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Thèse

En vue de l’obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option : ***Mathématique***

Sur la Stabilité de la poutre de Bresse

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ملخص

في هذه الرسالة، قمنا بدراسة استقرار بعض أنظمة بريس Bresse الحرارية أحادية البعد الخطية حيث تعطى ناقلية الحرارة بواسطة نظريتي جرين Green ونغدي Naghdi (المرونة الحرارية نوع III) مع وجود آليات تبديد مختلفة. الأول هو نظام من خمس معادلات تفاضلية جزئية زائدية مع ثلاث ذاكرات لا نهائية. الثاني له نفس شكل النظام السابق، لكننا استبدلنا الحدود الثلاثة للذاكرات اللانهائية بآخرين بذاكرة منتهية. والأخير هو نظام من أربع معادلات تفاضلية جزئية زائدية مع حدين مختلفين من التخميد، وهما ثابت التأخير و الذاكرة المحدودة.

لإظهار استقرار هذه الأنظمة الثلاثة، استخدمنا طريقة أنصاف الزمر، وهي تعتمد على بناء دالة Lyapunov (L) المكافئة لطاقة الحلول(E) . في البرهان، استخدمنا دالة الطاقة من الدرجة الثانية لتقريب الحدين ($\int_0^1 \theta_{1tx}(\varphi_x + \psi + l\omega)dx$) و ($\int_0^1 \theta_{2tx}(\varphi_x + \psi + l\omega)dx$).

الكلمات المفتاحية : التأخير، المرونة الحرارية من النوع الثالث، نظام Bresse، الإستقرار، نظرية الزمر الجزئية، اللزوجة-المرنة.

CONTENTS

Dedication

I dedicate this humble work to:

My parents whose their love for me has no bounds, who taught me the value of hard work, my dear Mother. THANK you for not only being a parent for me but also being a friend, a teacher and a mentor.

To my husband for his motivation and moral encouragement.

To my children, the source of my inspiration.

To my lovely sisters and dear brothers, who stand by my side when things look bleak.

To my lovely teachers who made me an ambitious person.

A mes amis qui m'ont soutenu et aidé à accomplir ce travail

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Abstract

In this thesis, we studied the stability of some one-dimensional linear thermoelastic Bresse systems where the heat conduction is given by Green and Naghdi theories (thermoelasticity type III) with the presence of different mechanisms of dissipation. The first is a system of five hyperbolic partial differential equations with three infinite memories. The second has the same form as the previous system, but we replaced the three infinite memory terms by two finite memory terms. The last one is a system of four hyperbolic partial differential equations with two different damping, they are constant delay and finite memory.

To show the stabilization of these three systems, we use a multipliers method, it is based on the construction of a Lyapunov function L equivalent to energy E of the solutions. In the proof, we used the second order energy function to estimate the terms $\int_0^1 \theta_{1tx} \left(\varphi_x + \psi + l\omega\right) dx$ and $\int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx$.

Keywords: Delay, Thermoelasticity type III, Bress system, Lyapunov Function, Stability, Semigroups theory, Viscoelasticity.

Résumé

Dans cette thèse, nous avons étudié la stabilité de certains systèmes de Bresse thermoélastiques linéaires unidimensionnels où la conduction thermique est donnée par les théories de Green et Naghdi (thermoélasticité de type III) avec la présence de différents mécanismes de dissipation. Le premier est un système de cinq équations aux dérivées partielles hyperboliques avec trois mémoires infinies. Le second a la même forme que le système précédent, cependant, les trois termes de mémoire infinie ont été remplacés par deux termes de mémoire finie. Le dernier est un système de quatre équations aux dérivées partielles hyperboliques avec deux amortissements différents: constante de retard et mémoire finie.

Pour montrer la stabilisation de ces trois systèmes, nous avons utilisé la méthode des multiplicateurs, elle est basée sur la construction d'une fonction de Lyapunov L équivalente à l'énergie des solutions (E). Dans la démonstration, nous avons utilisé la fonction d'énergie du second ordre pour estimer les termes $\int_0^1 \theta_{1tx} \left(\varphi_x + \psi + l\omega\right) dx$ et $\int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx$.

Mots-clé: Retard, Thermoélasticité de type III, Système de Bress, Fonctionnelle de Lyapunov, Stabilité, Théorie des semigroupes, Viscoélasticité.

Introduction

In mechanics, vibratory movements are often harmful to the structure. To reduce these vibrations as much as possible, various types of damping have been developed and integrated directly into the system, on the border or part of the border such as friction, thermal or viscoelastic damping. Therefore, in applications, stabilization must occur very quickly to prevent possible damage to the structure or malfunctioning of thedevice.In the thermoelasticity theory, as the name suggests, it is concerned with the effects of heat on constraints and deformations in elastic solid bodies and vice versa. In other words, thermoelasticity deals with the relationship between the elastic properties of a material and its temperature, or between its thermal conductivity and its constraints.

In this thesis we will expose the mathematical formalism of the theory of thermoelasticity and we recall some results concerning the existence and the asymptotic behavior of thermoelastic Bresse systems.

The Bresse systems

In 1856, the isothermal system [6], known as the circular arc problem, was created by Bresse, which consists of three wave equations where the main variables φ, ω, and ψ describe, respectively, the vertical, longitudinal and shear angle displacements. This system takes the following form

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases} \tag{1}$$

where F_1, F_2 and F_3 represent the external forces. N, Q and M are used to denote the axial force, the shear force and the bending moment, respectively. The last three forces are defined as follows

$$N=k_0\left(\omega_x-l\varphi\right),Q=k\left(\varphi_x+\psi+l\omega\right),M=b\psi_x.$$

Here $\rho_1=\rho A,\ \rho_2=\rho I,\ k=\bar{k}AG,\ k_0=EA,\ b=EI$ and $l=R^{-1}$ where ρ is the density of the material, E is the modulus of elasticity, G is the shear modulus, $\bar{k}$ is the shear factor, A is the cross-sectional area, I is the second moment of the cross-section, and R is the radius of curvature.

System (1) is an undamped system and its associated energy remains constant when the time t evolves. For this reason, dampings of different kinds must be added to the equations or at the boundary to stabilize the system that have been considered by several authors [1, 3, 4, 5, 13, 25, 35, 43], among these damping we mention the viscoelastic damping of the form $\displaystyle\int_0^{+\infty} g\left(s\right)\omega_{xx}\left(x,t-s\right)ds$, which Guesmia [20] added in the last equation of the Bresse system (1) (See as well [21, 42]).

In what concerns the Bresse system in the classical thermoelasticity, Liu and Rao [33] examined the system

$$\begin{cases} \rho_1\varphi_{tt}-k\left(\varphi_x+\psi+l\omega\right)_x-k_0l\left(\omega_x-l\varphi\right)+l\gamma\theta_1=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_2\psi_{tt}-b\psi_{xx}+k\left(\varphi_x+\psi+l\omega\right)+\gamma\theta_x=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_1\omega_{tt}-k_0\left(\omega_x-l\varphi\right)_x+kl\left(\varphi_x+\psi+l\omega\right)+\gamma\theta_{1x}=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_3\theta_t-\theta_{xx}+\gamma\psi_{tx}=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_3\theta_{1t}-\theta_{1xx}+\gamma(\omega_{tx}-l\varphi_t)=0, & (x,t)\in(0,L)\times(0,+\infty). \end{cases} \tag{2}$$

The authors showed, under initial and specific boundary conditions, that the system is exponentially stable provided

$$\frac{k}{\rho_1}=\frac{b}{\rho_2}\quad\text{and}\quad k=k_0. \tag{3}$$

Otherwise, in the case of non-equal speeds they showed that the energy of the system decays polynomially. Later, Fatori and Muñoz Rivera in [14] obtained a similar result where they are studying the stability of System (2) but without the last equation and θ_1.

In the above thermoelasticity Bresse system, the heat flux is given by the Fourier's law of heat conduction, which assumes that this flux is proportional to the gradient of the temperature. Further, it is well known that this theory predicts a physical paradox of the infinite propagation speed of thermal signals, that is, any sudden thermal disturbance at one point will be instantaneously transferred to the other parts of the body. Different models, removing this physical paradox, have been introduced such as thermoelasticity by second sound or thermoelasticity type III. This latter was introduced by Green and Naghdi [17, 18, 19] in the end of the last century, where they proposed a model of thermoelasticity that includes temperature gradient and thermal displacement gradient among the constitutive variables and proposed a heat conduction law as

$$q(x,t) = -\left[k\theta_x(x,t) + k^* v_x(x,t)\right], \tag{4}$$

where $v_t = \theta$ and v is the thermal displacement gradient, k and k^* are two positive constants.

Combination of (4) with the following energy balance law

$$\rho\theta_t + \varrho \operatorname{div} q = 0$$

leads to the equation

$$\rho\theta_{tt} - \varrho k\theta_{txx} - \varrho k^* \theta_{xx} = 0$$

which permits propagation of thermal waves at finite speed.

To our knowledge, few authors are interested in studying the coupling of this theory with a system of Bresse, we mention Gallego and Muñoz Rivera [16] and, in 2016 M.L. Santos [41] considered a Bresse system in thermoelasticity of type III acting on shear force of the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + k\theta_{tx} = 0, & in\ (0,L)\times(0,+\infty),\\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) - k\theta_t = 0, & in\ (0,L)\times(0,+\infty),\\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) - kl\theta_t = 0, & in\ (0,L)\times(0,+\infty),\\ \rho_3\theta_{tt} - \delta\theta_{xx} + k\left(\varphi_x + \psi + l\omega\right)_t - \gamma\theta_{txx} = 0, & in\ (0,L)\times(0,+\infty), \end{cases}$$

where they have shown that the corresponding semigroup is exponentially stable if and only if the wave velocities associated with the hyperbolic part of the system are equal. Otherwise, the solution decreases polynomially and they have proved that the rate of decay is optimal.

The Bresse system (1) is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered ($l = 0$). There exist some publications concerning the stabilization of the Timoshenko type III thermoelasticity system with different kinds of damping such as the delay time (see for instance [27, 30]). Recently, the control of PDEs with this delay time has become widely known because many physical phenomena that require knowledge of certain events in the past, that is why we have introduced this term in our system, such as that considered by Kafini et al. in [31] where they studied a general decay of energy for a Timoshenko system of thermoelasticity of type III

$$\begin{cases} \rho_1\phi_{tt} - k\left(\phi_x + \psi\right)_x = 0, & in\ (0,1)\times(0,+\infty),\\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\phi_x + \psi\right) + \beta\theta_{tx} = 0, & in\ (0,1)\times(0,+\infty),\\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - \mu_1\theta_{txx}\left(x,t\right) - \mu_2\theta_{txx}\left(x,t-\tau\right) = 0, & in\ (0,1)\times(0,+\infty), \end{cases} \tag{5}$$

where $\rho_1, \rho_2, \rho_3, b, k, \beta, \gamma, \delta, \mu_1$ are positive constants, μ_2 is a real number and $\tau > 0$ represents the time delay, under Dirichlet-Dirichlet–Neumann boundary conditions and the assumption $\mu_1 > |\mu_2|$.

A. Djebabla and N. Tatar in [11] studied a linear Timoshenko system with much weak dissipation of the viscoelastic type of the form $\int_0^t g(t-s)\,\theta_{xx}(x,s)\,ds$, more precisely, they studied the system

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x+\psi)_x = 0, & in\ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x+\psi) + \gamma\theta_x = 0, & in\ (0,L)\times(0,+\infty), \\ \rho_3\theta_{tt} - l\theta_{xx} + \beta\displaystyle\int_0^t g(t-s)\,\theta_{xx}(x,s)\,ds + \gamma\psi_{ttx} = 0, & in\ (0,L)\times(0,+\infty), \end{cases}$$

where they prove the exponential decay of the solutions in the energy norm, and they introduced two new numbers of exponential stability.

Description and objective of the thesis

In this thesis, we consider three problems. The first is a one-dimensional linear Bresse system of thermoelasticity of type III formed by five hyperbolic partial differential equations, with three infinite memories acting in the second, fourth and fifth equations. We use the semigroup theory and the energy method to prove the well-posedness and stability results, respectively. In order to prove the stability of the system, we used the second order energy function to estimate the terms $\int_0^1 \theta_{1tx}(\varphi_x+\psi+l\omega)\,dx$ and $\int_0^1 \theta_{2tx}(\varphi_x+\psi+l\omega)\,dx$.

The second is also a one-dimensional linear Bresse system of thermoelasticity of type III but we have replaced the three infinite memory terms by two finite memory terms where they are positioned in the last two equations. We establish the stability of the system with Drichlet boundary conditions under a smallness condition on l, this is done by selecting suitable Lyapunov functionals.

The last problem is a linear Bresse system of thermoelasticity of type III formed by four hyperbolic partial differential equations, with a finite memory term in the second equation and a constant delay in the last equation. We prove the stability of the system in the case of equal wave speeds when l is small enough, under the homogeneous Dirichlet–Neumann–Neumann–Dirichlet boundary conditions.

This thesis consists of four chapters:

Chapter 1:

In this chapter, we present preliminary notions.

Chapter 2:

Chapter 2 deals with energy decay for a thermoelastic Bresse system of type III with three infinite memories. It consists of three sections: preliminaries, result of existence and uniqueness and result of general stability of this system.

Chapter 3:

In this chapter, we study a thermoelastic Bresse system with two thermo-viscoelastic dampings where the heat conduction is given by Green and Naghdi theories, using the multiplier method we demonstrate the stability for the case of equal speeds of wave propagation.

Chapter 4:

In this last chapter, we obtain an exponential rate of decay for the solution of the Bresse system with a constant delay and viscoelastic terme.

Publications

[1] S. Boulechfar, S. Zitouni, A. Djebabla, A. Guesmia, Energy decay of the Bresse system by two thermo-viscoelastic dampings, *NONLINEAR STUDIES,* Vol. 27, No. 4, pp. 957-974, 2020.

CHAPTER 1

Preliminaries

In this chapter, we recall some necessary materials needed in the proof of our results, such as basic results which concerning the fontamotal spaces some theorems on these last and existence and uniqueness theorem.

1.1 Fondamental spaces

1.1.1 Hilbert Spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. We will suffice to mention its definition.

Definition 1.1 [7] A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.

1.1.2 Sobolev Spaces

The $L^p(\Omega)$ spaces:

Definition 1.2 [7] Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

If $p = \infty$, we have

$$\begin{aligned} L^\infty(\Omega) &= \{f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists a constant } C \\ &\quad \text{such that, } |f(x)| \leq C \text{ a.e in } \Omega\}. \end{aligned}$$

Notation 1.1 *For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Also, we denote by

$$\|f\|_\infty = \inf \{C, \ |f(x)| \leq C \text{ a.e in } \Omega\}.$$

Notation 1.2 *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e.* $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1.1 In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) \, g(x) \, dx,$$

is a Hilbert space.

The Sobolev spaces H^m :

Definition 1.3 The Sobolev space $H^m(\Omega)$ $(m \in \mathbb{N}$)is defined to be

$$\begin{aligned} H^m(\Omega) &= \Big\{ u \in L^2(\Omega) : \ D^\alpha u \in L^2(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| = \sum_{j=1}^n \alpha_j \leq m, \\ &\quad \text{where the derivatives } D^\alpha u \text{ are taken in the weak sense } \Big\}, \end{aligned}$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx,$$

with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha u\|_{L^2})^2 \right)^{\frac{1}{2}}.$$

Remark 1.2 We have the following characterization of

$$H_0^m(\Omega) = \{u \in H^m(\Omega), u = u' = ... = u^{(m-1)} = 0 \text{ on } \partial\Omega\}.$$

It is essential to notice the distinction between

$$H_0^2(\Omega) = \{u \in H^2(\Omega), \ u = u' = 0 \text{ on } \partial\Omega\},$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega), \ u = 0 \text{ on } \partial\Omega\}.$$

1.2 Some inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.1 *[7] (Hölder's Inequality) Let $1 \leq p \leq \infty$, assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg| \, dx \leq \|f\|_p \|g\|_q . \tag{1.1}$$

Lemma 1.2 *[7] (Poincaré's inequality) Suppose I is a bounded interval. Then there exists a positive constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{H^1(I)} \leq C \|u'\|_{L^2(I)}, \text{ for all } u \in H_0^1(I) . \tag{1.2}$$

Lemma 1.3 *[38] (Poincaré type Scheeffer's inequality): Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2 \, dx \leq \ l \int_0^L |h_x|^2 \, dx, \quad l = \frac{L^2}{\pi^2}. \tag{1.3}$$

Lemma 1.4 *[7] (Cauchy-Schwarz Inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\| . \tag{1.4}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Lemma 1.5 *[7](Young's Inequality) For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{1.5}$$

where ϵ is any positive constant.

1.3 Existence and uniqueness theorem

1.3.1 C_0-Semigroup of bounded linear operators

Throughout this section $\mathcal{H}$ denotes a Hilbert space.

Definition 1.4 [40] Let X be a Banach space. A one parameter family $\{S(t)\}_{t \geq 0}$ of bounded linear operators defined from X into X is a strongly continuous semigroup of bounded linear operators on X if:

- $S(0) = I$ (I identity operator on X).
- $S(t+s) = S(t) S(s)$ for every $t, s \geq 0$.
- $S(t) x \to x$, as $t \to 0$, $\forall x \in X$.

Such a semigroup is called a C_0-semigroup.

