute the determinant of P :

$$\det(P) = \begin{vmatrix} -2 & -4 & 2 \\ \frac{3}{2} & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1$$

We now compute the adjugate matrix $\mathrm{adj}(P)$, which is the transpose of the cofactor matrix $\mathrm{com}(A)$.

The cofactor matrix is

$$\mathrm{com}(P) = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \frac{3}{2} & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} \frac{3}{2} & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} -2 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -2 & -4 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -4 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ \frac{3}{2} & 1 \end{vmatrix} & \begin{vmatrix} -2 & -4 \\ \frac{3}{2} & 0 \end{vmatrix} \end{pmatrix}$$

So

$$\mathrm{com}(P) = \begin{pmatrix} -1 & 1 & \frac{3}{2} \\ 2 & -2 & -2 \\ -4 & 5 & 6 \end{pmatrix}.$$

The transpose of the cofactor matrix (i.e. the adjugate $\mathrm{adj}(P)$) is

$$\mathrm{adj}(P) = (\mathrm{com}(P))^T = \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & 5 \\ \frac{3}{2} & -2 & 6 \end{pmatrix}.$$

Now

$$P^{-1} = \frac{1}{\det(P)}\mathrm{adj}(P) = \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & 5 \\ \frac{3}{2} & -2 & 6 \end{pmatrix}.$$

So

$$P^{-1} = \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & 5 \\ \frac{3}{2} & -2 & 6 \end{pmatrix}.$$

Now we compute the diagonal matrix $P^{-1}AP$:

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & 5 \\ \frac{3}{2} & -2 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} -2 & -4 & 2 \\ \frac{3}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

So, A is diagonalizable as

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

5. Verify that A satisfies its characteristic polynomial (Cayley-Hamilton theorem): We check whether

$$P_A(A) = A(A-1)(2-A) = -A^3 + 3A^2 - 2A = 0.$$

We have

$$-A^3 = -\begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix}.$$

So

$$-A^3 = \begin{pmatrix} -20 & 24 & -76 \\ -12 & 16 & -48 \\ -1 & 2 & -5 \end{pmatrix},$$

and

$$3A^2 = 3\begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix}.$$

So

$$3A^2 = \begin{pmatrix} 24 & -24 & 84 \\ 18 & -24 & 72 \\ 3 & -6 & 15 \end{pmatrix}.$$

Therefore

$$\begin{aligned} -A^3 + 3A^2 - 2A &= \begin{pmatrix} -20 & 24 & -76 \\ -12 & 16 & -48 \\ -1 & 2 & -5 \end{pmatrix} + \begin{pmatrix} 24 & -24 & 84 \\ 18 & -24 & 72 \\ 3 & -6 & 15 \end{pmatrix} - 2\begin{pmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, matrix A satisfies its characteristic polynomial, confirming the Cayley-Hamilton theorem.

Exercise 13

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear endomorphism, whose matrix in the standard (canonical) basis is

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

1. Find the eigenvalues of the matrix A.

2. Find the eigenvectors of the matrix A.

3. Show that there exists a basis of eigenvectors in which the matrix of the transformation f is diagonalizable.

4. Determine the change of basis matrix P, from the standard basis to the eigenbasis.

5. Compute $P^{-1}$and evaluate $P^{-1}AP$.

6. Show, by direct computation, that the matrix A satisfies its characteristic polynomial.

Solution

1. Find the eigenvalues of the matrix A: To determine the eigenvalues of A, we compute its characteristic polynomial, defined as

$$P_A(\lambda) = \det(A - \lambda I_3)$$

Given the matrix A, we have

$$\det(A - \lambda I_3) = \begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

To compute the determinant, we expand along the first row

$$\begin{aligned} \det(A - \lambda I_3) &= (-1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= (-1-\lambda)\left(\lambda^2 + 2\lambda\right) + \lambda + 2 + \lambda + 2 \\ &= (\lambda+2)\left((-1-\lambda)\lambda + 2\right) \\ &= (1-\lambda)(\lambda+2)^2 . \end{aligned}$$

Thus, the characteristic polynomial is

$$\boxed{P_A(\lambda) = (1-\lambda)(\lambda+2)^2.}$$

So the eigenvalues are

$$\boxed{\lambda = 1 \text{ (algebraic multiplicity 1)}, \quad \lambda = -2 \text{ (algebraic multiplicity 2)}.}$$

2. Find the eigenvectors of the matrix A.

$\underline{\lambda = 1}$: Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = 1$. We solve the equation

$$(A - (1)\, I_3)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} -1-1 & 1 & 1 \\ 1 & -1-1 & 1 \\ 1 & 1 & -1-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{cases}$$

From the first equation

$$-2x + y + z = 0 \Longrightarrow y = 2x - z.$$

Substitute $y = 2x - z$ into the second equation

$$x - 2\,(2x - z) + z = 0 \Longrightarrow x = z$$

So

$$y = 2z - z = z.$$

Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So, the eigenvector for $\lambda = 1$ is

$$v_1 = \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

The geometric multiplicity is 1.

$\underline{\lambda = -2}$: Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda = 1$.

We solve the equation

$$(A - 2I_3)\, X = 0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix} -1-(-2) & 1 & 1 \\ 1 & -1-(-2) & 1 \\ 1 & 1 & -1-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} x + y + z = 0 \\ x + y + z = 0 \\ x + y + z = 0 \end{cases}$$

From one of the equation

$$x = -y - z.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

So, the eigenvectors for $\lambda = -2$ are

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

The geometric multiplicity is 2.

3. Diagonalizability of A: We found two linearly independent eigenvectors. The geometric multiplicities equal the algebraic multiplicities for all eigenvalues. Therefore A is **Diagonalizable.**

4. Determine the change of basis matrix P, from the standard basis to the eigenbasis: The matrix P is formed by placing the eigenvectors as columns. So

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

5. Compute $P^{-1}$and evaluate $P^{-1}AP$:

<u>The inverse of P</u>: To calculate P^{-1}**:** We use the formula

$$A^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P).$$

We compute the determinant of P :

$$\det(P) = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 3.$$

We now compute the adjugate matrix adj(P), which is the transpose of the cofactor matrix com(A).

The cofactor matrix is

$$\operatorname{com}(P) = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \end{pmatrix}$$

: -1 : -1 So

$$\operatorname{com}(P) = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

The transpose of the cofactor matrix (i.e. the adjugate adj(P)) is

$$\operatorname{adj}(P) = (\operatorname{com}(P))^T = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Now

$$P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

So

$$P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Now we compute the diagonal matrix $P^{-1}AP$:

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

So, A is diagonalizable as

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

6. Verify that A satisfies its characteristic polynomial (Cayley-Hamilton theorem): We check whether

$$P_A(A) = (1-A)(A+2)^2 = -A^3 - 3A^2 + 4I_3 = 0.$$

We have

$$-A^3 = -\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

So

$$-A^3 = \begin{pmatrix} 5 & -3 & -3 \\ -3 & 5 & -3 \\ -3 & -3 & 5 \end{pmatrix},$$

and

$$-3A^2 = -3\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

So

$$-3A^2 = \begin{pmatrix} -9 & 3 & 3 \\ 3 & -9 & 3 \\ 3 & 3 & -9 \end{pmatrix}.$$

Therefore

$$\begin{aligned} -A^3 - 3A^2 + 4I_3 &= \begin{pmatrix} 5 & -3 & -3 \\ -3 & 5 & -3 \\ -3 & -3 & 5 \end{pmatrix} + \begin{pmatrix} -9 & 3 & 3 \\ 3 & -9 & 3 \\ 3 & 3 & -9 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, matrix A satisfies its characteristic polynomial, confirming the Cayley-Hamilton theorem.

Exercise 14 Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear endomorphism, whose matrix in the standard (canonical) basis is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Find the eigenvalues of the matrix A.

2. Find the eigenvectors of the matrix A.

3. Determine whether the matrix A is diagonalizable.

Solution

1. Find the eigenvalues of the matrix A: To determine the eigenvalues of A, we compute its characteristic polynomial

$$P_A(\lambda) = \det(A - \lambda I_3)$$

Given the matrix A, we have

$$\det(A-\lambda I_3)=\begin{vmatrix}1-\lambda & 0 & 0\\ 0 & 1-\lambda & -1\\ 0 & 0 & 1-\lambda\end{vmatrix}.$$

This is an upper triangular matrix, so its determinant is the product of the diagonal elements

$$\det(A-\lambda I_3)=(1-\lambda)^3.$$

Thus, the characteristic polynomial is

$$P_A(\lambda)=(1-\lambda)^3.$$

So the only eigenvalue is

$$\lambda=1 \text{ (algebraic multiplicity 3)}$$

2. Find the eigenvectors of the matrix A: Let $\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ be the eigenvector associate to the eigenvalue $\lambda=1$. We solve the system

$$(A-(1)I_3)X=0_{\mathbb{R}^3}.$$

That is

$$\begin{pmatrix}1-1 & 0 & 0\\ 0 & 1-1 & -1\\ 0 & 0 & 1-1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

Which simplifies to

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

This yields the system of equations

$$\begin{cases} 0 = 0 \\ -z = 0 \\ 0 = 0 \end{cases}$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So, The eigenvectors corresponding to $\lambda = 1$ are

$$\boxed{v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.}$$

3. Diagonalizability of A: The algebraic multiplicity of $\lambda = 1$ is 3, but the geometric multiplicity is only 2. Since the geometric multiplicity is strictly less than the algebraic multiplicity, the matrix A is **not diagonalizable**.

CHAPTER 3

Exercises Related to the Chapter: Taylor Expansions

Contents

This chapter offers a collection of fully solved exercises on Taylor expansions, with a focus on developing approximation techniques and recognizing their significance in analysis. Exercises explore expansions around zero, expansions at arbitrary points, and generalized series representations. Through detailed step-by-step solutions, students will gain proficiency in constructing and applying Taylor series in various mathematical contexts, particularly in approximating functions and analyzing local behavior.

3.1 Course Recap: Taylor Expansions (Limited Developments)

3.1.1 Limited Developments of Order n Near 0

Definition. Let f be a real-valued function defined on a neighborhood of 0. We say that f admits a **limited development of order** n at 0 if there exist real numbers $a_0, a_1, \ldots, a_n$ such that:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n) \quad \text{as } x \to 0.$$

Here, $o(x^n)$ denotes a function that tends to zero faster than x^n as $x \to 0$; that is

$$\lim_{x \to 0} \frac{o(x^n)}{x^n} = 0.$$

This may also be written using summation notation as

$$f(x) = \sum_{k=0}^{n} a_k x^k + o(x^n).$$

If f is n-times differentiable at 0, the coefficients a_k are given by:

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Hence, the Taylor expansion of order n at 0 (also called the Maclaurin expansion) is

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + o(x^n).$$

Remark.

- In the expression

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n),$$

the sum $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is called the **main part** (or t**he principal part**) of the expansion, and it is a polynomial of degree at most n. The term $o(x^n)$ is the negligible part or error term, which becomes insignificant compared to x^n as $x \to 0$.

- The order of the development is determined by the exponent of the lowest power of x for which the remainder is strictly of smaller order, not merely by the degree of the polynomial part.

3.1.2 Uniqueness

Theorem (Uniqueness): If a function f admits a limited development of order n near 0, then this expansion is **unique**.

3.1.3 Common Limited Developments via Maclaurin Formula

$$
\begin{aligned}
&e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)\\
&\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)\\
&\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n), \quad |x| < 1\\
&(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + o(x^n), \quad |x| < 1\\
&\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + o(x^n)\\
&\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + o(x^n)\\
&\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + o(x^n)
\end{aligned}
$$

3.1.4 Operations on Limited Developments

Limited Developments by Restriction

If f admits a limited development, we write:

$$f(x) \approx a_0 + a_1 x + \cdots + a_n x^n$$

This is used for polynomial approximations.

3.1.5 Algebraic Operations

Given

$$f(x) = \sum a_k x^k + o(x^n) \ g(x) = \sum b_k x^k + o(x^n),$$

the following operations hold, keeping only terms of degree $\leq n$ in the final expression

- $f(x) + g(x) = \sum (a_k + b_k) x^k + o(x^n)$
- $f(x) \cdot g(x) = \sum c_k x^k + o(x^n)$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$
- If $a_0 \neq 0$, then $\dfrac{1}{f(x)}$ can also be expanded as a limited development of order n.

3.1.6 Composition of Functions

If $f(x)$ admits a limited development at 0 and $g(x) \to 0$ as $x \to 0$, then

$$f(g(x)) = \text{the expansion of } f(x) \text{ with } x \text{ is replaced with } g(x),$$

retaining terms up to degree n, if appropriate.

3.1.7 Integration and Derivation

If

$$f(x) = \sum a_k x^k + o(x^n),$$

then

- Its integral is given by:

$$\int f(x)dx = C + \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1} + o(x^{n+1}),$$

  where C is the constant of integration.

- Its derivative is

$$f'(x)dx = \sum_{k=1}^{n} k a_k x^{k-1} + o(x^{n-1}).$$

Note: The derivative starts from $k = 1$ because the derivative of the constant term a_0 is zero.

3.1.8 Limited Developments Near a Point x_0

If f is n-times differentiable at a point x_0, then the Taylor expansion of order n near x_0 is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n).$$

3.1.9 Generalized Limited Developments

To study the behavior of a function $f(x)$ as $x \to \infty$, we apply a change of variable $t = \frac{1}{x}$, so that $t \to 0$ as $x \to \infty$. If the function $f\left(\frac{1}{t}\right)$ admits a limited development near 0 of the form

$$f\left(\frac{1}{t}\right) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + o(t^n),$$

then by replacing $t = \frac{1}{x}$, we obtain the **asymptotic expansion of $f(x)$ at infinity**

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + o(\frac{1}{x^n}), \text{ as } x \to \infty.$$

Such an expansion provides a **polynomial approximation in terms of negative powers of x**, valid for large values of x.

3.1.10 Applications of Limited Developments

Taylor expansions (limited developments) are powerful tools used in various areas of mathematics and applied sciences. They allow us to:

- Approximate functions near a given point using polynomials.
- Compute limits, especially indeterminate forms.
- Study local behavior (convexity, inflection points, etc.)
- Solve differential equations approximately
- Develop numerical methods (Newton-Raphson, interpolation, etc.)

3.2 Solved Exercises

Exercise 01

Find the Taylor expansion near 0 of the following functions, up to the indicated order:

1) The third-degree Taylor expansion of $e^x \cos x$

2) The fourth-degree Taylor expansion of $e^{\sin x}$

3) The fifth-degree Taylor expansion of $e^{\cos x}$

4) The third-degree Taylor expansion of $\sin(\ln(1+x))$

5) The third-degree Taylor expansion of $\ln(1+\sin 2x)$

Solution

1) The third-degree Taylor expansion near 0 of $e^x \cos x$: We begin with the Taylor expansions near $x = 0$ for the functions e^x and $\cos x$. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o\left(x^3\right),$$
$$\cos x = 1 - \frac{x^2}{2!} + o\left(x^3\right).$$

Multiply e^x and $\cos x$ up to terms of order x^3, we get

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o\left(x^3\right)\right)\left(1 - \frac{x^2}{2!} + o\left(x^3\right)\right).$$

Multiply and keep terms up to x^3, we obtain

$$e^x \cos x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^2}{2!} - \frac{x^3}{2!} + o\left(x^3\right).$$

Group like terms, we get

$$\boxed{e^x \cos x = 1 + x - \frac{x^3}{3} + o\left(x^3\right)}$$

2) The fourth-degree Taylor expansion near 0 **of** $e^{\sin x}$: Let us define the functions

$$f(x) = e^x \text{ and } g(x) = \sin x,$$

so that

$$h(x) = e^{\sin x} = f(g(x)) = (f \circ g)(x).$$

Since $g(0) = \sin 0 = 0$, we can proceed by substituting the Taylor expansions of $f(x)$ and $g(0)$ around 0

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o\left(x^4\right),$$
$$\sin x = x - \frac{x^3}{3!} + o\left(x^4\right).$$

Now substitute $g(x) = \sin x$ into the expansion of $f(x) = e^x$, and expand up to degree 4

$$\begin{aligned} h(x) &= e^{\sin x} = (f \circ g)(x) \\ &= 1 + \left(x - \frac{x^3}{3!}\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!}\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!}\right)^3 + \frac{1}{4!}\left(x - \frac{x^3}{3!}\right)^4 + o\left(x^4\right). \end{aligned}$$

Simplifying the terms yields

$$\boxed{e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + o\left(x^4\right).}$$

3) The fifth-degree Taylor expansion near zero of $e^{\cos x}$: We begin by writing

$$e^{\cos x} = e.e^{\cos -1}.$$

Let us define

$$f(x) = e^x \text{ and } g(x) = \cos x - 1,$$

so that

$$h(x) = e^{\cos x} = e.e^{\cos -1} = e\,(f \circ g)(x).$$

Since $g(0) = -1 + \cos 0 = 0$, we can expand $(f \circ g)(x)$ using the Taylor series of $f(x) = e^x$ and $g(x)$ up to degree 5.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + o\left(x^5\right), \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^5\right), \end{aligned}$$

so,

$$g(x) = \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^5\right)$$

Now substitute $g(x)$ into the expansion of $f(x)$

$$\begin{aligned} e^{g(x)} &= e^{\cos x - 1} = (f \circ g)(x) \\ &= 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \frac{1}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^3 \\ &\quad + \frac{1}{4!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^4 + \frac{1}{5!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^5 + o\left(x^5\right). \end{aligned}$$

Simplifying the series gives

$$e^{g(x)} = 1 - \frac{x^2}{2} + \frac{x^4}{6} + o\left(x^5\right).$$

Finally, multiply by e to obtain

$$\boxed{e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6}\right) + o\left(x^5\right).}$$

4. The third-degree Taylor expansion near 0 **of** $\sin\left(\ln\left(1+x\right)\right)$: The third-order Taylor expansion of $\ln\left(1+x\right)$ near $x=0$ is

$$\ln\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^3\right),$$

and Taylor expansion of $\sin y$ near $y=0$ is

$$\sin y = y - \frac{y^3}{3!} + o\left(y^3\right).$$

Since $\ln 1 = 0$, we substitute the expansion of $\ln\left(1+x\right)$ into the expansion of $\sin x$ to obtain

$$\sin\left(\ln\left(1+x\right)\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^3}{3!} + o\left(x^3\right).$$

Keeping terms up to degree 3, we get

$$\boxed{\sin\left(\ln\left(1+x\right)\right) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o\left(x^3\right).}$$

5. The third-degree Taylor expansion near 0 **of** $\ln\left(1+\sin 2x\right)$: Let us consider the function

$$\sin 2x = \sin u, \quad \text{where } u = 2x.$$

The third-order Taylor expansion of $\sin u$ near 0 is

$$\sin u = u - \frac{u^3}{3!} + o\left(u^3\right).$$

By substituting $u = 2x$, we obtain

$$\begin{aligned} \sin u &= \sin 2x = 2x - \frac{(2x)^3}{3!} + o\left(u^3\right) \\ &= 2x - \frac{4}{3}x^3 + o\left(x^3\right). \end{aligned}$$

Furthermore, since $\sin 0 = 0$, we can write

$$\ln(1+\sin 2x) = \ln(1+v), \text{ where } v = \sin 2x.$$

We know that the third-order Taylor expansion of $\ln(1+v)$ near 0 is

$$\ln(1+v) = v - \frac{v^2}{2} + \frac{v^3}{3} + o\left(v^3\right)$$

Substituting the expansion of $\sin 2x$ into this, we get

$$\ln(1+\sin 2x) = \left(2x - \frac{4}{3}x^3\right) - \frac{\left(2x - \frac{4}{3}x^3\right)^2}{2} + \frac{\left(2x - \frac{4}{3}x^3\right)^3}{3} + o\left(x^3\right).$$

Now, neglecting all terms of degree higher than 3, we obtain

$$\boxed{\ln(1+\sin 2x) = 2x - 2x^2 + \frac{4}{3}x^3 + o\left(x^3\right).}$$

Exercise 02

Calculate the second-order Taylor expansion near 0 of the following functions:

$$\textbf{1)}\ \frac{e^x}{\sqrt{1+2x}}, \quad \textbf{2)}\ \frac{\ln(1+2x)}{\sin 2x}.$$

Solution

1) The second-order Taylor expansion near 0 **of** $\frac{e^x}{\sqrt{1+2x}}$: We begin by rewriting the given expression in a more convenient form

$$\frac{e^x}{\sqrt{1+2x}} = e^x (1+2x)^{-\frac{1}{2}}$$

Next, we compute the second-order Taylor expansions of each factor separately.

The Taylor expansion of e^x near $x = 0$ is

$$e^x = 1 + x + \frac{x^2}{2!} + o\left(x^2\right),$$

The general formula for the Taylor expansion of $(1+u)^\alpha$ near $u = 0$ is

$$(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + o\left(u^2\right)$$

Applying it with $u = 2x$ and $\alpha = -\frac{1}{2}$, we get

$$\begin{aligned}
(1+u)^\alpha &= (1+2x)^{-\frac{1}{2}} \\
&= 1 + \left(-\frac{1}{2}\right)(2x) + \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)}{2}(2x)^2 + o\left(x^2\right) \\
&= 1 - x + \frac{3}{2}x^2 + o\left(x^2\right)
\end{aligned}$$

Now multiply the two series

$$\left(1 + x + \frac{x^2}{2!}\right)\left(1 - x + \frac{3}{2}x^2\right) + o\left(x^2\right)$$

Expanding and keeping terms up to degree 2, we get

$$\boxed{e^x(1+2x)^{-\frac{1}{2}} = \frac{e^x}{\sqrt{1+2x}} = 1 + x^2 + o\left(x^2\right).}$$

2) The second-order Taylor expansion near 0 **of** $\frac{\ln(1+2x)}{\sin 2x}$**:**

We begin with the Taylor expansion of $\ln(1+u)$ near $u = 0$

$$\ln(1+u) = u - \frac{(u)^2}{2} + \frac{(u)^3}{3} + o\left(u^3\right).$$

Applying this with $u = 2x$, we get

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + o\left(x^3\right) = 2x - 2x^2 + \frac{8}{3}x^3 + o\left(x^3\right)$$

Next, we use the Taylor expansion of $\sin v$ near $v = 0$

$$\sin v = v - \frac{v^3}{3!} + o\left(v^3\right)$$

Applying this with $v = 2x$, we obtain

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + o\left(x^3\right) = 2x - \frac{4}{3}x^3 + o\left(x^3\right).$$

We now factor out $2x$ from both the numerator and denominator

$$\frac{\ln(1+2x)}{\sin 2x} = \frac{2x\left(1 - x + \frac{4}{3}x^2\right)}{2x\left(1 - \frac{2x^2}{3}\right)} + o\left(x^2\right) = \frac{1 - x + \frac{4}{3}x^2}{1 - \frac{2x^2}{3}} + o\left(x^2\right).$$

We now apply the Taylor expansion of $\frac{1}{1+s}$ near $s = 0$

$$\frac{1}{1+s} = 1 + s + s^2 + o\left(s^2\right),$$

but since we have $1 - \frac{2x^2}{3}$, we treat this as

$$\frac{1}{1 - \frac{2x^2}{3}} = 1 + \left(\frac{2x^2}{3}\right) + \left(\frac{2x^2}{3}\right)^2 + o\left(x^2\right).$$

Keeping terms up to degree 2, we get

$$\frac{1}{1 - \frac{2x^2}{3}} = 1 + \frac{2}{3}x^2 + o\left(x^2\right).$$

Now multiply the numerator and the reciprocal of the denominator

$$\frac{\ln(1+2x)}{\sin 2x} = \left(1 - x + \frac{4}{3}x^2\right)\left(1 + \frac{2}{3}x^2\right) + o\left(x^2\right)$$

Multiplying and keeping terms up to order x^2, we obtain

$$\boxed{\frac{\ln(1+2x)}{\sin 2x} = 1 - x + 2x^2 + o\left(x^2\right).}$$

Exercise 03

Calculate the second-order Taylor expansion near 0 of the following functions by reducing them to known expansions:

$$\textbf{1)}\ \frac{1}{4+3x}, \quad \textbf{2)}\ \ln(5+3x), \quad \textbf{3)}\ e^{3+2x}.$$

Solution

1) Expansion of $\frac{1}{4+3x}$ near $x=0$, up to order 2: We start by rewriting the expression in a more convenient form

$$\frac{1}{4+3x} = \frac{1}{4\left(1+\frac{3x}{4}\right)} = \frac{1}{4}\frac{1}{\left(1+\frac{3x}{4}\right)}.$$

Let $u = \frac{3x}{4}$, which tends to 0 as $x \to 0$, so we can use the known second-order Taylor expansion of $\frac{1}{1+u}$ near $u=0$. We have

$$\frac{1}{1-u} = 1 + u + u^2 + o\left(u^2\right),$$

so

$$\frac{1}{1+u} = \frac{1}{1-(-u)} = 1-(-u)+(-u)^2+o\left(u^2\right) = 1-u+u^2+o\left(u^2\right).$$

Substituting $\frac{3x}{4}$ into the expansion, we get

$$\frac{1}{1+\frac{3x}{4}} = 1-\frac{3x}{4}+\left(\frac{3x}{4}\right)^2+o\left(x^2\right).$$

Multiplying by $\frac{1}{4}$, we obtain

$$\frac{1}{4+3x} = \frac{1}{4}\left(1-\frac{3x}{4}+\left(\frac{3x}{4}\right)^2\right)+o\left(x^2\right).$$

Thus

$$\boxed{\frac{1}{4+3x}=\frac{1}{4}-\frac{3}{16}x+\frac{9}{64}x^2+o\left(x^2\right).}$$

2) Expansion of $\ln(5+3x)$ **near** $x=0$**, up to order 2:** We start by rewriting the expression

$$\ln(5+3x)=\ln\left(5\left(1+\frac{3}{5}x\right)\right)=\ln 5+\ln\left(1+\frac{3}{5}x\right).(\text{since } \ln(a.b)=\ln a+\ln b)$$

Let $u=\frac{3}{5}x$, which tends to 0 as $x\to 0$. Using the known Taylor expansion of $\ln(1+u)$ near $u=0$, up to second order

$$\ln(1+u)=u-\frac{u^2}{2}+o\left(u^2\right),$$

we substitute $u=u=\frac{3}{5}x$ to obtain

$$\ln\left(1+\frac{3}{5}x\right)=\frac{3}{5}x-\frac{1}{2}\left(\frac{3}{5}x\right)^2+o\left(x^2\right)=\frac{3}{5}x-\frac{9}{50}x^2+o\left(x^2\right).$$

Adding $\ln 5$, we get

$$\boxed{\ln(5+3x)=\ln 5+\frac{3}{5}x-\frac{9}{50}x^2+o\left(x^2\right).}$$

3) Expansion of e^{3+2x} **near** $x=0$**, up to order 2:** We start the expression using the exponential identity $e^{a.b}=\left(e^a\right)\left(e^b\right)$

$$e^{3+2x}=\left(e^3\right)\left(e^{2x}\right).$$

Let $u=2x$, which tends to 0 as $x\to 0$. We now using the second-order Taylor expansion of e^u near $u=0$, up to second order

$$e^u=1+u+\frac{u^2}{2!}+o\left(u^2\right).$$

Substitute $u=2x$ into the expansion, we get

$$e^{2x}=1+2x+\frac{1}{2!}(2x)^2+o\left(x^2\right)=1+2x+2x^2+o\left(x^2\right).$$

Multiplying by e^3, we find

$$\boxed{e^{3+2x} = e^3 + 2e^3x + 2e^3x^2 + o\left(x^2\right).}$$

Exercise 04

1) Compute the Taylor expansion near 0, up to order 4, of the function $\ln\left(\dfrac{\sin x}{x}\right)$.