Definition 1.5 [40] We call infinitesimal generator of the C_0-semigroup $\{S(t)\}_{t\geq 0}$ any operator $\mathcal{A}$ defined on the set

$$D(\mathcal{A}) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\},$$

by

$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t}, \ x \in D(\mathcal{A}).$$

Definition 1.6 A C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$ is said to be of contractions if

$$\|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq 1, \forall t \geq 0.$$

Definition 1.7 [7] An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \ \to \mathcal{H}$ is said to be dissipative if it satisfies

$$(\mathcal{A}u, u) \leq 0, \ \forall u \in D(\mathcal{A}),$$

It is called maximal dissipative if, in addition

$$Im(\mathcal{I} - \mathcal{A}) = \mathcal{H}, \text{ i.e.,}$$

$$\forall f \in \mathcal{H}, \exists u \in D(\mathcal{A}) \text{ such that } u - Au = f.$$

Proposition 1.1 *[7] Let $\mathcal{A}$ be a maximal dissipative operator. Then $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

Generally speaking, the first step in dealing with the study of the well-posedness of the solution is to rewrite our evolution system of partial differential equations as a Cauchy problem on some appropriate Hilbert space $\mathcal{H}$ called the energy space

$$\begin{cases} u' + \mathcal{A}(t)u = 0, \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}(t)$ is an unbounded operator on $\mathcal{H}$. Then we prove that $\mathcal{A}(t)$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$ in order to deduce the existence of a solution in a certain Hilbert space. The solution is hence of the form $u(t) = S(t)u_0$. We mention here Hille–Yosida Theorem: Lumer-Phillips form which is applied to justify the existence and uniqueness of solutions of some partial differential equations.

Theorem 1.1 *[7] (Hille–Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C([0,\infty[, D(\mathcal{A})) \cap C^1([0,\infty[, \mathcal{H})$$

satisfying

$$\begin{cases} u' + \mathcal{A}(t)u = 0 \quad on \ [0,\infty[\\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0|, \ \forall t \geq 0 \ and \ |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0|, \, \forall t > 0.$$

Theorem 1.2 *(Lumer-Phillips) Let $\mathcal{A}$:$D(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ a linear operator and $D(\mathcal{A})$ is dense in $\mathcal{H}$. Then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if*

i) $\mathcal{A}$ is dissipative,

ii) There exists $\lambda > 0$ such that $Im(\lambda I - \mathcal{A}) = \mathcal{H}$ ($\mathcal{A}$ is maximal).

1.3.2 Lax-Milgram theorem

The existence and uniqueness of a solution to the weak formulation of the problem can be proved using the Lax-Milgram Lemma. This states that the weak formulation admits a unique solution.

Theorem 1.3 *[7](Lax-Milgram theorem). Let $a(\cdot,\cdot)$ be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with norm $\|\cdot\|_{\mathcal{H}}$ and the following properties:*

i) $a(\cdot,\cdot)$ is continuous, that is

$$\exists \gamma_1 > 0 \ such \ that \ |a(u,v)| \leq \gamma_1 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \ \forall u, v \in \mathcal{H}.$$

ii) $a(\cdot,\cdot)$ coercive (or $\mathcal{H}$-elliptic), that is

$$\exists \alpha > 0 \ such \ that \ a(v,v) \geq \alpha \|v\|_{\mathcal{H}}^2, \ \forall v \in \mathcal{H}.$$

iii) L is a linear mapping on $\mathcal{H}$ (thus L is continuous), that is

$$\exists \gamma_2 > 0 \ such \ that \ |L(v)| \leq \gamma_2 \|v\|_{\mathcal{H}}, \ \forall v \in \mathcal{H}.$$

Then there exists a unique $u \in \mathcal{H}$ such that

$$a(u,v) = L(v), \ \forall v \in \mathcal{H}.$$

1.4 Convolution product

Definition 1.8 (Convolution product) The convolution product of two real or complex functions $f$and g is a function, which is generally denoted $(f * g)$ and which is defined by:

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds.$$

Definition 1.9 Let f and g be two real or complex functions. We define the binary operators $\diamond$ and $\circ$ respectively by

$$(f \diamond g)(t) = \int_0^t |f(t-s)|\, |g(t) - g(s)|\, ds,$$

and

$$(f \circ g)(t) = \int_0^t |f(t-s)|\, |g(s) - g(t)|^2\, ds.$$

Lemma 1.6 *[2] Let f and g be two functions of $C^1([0,\infty), \mathbb{R})$, then we have*

$$(g * f)\frac{df}{dt} = -\frac{1}{2}g(t)\,|f(t)|^2 + \frac{1}{2}\left(g^{'} \circ f\right)(t) - \frac{1}{2}\frac{d}{dt}\left((g \circ f)(t) - \int_0^t g(s)ds\,|f(t)|^2\right). \tag{1.6}$$

Lemma 1.7 *[11] For any function $g \in C([0,+\infty), \mathbb{R}_+)$ and any $v \in L^2(0,1)$ we have*

$$\int_0^1 (g \diamond v)^2 dx \leq \left(\int_0^t g(s)ds\right) \int_0^1 (g \circ v)(t)dx, \qquad \forall t \geq 0. \tag{1.7}$$

Remark 1.3 Using Lemma 1.7 and the Poincaré inequality, with $-g^{'}$, instead of g, we obtain

$$\int_0^1 (-g^{'} \diamond v)^2 dx \leq c_p g(0) \int_0^1 (g \circ v)(t)dx, \tag{1.8}$$

where c_p is a positive constant.

Notation 1.3 *Along this thesis, for $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $g(0) > 0$, we use the following notation*

$$\bar{g} = \int_0^\infty g(s)ds. \tag{1.9}$$

Lemma 1.8 *[13] There exists a positive constant c such that the following inequality holds for every $(\varphi, \psi, \omega) \in [H_0^1(0,L)]^3$*

$$\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx \leq c \int_0^L \left[b\psi_x^2 + k(\varphi_x + \psi + l\omega)^2 + k_0(\omega_x - l\varphi)^2\right] dx. \tag{1.10}$$

CHAPTER 2

Energy decay for a thermoelastic Bresse system of type III with three infinite memories

2.1 Introduction

Bresse system takes into account the arc deformations of a circle subjected to longitudinal and vertical displacements and the angle of rotation of a filament, denoted by ω, φ and ψ, respectively. The system is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) = F_1, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) = F_2, \\ \rho_3\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = F_3, \end{cases} \tag{2.1}$$

where F_i, $i = 1, 2, 3$ denote the external forces exerted on the object and the coefficients $\rho_1, \rho_2, \rho_3, k, k_0, l$ and b are positive constants characterizing the elastic properties of materials. The Bresse system (2.1) is a linear model coupling three wave equations and it was initially introduced by Bresse [6]. When $F_1 = F_2 = F_3 = 0$, (2.1) is purely conservative. In other words, by taking into account the conditions at the edges, its associated energy, defined by the functional

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\omega_t^2 + b\psi_x^2 + k\left(\varphi_x + \psi + l\omega\right)^2 + k_0\left(\omega_x - l\varphi\right)^2\right] dx,$$

satisfies $E^{'}(t) = 0$. Therefore, the identity given by $E(t) = E(0)$ remains true for all $t \geq 0$. This identity is called the energy conservation property.

There is a large number of publications concerning the stabilization of Bresse systems with dissipative mechanisms of several types, such as viscoelastic or memory type of the form $\int_0^{+\infty} g_1\left(s\right)\psi_{xx}\left(t-s\right) ds$, in which we are interested throughout this chapter.

The issue of existence and stability of Bresse system with the infinite memory terms has attracted a great deal of attention in the last decades (e.g. [20, 21, 22, 42]), in

[21], Guesmia and Kafini studied system (2.1) when $F_1 = -\int\limits_0^{+\infty} g_1(s)\,\varphi_{xx}(t-s)\,ds, F_2 = -\int\limits_0^{+\infty} g_1(s)\,\psi_{xx}(t-s)\,ds$ and $F_3 = -\int\limits_0^{+\infty} g_1(s)\,\omega_{xx}(t-s)\,ds$, where g_i are decreasing differentiable functions satisfying some hypotheses. Precisely, they established the existence and uniqueness of the solution and the asymptotic stability of this system, but without imposing any conditions on the coefficients. When $F_1 = 0, F_2 = 0$ and $F_3 = -\int\limits_0^{+\infty} g_1(s)\,\omega_{xx}(t-s)\,ds$,under Dirichlet–Neumann–Neumann boundary conditions, Guesmia [20] proved the asymptotic stability of the system in the case of equal propagation speeds $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right.$ and $\left.k = k_0\right)$ as well as for different speeds of wave propagation $\left(\frac{k}{\rho_1} \neq \frac{b}{\rho_2}\right.$ and $\left.k = k_0\right)$, where the author used, in this last case, the second order energy function to estimate the term $\int\limits_0^L \varphi_t \omega_{tx} dx$.

As for the Bresse thermoelastic system, it has been considered by several authors [14, 16, 28, 33, 41], where they used, in these works, the theory of classical in which the heat flux is given by the Fourier's law of heat conduction or the non-classical thermoelasticity where the heat conduction is given by Green and Naghdi theories.

In this chapter we are concerned with the thermoelastic Bresse system of type III with three infinite memories, which has the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + l\gamma\theta_{1t} = 0, \\ \rho_2\psi_{tt} - b_1\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \beta_1 \int\limits_0^{+\infty} g_1(s)\,\psi_{xx}(t-s)\,ds + \gamma\theta_{2tx} = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_{1tx} = 0, \\ \rho_3\theta_{1tt} - b_2\theta_{1xx} + \beta_2 \int\limits_0^{+\infty} g_2(s)\,\theta_{1xx}(t-s)\,ds + m_1\left(w_x - l\varphi\right)_t = 0, \\ \rho_4\theta_{2tt} - b_3\theta_{2xx} + \beta_3 \int\limits_0^{+\infty} g_3(s)\,\theta_{2xx}(t-s)\,ds + m_2\psi_{xt} = 0. \end{cases} \tag{2.2}$$

in $\Omega \times \mathbb{R}^+$, where $\Omega = [0,1]$. We supplement system (2.2) by the following initial data and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,1), \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in (0,1), \\ \theta_1(x,0) = \theta_{10}(x),\ \theta_{1t}(x,0) = \theta_{11}(x), & x \in (0,1), \\ \theta_2(x,0) = \theta_{20}(x),\ \theta_{2t}(x,0) = \theta_{21}(x), & x \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \omega(0,t) = \omega\left(1,t\right) = 0, & t \in (0,+\infty), \\ \theta_1\left(0,t\right) = \theta_1\left(1,t\right) = \theta_2\left(0,t\right) = \theta_2\left(1,t\right) = 0, & t \in (0,+\infty). \end{cases} \tag{2.3}$$

$\rho_1, \rho_2, \rho_3, \rho_4, k, k_0, l, \gamma, b_1, b_2, b_3, \beta_1, \beta_2, \beta_3, m_1, m_2$ are positive constants, and g is a positive function satisfying some conditions to be specified later.

Motivated by works mentioned above, we investigate (2.2)–(2.3) under suitable condition and establish the well-posedness of the problem using semi-group theory, as well as the stability result of the solution using the multiplier method. Our purpose, here, is to obtain a general decay rate estimates of the energy function of (2.2) in the case of equal and nonequal wave propagation speeds, without any restriction on the coefficients of the system.

2.2 Preliminaries and well-posedness result

In this section, we present some materials needed in the proof of our results. We also state, with proof, a result of existence and uniqueness of the solution for the problem (2.2)–(2.3). The proof is established by using semi-group method.

We shall use the following assumptions:

$(H1)$ $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function for $i = 1, 2, 3$, such that

$$g_i(0) > 0, \quad b_i - \beta_i \int_0^{+\infty} g_i(s)\, ds = b_i - \beta_i g_i^0 = \lambda_i > 0. \tag{2.4}$$

$(H2)$ There exists a nonincreasing function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy

$$g_i'(t) \leq -\zeta(t)\, g_i(t), \quad \forall\, t > 0. \tag{2.5}$$

We first introduce these new functionals

$$\begin{cases} \eta_1(x,t,s) = \psi(x,t) - \psi(x,t-s) & in\ (0,1) \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \eta_2(x,t,s) = \theta_1(x,t) - \theta_1(x,t-s) & in\ (0,1) \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \eta_3(x,t,s) = \theta_2(x,t) - \theta_2(x,t-s) & in\ (0,1) \times \mathbb{R}^+ \times \mathbb{R}^+. \end{cases} \tag{2.6}$$

then system (2.2) is rewritten as

$$\left\{\begin{array}{l}
\rho_1\varphi_{tt}-k\left(\varphi_x+\psi+l\omega\right)_x-k_0l\left(\omega_x-l\varphi\right)+l\gamma\theta_{1t}=0,\\
\rho_2\psi_{tt}-\lambda_1\psi_{xx}+k\left(\varphi_x+\psi+l\omega\right)-\beta_1\int\limits_0^{+\infty}g_1\left(s\right)\eta_{1xx}\left(s\right)ds+\gamma\theta_{2tx}=0,\\
\rho_1\omega_{tt}-k_0\left(\omega_x-l\varphi\right)_x+kl\left(\varphi_x+\psi+l\omega\right)+\gamma\theta_{1tx}=0,\\
\rho_3\theta_{1tt}-\lambda_2\theta_{1xx}-\beta_2\int\limits_0^{+\infty}g_2\left(s\right)\eta_{2xx}\left(s\right)ds+m_1\left(w_x-l\varphi\right)_t=0,\\
\rho_4\theta_{2tt}-\lambda_3\theta_{2xx}-\beta_3\int\limits_0^{+\infty}g_3\left(s\right)\eta_{3xx}\left(s\right)ds+m_2\psi_{xt}=0.\\
\eta_{1t}+\eta_{1s}-\psi_t=0,\\
\eta_{2t}+\eta_{2s}-\theta_{1t}=0,\\
\eta_{3t}+\eta_{3s}-\theta_{2t}=0,\\
\varphi(x,0)=\varphi_0(x),\varphi_t(x,0)=\varphi_1(x),\psi(x,0)=\psi_0(x),\psi_t(x,0)=\psi_1(x),\quad x\in(0,1),\\
\omega(x,0)=\omega_0(x),\omega_t(x,0)=\omega_1(x),\theta_1(x,0)=\theta_{10}(x),\quad x\in(0,1),\\
\ \theta_{1t}(x,0)=\theta_{11}(x),\ \theta_2(x,0)=\theta_{20}(x),\theta_{2t}(x,0)=\theta_{21}(x),\quad x\in(0,1),\\
\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=\omega(0,t)=\omega\left(1,t\right)=0,\ t\in(0,+\infty),\\
\theta_1\left(0,t\right)=\theta_1\left(1,t\right)=\theta_2\left(0,t\right)=\theta_2\left(1,t\right)=0,\ t\in(0,+\infty),\\
\eta_i\left(0,t,s\right)=\eta_i\left(1,t,s\right)=0,\quad for\quad i=1,2,3,\\
\eta_i\left(x,t,0\right)=0,\quad for\quad i=1,2,3,\\
\eta_i^0\left(x,s\right)=\eta_i\left(x,0,s\right),\quad for\quad i=1,2,3.
\end{array}\right. \tag{2.7}$$

Now, we introduce the vector function $U=(\varphi,\varphi_t,\psi,\psi_t,\omega,\omega_t,\theta_1,\theta_{1t},\theta_2,\theta_{2t},\eta_1,\eta_2,\eta_3)^T$ and the new dependent variables $\varphi_t=\widetilde{\varphi},\ \psi_t=\widetilde{\psi},\ \omega_t=\widetilde{\omega},\ \theta_{1t}=\widetilde{\theta}_1,\ \theta_{2t}=\widetilde{\theta}_2$

Then, the system (2.7) is equivalent to

$$\left\{\begin{array}{l}
U_t=AU,\\
U(t=0)=U_0,
\end{array}\right. \tag{2.8}$$

where $U_0=(\varphi_0,\varphi_{1,}\psi_0,\psi_{1,}\omega_0,\omega_1,\theta_{10},\theta_{11},\theta_{20},\theta_{21},\eta_{10},\eta_{20},\eta_{30})^T$ and A is a linear operator such that

$$AU = \begin{pmatrix} \widetilde{\varphi} \\ \frac{k}{\rho_1}\left(\varphi_x+\psi+l\omega\right)_x+\frac{k_0 l}{\rho_1}\left(\omega_x-l\varphi\right)-\frac{l\gamma}{\rho_1}\widetilde{\theta}_1 \\ \widetilde{\psi} \\ \frac{\lambda_1}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\varphi_x+\psi+l\omega\right)+\frac{\beta_1}{\rho_2}\int\limits_0^{+\infty} g_1\left(s\right)\eta_{1xx}\left(x,t,s\right)ds-\frac{\gamma}{\rho_2}\widetilde{\theta}_{2x} \\ \widetilde{\omega} \\ \frac{k_0}{\rho_1}\left(\omega_x-l\varphi\right)_x-\frac{kl}{\rho_1}\left(\varphi_x+\psi+l\omega\right)-\frac{\gamma}{\rho_1}\widetilde{\theta}_{1x} \\ \widetilde{\theta}_1 \\ \frac{\lambda_2}{\rho_3}\theta_{1xx}+\frac{\beta_2}{\rho_3}\int\limits_0^{+\infty} g_2\left(s\right)\eta_{2xx}\left(x,t,s\right)ds-\frac{m_1}{\rho_3}\left(\widetilde{\omega}_x-l\widetilde{\varphi}\right) \\ \widetilde{\theta}_2 \\ \frac{\lambda_3}{\rho_4}\theta_{2xx}+\frac{\beta_3}{\rho_4}\int\limits_0^{+\infty} g_3\left(s\right)\eta_{3xx}\left(x,t,s\right)ds-\frac{m_2}{\rho_4}\widetilde{\psi}_x \\ \widetilde{\psi}-\eta_{1s} \\ \widetilde{\theta}_1-\eta_{2s} \\ \widetilde{\theta}_2-\eta_{3s} \end{pmatrix}.$$