2) Compute the Taylor expansion near 0, up to order 5, of the function $(\cos x)^{\sin x}$.

Solution

1) Expansion of $\ln\left(\dfrac{\sin x}{x}\right)$ **near** 0**, up to order 4:** We start from the Taylor expansion of $\sin x$ near $x = 0$, up to order 5. We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o\left(x^5\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o\left(x^5\right).$$

Dividing by x (for $x \neq 0$), we obtain

$$\frac{\sin x}{x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5}{x} + o\left(x^4\right) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + o\left(x^4\right).$$

Let us now expand the logarithm. We use the Taylor expansion of $\ln(1+u)$ near $u = 0$, up to order 3

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} o\left(u^3\right).$$

Let

$$u = \frac{\sin x}{x} - 1 = \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - 1 + o\left(x^4\right)\right) - 1 = -\frac{1}{6}x^2 + \frac{1}{120}x^4 + o\left(x^4\right).$$

Then, substituting into the logarithm expansion gives

$$\ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6}x^2 - \frac{1}{2}\left(-\frac{1}{6}x^2 + \frac{1}{120}x^4\right)^2 + o\left(x^4\right).$$

Keeping terms up to degree 4, we get

$$\boxed{\ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6}x^2 - \frac{1}{72}x^4 + o\left(x^4\right).}$$

2) Expansion of $(\cos x)^{\sin x}$ **near** x**= 0, up to order 5**: Since we are dealing with an exponential of a function, we apply logarithms first

$$(\cos x)^{\sin x} = e^{\ln\left((\cos x)^{\sin x}\right)} = e^{\sin x \ln(\cos x)}$$

Define

$$h(x) = \sin x \ln(\cos x),$$

so

$$(\cos x)^{\sin x} = e^{h(x)}.$$

The Taylor expansion of $\sin x$ near $x = 0$ to order 5 is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o\left(x^5\right),$$

the Taylor expansion of $\cos x$ near $x = 0$, up to order 5 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^5\right),$$

and the Taylor expansion of $\ln(1+u)$ near $u = 0$, up to order 3 is

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} o\left(u^3\right).$$

Let

$$u = \cos x - 1 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^5\right)\right) - 1 = -\frac{1}{2}x^2 + \frac{1}{24}x^4 + o\left(x^5\right).$$

Then, substituting into the logarithm expansion gives

$$\ln(\cos x) = \ln(1+u) = \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) - \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right)^2 + o\left(x^5\right)$$

Keeping terms up to degree 5, we get

$$\ln(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o\left(x^5\right).$$

Multiply $\sin x$ and $\ln(\cos x)$, we obtain

$$h(x) = \sin x.\ln(\cos x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(-\frac{1}{2}x^2 - \frac{1}{12}x^4\right) + o\left(x^5\right).$$

Keeping terms up to degree 5, we get

$$h(x) = -\frac{1}{2}x^3 + o\left(x^5\right).$$

We now expand the exponential

$$(\cos x)^{\sin x} = e^{h(x)} = e^{-\frac{1}{2}x^3 + o\left(x^5\right)}.$$

Using the Taylor expansion of e^v near $v = 0$, up to order 5

$$e^v = 1 + v + \frac{v^2}{2!} + \frac{v^3}{3!} + \frac{v^4}{4!} + \frac{v^5}{5!} + o\left(v^5\right),$$

Substitute $v = -\frac{1}{2}x^3$

$$e^{-\frac{1}{2}x^3} = 1 - \frac{1}{2}x^3 + o\left(x^5\right)$$

Thus

$$\boxed{(\cos x)^{\sin x} = 1 - \frac{1}{2}x^3 + o\left(x^5\right).}$$

Exercise 05

1) Using two different methods, find the third-order Taylor expansion of the function

$$\ln(\sin x)$$

near $x = \frac{\pi}{3}$.

2) Find the fourth-order Taylor expansion near $x = 1$ of the function

$$\frac{\ln x}{x^2}$$

Solution

I. The third-order Taylor expansion of the function $\ln(\sin x)$ near $x = \frac{\pi}{3}$.

Method 1: Direct Taylor expansion using derivatives: The third-order Taylor expansion of a function $f(x)$ near a point x_0 up to order 3 is given by:

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\frac{f'''(x_0)}{6}(x-x_0)^3+o\left((x-x_0)^3\right).$$

In our case, we take $x_0 = \frac{\pi}{3}$, and $f(x) = \ln(\sin x)$.

Now, we compute the derivatives of $f(x)$. Using the chain rule and standard derivatives of trigonometric identities, we get

$$\begin{aligned} f'(x) &= (\ln(\sin x))' = \frac{\cos x}{\sin x} = \cot x, \\ f''(x) &= (\cot x)' = -\cot^2 x - 1, \\ f'''(x) &= \left(-\cot^2 x - 1\right)' = 2(\cot x)\left(\cot^2 x + 1\right). \end{aligned}$$

We now evaluate the function and its derivatives at $\frac{\pi}{3}$, using known trigonometric values

$$f\left(\frac{\pi}{3}\right) = \ln\frac{\sqrt{3}}{2}, \quad f'\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}, \quad f''\left(\frac{\pi}{3}\right) = -\frac{4}{3}, \quad f'''\left(\frac{\pi}{3}\right) = \frac{8}{3\sqrt{3}}.$$

Substituting all values into the expansion formula

$$\ln(\sin x) = \ln\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}(x-x_0)-\frac{4}{3}\frac{1}{2}\left(x-\frac{\pi}{3}\right)^2+\frac{8}{3\sqrt{3}}\frac{1}{6}\left(x-\frac{\pi}{3}\right)^3+o\left(\left(x-\frac{\pi}{3}\right)^3\right),$$

which simplifies to

$$\boxed{\ln(\sin x) = \ln\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\left(x-\frac{\pi}{3}\right) - \frac{2}{3}\left(x-\frac{\pi}{3}\right)^2 + \frac{4}{9\sqrt{3}}\left(x-\frac{\pi}{3}\right)^3 + o\left(\left(x-\frac{\pi}{3}\right)^3\right).}$$

Method 2: Expansion by Change of Variables: We begin by introducing a change of variable to simplify the expansion. Let

$$u = x - \frac{\pi}{3} \quad (\text{so that } x = u + \frac{\pi}{3})$$

We then rewrite

$$\sin x = \sin\left(u + \frac{\pi}{3}\right)$$

Using the angle addition formula

$$\sin x = \sin(u)\cos\left(\frac{\pi}{3}\right) + \cos(u)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\cos u + \frac{1}{2}\sin u$$

We now use the standard Taylor expansions of $\cos u$ and $\sin u$ near $u = 0$. We have

$$\cos u = 1 - \frac{u^2}{2!} + o\left(u^3\right),$$

and

$$\sin u = u - \frac{u^3}{3!} + o\left(u^3\right)$$

Substituting into the expression for $\sin x$, we obtain

$$\begin{aligned}\sin x &= \frac{\sqrt{3}}{2}\left(1 - \frac{u^2}{2!}\right) + \frac{1}{2}\left(u - \frac{u^3}{3!}\right) + o\left(u^3\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}u - \frac{\sqrt{3}}{4}u^2 - \frac{1}{12}u^3 + o\left(u^3\right)\end{aligned}$$

Rewriting in terms of x, recall that $u = x - \frac{\pi}{3}$. Thus

$$\sin x = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3 + o\left(\left(x - \frac{\pi}{3}\right)^3\right)$$

Now, expand $\ln(\sin x)$ using the expansion of $\ln(a + v)$. Let

$$\begin{aligned}\ln(a+v) &= \ln\left(a\left(1 + \frac{v}{a}\right)\right) = \ln(a) + \ln\left(1 + \frac{v}{a}\right) \\ &= \ln(a) + \frac{v}{a} - \frac{1}{2}\left(\frac{v}{a}\right)^2 + \frac{1}{3}\left(\frac{v}{a}\right)^3 + o\left(v^3\right)\end{aligned}$$

We use this expansion for

$$\ln(\sin x) = \ln\left(\frac{\sqrt{3}}{2} + \frac{1}{2}u - \frac{\sqrt{3}}{4}u^2 - \frac{1}{12}u^3 + o\left(u^3\right)\right)$$

Let

$$a = \frac{\sqrt{3}}{2}, \quad v = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}u - \frac{\sqrt{3}}{4}u^2 - \frac{1}{12}u^3 + o\left(u^3\right)\right)$$

Now apply the expansion

$$\begin{aligned} \ln(\sin x) &= \ln\left(\frac{\sqrt{3}}{2} + v\right) \\ &= \ln\left(\frac{\sqrt{3}}{2}\right) + \frac{v}{\frac{\sqrt{3}}{2}} - \frac{1}{2}\left(\frac{v}{\frac{\sqrt{3}}{2}}\right)^2 + \frac{1}{3}\left(\frac{v}{\frac{\sqrt{3}}{2}}\right)^3 + o\left(u^3\right) \end{aligned}$$

Since

$$\frac{v}{\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}u - \frac{\sqrt{3}}{4}u^2 - \frac{1}{12}u^3}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}u - \frac{1}{2}u^2 - \frac{1}{6\sqrt{3}}u^3$$

The remaining terms lead to simplifications (via binomial expansion) giving

$$\begin{aligned} \ln(\sin x) &= \ln\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{3}}u - \frac{1}{2}u^2 - \frac{1}{6\sqrt{3}}u^3\right) \\ &\quad -\frac{1}{2}\left(\left(\frac{1}{\sqrt{3}}u - \frac{1}{2}u^2 - \frac{1}{6\sqrt{3}}u^3\right)\right)^2 \\ &\quad +\frac{1}{3}\left(\frac{1}{\sqrt{3}}u - \frac{1}{2}u^2 - \frac{1}{6\sqrt{3}}u^3\right)^3 + o\left(u^3\right) \\ &= \ln\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{3}}u - \frac{2}{3}u^2 + \frac{4}{9\sqrt{3}}u^3 \end{aligned}$$

Since $u = x - \frac{\pi}{3}$, the final third-order Taylor expansion becomes

$$\boxed{\ln(\sin x) = \ln\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\left(x - \frac{\pi}{3}\right) - \frac{2}{3}(x - x_0)^2 + \frac{4}{9\sqrt{3}}\left(x - \frac{\pi}{3}\right)^3 + o\left(\left(x - \frac{\pi}{3}\right)^3\right).}$$

2) The fourth-order Taylor expansion near $x = 1$ of $\frac{\ln x}{x^2}$: To simplify the expansion near $x = 1$, we make the substitution

$$y = x - 1 \Longrightarrow x = y + 1.$$

As $x \to 1$, we have $y \to 0$. This allows us to rewrite the function in terms on y

$$\frac{\ln x}{x^2} = \frac{\ln (y+1)}{(y+1)^2} = \frac{\ln (y+1)}{1+2y+y^2}.$$

The Taylor expansion of $\ln (y+1)$ near $y=0$, up to the fourth order, is

$$\ln (y+1) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + o\left(y^4\right).$$

Divide the series by $1+2y+y^2$, we get

$$\frac{\ln (y+1)}{(y+1)^2} = \frac{y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}}{1+2y+y^2} = \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}\right)\frac{1}{1+2y+y^2}$$

We express the denominator using the identity

$$\frac{1}{1+2y+y^2} = \frac{1}{1+v} \text{ where } v = -\left(2y+y^2\right) = -2y-y^2$$

Using the fourth-order Taylor expansion near $v=0$ of $\frac{1}{1+v}$

$$\frac{1}{1+v} = \frac{1}{1-(-v)} = 1 + v + v^2 + v^3 + v^4 + o\left(v^4\right),$$

we substitute $v = -\left(2y+y^2\right) = -2y-y^2$, and expand each power, we obtain

$$\begin{aligned} \frac{1}{1+2y+y^2} &= 1 + \left(-y^2-2y\right) + \left(-y^2-2y\right)^2 + \left(-y^2-2y\right)^3 \\ &\quad + \left(-y^2-2y\right)^4 + o\left(v^4\right) \\ &= 1 - 2y + 3y^2 - 4y^3 + o\left(y^4\right). \end{aligned}$$

Now, multiply the two expansions

$$\frac{\ln (y+1)}{(y+1)^2} = \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}\right)\left(1 - 2y + 3y^2 - 4y^3\right)$$

Multiplying and collecting terms up to y^4, we get

$$\frac{\ln (y+1)}{(y+1)^2} = y - \frac{5}{2}y^2 + \frac{13}{3}y^3 - \frac{77}{12}y^4 + o\left(y^4\right)$$

Finally, substituting back $y = x - 1$, the final third-order Taylor expansion becomes

$$\boxed{\frac{\ln x}{x^2} = x - 1 - \frac{5}{2}(x-1)^2 + \frac{13}{3}(x-1)^3 - \frac{77}{12}(x-1)^4 + o\left((x-1)^4\right).}$$

Exercise 06

1) Give the second-order Taylor expansion of the function

$$f(x) = \frac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}}$$

near $x = 0$.

2) Deduce from this result a second-order expansion of the function as $x \to +\infty$.

3) Compute a second-order expansion of the function as $x \to -\infty$.

Solution

1) Second-order Taylor Expansion of $f(x)$ at $x = 0$: We begin by expanding the square root function

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + o\left(x^2\right).$$

Now substitute into the original function

$$\begin{aligned} f(x) &= \frac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}} = \frac{\left(1+\frac{x^2}{2}\right)}{1+x+\left(1+\frac{x^2}{2}\right)} + o\left(x^2\right) \\ &= \frac{1+\frac{x^2}{2}}{2+x+\frac{x^2}{2}} + o\left(x^2\right). \end{aligned}$$

Let's simplify the denominator by setting

$$u = x + \frac{x^2}{2}.$$

So we have

$$f(x) = \frac{1+\frac{x^2}{2}}{2+u} + o\left(x^2\right) = \left(1+\frac{x^2}{2}\right)\frac{1}{2+u} + o\left(x^2\right).$$

To expand $\frac{1}{2+u}$, we write

$$\frac{1}{2+u} = \frac{1}{2\left(1+\frac{u}{2}\right)} = \frac{1}{2}.\frac{1}{1+\frac{u}{2}} = \frac{1}{2}.\frac{1}{1-\left(-\frac{u}{2}\right)}.$$

Using the second-order Taylor expansion near $v=0$ of $\frac{1}{1+v}$

$$\frac{1}{1+v} = \frac{1}{1-(-v)} = 1+v+v^2+o\left(v^2\right),$$

we substitute $v = -\frac{u}{2}$, and expand each power, we obtain

$$\begin{aligned}\frac{1}{2+u} &= \frac{1}{2}\left(1+-\frac{u}{2}+\left(-\frac{u}{2}\right)^2\right)+o\left(u^2\right)\\ &= \frac{1}{2}-\frac{1}{4}u+\frac{1}{8}u^2+o\left(u^2\right).\end{aligned}$$

Substitute $u = x+\frac{x^2}{2}$, we get

$$\begin{aligned}\frac{1}{2+x+\frac{x^2}{2}} &= \frac{1}{2}-\frac{1}{4}\left(x+\frac{x^2}{2}\right)+\frac{1}{8}\left(x+\frac{x^2}{2}\right)^2+o\left(x^2\right)\\ &= \frac{1}{2}-\frac{1}{4}x+o\left(x^2\right)\end{aligned}$$

Now, multiply the two expansions $1+\frac{x^2}{2}$ and $\frac{1}{2}-\frac{1}{4}x$

$$f(x) = \left(1+\frac{x^2}{2}\right)\left(\frac{1}{2}-\frac{1}{4}x\right)+o\left(x^2\right)$$

Finally, multiplying and collecting terms up to x^2, the final second-order Taylor expansion becomes

$$\boxed{f(x) = \frac{1}{2}-\frac{1}{4}x+\frac{1}{4}x^2+o\left(x^2\right).}$$

2. The second-order expansion of the f as $x \to \infty$: To analyze the behavior of f as $x \to \infty$, we perform a change of variable

$$h = \frac{1}{x} \Longrightarrow x = \frac{1}{h} \text{ and } h \to 0^+.$$

Rewrite the function

$$\begin{aligned} f(x) &= \frac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}} = \frac{\sqrt{x^2\left(\frac{1}{x^2}+1\right)}}{1+x+\sqrt{x^2\left(\frac{1}{x^2}+1\right)}} \\ &= \frac{|x|\sqrt{\frac{1}{x^2}+1}}{1+x+|x|\sqrt{\frac{1}{x^2}+1}} = \frac{x\sqrt{\frac{1}{x^2}+1}}{1+x+x\sqrt{\frac{1}{x^2}+1}}. \end{aligned}$$

Factor out x from the numerator and denominator

$$\begin{aligned} f(x) &= \frac{x\sqrt{\frac{1}{x^2}+1}}{1+x+x\sqrt{\frac{1}{x^2}+1}} = \frac{x\sqrt{h^2+1}}{x\left(h+1+\sqrt{1+h^2}\right)} \\ &= \frac{\sqrt{1+h^2}}{\left(h+1+\sqrt{1+h^2}\right)\sqrt{1+h^2}} = \frac{\sqrt{1+h^2}}{h+1+\sqrt{1+h^2}} = f(h). \end{aligned}$$

We already computed the expansion of this expression in part 1 (in terms of h), so we can reuse the result

$$f(h) = \frac{1}{2} - \frac{1}{4}h + \frac{1}{4}h^2 + o\left(h^2\right).$$

Thus, substituting back $h = \frac{1}{x}$, we obtain the second-order expansion at $+\infty$

$$\boxed{f(x) = \frac{1}{2} - \frac{1}{4x} + \frac{1}{4x^2} + o\left(\frac{1}{x^2}\right).}$$

3. The second-order expansion of f as $x \to -\infty$: The previous argument assumes $x > 0$, so it does not holds for $x \to -\infty$. In this case, we must be careful with the sign. For large negative x.

We have

$$\begin{aligned} f(x) &= \frac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}} = \frac{\sqrt{x^2\left(\frac{1}{x^2}+1\right)}}{1+x+\sqrt{x^2\left(\frac{1}{x^2}+1\right)}} = \frac{|x|\sqrt{\frac{1}{x^2}+1}}{1+x+|x|\sqrt{\frac{1}{x^2}+1}} \\ &= \frac{-x\sqrt{\frac{1}{x^2}+1}}{1+x-x\sqrt{\frac{1}{x^2}+1}}. \end{aligned}$$

Then

$$\begin{aligned} f(x) &= \frac{-x\sqrt{h^2+1}}{1+x-x\sqrt{h^2+1}} = \frac{-x\sqrt{h^2+1}}{x\left(h+1-\sqrt{1+h^2}\right)} \\ &= \frac{-\sqrt{1+h^2}}{h+1-\sqrt{1+h^2}} = -\frac{\sqrt{1+h^2}}{h+1-\sqrt{1+h^2}}. \end{aligned}$$

Now expand numerator and denominator

$$\sqrt{1+h^2} = 1+\frac{h^2}{2}+o\left(h^2\right),$$

and

$$\begin{aligned} h+1-\sqrt{1+h^2} &= h+1-\left(1+\frac{h^2}{2}\right)+o\left(h^2\right) \\ &= h-\frac{h^2}{2}+o\left(h^2\right). \end{aligned}$$

So

$$f(x) = \frac{-\left(1+\frac{h^2}{2}\right)}{h-\frac{h^2}{2}}+o\left(h^2\right) = \frac{1}{h}\left(-\frac{\left(1+\frac{h^2}{2}\right)}{1-\frac{h}{2}}\right)+o\left(h^2\right).$$

Now expand

$$\frac{1}{1-\frac{h}{2}} = 1+\frac{h}{2}+\frac{h^2}{4}+o\left(h^2\right)$$

Multiplying

$$\begin{aligned} f(x) &= -\frac{1}{h}\left(1+\frac{h^2}{2}\right)\left(1+\frac{h}{2}+\frac{h^2}{4}\right)+o\left(h^2\right) \\ &= -\frac{1}{h}\left(1+\frac{1}{2}h+\frac{3}{4}h^2\right)+o\left(h^2\right). \end{aligned}$$

Now substitute back $h = \frac{1}{x}$, we obtain

$$\boxed{f(x) = -\frac{1}{2} - x - \frac{3}{4x} + o\left(\frac{1}{x}\right).}$$

Exercise 07

1) Determine the $6th$-order Taylor expansion of the function

$$f(x) = \frac{1}{1+x^2}.$$

near $x = 0$.

2) Deduce the $7th$-order Taylor expansion of the function

$$g(x) = \arctan x.$$

near $x = 0$.

Solution

1) The $6th$-order Taylor expansion near $x = 0$ of $f(x)$: We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + o\left(x^6\right).$$

Let $u = -x^2$, so

$$\frac{1}{1+x^2} = \frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + u^5 + u^6 + o\left(u^6\right)$$

Now substitute back $u = -x^2$, and collecting terms up to x^6, we get

$$\begin{aligned} \frac{1}{1+x^2} &= 1 + \left(-x^2\right) + \left(-x^2\right)^2 + \left(-x^2\right)^3 + \left(-x^2\right)^4 + \left(-x^2\right)^5 + \left(-x^2\right)^6 + o\left(x^6\right) \\ &= 1 - x^2 + x^4 - x^6 + o\left(x^6\right). \end{aligned}$$

Thus

$$\boxed{f(x) = 1 - x^2 + x^4 - x^6 + o\left(x^6\right).}$$

2) The $7th$-order Taylor expansion near $x = 0$ of $\arctan x$: We know

$$(\arctan x)' = \frac{1}{1+x^2}$$

From Part 1, we already have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + o\left(x^6\right).$$

To find $\arctan x$, we integrate term-by-term from 0 to x

$$\arctan x = \int_0^x \left(1 - t^2 + t^4 - t^6\right) dt + o\left(x^7\right).$$

Thus

$$\boxed{\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + o\left(x^7\right).}$$

Exercise 08 Using Taylor series (limited development), evaluate the limit

$$\textbf{1)} \lim_{x\to 0} \frac{(1-\cos x)(\tan x)^2}{x^3 \ln(1+x)}, \quad \textbf{2)} \lim_{x\to 0} \left[\ln(e+x)\right]^{\frac{1}{x}}$$

$$\textbf{3)} \lim_{x\to\infty} \left(\frac{\ln(1+x)}{\ln x}\right)^{x\ln x}, \quad \textbf{4)} \lim_{x\to\infty} \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x$$

Solution

1) The fourth-order Taylor expansions near $x = 0$ of $\cos x$, $\tan x$, and $\ln(1+x)$ are given by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^4\right),$$
$$\tan x = x + \frac{x^3}{3} + o\left(x^4\right),$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o\left(x^4\right).$$

Now, we expand the numerator and denominator

$$(1-\cos x)(\tan x)^2 = \left(1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right)\left(x + \frac{x^3}{3}\right)^2 + o\left(x^4\right),$$

and

$$x^3 \ln(1+x) = x^3\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) + o\left(x^4\right).$$

Multiplying the expressions and keeping terms up to order 4, we get

$$(1-\cos x)(\tan x)^2 = \frac{1}{2}x^4 + o\left(x^4\right),$$

and

$$x^3 \ln(1+x) = x^4 + o\left(x^4\right).$$

Now substitute both into the original limit

$$\lim_{x\to 0} \frac{(1-\cos x)(\tan x)^2}{x^3 \ln(1+x)} = \lim_{x\to 0} \frac{\frac{1}{2}x^4}{x^4}$$

Thus

$$\boxed{\lim_{x\to 0} \frac{(1-\cos x)(\tan x)^2}{x^3 \ln(1+x)} = \frac{1}{2}.}$$

2) We begin with the Taylor expansion

$$\ln(1+x) = x - \frac{x^2}{2} + o\left(x^2\right).$$

Now consider

$$\ln(e+x) = \ln\left(e\left(1+\frac{x}{e}\right)\right) = \ln(e) + \ln\left(1+\frac{x}{e}\right) = 1 + \ln\left(1+\frac{x}{e}\right)$$

Using the expansion of $\ln(1+x)$, we get

$$\ln(e+x) = 1 + \frac{x}{e} - \frac{\left(\frac{x}{e}\right)^2}{2} + o\left(x^2\right) = 1 + \frac{1}{e}x - \frac{1}{2e^2}x^2 + o\left(x^2\right).$$

Therefore

$$[\ln(e+x)]^{\frac{1}{x}} = \left(1 + \frac{1}{e}x - \frac{1}{2e^2}x^2 + o\left(x^2\right)\right)^{\frac{1}{x}}.$$

We now apply the logarithm to handle the exponent

$$[\ln(e+x)]^{\frac{1}{x}} = e^{\ln\left(1+\frac{1}{e}x-\frac{1}{2e^2}x^2+o\left(x^2\right)\right)^{\frac{1}{x}}} = e^{\frac{1}{x}\ln\left(1+\frac{1}{e}x-\frac{1}{2e^2}x^2+o\left(x^2\right)\right)}.$$

Let

$$u = \frac{1}{e}x - \frac{1}{2e^2}x^2,$$

so,

$$\ln(1+u) = u - \frac{u^2}{2} + o\left(u^2\right).$$

Now substitute u and compute

$$\begin{aligned}\ln\left(1 + \frac{1}{e}x - \frac{1}{2e^2}x^2\right) &= \left(\frac{1}{e}x - \frac{1}{2e^2}x^2\right) \\ &\quad - \frac{\left(\frac{1}{e}x - \frac{1}{2e^2}x^2\right)^2}{2} + o\left(x^2\right) \\ &= \frac{1}{e}x - \frac{1}{e^2}x^2 + o\left(x^2\right).\end{aligned}$$

So the original expression becomes

$$[\ln(e+x)]^{\frac{1}{x}} = e^{\frac{1}{x}\left(\frac{1}{e}x - \frac{1}{e^2}x^2 + o\left(x^2\right)\right)} = e^{\frac{1}{e} - \frac{x}{e^2}} e^{o\left(x^2\right)}.$$

Taking the limit as $x \to 0$

$$\lim_{x\to 0}[\ln(e+x)]^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{e} - \frac{x}{e^2}} = \lim_{x\to 0} e^{\frac{1}{e} - \frac{x}{e^2}} = e^{\frac{1}{e}}.$$

Thus

$$\boxed{\lim_{x\to 0}[\ln(e+x)]^{\frac{1}{x}} = e^{\frac{1}{e}}.}$$

3) As $x \to \infty$, we write

$$\ln(1+x) = \ln\left(x\left(1 + \frac{1}{x}\right)\right) = \ln x + \ln\left(1 + \frac{1}{x}\right) = \ln x + \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

Now expand $\ln(1+z)$ using the Taylor expansion

$$\ln(1+z) = z - \frac{z^2}{2} + o\left(z^2\right)$$

Let $z = \frac{1}{x}$, then

$$\ln(1+x) = \ln x + \ln\left(1 + \frac{1}{x}\right) = \ln x + \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

Now

$$\begin{aligned}\frac{\ln(1+x)}{\ln x} &= \frac{\ln x+\frac{1}{x}-\frac{1}{2x^2}}{\ln x}+o\left(\frac{1}{x^2}\right)\\ &= \frac{\ln x\left(1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}\right)}{\ln x}+o\left(\frac{1}{x^2}\right)\\ &= 1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}+o\left(\frac{1}{x^2}\right).\end{aligned}$$

We have

$$\left(\frac{\ln(1+x)}{\ln x}\right)^{x\ln x}=e^{\ln\left(\frac{\ln(1+x)}{\ln x}\right)^{x\ln x}}=e^{x(\ln x)\ln\left(\frac{\ln(1+x)}{\ln x}\right)}=e^{x(\ln x)\ln\left(1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}+o\left(\frac{1}{x^2}\right)\right)},$$