Let $L_{g_i}=\left\{v:\mathbb{R}^+\longrightarrow H_0^1(0,1),\int\limits_0^1\int\limits_0^{+\infty} g_i\left(s\right)v_x^2dsdx<+\infty\right\}$ is endowed with the following inner product

$$\left\langle v_1,v_2\right\rangle_{L_{g_i}}=\int\limits_0^1\int\limits_0^{+\infty} g_i\left(s\right)v_{1x}\left(s\right)v_{2x}\left(s\right)dsdx,\ \ for\ i=1,2,3,$$

and the energy space defined by

$$H=\left(H_0^1(0,1)\times L^2(0,1)\right)^5\times L_{g_1}\times L_{g_2}\times L_{g_3}, \tag{2.9}$$

endowed with the inner product

$$\begin{aligned} \left\langle U,\bar{U}\right\rangle_H \ = \ & \int\limits_0^1\left[\rho_1\tilde{\varphi}\bar{\tilde{\varphi}}+\rho_2\tilde{\psi}\bar{\tilde{\psi}}+\rho_1\tilde{\omega}\bar{\tilde{\omega}}+\frac{\gamma\rho_3}{m_1}\tilde{\theta}_1\bar{\tilde{\theta}}_1+\frac{\gamma\rho_4}{m_2}\tilde{\theta}_2\bar{\tilde{\theta}}_2\right]dx \\ & +k_0\int\limits_0^1(\omega_x-l\varphi)(\bar{\tilde{\omega}}_x-l\bar{\varphi})dx+\lambda_1\int\limits_0^1\psi_x\bar{\psi}_xdx \\ & +\frac{\gamma\lambda_2}{m_1}\int\limits_0^1\theta_{1x}\bar{\theta}_{1x}dx+\frac{\gamma\lambda_3}{m_2}\int\limits_0^1\theta_{2x}\bar{\theta}_{2x}dx \\ & +k\int\limits_0^1(\varphi_x+\psi+l\omega)(\bar{\varphi}_x+\bar{\psi}+l\bar{\omega})dx+\beta_1\int\limits_0^1\int\limits_0^{+\infty} g_1\left(s\right)\eta_{1x}\bar{\eta}_{1x}dsdx \\ & +\frac{\gamma\beta_2}{m_1}\int\limits_0^1\int\limits_0^{+\infty} g_3\left(s\right)\eta_{2x}\bar{\eta}_{2x}dsdx+\frac{\gamma\beta_3}{m_2}\int\limits_0^1\int\limits_0^{+\infty} g_3\left(s\right)\eta_{3x}\bar{\eta}_{3x}dsdx, \end{aligned} \tag{2.10}$$

where $U=(\varphi,\tilde{\varphi},\psi,\tilde{\psi},\omega,\tilde{\omega},\theta_1,\tilde{\theta}_1,\theta_2,\tilde{\theta}_2,\eta_1,\eta_2,\eta_3)^T\in H$
and $\bar{U}=(\bar{\varphi},\tilde{\bar{\varphi}},\bar{\psi},\tilde{\bar{\psi}},\bar{\omega},\tilde{\bar{\omega}},\bar{\theta}_1,\tilde{\bar{\theta}}_1,\bar{\theta}_2,\tilde{\bar{\theta}}_2,\bar{\eta}_1,\bar{\eta}_2,\bar{\eta}_3)^T\in H.$
The operator A has the domain defined by

$$\begin{aligned} D(A) \;=\; & \Big\{U\in H:\varphi,\psi,\omega,\theta_1,\theta_2\in H^2(0,1)\times H_0^1(0,1),\\ & \tilde{\varphi},\tilde{\psi},\tilde{\omega},\tilde{\theta}_1,\tilde{\theta}_2\in H_0^1(0,1),\eta_1\in L_{g_1},\eta_2\in L_{g_2},\eta_3\in L_{g_3}\Big\}. \end{aligned}$$

Theorem 2.1 *Assume that g_i satisfies $(H1)-(H2)$ and that for any initial data $U_0\in H$, then there exists a unique solution $U\in C(\mathbb{R}^+,H)$. Moreover, if $U_0\in D(A)$, then $U\in C(\mathbb{R}^+,D(A))\cap C^1(\mathbb{R}^+,H)$.*

Proof. The result follows from semigroup approach, we show that the operator A generates a C_0- semigroup of contractions in H. First, we prove that the operator A is dissipative. For any $U\in D(A)$, and using the inner product (2.10), we obtain

$$\begin{aligned} \langle AU,U\rangle_H \;=\; & \rho_1\int_0^1\left[\frac{k}{\rho_1}\left(\varphi_x+\psi+l\omega\right)_x+\frac{k_0 l}{\rho_1}\left(\omega_x-l\varphi\right)-\frac{l\gamma}{\rho_1}\widetilde{\theta}_1\right]\widetilde{\varphi}dx\\ & +\rho_2\int_0^1\left[\frac{\lambda_1}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\varphi_x+\psi+l\omega\right)\right.\\ & \left.+\frac{\beta_1}{\rho_2}\int_0^{+\infty}g_1\left(s\right)\eta_{1xx}\left(x,t,s\right)ds-\frac{\gamma}{\rho_2}\widetilde{\theta}_{2x}\right]\widetilde{\psi}dx\\ & +\rho_1\int_0^1\left[\frac{k_0}{\rho_1}\left(\omega_x-l\varphi\right)_x-\frac{kl}{\rho_1}\left(\varphi_x+\psi+l\omega\right)-\frac{\gamma}{\rho_1}\widetilde{\theta}_{1x}\right]\widetilde{\omega}dx\\ & +\frac{\gamma\rho_3}{m_1}\int_0^1\left[\frac{\lambda_2}{\rho_3}\theta_{1xx}+\frac{\beta_2}{\rho_3}\int_0^{+\infty}g_2\left(s\right)\eta_{2xx}\left(x,t,s\right)ds-\frac{m_1}{\rho_3}\left(\widetilde{\omega}_x-l\widetilde{\varphi}\right)\right]\widetilde{\theta}_1dx\\ & +\frac{\gamma\rho_4}{m_2}\int_0^1\left[\frac{\lambda_3}{\rho_4}\theta_{2xx}+\frac{\beta_3}{\rho_4}\int_0^{+\infty}g_3\left(s\right)\eta_{3xx}\left(x,t,s\right)ds-\frac{m_2}{\rho_4}\widetilde{\psi}_x\right]\widetilde{\theta}_2dx\\ & +k_0\int_0^1\left(\widetilde{\omega}_x-l\widetilde{\varphi}\right)\left(\omega_x-l\varphi\right)dx+k\int_0^1\left(\widetilde{\varphi}_x+\widetilde{\psi}+l\widetilde{\omega}\right)\left(\varphi_x+\psi+l\omega\right)dx\\ & +\lambda_1\int_0^1\widetilde{\psi}_x\psi_xdx+\frac{\gamma\lambda_2}{m_1}\int_0^1\widetilde{\theta}_{1x}\theta_{1x}dx+\frac{\gamma\lambda_3}{m_2}\int_0^1\widetilde{\theta}_{2x}\theta_{2x}dx\\ & +\beta_1\int_0^1\int_0^{+\infty}g_1(s)(\widetilde{\psi}-\eta_{1s})_x(s)\eta_{1x}(s)dsdx\\ & +\frac{\gamma\lambda_2}{m_1}\int_0^1\int_0^{+\infty}g_2(s)(\widetilde{\theta}_1-\eta_{2s})_x(s)\eta_{2x}(s)dsdx\\ & +\frac{\gamma\lambda_3}{m_2}\int_0^1\int_0^{+\infty}g_3(s)(\widetilde{\theta}_2-\eta_{3s})_x(s)\eta_{3x}(s)dsdx\\ \;=\; & \frac{1}{2}\int_0^1\int_0^{+\infty}\left[\beta_1g_1^{'}(s)\eta_{1x}^2+\frac{\gamma\beta_2}{m_1}g_2^{'}(s)\eta_{2x}^2+\frac{\gamma\beta_3}{m_2}g_3^{'}(s)\eta_{3x}^2\right]dsdx. \end{aligned}$$

Since g_i is nonincreasing, we deduce that

$$\langle AU, U \rangle_H \leq 0.$$

So, A is dissipative. Next, we prove that the operator $I - A$ is surjective. Let $F = (f_1, f_2, ..., f_{13})^T \in H$, we prove that there exists a unique $U \in D(A)$ satisfying

$$(I - A)U = F, \tag{2.11}$$

that is,

$$\begin{cases}
\varphi - \widetilde{\varphi} = f_1 \in H_0^1(0,1), \\
\widetilde{\varphi} - \frac{k}{\rho_1}\varphi_{xx} + \frac{k_0 l^2}{\rho_1}\varphi - \frac{k}{\rho_1}\psi_x - \frac{(k+k_0)l}{\rho_1}\omega_x + \frac{l\gamma}{\rho_1}\widetilde{\theta}_1 = f_2 \in L^2(0,1), \\
\psi - \widetilde{\psi} = f_3 \in H_0^1(0,1), \\
\widetilde{\psi} + \frac{k}{\rho_2}\varphi_x - \frac{\lambda_1}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}\psi + \frac{kl}{\rho_2}\omega - \frac{\beta_1}{\rho_2}\int\limits_0^{+\infty} g_1(s)\,\eta_{1xx}(s)\,ds + \frac{\gamma}{\rho_2}\widetilde{\theta}_{2x} = f_4 \in L^2(0,1), \\
\omega - \widetilde{\omega} = f_5 \in H_0^1(0,1), \\
\widetilde{\omega} + \frac{(k+k_0)l}{\rho_1}\varphi_x + \frac{kl}{\rho_1}\psi - \frac{k_0}{\rho_1}\omega_{xx} + \frac{kl^2}{\rho_1}\omega + \frac{\gamma}{\rho_1}\widetilde{\theta}_{1x} = f_6 \in L^2(0,1), \\
\theta_1 - \widetilde{\theta}_1 = f_7 \in H_0^1(0,1), \\
\widetilde{\theta}_1 - \frac{\lambda_2}{\rho_3}\theta_{1xx} - \frac{\beta_2}{\rho_3}\int\limits_0^{+\infty} g_2(s)\,\eta_{2xx}(s)\,ds + \frac{m_1}{\rho_3}\widetilde{\omega}_x - \frac{m_1 l}{\rho_3}\widetilde{\varphi} = f_8 \in L^2(0,1), \\
\theta_2 - \widetilde{\theta}_2 = f_9 \in H_0^1(0,1), \\
\widetilde{\theta}_2 - \frac{\lambda_3}{\rho_4}\theta_{2xx} - \frac{\beta_3}{\rho_4}\int\limits_0^{+\infty} g_3(s)\,\eta_{3xx}(s)\,ds + \frac{m_2}{\rho_4}\widetilde{\psi}_x = f_{10} \in L^2(0,1), \\
\eta_1 - \widetilde{\psi} + \eta_{1s} = f_{11} \in L_{g_1}, \\
\eta_2 - \widetilde{\theta}_1 + \eta_{2s} = f_{12} \in L_{g_2}, \\
\eta_3 - \widetilde{\theta}_2 + \eta_{3s} = f_{13} \in L_{g_3}.
\end{cases} \tag{2.12}$$

Integrating lines 11, 12 and 13 in the above equation, we obtain

$$\eta_1(s) = \widetilde{\psi}(1 - e^{-s}) + \int\limits_0^s e^{\tau - s} f_{11}(\tau)d\tau. \tag{2.13}$$

$$\eta_2(s) = \widetilde{\theta}_1(1 - e^{-s}) + \int\limits_0^s e^{\tau - s} f_{12}(\tau)d\tau. \tag{2.14}$$

$$\eta_3(s) = \widetilde{\theta}_2(1 - e^{-s}) + \int\limits_0^s e^{\tau - s} f_{13}(\tau)d\tau. \tag{2.15}$$

Substituting $\widetilde{\varphi} = \varphi - f_1, \widetilde{\psi} = \psi - f_3, \widetilde{\omega} = \omega - f_5, \widetilde{\theta}_1 = \theta_1 - f_7, \widetilde{\theta}_2 = \theta_2 - f_9$,(2.13), (2.14) and (2.15) in $(2.12)_2$,$(2.12)_4$,$(2.12)_6$,$(2.12)_8$ and $(2.12)_{10}$, we obtain

$$\begin{cases} \rho_1\varphi - k\varphi_{xx} + k_0 l^2\varphi - k\psi_x - (k+k_0)l\omega_x + l\gamma\theta_1 = h_1 \in L^2(0,1), \\ \rho_2\psi + k\varphi_x - C_{g1}\psi_{xx} + k\psi + kl\omega + \gamma\theta_{2x} = h_2 \in L^2(0,1), \\ \rho_1\omega + (k+k_0)l\varphi_x + kl\psi - k_0\omega_{xx} + kl^2\omega + \gamma\theta_{1x} = h_3 \in L^2(0,1), \\ \rho_3\theta_1 - C_{g_2}\theta_{1xx} + m_1\omega_x - m_1 l\varphi = h_4 \in L^2(0,1), \\ \rho_4\theta_2 - C_{g3}\theta_{2xx} + m_2\psi_x = h_5 \in L^2(0,1), \end{cases} \tag{2.16}$$

where

$$\begin{cases} h_1 = \rho_1(f_1+f_2) + l\gamma f_7, \\ h_2 = \rho_2(f_3+f_4) + \gamma f_{9x} - \beta_1 \int\limits_0^{+\infty} g_1(s)\left[f_{3xx}(1-e^{-s}) - \int\limits_0^s e^{\tau-s} f_{11xx}(\tau)d\tau \right] ds, \\ h_3 = \rho_1(f_5+f_6) + \gamma f_{7x}, \\ h_4 = \rho_3(f_7+f_8) + m_1(f_{5x} - lf_1) - \beta_2 \int\limits_0^{+\infty} g_2(s)\left[f_{7xx}(1-e^{-s}) - \int\limits_0^s e^{\tau-s} f_{12xx}(\tau)d\tau \right] ds, \\ h_5 = \rho_4(f_9+f_{10}) + m_2 f_{3x} - \beta_3 \int\limits_0^{+\infty} g_3(s)\left[f_{9xx}(1-e^{-s}) - \int\limits_0^s e^{\tau-s} f_{13xx}(\tau)d\tau \right] ds, \end{cases}$$

and

$$C_{g_i} = \lambda_i + \beta_i \int\limits_0^{+\infty} g_i(s)(1-e^{-s})ds > 0, \text{ for } i = 1,2,3.$$

The variational formulation associated with (2.16) takes the form

$$B\left((\varphi,\psi,\omega,\theta_1,\theta_2),(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2)\right) = L\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right), \tag{2.17}$$

where B is a bilinear form of $\left[H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\right]^2$ in $\mathbb{R}$

defined by

$$
\begin{aligned}
& B\left((\varphi,\psi,\omega,\theta_1,\theta_2),\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\right) \\
= {} & (\rho_1+k_0l^2)\int_0^1\varphi\bar{\varphi}dx+k\int_0^1\varphi_x\bar{\varphi}_xdx+k\int_0^1\psi\bar{\varphi}_xdx+(k+k_0)l\int_0^1\omega\bar{\varphi}_xdx+l\gamma\int_0^1\theta_1\bar{\varphi}_xdx \\
& +(\rho_2+k)\int_0^1\psi\bar{\psi}dx+k\int_0^1\varphi_x\bar{\psi}dx+C_{g1}\int_0^1\psi_x\bar{\psi}_xdx+kl\int_0^1\omega\bar{\psi}dx+\gamma\int_0^1\theta_{2x}\bar{\psi}dx \\
& +(\rho_1+kl^2)\int_0^1\omega\bar{\omega}dx+(k+k_0)l\int_0^1\varphi_x\bar{\omega}dx+kl\int_0^1\psi\bar{\omega}dx+k_0\int_0^1\omega_x\bar{\omega}_xdx \\
& +\gamma\int_0^1\theta_{1x}\bar{\omega}dx+\frac{\gamma\rho_3}{m_1}\int_0^1\theta_1\bar{\theta}_1dx+\frac{\gamma}{m_1}C_{g2}\int_0^1\theta_{1x}\bar{\theta}_{1x}dx+\gamma\int_0^1\omega_x\bar{\theta}_1dx-\gamma l\int_0^1\varphi\bar{\theta}_1dx \\
& +\frac{\gamma\rho_4}{m_2}\int_0^1\theta_2\bar{\theta}_2dx+\frac{\gamma}{m_2}C_{g3}\int_0^1\theta_{2x}\bar{\theta}_{2x}dx+m_2\int_0^1\psi_x\bar{\theta}_2dx \\
= {} & \rho_1\int_0^1\varphi\bar{\varphi}dx+\rho_2\int_0^1\psi\bar{\psi}dx+\rho_1\int_0^1\omega\bar{\omega}dx+\frac{\gamma\rho_3}{m_1}\int_0^1\theta_1\bar{\theta}_1dx+\frac{\gamma\rho_4}{m_2}\int_0^1\theta_2\bar{\theta}_2dx \\
& +k\int_0^1(\varphi_x+\psi+l\omega)\left(\bar{\varphi}_x+\bar{\psi}+l\bar{\omega}\right)dx+k_0\int_0^1(\omega_x-l\varphi)(\bar{\omega}_x-l\bar{\varphi})dx+C_{g1}\int_0^1\psi_x\bar{\psi}_xdx \\
& +\frac{\gamma}{m_1}C_{g2}\int_0^1\theta_{1x}\bar{\theta}_{1x}dx+\frac{\gamma}{m_2}C_{g3}\int_0^1\theta_{2x}\bar{\theta}_{2x}dx+\gamma\int_0^1(\omega_x-l\varphi)\bar{\theta}_1dx \\
& -\gamma\int_0^1\theta_1(\bar{\omega}_x-l\bar{\varphi})dx+\gamma\int_0^1\psi_x\bar{\theta}_2dx-\gamma\int_0^1\theta_2\bar{\psi}_xdx,
\end{aligned}
$$

and L is a linear form of $H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$ in $\mathbb{R}$ defined by

$$
L\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)=\int_0^1h_1\bar{\varphi}dx+\int_0^1h_2\bar{\psi}dx+\int_0^1h_3\bar{\omega}dx+\frac{\gamma}{m_1}\int_0^1h_4\bar{\theta}_1dx+\frac{\gamma}{m_2}\int_0^1h_5\bar{\theta}_2dx.
$$