Use the expansion

$$\ln(1+u)=u-\frac{1}{2}u^2+o\left(u^2\right)$$

Here, $u=\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}$, so

$$\begin{aligned}\ln\left(1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}\right) &= \ln(1+u)=\left(\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}\right)\\ &\quad -\frac{1}{2}\left(\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}\right)^2+o\left(\frac{1}{x^2}\right)\\ &= \frac{1}{x\ln x}-\frac{1}{2x^2\ln x}-\frac{1}{2x^2\ln^2 x}+o\left(\frac{1}{x^2}\right),\end{aligned}$$

and

$$\begin{aligned}x(\ln x)\ln\left(1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}\right) &= x(\ln x)\left(\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}-\frac{1}{2x^2\ln^2 x}\right)+o\left(\frac{1}{x^2}\right)\\ &= 1-\frac{1}{2x}-\frac{1}{2x\ln x}+o\left(\frac{1}{x^2}\right).\end{aligned}$$

Therefore

$$\left(\frac{\ln(1+x)}{\ln x}\right)^{x\ln x}=e^{\left(1-\frac{1}{2x}-\frac{1}{2x\ln x}+o\left(\frac{1}{x^2}\right)\right)}=e^1e^{-\frac{1}{2x}-\frac{1}{2x\ln x}+o\left(\frac{1}{x^2}\right)}.$$

Since $-\frac{1}{2x}-\frac{1}{2x\ln x}\to 0$, as $x\to\infty$, then

$$\boxed{\lim_{x\to\infty}\left(\frac{\ln(1+x)}{\ln x}\right)^{x\ln x}=e.}$$

4) As $x \to \infty$, then $\frac{1}{x} \to 0$. So $\sin \frac{1}{x} \to 0$, and $\cos \frac{1}{x} \to 0$. Now, We use

$$\sin \frac{1}{x} = \frac{1}{x} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right),$$

and

$$\cos \frac{1}{x} = 1 - \frac{1}{2x^2} + o\left(\frac{1}{x^3}\right),$$

Compute the sum

$$\sin \frac{1}{x} + \cos \frac{1}{x} = 1 + \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)$$

So

$$\begin{aligned}
\left(\sin \frac{1}{x} + \cos \frac{1}{x}\right)^x &= \left(1 + \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)\right)^x \\
&= e^{\ln\left(1+\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{6x^3}+o\left(\frac{1}{x^3}\right)\right)^x} \\
&= e^{x\ln\left(1+\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{6x^3}+o\left(\frac{1}{x^3}\right)\right)}
\end{aligned}$$

We now expand $\ln(1+u)$ as

$$\ln(1+u) = u - \frac{1}{2}u^2 + o\left(u^2\right), \quad \text{where } u = \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3}.$$

So

$$\begin{aligned}
\ln\left(1 + \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3}\right) &= \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} - \frac{1}{2}\left(\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3}\right)^2 + o\left(\frac{1}{x^3}\right) \\
&= \frac{1}{x} - \frac{1}{x^2} + \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right)
\end{aligned}$$

Now multiply by x

$$\begin{aligned}
x\ln\left(1 + \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3}\right) &= x\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{3x^3}\right) + o\left(\frac{1}{x^3}\right) \\
&= 1 - \frac{1}{x} + \frac{1}{3x^2} + o\left(\frac{1}{x^3}\right).
\end{aligned}$$

Now

$$\left(\sin \frac{1}{x} + \cos \frac{1}{x}\right)^x = e^{1-\frac{1}{x}+\frac{1}{3x^2}+o\left(\frac{1}{x^3}\right)}$$

Since $-\frac{1}{x} + \frac{1}{3x^2} \to 0$, as $x \to \infty$, we obtain

$$\boxed{\lim_{x\to\infty} \left(\sin\tfrac{1}{x} + \cos\tfrac{1}{x}\right)^x = e.}$$

CHAPTER 4

Exercises Related to the Chapter: Functions of Two Variables

Contents

This chapter presents a coherent set of exercises on functions of two variables, aiming at fostering an understanding of multivariable calculus. It includes exercises on limits, differentiability, partial derivatives, differential operators. many exercises are designed to blend theoretical insight with computational practice, helping students to apply abstract concepts to practical problems across science and engineering disciplines.

4.1 Course Recap: Functions of Two Variables

4.1.1 Definition of a Function of Two Variables

Definition. A function of two variables maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The standard notation for functions of two variables is

$$z = f(x, y).$$

This defines a surface in three-dimensional space (x, y, z), where z depends on the two independent variables x and y.

4.1.2 Domain and Range

Domain: The set of all input pairs (x, y) for which the function is defined.
Range: The set of all output values $z = f(x, y)$.

4.1.3 Graphical Representation of Functions of two Variables

Functions of two variables, $z = f(x, y)$, can be visualized as three-dimensional surfaces in a Cartesian coordinate system, where each pair (x, y) corresponds to a specific z value. To better interpret these 3D surfaces, one often uses level curves (contours representing constant z values) and cross-sections (intersections of the surface with horizontal or vertical planes).

Graph of the Function

Definition. The graph of a function of two variables is the surface in $\mathbb{R}^3$ given by the set:

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}.$$

Level Curves (Contour Lines)

Definition. A level curve of a function $f(x, y)$ is the set of all points (x, y) such that:

$$f(x, y) = c.$$

where c is a constant.

4.1.4 Limits and Continuity

Limit

Definition. We say that

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon.$$

This mean $f(x, y)$ approaches L as (x, y) approaches (a, b) from **any direction**.

Two-Path Test: If $f(x, y)$ approaches different values along two paths to (a, b), then the limit does not exist.

Continuity

$f(x, y)$ is continuous at (a, b) if:

- $f(a,b)$ is defined,
- the limit exists,
- the limit equals $f(a,b)$.

4.1.5 Differentiability

Definition. A function $f(x,y)$ is differentiable at (a,b) if:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Sufficient Condition: If f_x and f_y are continuous near (a,b), then f is differentiable at that point.

4.1.6 First Order Partial Derivatives

A first order partial derivative of a function of two variables is its derivative with respect to one of those variables, with the others held constant.

Definition 4.1 Let Ω be an open subset of $\mathbb{R}^2$ and let $f : \Omega \longrightarrow \mathbb{R}$. The first partial derivative of f at the point $(x,y) \in \Omega$ with respect to the x and y are defined as

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Remark 4.1 Let $f \in C^1(\Omega)$. We sometimes use the following alternate notations for the first partial derivative at $(x,y) \in \Omega$ with respect to x.

$$\frac{\partial f}{\partial x}(x,y),\ f_x(x,y),\ D_x f(x,y),\ D_1 f(x,y) \text{ or } \partial_x f(x,y).$$

4.1.7 Tangent Plane

The equation of the tangent plane to $z = f(x, y)$ at the point (a, b) is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The tangent plane is used for linear approximation, providing a good estimate for a function's value at points near the point of tangency.

4.1.8 Directional Derivative

The directional derivative of f at (x, y) along a vector $d = (d_1, d_2)$ is the function ∇f_d defined by the limit

$$\nabla f_d = f'\left((x_0, y_0), \overrightarrow{d}\right) = \lim_{h \longrightarrow 0} \left(\frac{f\left((x_0 + hd_1, y_0 + hd_2\right) - f(x_0, y_0)}{ht} \right) \quad \text{(if this limit exists).}$$

Furthermore, if the partial derivatives of f are continuous, then it is possible to demonstrate that

$$f'\left((x_0, y_0), \overrightarrow{d}\right) = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} d_1 + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} d_2 = \overbrace{\overrightarrow{\nabla} f\Big|_{(x_0,y_0)} \cdot \overrightarrow{d}}^{\text{dot product}}.$$

where the ∇ on the right denotes the gradient, $\cdot$ is the dot product (scalar product), and d is a vector

4.1.9 Second Order Partial Derivatives

Second-order partial derivatives are obtained by differentiating the first-order partial derivatives. They describe how the rate of change of a function with respect to one variable varies when the other variable is held constant.
The second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are called the direct partial

derivatives whereas the second order partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are called the cross partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \text{ and } f_{yx} = \frac{\partial^2 f}{\partial y \partial x}.$$

Schwarz's Theorem (Clairaut's Theorem or Young's Theorem

Theorem 4.1 *If at a point (x_0, y_0) of Ω, the successive partial derivatives of a function f, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist and are continuous at this point, then these derivatives are equal*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

4.1.10 Differential Operators

Let Ω be an open subset of $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}$ be a differentiable function on Ω.

Gradient

The gradient of a function of two variables is a vector with first partial derivatives. i.e.

$$\overrightarrow{\text{grad}} f(x, y) = \vec{\nabla} f(x, y) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right).$$

Divergence

The divergence is defined as the sum of the two first order partial derivatives as follows:

$$\text{div}\, f(x, y) = \nabla . f = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y).$$

Laplacien

$$\begin{aligned}\Delta u(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y).\\ &= \nabla.\nabla f(x,y).\end{aligned}$$

The differential operator Δ is called the Laplacian or the Laplace operator.

4.1.11 Partial Differential Equation

A partial differential equation (PDE) is a mathematical equation involving an unknown function of two or more variables and its partial derivatives with respect to those variables.

4.2 Solved Exercises

Exercise 01

1. Let $f(x,y)=x^2-3y$ and

$$\begin{cases} x=2t \\ y=\dfrac{5t-1}{2}. \end{cases}$$

Calculate $\dfrac{\partial f}{\partial t}$.

2. Let $g(x,y)=x^2+xy+y^2$ where

$$\begin{cases} x=2u+v \\ y=u-2v. \end{cases}$$

Calculate $\dfrac{\partial g}{\partial u}(x,y)$ and $\dfrac{\partial g}{\partial v}(x,y)$.

Solution

1. One has

$$\begin{aligned}\frac{\partial f}{\partial t}(x,y) &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right)+\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)\\ &= (2x)(2)+(-3)\left(\frac{5}{2}\right)\\ &= 4x-\frac{15}{2}.\end{aligned}$$

2. One has

$$\begin{aligned}&\frac{\partial g}{\partial x}(x,y)=2x+y\\ &\frac{\partial g}{\partial y}(x,y)=x+2y\\ &\frac{\partial x}{\partial u}=2,\quad \frac{\partial x}{\partial v}=1\\ &\frac{\partial y}{\partial u}=1,\quad \frac{\partial y}{\partial v}=-2,\end{aligned}$$

which gives

$$\begin{aligned}\frac{\partial g}{\partial u}(x,y)&=\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right)+\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right)\\ \frac{\partial g}{\partial v}(x,y)&=\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right)+\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right).\end{aligned}$$

So

$$\boxed{\frac{\partial g}{\partial u}(x,y)=(2x+y)(2)+(x+2y)(1)=5x+4y,}$$

and

$$\boxed{\frac{\partial g}{\partial v}(x,y)=(2x+y)(1)+(x+2y)(-2)=-3y.}$$

Exercise 02

1. Prove that the PDE

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}+2\lambda\frac{\partial u}{\partial t}=0,$$

can be written in the following form:

$$\left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\,\frac{\partial}{\partial x}+2\lambda\right)u-2\lambda c\,\frac{\partial u}{\partial x}=0,$$

where $u=u\,(x,t)$.

2. Conclude from this that this PDE is equivalent to the following system:

$$\begin{cases}\dfrac{\partial u}{\partial t}-c\,\dfrac{\partial u}{\partial x}=v-2\lambda u\\ \dfrac{\partial v}{\partial t}+c\,\dfrac{\partial v}{\partial x}-2\lambda c\,\dfrac{\partial u}{\partial x}=0\end{cases}.$$

Solution

1.

$$\begin{aligned}\left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\,\frac{\partial}{\partial x}+2\,\lambda\right)u-2\,c\,\lambda\,\frac{\partial u}{\partial x} &= \left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t}-c\,\frac{\partial u}{\partial x}+2\,\lambda\,u\right)\\ &\quad -2\,c\,\lambda\,\frac{\partial u}{\partial x}.\end{aligned}$$

Assuming that the second-order partial derivatives of u are continuous, we will have

$$\begin{aligned}\left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\,\frac{\partial}{\partial x}+2\,\lambda\right)u-2\,c\,\lambda\,\frac{\partial u}{\partial x} &= \left(\frac{\partial^2 u}{\partial t^2}-c\,\frac{\partial^2 u}{\partial t\partial x}+2\,\lambda\,\frac{\partial u}{\partial t}\right)\\ &\quad +c\left(\frac{\partial^2 u}{\partial x\partial t}-c\,\frac{\partial^2 u}{\partial x^2}+2\,\lambda\,\frac{\partial u}{\partial x}\right)\\ &\quad -2c\lambda\frac{\partial u}{\partial x}\\ &= \frac{\partial^2 u}{\partial t^2}-c^2\,\frac{\partial^2 u}{\partial x^2}+2\,\lambda\,\frac{\partial u}{\partial t}.\end{aligned}$$

Therefore

$$\boxed{\left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\,\frac{\partial}{\partial x}+2\,\lambda\right)u-2\,c\,\lambda\,\frac{\partial u}{\partial x}=0.}$$

2. If we consider

$$\left(\frac{\partial}{\partial t}+c\,\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t}-c\,\frac{\partial}{\partial x}+2\,\lambda\right)u}_{v}-2\,c\,\lambda\,\frac{\partial u}{\partial x}=0,$$

we get

$$\left(\frac{\partial v}{\partial t}+c\,\frac{\partial v}{\partial x}\right)-2\,c\,\lambda\,\frac{\partial u}{\partial x}=0.$$

Then we obtain the following system of two equations with two dependent variables :

$$\begin{cases}\dfrac{\partial u}{\partial t}-c\,\dfrac{\partial u}{\partial x}+2\,\lambda\,u=v\\ \dfrac{\partial v}{\partial t}+c\,\dfrac{\partial v}{\partial x}-2\,c\,\lambda\,\dfrac{\partial u}{\partial x}=0,\end{cases}$$

which in turn is equivalent to

$$\boxed{\begin{cases}\dfrac{\partial u}{\partial t}-c\,\dfrac{\partial u}{\partial x}=v-2\,\lambda\,u\\ \dfrac{\partial v}{\partial t}+c\,\dfrac{\partial v}{\partial x}-2\,c\,\lambda\,\dfrac{\partial u}{\partial x}=0.\end{cases}}$$

Exercise 03

Prove that the general solution of

$$u_{xy}-\left(\frac{1}{x-y}\right)u_x+\left(\frac{1}{x-y}\right)u_y=0,$$

has the following form:

$$u\,(x,y)=\frac{F\,(x)+G\,(y)}{x-y},$$

where F and G are functions of class C^2.

Solution

Let's make the following change of variables:

$$v\,(x,y)=(x-y)\,u\,(x,y)\,,$$

so

$$v_x(x,y) = (x-y)\,u_x(x,y) + u(x,y),$$

and

$$v_{xy}(x,y) = -u_x(x,y) + (x-y)\,u_{xy}(x,y) + u_y(x,y) = 0.$$

This implies that

$$v_x(x,y) = f(x).$$

It follows that

$$v(x,y) = \int f(x)\,dx + G(y),$$

and hence

$$v(x,y) = F(x) + G(y),$$

where f, F and G are functions of class C^2.

Finally

$$\boxed{u(x,y) = \frac{F(x)+G(y)}{x-y}.}$$

Exercise 04

Show that by the following change of variables:

$$u(x,y) = v(x,y)\;e^{-bx-ay},$$

the PDE

$$u_{xy} + au_x + bu_y + cu = 0,$$

may be put in the following form:

$$v_{xy} + (c-ab)\,v = 0.$$

Solution

If

$$u(x,y) = v(x,y)\ e^{-bx-ay},$$

then

$$\begin{aligned}\frac{\partial u}{\partial x} &= v_x(x,y)\ e^{-bx-ay} - b\ v(x,y)\ e^{-bx-ay}\\ \frac{\partial u}{\partial y} &= v_y(x,y)\ e^{-bx-ay} - a\ v(x,y)\ e^{-bx-ay}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(v_y(x,y)\ e^{-bx-ay} - a\ v(x,y)\ e^{-bx-ay}\right)\\ &= v_{xy}(x,y)\ e^{-bx-ay} - b\ v_y(x,y)\ e^{-bx-ay}\\ &\quad -a\ v_x(x,y)\ e^{-bx-ay} + ab\ v(x,y)\ e^{-bx-ay}.\end{aligned}$$

By substituting these calculations into the proposed partial differential equation

$$u_{xy} + au_x + bu_y + cu = 0,$$

we get

$$\begin{aligned}& v_{xy}(x,y)\ e^{-bx-ay} - b\ v_y(x,y)\ e^{-bx-ay}\\ & -a\ v_x(x,y)\ e^{-bx-ay} + ab\ v(x,y)\ e^{-bx-ay}\\ & +av_x(x,y)\ e^{-bx-ay} - ab\ v(x,y)\ e^{-bx-ay} + bv_y(x,y)\ e^{-bx-ay}\\ & -ab\ v(x,y)\ e^{-bx-ay} + c\ v(x,y)\ e^{-bx-ay}\\ = \ & v_{xy}(x,y)\ e^{-bx-ay} - ab\ v(x,y)\ e^{-bx-ay} + c\ v(x,y)\ e^{-bx-ay}.\end{aligned}$$

Thus

$$\boxed{v_{xy} + (c - ab)\, v = 0.}$$

Exercise 05

Using the new coordinates $\alpha = x + y$ and $\beta = x - y$. Determine the general solution of the following PDE:

$$\frac{\partial^2 u\,(x,y)}{\partial x^2} - \frac{\partial^2 u\,(x,y)}{\partial y^2} = 0.$$

Solution

If $\alpha = x + y$ and $\beta = x - y$, then

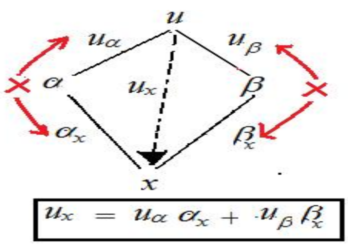
$$x = \frac{\alpha+\beta}{2} \text{ and } y = \frac{\alpha-\beta}{2}.$$

Moreover

$$\begin{cases} \dfrac{\partial \alpha}{\partial x} = 1 \\ \dfrac{\partial \beta}{\partial x} = 1 \end{cases} \qquad \begin{cases} \dfrac{\partial \alpha}{\partial y} = 1 \\ \dfrac{\partial \beta}{\partial y} = -1. \end{cases}$$

$$u(x,y) = u(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}).$$

which means that u depends on α and β. By using the chain rule (the derivative of the composition of two differentiable functions), we obtain



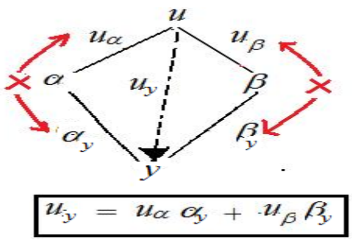
Chain Rule

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \left(\frac{\partial u}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial x}\right) + \left(\frac{\partial u}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial x}\right) \\
&= \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}.
\end{aligned}$$

If we put $v = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$, we get

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial v}{\partial x} = \left(\frac{\partial v}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial x}\right) + \left(\frac{\partial v}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial x}\right) \\
&= \frac{\partial v}{\partial \alpha} + \frac{\partial v}{\partial \beta} = \frac{\partial}{\partial \alpha}\left[\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right] + \frac{\partial v}{\partial \beta}\left[\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right] \\
&= \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

One has



Chain Rule

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \left(\frac{\partial u}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial y}\right) + \left(\frac{\partial u}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial y}\right) \\
&= \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}.
\end{aligned}$$

If we put $w = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}$, we arrive at

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial y}\right) + \left(\frac{\partial w}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial y}\right) \\
&= \frac{\partial w}{\partial \alpha} - \frac{\partial w}{\partial \beta} \\
&= \frac{\partial}{\partial \alpha}\left[\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right] - \frac{\partial v}{\partial \beta}\left[\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right] \\
&= \frac{\partial^2 u}{\partial \alpha^2} - 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Consequently

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

becomes

$$4\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

It suffices to solve the obtained equation

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

We have

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = \frac{\partial}{\partial \alpha}\left[\frac{\partial u}{\partial \beta}\right] = 0,$$

which implies that

$$\frac{\partial u}{\partial \beta} = f(\beta),$$

and therefore

$$\begin{aligned} u(\alpha, \beta) &= \int f(\beta)\ d\beta + g(\alpha) \\ &= h(\beta) + g(\alpha), \end{aligned}$$

where f , g and h are arbitrary continuous and differentiable functions. Accordingly, the solution is given by

$$\boxed{u(x, y) = h(x - y) + g(x + y).}$$

Exercise 06

We are interested in studying the limit of the function

$$f(x, y) = \frac{x^2 y}{x^4 + y^2},$$

as $(x, y) \to (0, 0)$.

To investigate this limit, examine the behavior of $f(x, y)$ along the following curves:

1) Along a straight line through the origin: $y = \lambda x$

2) Along the parabola: $y = x^2$.

Solution

1) Along the line $y = \lambda x$: Assuming $x \neq 0$, substitute $y = \lambda x$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} f(x, \lambda x) = \lim_{x\to 0} \left(\frac{\lambda x^3}{x^4 + \lambda^2 x^2} \right) = \lim_{x\to 0} \left(\frac{\lambda x}{x^2 + \lambda^2} \right).$$

So

$$\boxed{\lim_{(x,y)\to(0,0)} f(x, y) = 0.}$$

2) Along the parabola $y = x^2$**:** Assuming $x \neq 0$, substitute $y = x^2$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} f(x, x^2) = \lim_{x\to 0} \left(\frac{x^4}{x^4 + x^4} \right) = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

Since the function $f(x, y)$ approaches different values depending on the path taken as $(x, y) \to (0, 0)$. Therefore, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ **does not exist**.

Exercise 07

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:

$$f(x, y) = \begin{cases} \dfrac{xy}{|xy| + (x + y)^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

1) Study the existence of

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

2) Verify that the iterated limits

$$\lim_{x\to 0} \left(\lim_{y\to 0} f(x, y) \right) = \lim_{y\to 0} \left(\lim_{x\to 0} f(x, y) \right) = 0.$$

Solution

1) Study the existence of $\lim\limits_{(x,y)\to(0,0)} f(x,y)$: Let $y = \lambda x$, with $\lambda \in \mathbb{R}$, and assume $x \neq 0$. Then

$$f(x, \lambda x) = \frac{\lambda x^2}{|\lambda x^2| + (x + \lambda x)^2} = \frac{\lambda x^2}{|\lambda| x^2 + (1+\lambda)^2 x^2} = \frac{\lambda}{|\lambda| + (1+\lambda)^2}$$

This expression depends on λ, hence the limit varies with the direction. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) \ \textbf{does not exist.}$$

2) Compute the iterated limits: We have

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0, \ \text{and} \ \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$$

Hence, both iterated limits are equal to 0. However, based on the first part of the exercise, the full limit does not exist.

Exercise 08

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

1) Compute $g(x) = \dfrac{\partial f}{\partial y}(x,0)$ and $h(x) = \dfrac{\partial f}{\partial x}(0,y)$.

2) Compare the derivatives of $g(x)$ at $x = 0$ and of $h(y)$ at $y = 0$. What do you observe?

Solution

1) Compute $g(x) = \dfrac{\partial f}{\partial y}(x,0)$ and $h(x) = \dfrac{\partial f}{\partial x}(0,y)$:

We compute $g(x)$ by the definition of a partial derivative

$$g(x)=\frac{\partial f}{\partial y}(x,0)=\lim_{t\to 0}\frac{f(x,t)-\overbrace{f(x,0)}^{=0}}{t}=\lim_{t\to 0}\frac{xt\dfrac{x^2-t^2}{x^2+t^2}}{t}=\lim_{t\to 0}x\frac{x^2-t^2}{x^2+t^2}.$$

Therefore

$$\boxed{g(x)=x.}$$

Similarly

$$h(y)=\frac{\partial f}{\partial x}(0,y)=\lim_{t\to 0}\frac{f(t,y)-\overbrace{f(0,y)}^{=0}}{t}=\lim_{t\to 0}\frac{ty\dfrac{t^2-y^2}{t^2+y^2}}{t}=\lim_{t\to 0}y\frac{t^2-y^2}{t^2+y^2}.$$

Thus

$$\boxed{h(y)=-y.}$$

2) Compare $g'(0)$ and $h'(0)$: We find

$$\begin{aligned} g'(x) &= 1\Longrightarrow g'(0)=1=\frac{\partial^2}{\partial x\partial y}f(0,0),\\ h'(x) &= -1\Longrightarrow h'(0)=1=\frac{\partial^2}{\partial y\partial x}f(0,0). \end{aligned}$$

Thus, the mixed partial derivatives exist at the origin but are not equal

$$\frac{\partial^2}{\partial x\partial y}f(0,0)\neq\frac{\partial^2}{\partial y\partial x}f(0,0).$$

The second-order partial derivatives exist at $(0,0)$, but are not continuous there. This discontinuity explains why the mixed partials are not equal.

Exercise 09

Consider the function $F:\mathbb{R}^3\longrightarrow\mathbb{R}$ defined by:

$$F(x,y,z)=\sqrt{x^2+y^2+z^2}$$

1) Show that the Laplacian of F satisfies

$$\Delta F=\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

2) Using the fourth-order partial derivatives of F, express the quantity

$$\Delta(\Delta F)=\frac{\partial^2(\Delta F)}{\partial x^2}+\frac{\partial^2(\Delta F)}{\partial y^2}+\frac{\partial^2(\Delta F)}{\partial z^2},$$

and show that $\Delta(\Delta F)=0$.

Solution

1) $\Delta F=\dfrac{2}{\sqrt{x^2+y^2+z^2}}$: We have

$$\frac{\partial^2 F}{\partial x^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \frac{\partial^2 F}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \frac{\partial^2 F}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

$$\begin{aligned}\Delta F &= \frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}\\ &= \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}.\end{aligned}$$

Hence

$$\boxed{\Delta F=\frac{2}{\sqrt{x^2+y^2+z^2}}}.$$

2) We have

$$\begin{aligned}\frac{\partial^2(\Delta F)}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)=\frac{2(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}},\\ \frac{\partial^2(\Delta F)}{\partial y^2} &= \frac{\partial^2}{\partial y^2}\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)=\frac{2(2y^2-x^2-z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}},\\ \frac{\partial^2(\Delta F)}{\partial z^2} &= \frac{\partial^2}{\partial z^2}\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)=\frac{2(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}\end{aligned}$$

So

$$\begin{aligned}\Delta(\Delta F) &= \frac{\partial^2(\Delta F)}{\partial x^2}+\frac{\partial^2(\Delta F)}{\partial y^2}+\frac{\partial^2(\Delta F)}{\partial z^2}\\ &= \frac{2(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}+\frac{2(2y^2-x^2-z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}+\frac{2(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}.\end{aligned}$$

Then

$$\boxed{\Delta(\Delta F) = 0.}$$

We can also use

$$\begin{aligned}
\Delta(\Delta F) &= \frac{\partial^2(\Delta F)}{\partial x^2} + \frac{\partial^2(\Delta F)}{\partial y^2} + \frac{\partial^2(\Delta F)}{\partial z^2} \\
&= \frac{\partial^4 F}{\partial x^4} + \frac{\partial^4 F}{\partial y^4} + \frac{\partial^4 F}{\partial z^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + 2\frac{\partial^4 F}{\partial y^2 \partial z^2} + 2\frac{\partial^4 F}{\partial x^2 \partial z^2}.
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial^4 F}{\partial x^4} &= -3\frac{(y^2+z^2)(-4x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}, \\
\frac{\partial^4 F}{\partial y^4} &= -3\frac{(x^2+z^2)(x^2-4y^2+z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}, \\
\frac{\partial^4 F}{\partial z^4} &= -3\frac{(x^2+y^2)(x^2+y^2-4z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}, \\
\frac{\partial^4 F}{\partial x^2 \partial y^2} &= \frac{(2x^4-11x^2y^2+x^2z^2+2y^4+y^2z^2-z^4)}{(x^2+y^2+z^2)^{\frac{7}{2}}}, \\
\frac{\partial^4 F}{\partial y^2 \partial z^2} &= \frac{(-x^4+x^2y^2+x^2z^2+2y^4-11y^2z^2+2z^4)}{(x^2+y^2+z^2)^{\frac{7}{2}}} \\
\frac{\partial^4 F}{\partial x^2 \partial z^2} &= \frac{(2x^4+x^2y^2-11x^2z^2-y^4+y^2z^2+2z^4)}{(x^2+y^2+z^2)^{\frac{7}{2}}}.
\end{aligned}$$

Exercise 10

Let f be a function of the variables x and y, and consider the change of variables defined by:

$$u(x,y) = x+y \text{ and } v(x,y) = x-y.$$

Calculate, in terms of the partial derivatives of f with respect to u and v, the Laplacian of f.