Now, for $V=H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$ equipped with the norm

$$
\begin{aligned}
\|(\varphi,\psi,\omega,\theta_1,\theta_2)\|_V = {} & \|\varphi_x+\psi+l\omega\|_2^2+\|\omega_x-l\varphi\|_2^2+\|\varphi\|_2^2+\|\psi\|_2^2 \\
& +\|\omega\|_2^2+\|\psi_x\|_2^2+\|\theta_1\|_2^2+\|\theta_2\|_2^2+\|\theta_{1x}\|_2^2+\|\theta_{2x}\|_2^2.
\end{aligned}
$$

Hence, using Holder's and Poincaré's inequalities, we get

$$
\left|B\left((\varphi,\psi,\omega,\theta_1,\theta_2),\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\right)\right|\leq c\,\|(\varphi,\psi,\omega,\theta_1,\theta_2)\|_V\left\|\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\right\|_V.
$$

Similarly

$$
\left|L\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\right|\leq c\left\|\left(\bar{\varphi},\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\right\|_V.
$$

Furthermore, from the definition of B, we get

$$\begin{aligned}
& B\left(\left(\varphi,\psi,\omega,\theta_1,\theta_2\right),\left(\varphi,\psi,\omega,\theta_1,\theta_2\right)\right) \\
= \; & \rho_1\int_0^1\varphi^2dx+\rho_2\int_0^1\psi^2dx+\rho_1\int_0^1\omega^2dx+\frac{\gamma\rho_3}{m_1}\int_0^1\theta_1^2dx+\frac{\gamma\rho_4}{m_2}\int_0^1\theta_2^2dx \\
& +C_{g1}\int_0^1\psi_x^2dx+\frac{\gamma}{m_1}C_{g_2}\int_0^1\theta_{1x}^2dx+\frac{\gamma}{m_2}C_{g3}\int_0^1\theta_{2x}^2dx \\
& +k_0\int_0^1\left(\omega_x-l\varphi\right)^2dx+k\int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx \\
\geq \; & \alpha\left\|\left(\varphi,\psi,\omega,\theta_1,\theta_2\right)\right\|_V.
\end{aligned}$$

for some $\alpha>0$. Thus, B is coercive. Consequently, by Lax-Milgram Lemma, system (2.16) has a unique solution

$$\left(\varphi,\psi,\omega,\theta_1,\theta_2\right)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1).$$

Substituting $\varphi,\psi,\omega,\theta_1$ and θ_2 into $(2.12)_1$,$(2.12)_3$,$(2.12)_5$,$(2.12)_7$ and $(2.12)_9$, respectively, we get

$$\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{\omega},\widetilde{\theta}_1,\widetilde{\theta}_2\right)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1).$$

Similarly, inserting $\widetilde{\psi},\widetilde{\theta}_1,\widetilde{\theta}_2$ in (2.13), (2.14), (2.15) and using $(2.12)_{11}$,$(2.12)_{12}$,$(2.12)_{13}$, we obtain $\eta_1\in L_{g_1},\eta_2\in L_{g_2}$and $\eta_3\in L_{g_3}$.

On the other hand, if $\left(\bar{\psi},\bar{\omega},\bar{\theta}_1,\bar{\theta}_2\right)\equiv(0,0,0,0)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$, then (2.17) reduces to

$$\rho_1\int_0^1\varphi\bar{\varphi}dx+k\int_0^1\left(\varphi_x+\psi+l\omega\right)\bar{\varphi}_xdx-lk_0\int_0^1\left(\omega_x-l\varphi\right)\bar{\varphi}dx+l\gamma\int_0^1\theta_1\bar{\varphi}dx=\int_0^1h_1\bar{\varphi}dx.$$

Hence we obtain

$$k\int_0^1\left(\varphi_x+\psi+l\omega\right)\bar{\varphi}_xdx=\int_0^1\left[-\rho_1\varphi+lk_0\left(\omega_x-l\varphi\right)-l\gamma\theta_1+h_1\right]\bar{\varphi}dx,\quad \bar{\varphi}\in H_0^1(0,1).$$

By noting that $-\rho_1\varphi+lk_0\left(\omega_x-l\varphi\right)-l\gamma\theta_1+h_1\in L^2(0,1)$, we get $\varphi\in H^2(0,1)\cap H_0^1(0,1)$.Consequently using integration by parts we have

$$\int_0^1\left[-k\varphi_{xx}-k\psi_x-kl\omega_x+\rho_1\varphi-lk_0\left(\omega_x-l\varphi\right)+l\gamma\theta_1-h_1\right]\bar{\varphi}dx=0,\quad \bar{\varphi}\in H_0^1(0,1).$$

Therefore,

$$-k\varphi_{xx}+k\psi_x+kl\omega_x+\rho_1\varphi-lk_0\left(\omega_x-l\varphi\right)+l\gamma\theta_1=h_1.$$

This gives $(2.16)_1$.

Similarly, we get

$$\psi, \omega, \theta_1, \theta_2 \in H^2(0,1) \cap H_0^1(0,1),$$

and $(2.16)_2$,$(2.16)_3$,$(2.16)_4$,$(2.16)_5$ are satisfied.

Finally, from (2.14), (2.13) and (2.15) we can get $(\eta_1, \eta_2, \eta_3) \in L_{g_1} \times L_{g_2} \times L_{g_3}$.Hence, there exists a unique $U \in D(A)$ such that (2.11) is satisfied.

Therefore, by the Lumer–Phillips' theorem (see [34] and [40]), it follows that A is the infinitesimal generator of a contraction C_0-semigroup. ■

2.3 Technical Lemmas

In this section we establish several lemmas needed for the proof of our main result.

Lemma 2.1 *The following inequalities hold,*

$$\left(\int_0^{+\infty} g_i(s)\eta_i(s)ds\right)^2 \leq c_p g_i^0 \int_0^{+\infty} g_i(s)\eta_{ix}^2(s)ds, \text{ for } i = 1,2,3. \tag{2.18}$$

$$\left(\int_0^{+\infty} g_i(s)\eta_{ix}(s)ds\right)^2 \leq g_i^0 \int_0^{+\infty} g_i(s)\eta_{ix}^2(s)ds, \text{ for } i = 1,2,3. \tag{2.19}$$

$$\left(\int_0^{+\infty} g_i'(s)\eta_i(s)ds\right)^2 \leq -c_p g_i(0) \int_0^{+\infty} g_i'(s)\eta_{ix}^2(s)ds, \text{ for } i = 1,2,3. \tag{2.20}$$

$$\left(\int_0^{+\infty} g_i'(s)\eta_{ix}(s)ds\right)^2 \leq -g_i(0) \int_0^{+\infty} g_i'(s)\eta_{ix}^2(s)ds, fori = 1,2,3. \tag{2.21}$$

in which c_p is Poincaré constant.

Lemma 2.2 *Let $(\varphi, \psi, \omega, \theta_1, \theta_2)$ be the solution of (2.7). Then the energy functional, defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \frac{\gamma\rho_3}{m_1}\theta_{1t}^2 + \frac{\gamma\rho_4}{m_2}\theta_{2t}^2 + \lambda_1\psi_x^2 \\ &\quad + \frac{\gamma\lambda_2}{m_1}\theta_{1x}^2 + \frac{\gamma\lambda_3}{m_2}\theta_{2x}^2 + k\left(\varphi_x + \psi + l\omega\right)^2 + k_0\left(\omega_x - l\varphi\right)^2 \\ &\quad + \beta_1\int_0^{+\infty} g_1(s)\eta_{1x}^2 ds + \frac{\gamma\beta_2}{m_1}\int_0^{+\infty} g_2(s)\eta_{2x}^2 ds + \frac{\gamma\beta_3}{m_2}\int_0^{+\infty} g_3(s)\eta_{3x}^2 ds\Big] dx. \end{aligned} \tag{2.22}$$

satisfies

$$E^{'}(t) = \frac{1}{2}\int_0^1 \left[\beta_1\int_0^{+\infty} g_1^{'}(s)\eta_{1x}^2 ds + \frac{\gamma\beta_2}{m_1}\int_0^{+\infty} g_2^{'}(s)\eta_{2x}^2 ds + \frac{\gamma\beta_3}{m_2}\int_0^{+\infty} g_3^{'}(s)\eta_{3x}^2 ds\right] dx \leq 0. \tag{2.23}$$

Proof. *Multiplying Equation* (2.7)$_1$ *by* φ_t , (2.7)$_2$ *by* ψ_t, (2.7)$_3$ *by* ω_t, (2.7)$_4$ *by* $\frac{\gamma}{m_1}\theta_{1t}$, (2.7)$_5$ *by* $\frac{\gamma}{m_2}\theta_{2t}$, (2.7)$_6$ *by* η_1, (2.7)$_7$ *by* η_2, *and* (2.7)$_8$ *by* η_3*and integrating over* $(0,1)$, *we get after summing up*

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\frac{\gamma\rho_3}{m_1}\theta_{1t}^2+\frac{\gamma\rho_4}{m_2}\theta_{2t}^2+\lambda_1\psi_x^2\right.\\
& \left.+\frac{\gamma\lambda_2}{m_1}\theta_{1x}^2+\frac{\gamma\lambda_3}{m_2}\theta_{2x}^2+k\left(\varphi_x+\psi+l\omega\right)^2+k_0\left(\omega_x-l\varphi\right)^2\right]dx\\
& -\beta_1\int_0^1\psi_t\int_0^{+\infty}g_1(s)\eta_{1xx}(s)dsdx-\frac{\gamma\beta_2}{m_1}\int_0^1\theta_{1t}\int_0^{+\infty}g_2(s)\eta_{2xx}(s)dsdx\\
& -\frac{\gamma\beta_3}{m_2}\int_0^1\theta_{2t}\int_0^{+\infty}g_3(s)\eta_{3xx}(s)dsdx+\int_0^1\left(\eta_{1t}+\eta_{1s}-\psi_t\right)\eta_1dx\\
& +\int_0^1\left(\eta_{2t}+\eta_{2s}-\theta_{1t}\right)\eta_2dx+\int_0^1\left(\eta_{3t}+\eta_{3s}-\theta_{2t}\right)\eta_3dx\\
= {} & 0. \qquad (2.24)
\end{aligned}$$

Now, we estimate the last six terms on the left-hand side of the above equation.

$$\begin{aligned}
-\beta_1\int_0^1\psi_t\int_0^{+\infty}g_1(s)\eta_{1xx}(s)dsdx &= \beta_1\int_0^{+\infty}g_1(s)\int_0^1\eta_{1x}(s)\psi_{tx}(t)dxds\\
&= \beta_1\int_0^{+\infty}g_1(s)\int_0^1\left[\psi_x(t)-\psi_x(t-s)\right]\psi_{tx}(t)dxds\\
&= \beta_1\int_0^{+\infty}g_1(s)ds\left(\frac{d}{2dt}\int_0^1\psi_x^2dx\right)+M,
\end{aligned}$$

where

$$
\begin{aligned}
M &= -\beta_1 \int_0^1 \psi_{tx}(t) \int_0^{+\infty} g_1(s)\psi_x(t-s)\,ds\,dx = -\beta_1 \int_0^1 \psi_{tx}(t) \int_{-\infty}^{t} g_1(t-s)\psi_x(s)\,ds\,dx \\
&= \beta_1 \int_{-\infty}^{t} g_1(t-s) \int_0^1 \psi_{tx}(t) \left(\psi_x(t) - \psi_x(s)\right) dx\,ds \\
&\quad -\beta_1 \int_{-\infty}^{t} g_1(t-s)\,ds \int_0^1 \psi_{tx}(t)\psi_x(t)\,dx \\
&= \beta_1 \int_{-\infty}^{t} g_1(t-s) \left(\frac{d}{2dt} \int_0^1 \left(\psi_x(t) - \psi_x(s)\right)^2 dx \right) ds \\
&\quad -\beta_1 \int_{-\infty}^{t} g_1(t-s)\,ds \left(\frac{d}{2dt} \int_0^1 \psi_x^2(t)\,dx \right) \\
&= \frac{\beta_1 d}{2dt} \int_{-\infty}^{t} g_1(t-s) \left(\int_0^1 \left(\psi_x(t) - \psi_x(s)\right)^2 dx \right) ds - \frac{\beta_1 d}{2dt} \int_{-\infty}^{t} g_1(t-s) \int_0^1 \psi_x^2(t)\,dx\,ds \\
&\quad - \frac{\beta_1}{2} \int_{-\infty}^{t} g_1'(t-s) \left(\int_0^1 \left(\psi_x(t) - \psi_x(s)\right)^2 dx \right) ds + \frac{\beta_1}{2} \int_{-\infty}^{t} g_1'(t-s)\,ds \int_0^1 \psi_x^2(t)\,dx \\
&\quad + \frac{\beta_1}{2} g_1(0) \int_0^1 \psi_x^2(t)\,dx \\
&= \frac{\beta_1 d}{2dt} \int_0^1 \int_0^{+\infty} g_1(s)\eta_{1x}^2(s)\,ds\,dx - \frac{\beta_1}{2} \int_0^1 \int_0^{+\infty} g_1'(s)\eta_{1x}^2(s)\,ds\,dx - \frac{\beta_1 d}{2dt} \int_0^{+\infty} g_1(s)\,ds \int_0^1 \psi_x^2\,dx.
\end{aligned}
$$

So

$$
\begin{aligned}
-\beta_1 \int_0^1 \psi_t \int_0^{+\infty} g_1(s)\eta_{1xx}(s)\,ds\,dx &= \frac{\beta_1}{2} \frac{d}{dt} \int_0^1 \int_0^{+\infty} g_1(s)\eta_{1x}^2(s)\,ds\,dx \\
&\quad - \frac{\beta_1}{2} \int_0^1 \int_0^{+\infty} g_1'(s)\eta_{1x}^2(s)\,ds\,dx. \qquad (2.25)
\end{aligned}
$$

The following two relationships can be treated similarly.

$$
\begin{aligned}
-\frac{\gamma\beta_2}{m_1} \int_0^1 \theta_{1t} \int_0^{+\infty} g_2(s)\eta_{2xx}(s)\,ds\,dx &= \frac{\gamma\beta_2}{2m_1} \frac{d}{dt} \int_0^1 \int_0^{+\infty} g_2(s)\eta_{2x}^2(s)\,ds\,dx \\
&\quad - \frac{\gamma\beta_2}{2m_1} \int_0^1 \int_0^{+\infty} g_2'(s)\eta_{2x}^2(s)\,ds\,dx. \qquad (2.26)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\gamma\beta_3}{m_2} \int_0^1 \theta_{2t} \int_0^{+\infty} g_3(s)\eta_{3xx}(s)\,ds\,dx &= \frac{\gamma\beta_3}{2m_2} \frac{d}{dt} \int_0^1 \int_0^{+\infty} g_3(s)\eta_{3x}^2(s)\,ds\,dx \\
&\quad - \frac{\gamma\beta_3}{2m_2} \int_0^1 \int_0^{+\infty} g_3'(s)\eta_{3x}^2(s)\,ds\,dx. \qquad (2.27)
\end{aligned}
$$

and

$$\begin{aligned}\int_0^1 (\eta_{1t}+\eta_{1s}-\psi_t)\,\eta_1 dx &= \int_0^1 (\eta_{2t}+\eta_{2s}-\theta_{1t})\,\eta_2 dx \\ &= \int_0^1 (\eta_{3t}+\eta_{3s}-\theta_{2t})\,\eta_3 dx = 0. \end{aligned} \tag{2.28}$$

We replace these last four equalities in (2.24)*, we find the result.* ■

Now, we introduce the multiplier p given by the solution of the Dirichlet problem

$$-p_{xx} = \psi_x, \quad p(0) = p(1) = 0,$$

then we can obtain the following inequality

$$\int_0^1 p_t^2 dx \le \int_0^1 p_{tx}^2 dx \le \int_0^1 \psi_t^2 dx, \tag{2.29}$$

$$\int_0^1 p^2 dx \le \int_0^1 p_x^2 dx \le \int_0^1 \psi^2 dx \le \int_0^1 \psi_x^2 dx, \tag{2.30}$$

and we define the functional

$$I_1(t) = -\rho_2 \int_0^1 \psi_t p_x dx - l\int_0^1 (\rho_2 l\psi_t - \rho_1\omega_t)\left(\int_0^x p(y)dy\right) dx + \rho_1 \int_0^1 \varphi_t p dx. \tag{2.31}$$

So, we have the following lemma.

Lemma 2.3 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be a solution of* (2.7)*. Then we have for any* $\varepsilon_1 > 0$,

$$\begin{aligned} I_1'(t) \le\ & -\lambda_1\left(1-3l^2\right)\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{2\lambda_1 l^2}(1+l^4)\int_0^1 \theta_{2t}^2 dx \\ & + C(\varepsilon_1)\int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 \omega_t^2 dx + \varepsilon_1\int_0^1 \varphi_t^2 dx \\ & + \frac{\beta_1^2}{2l^2}\left(1+l^4\right) g_1^0 \int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx. \end{aligned} \tag{2.32}$$

where $C(\varepsilon_1) = \rho_2 + \frac{\rho_1^2}{4\varepsilon_1} + \rho_2 l^2 + \frac{\rho_1^2 l^2}{4\varepsilon_1}$.