Solution From the chain rule, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(\frac{\partial f}{\partial u}\right)\overbrace{\left(\frac{\partial u}{\partial x}\right)}^{=1}+\left(\frac{\partial f}{\partial v}\right)\overbrace{\left(\frac{\partial v}{\partial x}\right)}^{=1}=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=f'_u+f'_v,\\ \frac{\partial f}{\partial y} &= \left(\frac{\partial f}{\partial u}\right)\overbrace{\left(\frac{\partial u}{\partial y}\right)}^{=1}+\left(\frac{\partial f}{\partial v}\right)\overbrace{\left(\frac{\partial v}{\partial y}\right)}^{=-1}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}=f'_u-f'_v.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial f'_x}{\partial x}=\left(\frac{\partial f'_x}{\partial u}\right)\overbrace{\left(\frac{\partial u}{\partial x}\right)}^{=1}+\left(\frac{\partial f'_x}{\partial v}\right)\overbrace{\left(\frac{\partial v}{\partial x}\right)}^{=1}\\ &= \left[\frac{\partial\left(f'_u+f'_v\right)}{\partial u}\right]+\frac{\partial\left(f'_u+f'_v\right)}{\partial v}=f_{uu}+f_{uv}+f_{vu}+f_{vv}\\ \frac{\partial^2 f}{\partial y^2} &= f_{uu}-f_{uv}-f_{vu}+f_{vv}.\end{aligned}$$

Assuming $f \in C^2$ (i.e., continuous second derivatives), we use the symmetry of mixed partials, i.e.,

$$f_{uv}=f_{vu},$$

which leads to

$$\begin{aligned}\Delta f &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\\ &= f_{uu}+2f_{uv}+f_{vv}+f_{uu}-2f_{uv}+f_{vv}.\end{aligned}$$

Hence

$$\boxed{\Delta f=2\left(f_{uu}+f_{vv}\right).}$$

Exercise 11

By definition, a reaction–diffusion equation consists of a reaction term and a diffusion term. The prototypical model is formulated typically as the following nonlinear reaction-diffusion equation (also called nonhomogeneous heat conduction equation):

$$\frac{\partial u(t,x)}{\partial t} - d\Delta u(t,x) = f(u(t,x)),$$

where

- $u(t,x) = (u_1(t,x), ..., u_m(t,x)) \in \mathbb{R}^m$ is the state variable and describes temperature, density, concentration of a substance, number of individual in a population at position $x \in \Omega \subset \mathbb{R}^n$ and time t (Ω is an open set).
- d is a diffusion coefficient
- Δ is the Laplacian operator
- $D\Delta u(t,x)$ represent the term of diffusion
- The reaction term f is a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^m$.

Express the reaction-diffusion equation in polar coordinates then in spherical polar coordinates.

Solution

To express the heat equation in polar coordinates then in spherical polar coordinates, we need to rewrite the Laplacian operator in these new coordinates.

1. The general form of the diffusion equation or the heat conduction equation in Cartesian coordinates (three-dimensional space) is given by

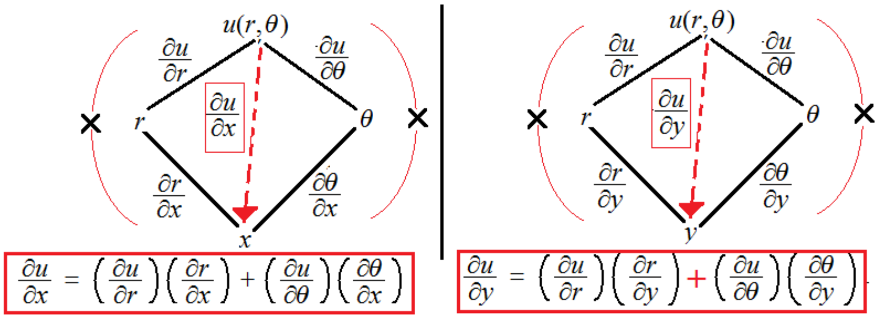
$$\frac{\partial u(t,x,y)}{\partial t} - d\Delta u(t,x,y) = f(u(t,x)),$$

where the Laplacian for a twice-differentiable scalar function $u(x,y)$ in a three-dimensional space, is given by

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

To convert from rectangular (Cartesian) to polar coordinates, we use the polar coordinates as follows:

$$\begin{cases} x = r\ \cos\theta \\ y = r\ \sin\theta \end{cases} \Longrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x}. \end{cases} \Longrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = arctg\left(\dfrac{y}{x}\right). \end{cases}$$



One has
$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right)$$

Calculating of $\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial r}{\partial x}$

One has $r = \sqrt{x^2+y^2}$ and $\theta = arctg\left(\dfrac{y}{x}\right)$, where

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= -\left(\frac{\sin\theta}{r}\right), \ \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \\ \frac{\partial r}{\partial x} &= \cos\theta, \ \frac{\partial r}{\partial y} = \sin\theta. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\overbrace{\frac{\partial u}{\partial x}}^{v}\right) = \left[\frac{\partial}{\partial r}\left(\overbrace{\frac{\partial u}{\partial x}}^{v}\right)\right]\left(\frac{\partial r}{\partial x}\right) + \left[\frac{\partial}{\partial \theta}\left(\overbrace{\frac{\partial u}{\partial x}}^{v}\right)\right]\left(\frac{\partial \theta}{\partial x}\right) \\ &= \left[\frac{\partial}{\partial r}\left(\left(\frac{\partial u}{\partial r}\right)(\cos\theta) + \left(\frac{\partial u}{\partial \theta}\right)\left(-\frac{\sin\theta}{r}\right)\right)\right](\cos\theta) \\ &\quad + \left[\frac{\partial}{\partial \theta}\left(\left(\frac{\partial u}{\partial r}\right)(\cos\theta) + \left(\frac{\partial u}{\partial \theta}\right)\left(-\frac{\sin\theta}{r}\right)\right)\right]\left(-\frac{\sin\theta}{r}\right) \\ &= \left[(\cos\theta)\frac{\partial^2 u}{\partial r^2} - \left(\frac{\sin\theta}{r}\right)\frac{\partial^2 u}{\partial r\partial\theta} + \left(\frac{\sin\theta}{r^2}\right)\frac{\partial u}{\partial \theta}\right](\cos\theta) \\ &\quad + \left[(\cos\theta)\frac{\partial^2 u}{\partial\theta\ \partial r} - (\sin\theta)\frac{\partial u}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial^2 u}{\partial\theta^2} - \left(\frac{\cos\theta}{r}\right)\frac{\partial u}{\partial\theta}\right]\left(-\frac{\sin\theta}{r}\right) \end{aligned}$$

Since $\dfrac{\partial^2 u}{\partial\theta\ \partial r} = \dfrac{\partial^2 u}{\partial r\ \partial\theta}$, then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \left(\frac{\sin^2\theta}{r^2}\right)\frac{\partial^2 u}{\partial\theta^2} - 2\left(\frac{\sin\theta\ \cos\theta}{r}\right)\frac{\partial^2 u}{\partial\theta\ \partial r} + \left(\cos^2\theta\right)\frac{\partial^2 u}{\partial r^2}\\ &\quad +2\left(\frac{\sin\theta\ \cos\theta}{r^2}\right)\frac{\partial u}{\partial\theta} + \left(\frac{\sin^2\theta}{r}\right)\frac{\partial u}{\partial r}.\end{aligned}$$

By the same calculation, we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \left(\frac{\cos^2\theta}{r^2}\right)\frac{\partial^2 u}{\partial\theta^2} + 2\left(\frac{\sin\theta\ \cos\theta}{r}\right)\frac{\partial^2 u}{\partial\theta\ \partial r} + \left(\sin^2\theta\right)\frac{\partial^2 u}{\partial r^2}\\ &\quad -2\left(\frac{\sin\theta\ \cos\theta}{r^2}\right)\frac{\partial u}{\partial\theta} + \left(\frac{\cos^2\theta}{r}\right)\frac{\partial u}{\partial r}.\end{aligned}$$

By adding up, we arrive at

$$\boxed{\Delta u = \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial r}.}$$

Finally, the reaction-diffusion equation in the polar coordinates is given by

$$\frac{\partial u}{\partial t} - \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial\theta^2} - \frac{\partial^2 u}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} = f\left(u\right).$$

2. One has

$$\begin{cases} x = r\ \sin\phi\cos\theta \\ y = r\ \sin\phi\sin\theta \\ z = r\ \cos\phi, \end{cases}$$

which gives

$$r = \sqrt{x^2+y^2+z^2},\ \ \theta = \arctan\left(\frac{y}{x}\right),\ \ \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

Consequently

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial u}{\partial r} - \frac{y}{\left(x^2+y^2\right)}\frac{\partial u}{\partial\theta} + \frac{xz}{\left(x^2+y^2+z^2\right)\sqrt{x^2+y^2}}\frac{\partial u}{\partial\phi},$$

and hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial u}{\partial r} - \frac{y}{(x^2+y^2)}\frac{\partial u}{\partial \theta} + \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\frac{\partial u}{\partial \phi}\right) \\
&= \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right)\frac{\partial u}{\partial r} + \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\right) \\
&\quad - \left(\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)}\right)\right)\frac{\partial u}{\partial \theta} - \left(\frac{y}{(x^2+y^2)}\right)\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\right) \\
&\quad + \left(\frac{\partial}{\partial x}\left(\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)\right)\frac{\partial u}{\partial \phi} \\
&\quad + \left(\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \phi}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{x^2}{(x^2+y^2+z^2)}\frac{\partial^2 u}{\partial r^2} + \frac{y^2}{(x^2+y^2)^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{x^2 z^2}{(x^2+y^2+z^2)^2(x^2+y^2)}\frac{\partial^2 u}{\partial \phi^2} \\
&\quad - \frac{2xy}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}\frac{\partial^2 u}{\partial \theta \partial r} + \frac{2x^2 z}{\sqrt{(x^2+y^2+z^2)^3}\sqrt{x^2+y^2}}\frac{\partial^2 u}{\partial \phi \partial r} \\
&\quad - \frac{2xyz}{(x^2+y^2+z^2)\sqrt{(x^2+y^2)^3}}\frac{\partial^2 u}{\partial \theta \partial \phi} + \frac{(y^2+z^2)}{\sqrt{(x^2+y^2+z^2)^3}}\frac{\partial u}{\partial r} \\
&\quad + \left(\frac{2xy}{(x^2+y^2)^2}\right)\frac{\partial u}{\partial \theta} + \left(\frac{y^2 z\,(x^2+y^2+z^2) - 2x^2 z\,(x^2+y^2)}{(x^2+y^2+z^2)^2\sqrt{(x^2+y^2)^3}}\right)\frac{\partial u}{\partial \phi}.
\end{aligned}$$

We have

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial u}{\partial r} + \frac{x}{(x^2+y^2)}\frac{\partial u}{\partial \theta} + \frac{yz}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2}}\frac{\partial u}{\partial \phi},$$

therefore

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{y^2}{(x^2+y^2+z^2)}\frac{\partial^2 u}{\partial r^2}+\frac{x^2}{(x^2+y^2)^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)}\frac{\partial^2 u}{\partial \phi^2}\\ &+\frac{2xy}{\sqrt{x^2+y^2+z^2}(x^2+y^2)}\frac{\partial^2 u}{\partial r\partial \theta}+\frac{2y^2z}{\sqrt{(x^2+y^2+z^2)^3}\sqrt{x^2+y^2}}\frac{\partial^2 u}{\partial r\partial \phi}\\ &+\frac{2xyz}{(x^2+y^2+z^2)\sqrt{(x^2+y^2)^3}}\frac{\partial^2 u}{\partial \theta\partial \phi}+\frac{(x^2+z^2)}{\sqrt{(x^2+y^2+z^2)^3}}\frac{\partial u}{\partial r}\\ &-\frac{2xy}{(x^2+y^2)^2}\frac{\partial u}{\partial \theta}+\frac{x^2z(x^2+y^2+z^2)-2y^2z(x^2+y^2)}{(x^2+y^2+z^2)^2\sqrt{(x^2+y^2)^3}}\frac{\partial u}{\partial \phi}.\end{aligned}$$

we have also

$$\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial u}{\partial r}-\frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)}\frac{\partial u}{\partial \phi},$$

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= \frac{z^2}{(x^2+y^2+z^2)}\frac{\partial^2 u}{\partial r^2}+\frac{(x^2+y^2)}{(x^2+y^2+z^2)^2}\frac{\partial^2 u}{\partial \phi^2}\\ &-\frac{2z\sqrt{x^2+y^2}}{\sqrt{(x^2+y^2+z^2)^3}}\frac{\partial^2 u}{\partial \phi\partial r}+\frac{(x^2+y^2)}{\sqrt{(x^2+y^2+z^2)^3}}\frac{\partial u}{\partial r}\\ &+\frac{2z\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2}\frac{\partial u}{\partial \phi}.\end{aligned}$$

From these calculations, it results that

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{(x^2+y^2)}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{(x^2+y^2+z^2)}\frac{\partial^2 u}{\partial \phi^2}\\ &+\frac{2}{\sqrt{(x^2+y^2+z^2)}}\frac{\partial u}{\partial r}+\frac{z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\frac{\partial u}{\partial \phi}.\end{aligned}$$

Since $x^2+y^2+z^2=r^2$ and $x^2+y^2=r^2\sin^2\phi$, we obtain

$$\boxed{\begin{aligned}\Delta u(x,y,z) = \frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}\\ +\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}.\end{aligned}}$$

Finally, the reaction-diffusion equation in the spherical coordinates is given by

$$\boxed{\begin{array}{l}\frac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial r^2}-\dfrac{1}{r^2\sin^2\phi}\dfrac{\partial^2 u}{\partial \theta^2}-\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2}\\ \quad -\dfrac{2}{r}\dfrac{\partial u}{\partial r}-\dfrac{\cot\phi}{r^2}\dfrac{\partial u}{\partial \phi}=f\left(u\right).\end{array}}$$

Exercise 12

Consider the following problem:

$$\begin{cases} u_{tt}-c^2\,u_{xx}=F\left(x,t\right)\\ u\left(x,0\right)=f\left(x\right)\\ \dfrac{\partial u}{\partial t}\left(x,0\right)=g\left(x\right), \end{cases}$$

where c is a real positive number, $F(x,t)$ is a given function and $u=u(x,t)$. Physically, $u(x,t)$ can be regarded as the vertical displacement of a vibrating string at position x and time t. We assume that the string is long enough so that its ends do not interfere during the time interval under consideration. $f(x)$ represents the initial displacement of the string and $g(x)$ stands for the initial (vertical) velocity.