Proof. By differentiating the expression of I_1, we get

$$\begin{aligned} I_1'(t) =\ & -\rho_2\int_0^1 \psi_{tt} p_x dx - \rho_2\int_0^1 \psi_t p_{tx} dx - l\int_0^1 (\rho_2 l\psi_{tt} - \rho_1\omega_{tt})\left(\int_0^x p(y)dy\right) dx \\ & - l\int_0^1 (\rho_2 l\psi_t - \rho_1\omega_t)\left(\int_0^x p_t(y)dy\right) dx + \rho_1\int_0^1 \varphi_{tt} p dx + \rho_1\int_0^1 \varphi_t p_t dx. \end{aligned}$$

We replace $\rho_1\varphi_{tt}$ by $k\left(\varphi_x+\psi+l\omega\right)_x+k_0l\left(\omega_x-l\varphi\right)-l\gamma\theta_{1t}$, $\rho_1\omega_{tt}$ by $k_0\left(\omega_x-l\varphi\right)_x-kl\left(\varphi_x+\psi+l\omega\right)-\gamma\theta_{1tx}$ and $-\rho_2\psi_{tt}$ by $-\lambda_1\psi_{xx}+k\left(\varphi_x+\psi+l\omega\right)-\beta_1\int\limits_0^{+\infty}g_1\left(s\right)\eta_{1xx}\left(s\right)ds+\gamma\theta_{2tx}$, that gives

$$\begin{aligned}
I_1^{'}\left(t\right) &= -\lambda_1\int\limits_0^1\psi_{xx}p_xdx+\int\limits_0^1k\left(\varphi_x+\psi+l\omega\right)p_xdx-\beta_1\int\limits_0^1\int\limits_0^{+\infty}g_1\left(s\right)\eta_{1xx}\left(s\right)dsp_xdx \\
&\quad+\gamma\int\limits_0^1\theta_{2tx}p_xdx-\rho_2\int\limits_0^1\psi_tp_{tx}dx-\lambda_1l^2\int\limits_0^1\psi_{xx}\left(\int\limits_0^xp(y)dy\right)dx \\
&\quad+kl^2\int\limits_0^1\left(\varphi_x+\psi+l\omega\right)\left(\int\limits_0^xp(y)dy\right)dx \\
&\quad-\beta_1l^2\int\limits_0^1\int\limits_0^{+\infty}g_1\left(s\right)\eta_{1xx}\left(s\right)ds\left(\int\limits_0^xp(y)dy\right)dx+\gamma l^2\int\limits_0^1\theta_{2tx}\left(\int\limits_0^xp(y)dy\right)dx \\
&\quad+k_0l\int\limits_0^1\left(\omega_x-l\varphi\right)_x\left(\int\limits_0^xp(y)dy\right)dx-kl^2\int\limits_0^1\left(\varphi_x+\psi+l\omega\right)\left(\int\limits_0^xp(y)dy\right)dx \\
&\quad-\gamma l\int\limits_0^1\theta_{1tx}\left(\int\limits_0^xp(y)dy\right)dx-l\int\limits_0^1\left(\rho_2l\psi_t-\rho_1\omega_t\right)\left(\int\limits_0^xp_t(y)dy\right)dx \\
&\quad+k\int\limits_0^1\left(\varphi_x+\psi+l\omega\right)_xpdx+k_0l\int\limits_0^1\left(\omega_x-l\varphi\right)pdx-l\gamma\int\limits_0^1\theta_{1t}pdx+\rho_1\int\limits_0^1\varphi_tp_tdx.
\end{aligned}$$

We come back to the boundary conditions which imply

$$\begin{aligned}
I_1^{'}\left(t\right) &= \lambda_1\int\limits_0^1\psi_xp_{xx}dx+\beta_1\int\limits_0^1\int\limits_0^{+\infty}g_1\left(s\right)\eta_{1x}\left(s\right)ds(p_{xx}+l^2p)dx \\
&\quad-\rho_2\int\limits_0^1\psi_tp_{tx}dx+\lambda_1l^2\int\limits_0^1\psi_xpdx-\gamma\int_0^1\theta_{2t}(p_{xx}+l^2p)dx \\
&\quad-l\int\limits_0^1\left(\rho_2l\psi_t-\rho_1\omega_t\right)\left(\int\limits_0^xp_t(y)dy\right)dx+\rho_1\int\limits_0^1\varphi_tp_tdx.
\end{aligned}$$

Using the fact that $p_{xx}=-\psi_x$,we get

$$\begin{aligned}
I_1^{'} &= -\lambda_1\int\limits_0^1\psi_x^2dx+\beta_1\int\limits_0^1\int\limits_0^{+\infty}g_1(s)\eta_{1x}(s)ds\left(l^2p-\psi_x\right)dx \\
&\quad+\gamma\int\limits_0^1\theta_{2t}\psi_xdx-\rho_2\int\limits_0^1\psi_tp_{tx}dx+l^2\lambda_1\int\limits_0^1p_x^2dx+\rho_1\int\limits_0^1\varphi_tp_tdx \\
&\quad-\gamma l^2\int\limits_0^1\theta_{2t}pdx-\rho_2l^2\int\limits_0^1\psi_t\left(\int\limits_0^xp_t(y)dy\right)dx+\rho_1l\int\limits_0^1\omega_t\left(\int\limits_0^xp_t(y)dy\right)dx. \qquad (2.33)
\end{aligned}$$

by using Young's inequality, Poincaré's inequality, (2.29), (2.30) and (2.18), we obtain

$$\begin{aligned}
\beta_1\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}(s)ds\psi_x dx &\leq \frac{\lambda_1 l^2}{2}\int_0^1\psi_x^2dx+\frac{\beta_1^2}{2\lambda_1 l^2}g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx,\\
l^2\beta_1\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}(s)dspdx &\leq \frac{\lambda_1 l^2}{2}\int_0^1\psi_x^2dx+\frac{\beta_1^2 l^2}{2\lambda_1}g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx,\\
\gamma\int_0^1\theta_{2t}\psi_x dx &\leq \frac{\lambda_1 l^2}{2}\int_0^1\psi_x^2dx+\frac{\gamma^2}{2\lambda_1 l^2}\int_0^1\theta_{2t}^2dx,\\
-\rho_2\int_0^1\psi_t p_{tx}dx &\leq \frac{\rho_2}{2}\int_0^1\psi_t^2dx+\frac{\rho_2}{2}\int_0^1 p_{tx}^2dx\leq\rho_2\int_0^1\psi_t^2dx,\\
\rho_1\int_0^1\varphi_t p_t dx &\leq \varepsilon_1\int_0^1\varphi_t^2dx+\frac{\rho_1}{4\varepsilon_1}\int_0^1\psi_t^2dx,\\
-\gamma l^2\int_0^1\theta_{2t}pdx &\leq \frac{\lambda_1 l^2}{2}\int_0^1\psi_x^2dx+\frac{\gamma^2 l^2}{2\gamma}\int_0^1\theta_{2t}^2dx,\\
-\rho_2 l^2\int_0^1\psi_t\left(\int_0^x p_t(y)dy\right)dx &\leq \frac{\rho_2 l^2}{2}\int_0^1\psi_t^2dx+\frac{\rho_2 l^2}{2}\int_0^1 p_{tx}^2dx\leq\rho_2 l^2\int_0^1\psi_t^2dx,
\end{aligned}$$

and

$$\rho_1 l\int_0^1\omega_t\left(\int_0^x p_t(y)dy\right)dx\leq\varepsilon_1\int_0^1\omega_t^2dx+\frac{\rho_1 l^2}{4\varepsilon_1}\int_0^1\psi_t^2dx.$$

By substituting these last inequalities in (2.33), we obtain the result. ■

Lemma 2.4 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be a solution of* (2.7). *Then the functional*

$$I_2(t)=-\rho_2\int_0^1\psi_t\int_0^{+\infty} g_1(s)\eta_1(s)dsdx, \tag{2.34}$$

satisfies for any $0<\delta<1$

$$\begin{aligned}
I_2'(t) \leq\ & -\rho_2\left(g_1^0-\delta\right)\int_0^1\psi_t^2dx+\delta\int_0^1\psi_x dx+\delta\int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx\\
&+\delta\int_0^1\theta_{2t}^2dx+C_1\left(\delta\right)g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx\\
&-\frac{\rho_2 g_1(0)}{4\delta}\int_0^1\int_0^{+\infty} g_1'(s)\eta_{1x}^2(s)dsdx,
\end{aligned}\tag{2.35}$$

where $C_1\left(\delta\right)=\beta_1+\frac{\lambda_1^2+k^2+\gamma^2}{4\delta}$.

Proof. *First, we note that*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_0^{+\infty} g_1(s)\eta_1(s)ds\right) &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{t} g_1(t-s)\left(\psi(t)-\psi(s)\right)ds\right)\\ &= \int_{-\infty}^{t} g_1'(t-s)\left(\psi(t)-\psi(s)\right)ds+\int_{-\infty}^{t} g_1(t-s)\psi_t(t)ds\\ &= \int_0^{+\infty} g_1'(s)\left(\psi(t)-\psi(t-s)\right)ds+g_1^0\psi_t\\ &= \int_0^{+\infty} g_1'(s)\eta_1(s)ds+g_1^0\psi_t. \qquad (2.36)\end{aligned}$$

By multiplying the second equation in (2.7) *by* $\int_0^{+\infty} g_1(s)\eta_1(s)ds$ *and integrating over* $(0,1)$, *we get*

$$\begin{aligned}-\rho_2\int_0^1\psi_{tt}\int_0^{+\infty} g_1(s)\eta_1(s)dsdx &= -\lambda_1\int_0^1\psi_{xx}\int_0^{+\infty} g_1(s)\eta_1(s)dsdx\\ &+k\int_0^1\left(\varphi_x+\psi+l\omega\right)\int_0^{+\infty} g_1(s)\eta_1(s)dsdx\\ &-\beta_1\int_0^1\int_0^{+\infty} g_1\left(s\right)\eta_{1xx}\left(s\right)ds\int_0^{+\infty} g_1(s)\eta_1(s)dsdx\\ &+\gamma\int_0^1\theta_{2tx}\int_0^{+\infty} g_1(s)\eta_1(s)dsdx,\end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned}\rho_2\int_0^1\psi_{tt}\int_0^{+\infty} g_1(s)\eta_1(s)dsdx &= +\lambda_1\int_0^1\psi_x\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx\\ &+k\int_0^1\left(\varphi_x+\psi+l\omega\right)\int_0^{+\infty} g_1(s)\eta_1(s)dsdx\\ &+\beta_1\int_0^1\left(\int_0^{+\infty} g_1\left(s\right)\eta_{1x}\left(s\right)ds\right)^2 dx\\ &-\gamma\int_0^1\theta_{2t}\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx. \qquad (2.37)\end{aligned}$$

By differentiating I_2, *using the equations* (2.37) *and* (2.36), *we obtain*

$$\begin{aligned}I_2'(t) &= -\rho_2\int_0^1\psi_{tt}\int_0^{+\infty} g_1(s)\eta_1(s)dsdx-\rho_2\int_0^1\psi_t\left(\int_0^{+\infty} g_1(s)\eta_1(s)ds\right)_t dx\\ &= \lambda_1\int_0^1\psi_x\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx+k\int_0^1\left(\varphi_x+\psi+l\omega\right)\int_0^{+\infty} g_1(s)\eta_1(s)dsdx\\ &+\beta_1\int_0^1\left(\int_0^{+\infty} g_1\left(s\right)\eta_{1x}\left(s\right)ds\right)^2 dx-\gamma\int_0^1\theta_{2t}\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx\\ &-\rho_2\int_0^1\psi_t\int_0^{+\infty} g_1'(s)\eta_1(s)dsdx-\rho_2 g_1^0\int_0^1\psi_t^2dx. \qquad (2.38)\end{aligned}$$

By using Young's inequality, (2.18), (2.19) *and* (2.20), *we get, for any* $0<\delta<1$

$$\begin{aligned}\lambda_1\int_0^1\psi_x\int_0^{+\infty}g_1(s)\eta_{1x}(s)dsdx &\leq \delta\int_0^1\psi_x^2dx+\frac{\lambda_1^2}{4\delta}\int_0^1\left(\int_0^{+\infty}g_1(s)\eta_{1x}(s)ds\right)^2dx\\ &\leq \delta\int_0^1\psi_x^2dx\\ &\quad+\frac{\lambda_1^2}{4\delta}g_1^0\int_0^1\int_0^{+\infty}g_1(s)\eta_{1x}^2(s)dsdx, \end{aligned}\tag{2.39}$$

$$\begin{aligned}k\int_0^1\left(\varphi_x+\psi+l\omega\right)\int_0^{+\infty}g_1(s)\eta_1(s)dsdx &\leq \delta\int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx\\ &\quad+\frac{k^2}{4\delta}g_1^0\int_0^1\int_0^{+\infty}g_1(s)\eta_{1x}^2(s)dsdx,\end{aligned}\tag{2.40}$$

$$-\gamma\int_0^1\theta_{2t}\int_0^{+\infty}g_1(s)\eta_{1x}(s)dsdx\leq\delta\int_0^1\theta_{2t}^2dx+\frac{\gamma^2}{4\delta}g_1^0\int_0^1\int_0^{+\infty}g_1(s)\eta_{1x}^2(s)dsdx,\tag{2.41}$$

and

$$-\rho_2\int_0^1\psi_t\int_0^{+\infty}g_1'(s)\eta_1(s)dsdx\leq\rho_2\delta\int_0^1\psi_t^2dx-\frac{\rho_2}{4\delta}g_1(0)\int_0^1\int_0^{+\infty}g_1'(s)\eta_{1x}^2(s)dsdx.\tag{2.42}$$

Combining (2.39)-(2.42), *the result follows.* ■

Lemma 2.5 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be a solution of* (2.7). *Then the functional*

$$I_3(t)=-\rho_3\int_0^1\theta_{1t}\int_0^{+\infty}g_2(s)\eta_2(s)dsdx,\tag{2.43}$$

satisfies for any $0<\delta<1$

$$\begin{aligned}I_3'(t) &\leq -\rho_3\left(g_2^0-\delta\right)\int_0^1\theta_{1t}^2dx+\delta\int_0^1\theta_{1x}dx+\delta\int_0^1\omega_t^2dx\\ &\quad+C_2\left(\delta\right)g_2^0\int_0^1\int_0^{+\infty}g_2(s)\eta_{2x}^2(s)dsdx+l\delta\int_0^1\varphi_t^2dx\\ &\quad-\frac{\rho_3g_2(0)}{4\delta}\int_0^1\int_0^{+\infty}g_2'(s)\eta_{2x}^2(s)dsdx,\end{aligned}\tag{2.44}$$

where $C_2\left(\delta\right)=\beta_2+\frac{\lambda_2^2+m_1^2+lm_1^2}{4\delta}$.

Proof. *First, we note that*

$$\frac{\partial}{\partial t}\left(\int_0^{+\infty}g_2(s)\eta_2(s)ds\right)=\int_0^{+\infty}g_2'(s)\eta_2(s)ds+g_2^0\theta_{1t}.\tag{2.45}$$

By multiplying the fourth equation in (2.7) *by* $\int_0^{+\infty} g_2(s)\eta_2(s)ds$ *and integrating over* $(0,1)$, *we get*

$$\begin{aligned}
-\rho_3\int_0^1\theta_{1tt}\int_0^{+\infty} g_2(s)\eta_2(s)dsdx &= -\lambda_2\int_0^1\theta_{1xx}\int_0^{+\infty} g_2(s)\eta_2(s)dsdx \\
&\quad -\beta_2\int_0^1\left(\int_0^{+\infty} g_2\left(s\right)\eta_{2xx}\left(s\right)ds\right)\left(\int_0^{+\infty} g_2(s)\eta_2(s)ds\right)dx \\
&\quad +m_1\int_0^1\left(w_x-l\varphi\right)_t\int_0^{+\infty} g_2(s)\eta_2(s)dsdx,
\end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned}
-\rho_3\int_0^1\theta_{1tt}\int_0^{+\infty} g_2(s)\eta_2(s)dsdx &= +\lambda_2\int_0^1\theta_{1x}\int_0^{+\infty} g_2(s)\eta_{x2}(s)dsdx \\
&\quad +\beta_2\int_0^1\left(\int_0^{+\infty} g_2\left(s\right)\eta_{2x}\left(s\right)ds\right)^2dx \\
&\quad +m_1\int_0^1\left(w_x-l\varphi\right)_t\int_0^{+\infty} g_2(s)\eta_2(s)dsdx. \qquad (2.46)
\end{aligned}$$

By differentiating I_3, *using the equations* (2.46) *and* (2.45), *we obtain*

$$\begin{aligned}
I_3'(t) &= -\rho_3\int_0^1\theta_{1tt}\int_0^{+\infty} g_2(s)\eta_2(s)dsdx-\rho_3\int_0^1\theta_{1t}\left(\int_0^{+\infty} g_2(s)\eta_2(s)ds\right)_t dx \\
&= \lambda_2\int_0^1\theta_{1x}\int_0^{+\infty} g_2(s)\eta_{2x}(s)dsdx+\beta_2\int_0^1\left(\int_0^{+\infty} g_2(s)\eta_{2x}(s)ds\right)^2dx \\
&\quad -m_1\int_0^1\omega_t\int_0^{+\infty} g_2(s)\eta_{2x}(s)dsdx-m_1l\int_0^1\varphi_t\int_0^{+\infty} g_2(s)\eta_2(s)dsdx \\
&\quad -\rho_3\int_0^1\theta_{1t}\int_0^{+\infty} g_2'(s)\eta_2(s)dsdx-\rho_3g_2^0\int_0^1\theta_{1t}^2dx. \qquad (2.47)
\end{aligned}$$

By using Young's inequality ,(2.18) *and* (2.20), *we get, for any* $0<\delta<1$

$$\lambda_2\int_0^1\theta_{1x}\int_0^{+\infty} g_2(s)\eta_{2x}(s)dsdx \leq \delta\int_0^1\theta_{1x}^2dx+\frac{\lambda_2^2}{4\delta}g_2^0\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx, \qquad (2.48)$$

$$-m_1\int_0^1\omega_t\int_0^{+\infty} g_2(s)\eta_{2x}(s)dsdx \leq \delta\int_0^1\omega_t^2dx+\frac{m_1^2}{4\delta}g_2^0\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx, \qquad (2.49)$$

$$-m_1l\int_0^1\varphi_t\int_0^{+\infty} g_2(s)\eta_2(s)dsdx \leq \delta l\int_0^1\varphi_t^2dx+\frac{lm_1^2}{4\delta}g_2^0\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx, \qquad (2.50)$$

and

$$-\rho_3\int_0^1\theta_{1t}\int_0^{+\infty} g_2'(s)\eta_2(s)dsdx \leq \rho_3\delta\int_0^1\theta_{1t}^2dx-\frac{\rho_3}{4\delta}g_2(0)\int_0^1\int_0^{+\infty} g_2'(s)\eta_{2x}^2(s)dsdx. \qquad (2.51)$$

Combining (2.47)-(2.51), *the result follows.* ■

2.3. Technical Lemmas

Lemma 2.6 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be a solution of* (2.7)*. Then the functional*

$$I_4(t) = -\rho_4 \int_0^1 \theta_{2t} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx, \tag{2.52}$$

satisfies for any $0 < \delta < 1$

$$\begin{aligned} I_4'(t) \leq & -\rho_4 \left(g_3^0 - \delta\right) \int_0^1 \theta_{2t}^2 dx + \delta \int_0^1 \theta_{2x} dx + \delta \int_0^1 \psi_t^2 dx \\ & + C_3(\delta)\, g_3^0 \int_0^1 \int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx \\ & - \frac{\rho_4 g_3(0)}{4\delta} \int_0^1 \int_0^{+\infty} g_3'(s)\eta_{3x}^2(s)dsdx, \end{aligned} \tag{2.53}$$

where $C_2(\delta) = \beta_3 + \frac{\lambda_3^2 + m_2^2}{4\delta}$.

Proof. *First, we have*

$$\frac{\partial}{\partial t}\left(\int_0^{+\infty} g_3(s)\eta_3(s)ds\right) = \int_0^{+\infty} g_3'(s)\eta_3(s)ds + g_3^0\theta_{2t}. \tag{2.54}$$

By multiplying the fifth equation in (2.7) *by* $\int_0^{+\infty} g_3(s)\eta_3(s)ds$ *and integrating over* $(0,1)$*, we get*

$$\begin{aligned} -\rho_4 \int_0^1 \theta_{2tt} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx = & -\lambda_3 \int_0^1 \theta_{2xx} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx \\ & -\beta_3 \int_0^1 \int_0^{+\infty} g_3(s)\,\eta_{3xx}(s)\,ds \int_0^{+\infty} g_3(s)\eta_3(s)dsdx \\ & + m_2 \int_0^1 \psi_{tx} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx. \end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned} -\rho_4 \int_0^1 \theta_{2tt} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx = & +\lambda_3 \int_0^1 \theta_{2x} \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx \\ & +\beta_3 \int_0^1 \left(\int_0^{+\infty} g_3(s)\,\eta_{3x}(s)\,ds\right)^2 dx \\ & - m_2 \int_0^1 \psi_t \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx. \end{aligned} \tag{2.55}$$

By differentiating I_4*, using* (2.55) *and* (2.54)*, we have*

$$\begin{aligned} I_4^{'}(t) &= -\rho_4 \int_0^1 \theta_{2tt} \int_0^{+\infty} g_3(s)\eta_3(s)dsdx - \rho_4 \int_0^1 \theta_{2t} \left(\int_0^{+\infty} g_3(s)\eta_3(s)ds \right)_t dx \\ &= \lambda_3 \int_0^1 \theta_{2x} \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx + \beta_3 \int_0^1 \left(\int_0^{+\infty} g_3(s)\eta_{3x}(s)ds \right)^2 dx \\ & \quad -m_2 \int_0^1 \psi_t \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx - \rho_4 g_3^0 \int_0^1 \theta_{2t}^2 dx. \\ & \quad -\rho_4 \int_0^1 \theta_{2t} \int_0^{+\infty} g_3^{'}(s)\eta_3(s)dsdx. \end{aligned} \tag{2.56}$$

By using Young's inequality ,(2.19) *and* (2.20)*, we get, for any* $0 < \delta < 1$

$$\lambda_3 \int_0^1 \theta_{2x} \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx \leq \delta \int_0^1 \theta_{2x}^2 dx + \frac{\lambda_3^2}{4\delta} g_3^0 \int_0^1 \int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx, \tag{2.57}$$

$$-m_2 \int_0^1 \psi_t \int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx \leq \delta \int_0^1 \psi_t^2 dx + \frac{m_2^2}{4\delta} g_3^0 \int_0^1 \int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx, \tag{2.58}$$

and

$$-\rho_4 \int_0^1 \theta_{2t} \int_0^{+\infty} g_3^{'}(s)\eta_3(s)dsdx \leq \rho_4 \delta \int_0^1 \theta_{2t}^2 dx - \frac{\rho_4}{4\delta} g_3(0) \int_0^1 \int_0^{+\infty} g_3'(s)\eta_{3x}^2(s)dsdx. \tag{2.59}$$

Combining (2.56)-(2.59)*, we obtain the desired result.* ■

Lemma 2.7 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be a solution of* (2.7)*. Then the functional*

$$I_5(t) = -\int_0^1 (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t + \rho_1 \omega\omega_t)\, dx,$$

satisfies

$$\begin{aligned} I_5^{'}(t) &\leq -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx - \rho_1 \int_0^1 \omega_t^2 dx + (k_0 + 2cl) \int_0^1 (\omega_x - l\varphi)^2 dx \\ & \quad + (k + 2cl) \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + (\lambda_1 + 2l + 2cl) \int_0^1 \psi_x^2 dx + \frac{\gamma^2}{4l} \int_0^1 \theta_{2t}^2 dx \\ & \quad + \frac{\gamma^2}{4l}(1 + l^2) \int_0^1 \theta_{1t}^2 dx + \frac{\beta_1^2}{4l} g_1^0 \int_0^1 \int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx. \end{aligned} \tag{2.60}$$

Proof. By multiplying the equation $(2.7)_1$ by φ and integrating over $(0, 1)$, we get

$$-\rho_1 \int_0^1 \varphi_{tt}\varphi dx = -k \int_0^1 (\varphi_x + \psi + l\omega)_x \varphi dx - k_0 l \int_0^1 (\omega_x - l\varphi)\varphi dx + l\gamma \int_0^1 \theta_{1t}\varphi dx,$$

then, by integrating by parts the term $-k\int_0^1 (\varphi_x+\psi+l\omega)_x\,\varphi dx$ and under the boundary conditions, we find

$$-\rho_1\int_0^1 \varphi_{tt}\varphi dx = -k\int_0^1 (\varphi_x+\psi+l\omega)\,\varphi_x dx - k_0 l\int_0^1 (\omega_x-l\varphi)\,\varphi dx + l\gamma\int_0^1 \theta_{1t}\varphi dx. \tag{2.61}$$

On the other hand, by multiplying the equation (2.7)$_2$ by ψ and integrating over $(0,1)$, we get

$$\begin{aligned}-\rho_2\int_0^1 \psi_{tt}\psi dx &= -\lambda_1\int_0^1 \psi_{xx}\psi dx + k\int_0^1 (\varphi_x+\psi+l\omega)\,\psi dx\\ &\quad -\beta_1\int_0^1\int_0^{+\infty} g_1(s)\,\eta_{1xx}(s)\,ds\psi dx + \gamma\int_0^1 \theta_{2tx}\psi dx,\end{aligned}$$

we conclude, by using the formula of integration by parts with the boundary conditions, that

$$\begin{aligned}-\rho_2\int_0^1 \psi_{tt}\psi dx &= +\lambda_1\int_0^1 \psi_x^2 dx + k\int_0^1 (\varphi_x+\psi+l\omega)\,\psi dx\\ &\quad +\beta_1\int_0^1\int_0^{+\infty} g_1(s)\,\eta_{1x}(s)\,ds\psi_x dx - \gamma\int_0^1 \theta_{2t}\psi_x dx.\end{aligned} \tag{2.62}$$

In the same way, by multiplying the equation (2.7)$_3$ by ω and integrating over $(0,1)$, we get

$$-\rho_1\int_0^1 \omega_{tt}\omega dx = -k_0\int_0^1 (\omega_x-l\varphi)_x\,\omega dx + kl\int_0^1 (\varphi_x+\psi+l\omega)\,\omega dx + \gamma\int_0^1 \theta_{1tx}\omega dx,$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned}-\rho_1\int_0^1 \omega_{tt}\omega dx &= +k_0\int_0^1 (\omega_x-l\varphi)\,\omega_x dx\\ &\quad +kl\int_0^1 (\varphi_x+\psi+l\omega)\,\omega dx - \gamma\int_0^1 \theta_{1t}\omega_x dx.\end{aligned} \tag{2.63}$$

By differentiating I_5 and using (2.61)-(2.63), we obtain

$$\begin{aligned}I_5^{'}(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - \rho_1\int_0^1 \varphi\varphi_{tt} dx - \rho_2\int_0^1 \psi_t^2 dx - \rho_2\int_0^1 \psi\psi_{tt} dx - \rho_1\int_0^1 \omega_t^2 dx - \rho_1\int_0^1 \omega\omega_{tt} dx\\ &= -\rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx - \rho_1\int_0^1 \omega_t^2 dx + k_0\int_0^1 (\omega_x-l\varphi)^2\, dx\\ &\quad +k\int_0^1 (\varphi_x+\psi+l\omega)^2\, dx + \lambda_1\int_0^1 \psi_x^2 dx + l\gamma\int_0^1 \theta_{1t}\varphi dx\\ &\quad +\beta_1\int_0^1 \psi_x\left(\int_0^{+\infty} g_1(s)\eta_{1x}(s)ds\right) dx - \gamma\int_0^1 \theta_{1t}\omega_x dx - \gamma\int_0^1 \theta_{2t}\psi_x dx.\end{aligned} \tag{2.64}$$

by using (1.10), Young's inequality and (2.19) to estimate the last four terms on the right-hand side of (2.64), we get

$$\begin{aligned} l\gamma\int_0^1 \theta_{1t}\varphi dx &\leq l\int_0^1 \varphi_x^2 dx + \frac{l\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx \\ &\leq cl\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx + \frac{l\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx, \end{aligned}$$

$$\beta_1\int_0^1 \psi_x\left(\int_0^{+\infty} g_1(s)\eta_{1x}(s)ds\right)dx \leq l\int_0^1 \psi_x^2 dx + \frac{\beta_1^2}{4l}g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx,$$

$$\begin{aligned} -\gamma\int_0^1 \theta_{1t}\omega_x dx &\leq l\int_0^1 \omega_x^2 dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{1t}^2 dx \\ &\leq cl\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx + \frac{\gamma^2}{4l}g_3^0\int_0^1 \theta_{1t}^2 dx, \end{aligned}$$

and

$$-\gamma\int_0^1 \theta_{2t}\psi_x dx \leq l\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{2t}^2 dx.$$

By substituting the last four inequalities in relation (2.64), we get the result. ■

Lemma 2.8 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be a solution of* (2.7)*, and let* $k = k_0$*. Then for any positive constant* ε_2*, the functional*

$$\begin{aligned} I_6(t) &= -\rho_1\int_0^1 (\omega_x - l\varphi)\int_0^x \omega_t(y,t)dydx \\ &\quad -\rho_1\int_0^1 \varphi_t\int_0^x (\varphi_x+\psi+l\omega)(y,t)dydx, \end{aligned} \tag{2.65}$$

satisfies

$$\begin{aligned} I_6'(t) &\leq -(k_0-\varepsilon_2)\int_0^1 (\omega_x - l\varphi)^2 dx - \rho_1(1-l)\int_0^1 \varphi_t^2 dx \\ &\quad +\frac{\rho_1}{4l}\int_0^1 \psi_t^2 dx + \rho_1\int_0^1 \omega_t^2 dx + \frac{\gamma^2}{2\varepsilon_2}\int_0^1 \theta_{1t}^2 dx \\ &\quad +(k+l^2\varepsilon_2)\int_0^1 (\varphi_x+\psi+l\omega)^2 dx. \end{aligned} \tag{2.66}$$

Proof. By multiplying the equation $(2.7)_1$ by $\int_0^x (\varphi_x+\psi+l\omega)(y,t)dy$ and integrating over $(0,1)$, we get

$$\begin{aligned} 0 = {} & \rho_1\int_0^1 \varphi_{tt}\int_0^x (\varphi_x+\psi+l\omega)(y,t)dydx - k\int_0^1 (\varphi_x+\psi+l\omega)_x\int_0^x (\varphi_x+\psi+l\omega)(y,t)dydx \\ & -k_0 l\int_0^1 (\omega_x - l\varphi)\int_0^x (\varphi_x+\psi+l\omega)(y,t)dydx + l\gamma\int_0^1 \theta_{1t}\int_0^x (\varphi_x+\psi+l\omega)(y,t)dydx, \end{aligned}$$

then, by integrating by parts the term $-k\int\limits_0^1 \left(\varphi_x+\psi+l\omega\right)_x \int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx$ and under the boundary conditions, we find

$$\begin{aligned}
0 &= \rho_1\int\limits_0^1 \varphi_{tt}\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx - k_0 l\int\limits_0^1 \left(\omega_x-l\varphi\right)\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx \\
&\quad -k\int\limits_0^1 \left(\varphi_x+\psi+l\omega\right)^2 dx + l\gamma\int\limits_0^1 \theta_{1t}\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx.
\end{aligned}$$

From the previous equality and under the boundary conditions, an integration by parts gives, we have

$$\begin{aligned}
&-\rho_1\frac{d}{dt}\int\limits_0^1 \varphi_t\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx \\
&= -\rho_1\int\limits_0^1 \varphi_{tt}\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx - \rho_1\int\limits_0^1 \varphi_t\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)_t(y,t)dydx \\
&= -k_0 l\int\limits_0^1 \left(\omega_x-l\varphi\right)\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx - k\int\limits_0^1 \left(\varphi_x+\psi+l\omega\right)^2 dx \\
&\quad +l\gamma\int\limits_0^1 \theta_{1t}\int\limits_0^x \left(\varphi_x+\psi+l\omega\right)(y,t)dydx - \rho_1\int\limits_0^1 \varphi_t^2 dx - \rho_1\int\limits_0^1 \varphi_t\int\limits_0^x \psi_t(y,t)dydx \\
&\quad -l\rho_1\int\limits_0^1 \varphi_t\int\limits_0^x \omega_t(y,t)dydx. \qquad (2.67)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-\rho_1\frac{d}{dt}\int\limits_0^1 \left(\omega_x-l\varphi\right)\int\limits_0^x \omega_t(y,t)dydx &= -\rho_1\int\limits_0^1 \left(\omega_x-l\varphi\right)_t\int\limits_0^x \omega_t(y,t)dydx \\
&\quad -\rho_1\int\limits_0^1 \left(\omega_x-l\varphi\right)\int\limits_0^x \omega_{tt}(y,t)dydx \\
&= -\rho_1\int\limits_0^1 \omega_{tx}\int\limits_0^x \omega_t(y,t)dydx + \rho_1 l\int\limits_0^1 \varphi_t\int\limits_0^x \omega_t(y,t)dydx \\
&\quad -\rho_1\int\limits_0^1 \left(\omega_x-l\varphi\right)\int\limits_0^x \omega_{tt}(y,t)dydx. \qquad (2.68)
\end{aligned}$$

In (2.68), we replace ω_{tt} by

$$-\rho_1\omega_{tt} = -k_0\left(\omega_x-l\varphi\right)_x + kl\left(\varphi_x+\psi+l\omega\right) + \gamma\theta_{1tx}.$$

Consequently using integration by parts and the boundary conditions we have

$$\begin{aligned}
& -\rho_1 \frac{d}{dt}\int_0^1 (\omega_x - l\varphi)\int_0^x \omega_t(y,t)dydx \\
= \; & -\rho_1\int_0^1 \omega_{tx}\int_0^x \omega_t(y,t)dydx + \rho_1 l\int_0^1 \varphi_t\int_0^x \omega_t(y,t)dydx - k_0\int_0^1 (\omega_x - l\varphi)\int_0^x (\omega_x - l\varphi)_x (y,t)dydx \\
& +kl\int_0^1 (\omega_x - l\varphi)\int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx + \gamma\int_0^1 (\omega_x - l\varphi)\int_0^x \theta_{1tx}(y,t)dydx \\
= \; & \rho_1\int_0^1 \omega_t^2 dx + \rho_1 l\int_0^1 \varphi_t\int_0^x \omega_t(y,t)dydx - k_0\int_0^1 (\omega_x - l\varphi)^2 dx \\
& +kl\int_0^1 (\omega_x - l\varphi)\int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx + \gamma\int_0^1 (\omega_x - l\varphi)\theta_{1t}dx. \qquad (2.69)
\end{aligned}$$