Show that the nonhomogeneous linear PDE

$$\frac{\partial^2 u}{\partial t^2}-c^2\,\frac{\partial^2 u}{\partial x^2}=F\left(x,t\right),\ \ c>0,$$

is equivalent to the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t}-c\,\dfrac{\partial u}{\partial x}=v\\ \dfrac{\partial v}{\partial t}+c\,\dfrac{\partial v}{\partial x}=F\left(x,t\right). \end{cases}$$

Solution

We can rewrite the PDE

$$u_{tt} - c^2\, u_{xx} = F\,(x,t)\,,$$

in the form

$$\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u = F\,(x,t)\,.$$

Indeed

$$\begin{aligned}\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u &= \left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\,\frac{\partial u}{\partial x}\right)\\ &= \frac{\partial^2 u}{\partial t^2} - c\,\frac{\partial^2 u}{\partial t \partial x} + c\,\frac{\partial^2 u}{\partial x \partial t} - c^2\frac{\partial^2 u}{\partial x^2}.\end{aligned}$$

Since the mixed partial derivatives are continuous, then

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t}.$$

If we consider $\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u = v$, then

$$\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u}_{v} = F\,(x,t)\,,$$

gives

$$\boxed{\begin{cases}\dfrac{\partial u}{\partial t} - c\,\dfrac{\partial u}{\partial x} = v\\[2pt] \dfrac{\partial v}{\partial t} + c\,\dfrac{\partial v}{\partial x} = F\,(x,t)\,.\end{cases}}$$

Exercise 13

Consider the following chemical reactions:

$$\sum_{i \in I} n_i R_i \underset{k}{\overset{h}{\rightleftharpoons}} \sum_{l \in L} n_l R_l$$

where

- $I = \{1, ..., p_0\}$ and $L = \{p_0 + 1, ..., p\}$ (*i.e.*., $I \bigcup L = \{1, ..., p\}$ and $I \bigcap L = \emptyset$).
- $n_1, n_2, n_3, ..., n_p$ are the stoichiometric coefficients (numbers of molecules of each substance),
- $R_1, R_2, R_3, ..., R_p$ are the substances,
- h and k are two positive constants that depend generally on the temperature, position x, and the time t. They stand for the rates or kinetic constants of each reaction.

By virtue of the law of mass action as well as the Fick's law, write the system of equations that describes this process.

Solution

Since the modelling of this process is based on the law of mass action, we obtain

$$\begin{cases} n_1 \dfrac{d}{dt}[R_1] = -h\ [R_1]^{n_1} ... [R_{p_0}]^{n_{p0}} + k\ [R_{p_0+1}]^{n_{p0+1}} ... [R_p]^{n_p} \\ . \\ n_{p_0} \dfrac{d}{dt}[R_{p_0}] = -h\ [R_1]^{n_1} ... [R_{p_0}]^{n_{p0}} + k\ [R_{p_0+1}]^{n_{p0+1}} ... [R_p]^{n_p} \\ n_{p_0+1} \dfrac{d}{dt}[R_{p_0+1}] = +h\ [R_1]^{n_1} ... [R_{p_0}]^{n_{p0}} - k\ [R_{p_0+1}]^{n_{p0+1}} ... [R_p]^{n_p} \\ . \\ n_p \dfrac{d}{dt}[R_p] = +h\ [R_1]^{n_1} ... [R_{p_0}]^{n_{p0}} - k\ [R_{p_0+1}]^{n_{p0+1}} ... [R_p]^{n_p} \end{cases},$$

where $[R_1] ... [R_p]$ denote the concentrations of substances $R_1, ..., R_p$. Thus the reaction term is given by

$$f_r = (-1)^m \left(-h \prod_{i \in I} [R_i]^{n_i} + k \prod_{l \in L} [R_l]^{n_l} \right) \text{ with } m \begin{cases} \text{is even if } r \in I \\ \text{is odd if } r \in L \end{cases}.$$

We choose a test volume V with boundary S. Since the rate of formation of the $i-$th reactant is equal to the quantity formed by the reaction removed from its total flux through the surface S, then

$$\frac{\partial}{\partial t}\int_V n_q c_q\left(t,x\right)dx=\int_V f_q\left(t,x,c_1\left(t,x\right),...,c_p\left(t,x\right)\right)dx-\int_S J_q d\sigma,$$

where c_q is the concentration of R_q. Thanks to the Gauss divergence theorem, we arrive at

$$\int_V\left(n_q\frac{\partial c_q}{\partial t}-f_q+\nabla J_q\right)dx=0.$$

Since this equation is satisfied for all V, the integral can be omitted, and hence

$$n_q\frac{\partial c_q}{\partial t}+\nabla J_q=f_q.$$

On one hand, the Fick's first law states that the movement of particles (diffusion flux) from a region of high concentration to a low concentration region is directly proportional to the magnitude of the concentration gradient of the substance. This law can be expressed mathematically as

$$J=-d\nabla c_q,$$

where d represents the diffusion coefficient and the negative sign indicates flux flows from high to low concentration.

Finally, if we assume that the coefficient of diffusion d_q is constant, the above equation becomes

$$\boxed{n_q\frac{\partial c_l}{\partial t}-d_q\Delta c_q=-h\prod_{i\in I}c_i^{n_i}+k\prod_{j\in J}c_J^{n_J}.}$$

Exercise 14

In $\mathbb{R}^d$, we consider a fluid moving with the velocity $C(x,t)$, into which pollutant particles are introduced. Let $u(x,t)$ denote the concentration of pollutant particles in the fluid at time t and position x, and let $f(x,t)$ be the pollutant source. The vector field $U(x,t) = u(x,t)C(x,t)$ stands for the flux of pollutant particles.

We assume that the pollutant particles are simply carried along by the fluid—that is, there is no diffusion of the pollutant—and that u, f, and C are smooth (regular) functions.

Write the mathematical model that can describe this phenomenon (transport equation).

Solution

Let us define the total quantity of particles at time t as follows:

$$m(t) = \int_{\Omega} u(x,t)dx.$$

So, the variation of the quantity of particles in Ω is given by

$$m'(t) = \int_{\partial\Omega} u_t(x,t)dx.$$

The variation of m is due to the loss of pollutant across the boundary $\partial\Omega$ and the appearance of particles in Ω. By virtue of the divergence theorem, we arrive at

$$\begin{aligned} m'(t) &= \int_{\Omega} f(x,t)dx - \int_{\partial\Omega} U(x,t).\eta\,(x)\,dS_x \\ &= \int_{\Omega} \left(f(x,t) - \operatorname{div}\left(U(x,t)\right)\right) dx. \end{aligned}$$

Thus

$$\int_{\Omega} \left(\operatorname{div} U + u_t\right) dx = \int_{\Omega} f(x,t)dx,$$

for all $\Omega \subset \mathbb{R}^d$, we deduce the equilibrium law of the pollutant

$$u_t + \operatorname{div} U = u_t + \nabla u.C + u \operatorname{div} C = f.$$

If we assume that C is a constant representing the transport speed and $\varphi(x)$ is an arbitrary function denoting the initial distribution of the pollutant, we obtain the following transport or advection (horizontal transport) equation:

$$\begin{cases} u_t + C.\nabla u = f(x,t) & \text{pour } x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = \varphi(x) & \text{pour } x \in \mathbb{R}^d. \end{cases}$$

Exercise 15

Solve the homogeneous transport equation in $\mathbb{R}$.

Solution

Solving the Equation for $d = 1$

Suppose that $f(x,t) = 0$, the problem of the previous exercise can be written as follows:

$$\begin{cases} u_t + cu_x = 0 & \text{pour } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \Phi(x) & \text{pour } x \in \mathbb{R}, \end{cases}$$

where $c \in \mathbb{R}$. If we put $s = \begin{pmatrix} 1 \\ c \end{pmatrix}$, the equation becomes $\dfrac{du}{ds}(x,t) = 0$, which means that u is constant in the direction of s. So the transport equation $u_t + cu_x = 0$ can be interpreted as a directional derivative, where the directional derivative of the function $u(x,t)$ in the direction $\begin{pmatrix} 1 \\ c \end{pmatrix}$ is zero. This means the solution $u(x,t)$ is constant along the lines with direction vector $\begin{pmatrix} 1 \\ c \end{pmatrix}$.

Step 1: Let $z = u(x,t)$.

Parametrizing of the Initial Curve

$\Gamma = \{(x,t,z) \in \mathbb{R}^3 : \ x = \xi,\ t = 0,\ z = \Phi(\xi),\ \xi \in \mathbb{R}\}$.

Calculating the Jacobian Determinant

It's worth noting that the Jacobian determinant is the determinant of the matrix containing all the first-order partial derivatives of a function with several. In this exercise

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial t}{\partial \xi} & \dfrac{\partial t}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Step 2:

We solve the following two ODEs:

$$\frac{dx}{ds} = c \text{ and } \frac{dt}{ds} = 1.$$

We have

$$dx = cds,$$

which implies that

$$x(s) = cs + K_1,$$

and from

$$dt = ds,$$

we obtain

$$t(s) = s + K_2.$$

The values $x(0) = \xi$ and $t(0) = 0$ give

$$\begin{cases} x = cs + \xi \\ t = s \end{cases} \Longrightarrow \begin{cases} \xi = x - ct \\ s = t. \end{cases}$$

Step 3:

We solve the following ODEs:

$$\frac{du}{ds} = 0,$$

where $0 < s < \infty$ with the initial condition

$$u(x,0) = u(ct+\xi,t) = u(\xi,0) = \Phi(\xi).$$

Then

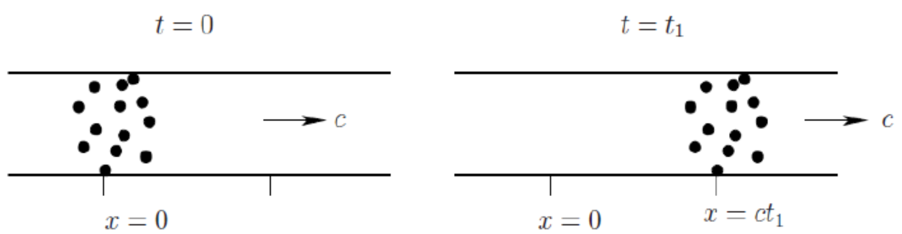
$$u(s,\xi) = K = \Phi(\xi).$$

Step 4:

Finally, by substituting the values of s and ξ as functions of x and t into the expression of $u(s,\xi)$, we deduce that

$$\boxed{u(x,t) = \Phi(x-ct).}$$

So if we consider a tube through which water flows at a speed $c > 0$ and which contains traces of pollutant. Therefore, the pollutant distribution at $t = 0$ is found be unchanged at time $t_1 > 0$, up to a translation by $+ct_1$.



BIBLIOGRAPHY

[1] Allab, K. (1990). *Éléments d'analyse: Fonctions d'une variable réelle* (3^{e} éd.). Entreprise Nationale des Livres.

[2] Appel, W. (2008). *Mathématiques pour la physique et les physiciens* (4^{e} éd.). H & K Éditions.

[3] Aslangul, C. (2011). *Des mathématiques pour les sciences : Concepts, méthodes et techniques pour la modélisation*. De Boeck.

[4] Aslangul, C. (2013). *Des mathématiques pour les sciences 2 : Corrigés détaillés et commentés des exercices et problèmes* (1^{e} éd.). De Boeck.

[5] Azoulay, É., & Avignant, J. (1984). *Mathématiques – Tome 2: Analyse : Cours et exercices* (Collection Enseignement supérieur technique). McGraw-Hill.

[6] Belorizky, É. (2007). *Outils mathématiques à l'usage des scientifiques et des ingénieurs. EDP Sciences.*

[7] Demidovich, B. P. (Ed.). (1965). *Problems in mathematical analysis* (G. Yankovsky, Trans.). Peace Publishers. (Original work published earlier in Russian)

[8] Bouakkaz, A. (2020). *Équations de la physique mathématique* (polycopié de cours), Université 20 Aout 1955, Skikda.

[9] Piskunov, N. S. (1969). *Calcul différentiel et intégral* (Tome II). Mir Publishers.

[10] Zitouni, M. (1993). *Algèbre* (2nd ed.). Alger, Algérie: Office des Publications Universitaires.

[11] University of Canterbury. Integration resource. Retrieved August 2025, from https://www.math.canterbury.ac.nz/php/resources/math100/integration/

[12] Maremathics Blog. (n.d.). Retrieved August 2025, from https://maremathics.pfst.hr/?p=119