By adding the two equalities (2.67) and (2.69), under the condition $k = k_0$ we get

$$\begin{aligned}
I_6^{'}(t) \; = \; & -\rho_1 \frac{d}{dt}\int_0^1 (\omega_x - l\varphi)\int_0^x \omega_t(y,t)dydx - \rho_1\frac{d}{dt}\int_0^1 \varphi_t\int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx \\
= \; & -\rho_1\int_0^1 \varphi_t^2 dx - k_0\int_0^1 (\omega_x - l\varphi)^2 dx + \rho_1\int_0^1 \omega_t^2 dx \\
& +k\int_0^1 (\varphi_x + \psi + l\omega)^2 dx - \rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y,t)dydx \\
& +\gamma\int_0^1 (\omega_x - l\varphi)\theta_{1t}dx + \gamma l\int_0^1 \theta_{1t}\int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx. \qquad (2.70)
\end{aligned}$$

By using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}
-\rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y,t)dydx \; & \leq \; \rho_1\varepsilon_2\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\varepsilon_2}\int_0^1 \psi_t^2 dx, \\
\gamma\int_0^1 (\omega_x - l\varphi)\theta_{1t}dx \; & \leq \; \varepsilon_2\int_0^1 (\omega_x - l\varphi)^2 dx + \frac{\gamma^2}{4\varepsilon_2}\int_0^1 \theta_{1t}^2 dx,
\end{aligned}$$

and

$$\gamma l\int_0^1 \theta_{1t}\int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx \leq l^2\varepsilon_2\int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \frac{\gamma^2}{4\varepsilon_2}\int_0^1 \theta_{1t}^2 dx.$$

By substituting the last three inequalities in (2.70), we get the result. ■

Lemma 2.9 *Let $(\varphi, \psi, \omega, \theta_1, \theta_2)$ be the solution of (2.7), and let $k = k_0$. Then, the functional*

$$I_7(t) = -\rho_1\int_0^1 \varphi_t(\omega_x - l\varphi)dx - \rho_1\int_0^1 \omega_t(\varphi_x + \psi + l\omega)dx, \qquad (2.71)$$

satisfies the estimate

$$\begin{aligned} I_7'(t) &\leq -l(k_0-1)\int_0^1 (\omega_x - l\varphi)^2\,dx - \frac{\rho_1 l}{2}\int_0^1 \omega_t^2 dx + \rho_1 l\int_0^1 \varphi_t^2 dx \\ &\quad +kl\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \frac{\rho_1}{2l}\int_0^1 \psi_t^2 dx + \frac{l\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx \\ &\quad +\gamma\int_0^1 \theta_{1tx}(\varphi_x+\psi+l\omega)\,dx. \qquad (2.72) \end{aligned}$$

Proof. By multiplying the equation $(2.7)_1$ by $(\omega_x - l\varphi)$ and integrating over $(0,1)$, we get

$$\begin{aligned} -\rho_1\int_0^1 \varphi_{tt}(\omega_x - l\varphi)\,dx &= -k\int_0^1 (\varphi_x+\psi+l\omega)_x(\omega_x - l\varphi)\,dx \\ &\quad -k_0 l\int_0^1 (\omega_x - l\varphi)^2\,dx + l\gamma\int_0^1 \theta_{1t}(\omega_x - l\varphi)\,dx. \end{aligned}$$

On the other hand, by multiplying the equation $(2.7)_3$ by $(\varphi_x+\psi+l\omega)$ and integrating over $(0,1)$, we get

$$\begin{aligned} -\rho_1\int_0^1 \omega_{tt}(\varphi_x+\psi+l\omega)\,dx &= -k_0\int_0^1 (\omega_x - l\varphi)_x(\varphi_x+\psi+l\omega)\,dx \\ &\quad +kl\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \gamma\int_0^1 \theta_{1tx}(\varphi_x+\psi+l\omega)\,dx, \end{aligned}$$

By differentiating I_7 and using these last two equations, under the condition $k = k_0$ we get, we get

$$\begin{aligned} I_7'(t) &= -\rho_1\int_0^1 \varphi_{tt}(\omega_x - l\varphi)\,dx - \rho_1\int_0^1 \varphi_t(\omega_x - l\varphi)_t\,dx \\ &\quad -\rho_1\int_0^1 \omega_{tt}(\varphi_x+\psi+l\omega)\,dx - \rho_1\int_0^1 \omega_t(\varphi_x+\psi+l\omega)_t\,dx \\ &= -k_0 l\int_0^1 (\omega_x - l\varphi)^2\,dx - \rho_1 l\int_0^1 \omega_t^2 dx + \rho_1 l\int_0^1 \varphi_t^2 dx \\ &\quad +kl\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \gamma\int_0^1 \theta_{1tx}(\varphi_x+\psi+l\omega)\,dx \\ &\quad +l\gamma\int_0^1 \theta_{1t}(\omega_x - l\varphi)\,dx - \rho_1\int_0^1 \omega_t\psi_t dx. \qquad (2.73) \end{aligned}$$

Use of Young's inequality for the last two terms in the right-hand side gives

$$\begin{aligned} l\gamma\int_0^1 \theta_{1t}(\omega_x - l\varphi)\,dx &\leq l\int_0^1 (\omega_x - l\varphi)^2\,dx + \frac{l\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx, \\ -\rho_1\int_0^1 \omega_t\psi_t dx &\leq \frac{\rho_1 l}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1}{2l}\int_0^1 \psi_t^2 dx. \end{aligned}$$

By substituting these inequalities in (2.73), we get the result. ■

2.3. Technical Lemmas

Lemma 2.10 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be the solution of* (2.7) *and let* $k = k_0$. *Then the functional*

$$\begin{aligned} I_8(t) &= \rho_2 \int_0^1 \psi_t \left(\varphi_x + \psi + l\omega\right) dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx \\ &\quad - \frac{\beta_1 \rho_1}{k} \int_0^1 \varphi_t \int_0^{+\infty} g_1(s) \psi_x(t-s) ds dx, \end{aligned} \tag{2.74}$$

satisfies, for any $\varepsilon_3 > 0$, *the estimate*

$$\begin{aligned} I_8'(t) &\leq -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_2 \left(1 + \frac{l}{4\varepsilon_3}\right) \int_0^1 \psi_t^2 dx + \rho_2 l \varepsilon_3 \int_0^1 \omega_t^2 dx \\ &\quad + l\varepsilon_3 \int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\lambda_1^2 l}{2\varepsilon_3} \left(1 + \gamma^2\right) \int_0^1 \psi_x^2 dx + \frac{l\varepsilon_3}{2k^2} \int_0^1 \theta_{1t}^2 dx \\ &\quad + \frac{l\beta_1^2}{\varepsilon_3} g_1^0 \int_0^1 \int_0^{+\infty} g_1(s) \eta_{1x}^2(s) ds dx - \frac{\rho_1^2 \beta_1^2}{4k\varepsilon_3} g_1(0) \int_0^1 \int_0^{+\infty} g_1'(s) \eta_{1x}^2(s) ds dx \\ &\quad - \gamma \int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx + \left(\rho_2 - \frac{b_1 \rho_1}{k}\right) \int_0^1 \psi_t \varphi_{tx} dx. \end{aligned} \tag{2.75}$$

Proof. First, we note that

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_0^{+\infty} g_1(s) \psi_x(t-s) ds \right) &= \frac{\partial}{\partial t} \left(\int_{-\infty}^{t} g_1(t-s) \psi_x(s) ds \right) \\ &= g(0)\psi_x(t) + \int_{-\infty}^{t} g_1'(t-s) \psi_x(s) ds \\ &= g(0)\psi_x(t) + \int_0^{+\infty} g_1'(s) \left(\psi_x(t-s) - \psi_x(t) + \psi_x(t)\right) ds \\ &= g(0)\psi_x(t) - \int_0^{+\infty} g_1'(s) \eta_{1x}(s) ds + \left(\int_0^{+\infty} g_1'(s) ds \right) \psi_x(t) \\ &= - \int_0^{+\infty} g_1'(s) \eta_{1x}(s) ds. \end{aligned} \tag{2.76}$$

By multiplying the equation $(2.7)_1$ by ψ_x then by $\int_0^{+\infty} g_1(s)\psi_x(t-s)ds$ and integrating over $(0,1)$, we get

$$\rho_1 \int_0^1 \varphi_{tt} \psi_x dx = k \int_0^1 \left(\varphi_x + \psi + l\omega\right)_x \psi_x dx + k_0 l \int_0^1 \left(\omega_x - l\varphi\right) \psi_x dx - l\gamma \int_0^1 \theta_{1t} \psi_x dx,$$

and

$$\begin{aligned} -\rho_1 \int_0^1 \varphi_{tt} \int_0^{+\infty} g_1(s) \psi_x(t-s) ds dx &= -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)_x \int_0^{+\infty} g_1(s) \psi_x(t-s) ds dx \\ &\quad - k_0 l \int_0^1 \left(\omega_x - l\varphi\right) \int_0^{+\infty} g_1(s) \psi_x(t-s) ds dx \\ &\quad + l\gamma \int_0^1 \theta_{1t} \int_0^{+\infty} g_1(s) \psi_x(t-s) ds dx. \end{aligned}$$

under the condition $k = k_0$, we deduce that

$$\frac{b_1\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx = b_1\int_0^1 (\varphi_x + \psi + l\omega)_x \psi_x dx + b_1 l\int_0^1 (\omega_x - l\varphi)\psi_x dx - \frac{b_1 l\gamma}{k}\int_0^1 \theta_{1t}\psi_x dx, \quad (2.77)$$

and

$$\begin{aligned} -\frac{\beta_1\rho_1}{k}\int_0^1 \varphi_{tt}\int_0^{+\infty} g_1(s)\psi_x(t-s)dsdx \quad = \quad & -\beta_1\int_0^1 (\varphi_x + \psi + l\omega)_x \int_0^{+\infty} g_1(s)\psi_x(t-s)dsdx \\ & -\beta_1 l\int_0^1 (\omega_x - l\varphi)\int_0^{+\infty} g_1(s)\psi_x(t-s)dsdx \\ & +\frac{\beta_1 l\gamma}{k}\int_0^1 \theta_{1t}\int_0^{+\infty} g_1(s)\psi_x(t-s)dsdx. \qquad (2.78) \end{aligned}$$

In the same way, by multiplying the equation (2.7)$_2$ by $(\varphi_x + \psi + l\omega)$ and integrating over $(0,1)$, we get

$$\begin{aligned} \rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi + l\omega)dx \quad = \quad & +\lambda_1\int_0^1 \psi_{xx}(\varphi_x + \psi + l\omega)dx - k\int_0^1 (\varphi_x + \psi + l\omega)^2 dx \\ & +\beta_1\int_0^1\int_0^{+\infty} g_1(s)\eta_{1xx}(s)ds(\varphi_x + \psi + l\omega)dx \\ & -\gamma\int_0^1 \theta_{2tx}(\varphi_x + \psi + l\omega)dx, \end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned} \rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi + l\omega)dx \quad = \quad & -\lambda_1\int_0^1 \psi_x(\varphi_x + \psi + l\omega)_x dx - k\int_0^1 (\varphi_x + \psi + l\omega)^2 dx \\ & -\beta_1\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}(s)ds(\varphi_x + \psi + l\omega)_x dx \\ & -\gamma\int_0^1 \theta_{2tx}(\varphi_x + \psi + l\omega)dx. \qquad (2.79) \end{aligned}$$

By differentiating I_8 and using (2.76)-(2.79), we obtain

$$\begin{aligned}
I_8'(t) &= \rho_2\int_0^1 \psi_{tt}\left(\varphi_x+\psi+l\omega\right)dx+\rho_2\int_0^1 \psi_t\left(\varphi_x+\psi+l\omega\right)_t dx+\frac{b\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx\\
&\quad +\frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_{tx}dx-\frac{\beta_1\rho_1}{k}\int_0^1 \varphi_{tt}\int_0^{+\infty} g_1(s)\psi_x(t-s)dsdx\\
&\quad -\frac{\beta_1\rho_1}{k}\int_0^1 \varphi_t\frac{\partial}{\partial t}\left(\int_0^{+\infty} g_1(s)\psi_x(t-s)ds\right)dx\\
&= -k\int_0^1 \left(\varphi_x+\psi+l\omega\right)^2dx-\gamma\int_0^1 \theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx+\rho_2\int_0^1 \psi_t^2dx\\
&\quad +\rho_2 l\int_0^1 \psi_t\omega_t dx+\lambda_1 l\int_0^1 \psi_x\left(\omega_x-l\varphi\right)dx-\frac{\lambda_1 l\gamma}{k}\int_0^1 \theta_{1t}\psi_x dx\\
&\quad +\beta_1 l\int_0^1 \left(\omega_x-l\varphi\right)\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx-\frac{\beta_1 l\gamma}{k}\int_0^1 \theta_{1t}\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx\\
&\quad +\frac{\beta_1\rho_1}{k}\int_0^1 \varphi_t\int_0^{+\infty} g_1'(s)\eta_{1x}(s)dsdx+\left(\rho_2-\frac{b_1\rho_1}{k}\right)\int_0^1 \psi_t\varphi_{tx}dx. \qquad (2.80)
\end{aligned}$$

By using Young's inequalitie, (2.19) and (2.21) we obtain

$$\begin{aligned}
\rho_2 l\int_0^1 \psi_t\omega_t dx &\leq \rho_2 l\varepsilon_3\int_0^1 \omega_t^2dx+\frac{\rho_2 l}{4\varepsilon_3}\int_0^1 \psi_t^2dx,\\
\lambda_1 l\int_0^1 \psi_x\left(\omega_x-l\varphi\right)dx &\leq \frac{l\varepsilon_3}{2}\int_0^1 \left(\omega_x-l\varphi\right)^2dx+\frac{\lambda_1^2 l}{2\varepsilon_3}\int_0^1 \psi_x^2dx,\\
-\frac{\lambda_1 l\gamma}{k}\int_0^1 \theta_{1t}\psi_x dx &\leq \frac{l\varepsilon_3}{2k^2}\int_0^1 \theta_{1t}^2dx+\frac{l\lambda_1^2\gamma^2}{2\varepsilon_3}\int_0^1 \psi_x^2dx,\\
\beta_1 l\int_0^1 \left(\omega_x-l\varphi\right)\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx &\leq \frac{l\varepsilon_3}{2}\int_0^1 \left(\omega_x-l\varphi\right)^2dx+\frac{l\beta_1^2 g_1^0}{2\varepsilon_3}\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx,\\
-\frac{\beta_1 l\gamma}{k}\int_0^1 \theta_{1t}\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx &\leq \frac{l\varepsilon_3}{2k^2}\int_0^1 \theta_{1t}^2dx+\frac{l\beta_1^2 g_1^0}{2\varepsilon_3}\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx,
\end{aligned}$$

and

$$\frac{\beta_1\rho_1}{k}\int_0^1 \varphi_t\int_0^{+\infty} g_1'(s)\eta_{1x}(s)dsdx\leq \varepsilon_3\int_0^1 \varphi_t^2dx-\frac{\rho_1^2\beta_1^2}{4k\varepsilon_3}g_1(0)\int_0^1\int_0^{+\infty} g_1'(s)\eta_{1x}^2(s)dsdx.$$

By substituting these inequalities in (2.80), we get the result. ■

Lemma 2.11 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be the solution of (2.7). Then the functional*

$$I_9(t)=\int_0^1 \left(\rho_3\theta_1\theta_{1t}+\rho_4\theta_2\theta_{2t}\right)dx. \qquad (2.81)$$

satisfies the estimate

$$\begin{aligned} I_9^{'}(t) \leq & -(\lambda_2-3l)\int_0^1\theta_{1x}^2dx-(\lambda_3-2l)\int_0^1\theta_{2x}^2dx+\frac{lm_1^2}{4}\int_0^1\varphi_t^2dx \\ & +\frac{m_2^2}{4l}\int_0^1\psi_t^2dx+\frac{m_1^2}{4l}\int_0^1\omega_t^2dx+\frac{\beta_2^2}{4l}g_2^0\int_0^1\left(\int_0^{+\infty}g_2(s)\eta_{2x}^2(s)ds\right)dx \\ & +\rho_3\int_0^1\theta_{1t}^2dx+\rho_4\int_0^1\theta_{2t}^2dx+\frac{\beta_3^2}{4l}g_3^0\int_0^1\left(\int_0^{+\infty}g_3(s)\eta_{3x}^2(s)ds\right)dx. \end{aligned} \tag{2.82}$$

Proof. By multiplying the equation (2.7)$_4$ by θ_1 and integrating over $(0,1)$, we get

$$\rho_3\int_0^1\theta_{1tt}\theta_1dx=\lambda_2\int_0^1\theta_{1xx}\theta_1dx+\beta_2\int_0^1\int_0^{+\infty}g_2\left(s\right)\eta_{2xx}\left(s\right)ds\theta_1dx-m_1\int_0^1\left(\omega_x-l\varphi\right)_t\theta_1dx.$$

On the other hand, by multiplying the equation (2.7)$_5$ by θ_2 and integrating over $(0,1)$, we get

$$\rho_4\int_0^1\theta_{2tt}\theta_2dx=\lambda_3\int_0^1\theta_{2xx}\theta_2dx+\beta_3\int_0^1\int_0^{+\infty}g_3\left(s\right)\eta_{3xx}\left(s\right)ds\theta_2dx-m_2\int_0^1\psi_{xt}\theta_2dx,$$

under the boundary conditions, an integration by parts gives

$$\rho_3\int_0^1\theta_{1tt}\theta_1dx=-\lambda_2\int_0^1\theta_{1x}^2dx-\beta_2\int_0^1\int_0^{+\infty}g_2\left(s\right)\eta_{2x}\left(s\right)ds\theta_{1x}dx-m_1\int_0^1\left(\omega_x-l\varphi\right)_t\theta_1dx,$$

and

$$\rho_4\int_0^1\theta_{2tt}\theta_2dx=-\lambda_3\int_0^1\theta_{2x}^2dx-\beta_3\int_0^1\int_0^{+\infty}g_3\left(s\right)\eta_{3x}\left(s\right)ds\theta_{2x}dx+m_2\int_0^1\psi_t\theta_{2x}dx.$$

By differentiating I_9 and using these last two equations, we get

$$\begin{aligned} I_9^{'}(t) &= \rho_3\int_0^1\theta_{1t}^2dx+\rho_3\int_0^1\theta_1\theta_{1tt}dx+\rho_4\int_0^1\theta_{2t}^2dx+\rho_4\int_0^1\theta_2\theta_{2tt}dx \\ &= -\lambda_2\int_0^1\theta_{1x}^2dx-\lambda_3\int_0^1\theta_{2x}^2dx+\rho_3\int_0^1\theta_{1t}^2dx+\rho_4\int_0^1\theta_{2t}^2dx \\ & \quad +m_1\int_0^1\theta_{1x}\omega_tdx+lm_1\int_0^1\theta_1\varphi_tdx+m_2\int_0^1\theta_{2x}\psi_tdx \\ & \quad -\beta_2\int_0^1\theta_{1x}\int_0^{+\infty}g_2(s)\eta_{2x}(s)dsdx-\beta_3\int_0^1\theta_{2x}\int_0^{+\infty}g_2(s)\eta_{3x}(s)dsdx. \end{aligned} \tag{2.83}$$

By using Young's, Poincaré's inequalities and (2.19), we get

$$m_1\int_0^1\theta_{1x}\omega_tdx \leq l\int_0^1\theta_{1x}^2dx+\frac{m_1^2}{4l}\int_0^1\omega_t^2dx, \tag{2.84}$$

$$lm_1\int_0^1\theta_1\varphi_tdx \leq l\int_0^1\theta_{1x}^2dx+\frac{lm_1^2}{4}\int_0^1\varphi_t^2dx, \tag{2.85}$$

$$m_2\int_0^1\theta_{2x}\psi_tdx \leq l\int_0^1\theta_{2x}^2dx+\frac{m_2^2}{4l}\int_0^1\psi_t^2dx, \tag{2.86}$$

$$-\beta_2\int_0^1\theta_{1x}\int_0^{+\infty} g_2(s)\eta_{2x}(s)dsdx \leq l\int_0^1\theta_{1x}^2dx + \frac{\beta_2^2}{4l}g_2^0\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx, \tag{2.87}$$

and

$$-\beta_3\int_0^1\theta_{2x}\int_0^{+\infty} g_3(s)\eta_{3x}(s)dsdx \leq l\int_0^1\theta_{2x}^2dx + \frac{\beta_3^2}{4l}g_3^0\int_0^1\int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx. \tag{2.88}$$

Combining (2.83)-(2.88), gives the desired result. ■

Lemma 2.12 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be the solution of (2.7). Then the functional*

$$I_{10}(t) = -\rho_2\int_0^1\psi_x\int_0^x\psi_t(y,t)dydx, \tag{2.89}$$

satisfies, for any $\varepsilon_4 > 0$, the estimate

$$\begin{aligned} I_{10}'(t) \ \leq\ & -(\lambda_1-\varepsilon_4(k+\beta_1+\gamma))\int_0^1\psi_x^2dx + \rho_2\int_0^1\psi_t^2dx \\ & +\frac{k}{4\varepsilon_4}\int_0^1(\varphi_x+\psi+l\omega)^2dx + \frac{\gamma}{4\varepsilon_4}\int_0^1\theta_{2t}^2dx \\ & +\frac{\beta_1}{4\varepsilon_4}g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx. \end{aligned} \tag{2.90}$$

Proof. By differentiating I_{10} and we replace $-\rho_2\psi_{tt}$ by $-\lambda_1\psi_{xx} + k(\varphi_x+\psi+l\omega) - \beta_1\int_0^{+\infty} g_1(s)\eta_{1xx}(s)ds + \gamma\theta_{2tx}$, we get

$$\begin{aligned} I_{10}'(t) &= -\rho_2\int_0^1\psi_{tx}\int_0^x\psi_t(y,t)dydx - \rho_2\int_0^1\psi_x\int_0^x\psi_{tt}(y,t)dydx \\ &= -\rho_2\int_0^1\psi_{tx}\int_0^x\psi_t(y,t)dydx - \lambda_1\int_0^1\psi_x^2dx \\ &\quad +k\int_0^1\psi_x\int_0^x(\varphi_x+\psi+l\omega)(y,t)dydx \\ &\quad -\beta_1\int_0^1\psi_x\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx + \gamma\int_0^1\psi_x\theta_{2t}dx, \end{aligned}$$

under the boundary conditions, an integration by parts gives

$$\begin{aligned} I_{10}'(t) &= -\lambda_1\int_0^1\psi_x^2dx + \rho_2\int_0^1\psi_t^2dx + \gamma\int_0^1\psi_x\theta_{2t}dx \\ &\quad -k\int_0^1\psi(\varphi_x+\psi+l\omega)dx - \beta_1\int_0^1\psi_x\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx. \end{aligned} \tag{2.91}$$

By using Young's, Poincaré's inequalities and (2.19), we get

$$\begin{aligned}
\gamma\int_0^1 \psi_x\theta_{2t}dx &\leq \gamma\varepsilon_4\int_0^1 \psi_x^2dx+\frac{\gamma}{4\varepsilon_4}\int_0^1 \theta_{2t}^2dx,\\
-k\int_0^1 \psi\left(\varphi_x+\psi+l\omega\right)dx &\leq k\varepsilon_4\int_0^1 \psi_x^2dx+\frac{k}{4\varepsilon_4}\int_0^1 \left(\varphi_x+\psi+l\omega\right)^2dx,
\end{aligned}$$

and

$$-\beta_1\int_0^1 \psi_x\int_0^{+\infty} g_1(s)\eta_{1x}(s)dsdx \leq \beta_1\varepsilon_4\int_0^1 \psi_x^2dx+\frac{\beta_1}{4\varepsilon_4}g_1^0\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx.$$

By substituting these inequalities in (2.91), we get the result. ■

Firstly, to estimate the terms $\left\{\left(\rho_2-\frac{b_1\rho_1}{k}\right)\int_0^1 \psi_t\varphi_{tx}dx, \gamma\int_0^1 \theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx,\right.$ $\left.-\gamma\int_0^1 \theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx\right\}$ appeared in (2.72),(2.75) respectively, we use the system obtained by differentiating (2.7) with respect to time, defined by

$$\begin{cases}
\rho_1\varphi_{ttt}-k\left(\varphi_x+\psi+l\omega\right)_{tx}-k_0l\left(\omega_x-l\varphi\right)_t+l\gamma\theta_{1tt}=0,\\
\rho_2\psi_{ttt}-\lambda_1\psi_{txx}+k\left(\varphi_x+\psi+l\omega\right)_t-\beta_1\int_0^{+\infty} g_1\left(s\right)\eta_{1txx}\left(s\right)ds+\gamma\theta_{2ttx}=0,\\
\rho_1\omega_{ttt}-k_0\left(\omega_x-l\varphi\right)_{tx}+kl\left(\varphi_x+\psi+l\omega\right)_t+\gamma\theta_{1ttx}=0,\\
\rho_3\theta_{1ttt}-\lambda_2\theta_{1txx}-\beta_2\int_0^{+\infty} g_2\left(s\right)\eta_{2txx}\left(s\right)ds+m_1\left(w_x-l\varphi\right)_{tt}=0,\\
\rho_4\theta_{2ttt}-\lambda_3\theta_{2txx}-\beta_3\int_0^{+\infty} g_3\left(s\right)\eta_{3txx}\left(s\right)ds+m_2\psi_{xtt}=0.\\
\eta_{1tt}+\eta_{1st}-\psi_{tt}=0,\\
\eta_{2tt}+\eta_{2st}-\theta_{1tt}=0,\\
\eta_{3tt}+\eta_{3st}-\theta_{2tt}=0,\\
\varphi_t(0,t)=\varphi_t(1,t)=\psi_t(0,t)=\psi_t(1,t)=\omega_t(0,t)=\omega_t\left(1,t\right)=0,\ t\in(0,+\infty),\\
\theta_{1t}\left(0,t\right)=\theta_{1t}\left(1,t\right)=\theta_{2t}\left(0,t\right)=\theta_{2t}\left(1,t\right)=0,\ t\in(0,+\infty),\\
\eta_{it}\left(0,t,s\right)=\eta_{it}\left(1,t,s\right)=0,\quad for\quad i=1,2,3.
\end{cases}$$

and we introduce the second-order energy functional

$$\begin{aligned}
\tilde{E}(t) = \frac{1}{2}\int_0^1 \Bigg[&\rho_1\varphi_{tt}^2+\rho_2\psi_{tt}^2+\rho_1\omega_{tt}^2+\frac{\rho_3\gamma}{m_1}\theta_{1tt}^2+\frac{\rho_4\gamma}{m_2}\theta_{2tt}^2+\lambda_1\psi_{xt}^2+\frac{\lambda_2\gamma}{m_1}\theta_{1tx}^2\\
&+\frac{\lambda_3\gamma}{m_2}\theta_{2tx}^2+k\left(\varphi_x+\psi+l\omega\right)_t^2+k_0\left(\omega_x-l\varphi\right)_t^2+\beta_1\left(\int_0^{+\infty} g_1(s)\eta_{1tx}^2(s)ds\right)\\
&+\frac{\gamma\beta_2}{m_1}\left(\int_0^{+\infty} g_2(s)\eta_{2tx}^2(s)ds\right)+\frac{\gamma\beta_3}{m_2}\left(\int_0^{+\infty} g_3(s)\eta_{3tx}^2(s)ds\right)\Bigg]dx. \qquad (2.92)
\end{aligned}$$

satisfies

$$\begin{aligned}\tilde{E}'(t) &= \frac{\beta_1}{2}\int_0^1 \left(\int_0^{+\infty} g_1'(s)\eta_{1tx}^2(s)ds\right)dx + \frac{\gamma\beta_2}{2m_1}\int_0^1 \left(\int_0^{+\infty} g_2'(s)\eta_{2tx}^2(s)ds\right)dx \\ &\quad + \frac{\gamma\beta_3}{2m_2}\int_0^1 \left(\int_0^{+\infty} g_3'(s)\eta_{3tx}^2(s)ds\right)dx. \qquad (2.93)\end{aligned}$$

Lemma 2.13 *For any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that*

$$\begin{aligned}\left(\rho_2 - \frac{b_1\rho_1}{k}\right) N_8\int_0^1 \psi_t\varphi_{tx}dx &\leq c_\varepsilon\int_0^1\int_0^{+\infty} g_1(s)\eta_{1tx}^2(s)dsdx \\ &\quad +\varepsilon E(t) - \frac{2c_\varepsilon}{\beta_1}E'(t). \qquad (2.94)\end{aligned}$$

$$\begin{aligned}\gamma N_7\int_0^1 (\varphi_x + \psi + l\omega)\,\theta_{1tx}dx &\leq c_\varepsilon\int_0^1\int_0^{+\infty} g_2(s)\eta_{2tx}^2(s)dsdx \\ &\quad +\varepsilon E(t) - \frac{2c_\varepsilon m_1}{\gamma\beta_2}E'(t). \qquad (2.95)\end{aligned}$$

$$\begin{aligned}-\gamma N_8\int_0^1 (\varphi_x + \psi + l\omega)\,\theta_{2tx}dx &\leq c_\varepsilon\int_0^1\int_0^{+\infty} g_3(s)\eta_{3tx}^2(s)dsdx \\ &\quad +\varepsilon E(t) - \frac{2c_\varepsilon m_2}{\gamma\beta_3}E'(t). \qquad (2.96)\end{aligned}$$

Proof. We have, by using the definition of η_1,

$$\begin{aligned}\left(\rho_2 - \frac{b_1\rho_1}{k}\right) N_8\int_0^1 \psi_t\varphi_{tx}dx &= -\left(\rho_2 - \frac{b_1\rho_1}{k}\right) N_8\int_0^1 \varphi_t\psi_{tx}dx \\ &= -\frac{1}{g_2^0}\left(\rho_2 - \frac{b_1\rho_1}{k}\right)\int_0^1 \varphi_t\int_0^{+\infty} g_1(s)\eta_{1tx}(s)dsdx \\ &\quad -\frac{1}{g_2^0}\left(\rho_2 - \frac{b_1\rho_1}{k}\right)\int_0^1 \varphi_t\int_0^{+\infty} g_1(s)\psi_{1tx}(t-s)dsdx.\end{aligned}$$

By using Young's inequality and (2.22), we obtain

$$\begin{aligned}\left|-\frac{1}{g_2^0}\left(\rho_2 - \frac{b_1\rho_1}{k}\right)\int_0^1 \varphi_t\int_0^{+\infty} g_1(s)\eta_{1tx}(s)dsdx\right| &\leq c\int_0^1 |\varphi_t|\int_0^{+\infty} g_1(s)\left|\eta_{1tx}(s)\right|dsdx \\ &\leq \frac{\varepsilon}{2}\int_0^1 E(t)dx + c\varepsilon\int_0^1\int_0^{+\infty} g_1(s)\eta_{1tx}^2(s)dsdx. \\ &\qquad (2.97)\end{aligned}$$

At the same time we have

$$\begin{aligned}\int_0^{+\infty} g_1(s)\psi_{1tx}(t-s)ds &= -\int_0^{+\infty} g_1(s)\psi_{sx}(t-s)ds = g(0)\psi_x(t) + \int_0^{+\infty} g_1'(s)\psi_x(t-s)ds \\ &= -\int_0^{+\infty} g_1'(s)\eta_{1x}(s)ds.\end{aligned}$$

So

$$\begin{aligned}&\left|-\frac{1}{g_2^0}\left(\rho_2-\frac{b_1\rho_1}{k}\right)\int_0^1 \varphi_t \int_0^{+\infty} g_1(s)\psi_{1tx}(t-s)dsdx\right| \\ &= \left|-\frac{1}{g_2^0}\left(\rho_2-\frac{b_1\rho_1}{k}\right)\int_0^1 \varphi_t \int_0^{+\infty} g_1'(s)\eta_{1x}(s)dsdx\right| \\ &\leq \frac{\varepsilon}{2}E(t) - \frac{2c_\varepsilon}{\beta_1}E'(t). \end{aligned} \tag{2.98}$$

Substituting these last two inequalities into the first equality, we obtain inequality (2.94).

We have, by using the definition of η_2,

$$\begin{aligned}\gamma N_7\int_0^1 (\varphi_x+\psi+l\omega)\,\theta_{1tx}dx &= \frac{\gamma N_7}{g_2^0}\int_0^1 (\varphi_x+\psi+l\omega)\int_0^{+\infty} g_2(s)\eta_{2tx}(s)dsdx \\ &\quad+\frac{\gamma N_7}{g_2^0}\int_0^1 (\varphi_x+\psi+l\omega)\int_0^{+\infty} g_2(s)\theta_{1tx}(t-s)dsdx.\end{aligned} \tag{2.99}$$

By using Young's inequality and (2.22), we obtain

$$\left|\frac{\gamma N_7}{g_2^0}\int_0^1 (\varphi_x+\psi+l\omega)\int_0^{+\infty} g_2(s)\eta_{2tx}(s)dsdx\right| \leq \frac{\varepsilon}{2}\int_0^1 E(t)dx + c\varepsilon\int_0^1\int_0^{+\infty} g_2(s)\eta_{2tx}^2(s)dsdx. \tag{2.100}$$

At the same time we have

$$\int_0^{+\infty} g_2(s)\theta_{1tx}(t-s)ds = -\int_0^{+\infty} g_2'(s)\eta_{2x}(s)ds.$$

So

$$\left|\frac{\gamma N_7}{g_2^0}\int_0^1 (\varphi_x+\psi+l\omega)\int_0^{+\infty} g_2(s)\theta_{1tx}(t-s)dsdx\right| \leq \frac{\varepsilon}{2}E(t) - \frac{2c_\varepsilon m_1}{\gamma\beta_2}E'(t). \tag{2.101}$$

By replacing inequalities (2.100) and (2.101) in equality (2.99), we obtain inequality (2.95).

Similarly we prove that

$$-\gamma N_8 \int_0^1 (\varphi_x + \psi + l\omega)\,\theta_{2tx} dx \leq c_\varepsilon \int\limits_0^1 \int\limits_0^{+\infty} g_3(s)\eta_{3tx}^2(s) ds dx + \varepsilon E(t) - \frac{2c_\varepsilon m_2}{\gamma \beta_3} E'(t).$$

■

2.4 Stability result

In this section we prove our stability result. First, we define a Lyapunov functional L by

$$L(t) = NE(t) + \sum_{i=1}^{10} N_i I_i(t), \tag{2.102}$$

for N_i, $i = 1, 2, ..., 10$ are positive constants to be properly chosen later.

Lemma 2.14 *For $N > 0$ large enough, there are two strictly positive constants δ_1 and δ_2 satisfying*

$$\delta_1 E(t) \leq L(t) \leq \delta_2 E(t), \qquad \forall t \geq 0. \tag{2.103}$$

In other words, the functions E and L are equivalent.

Proof. *We pose*

$$\pounds(t) = \sum_{i=1}^{10} N_i I_i(t),$$

so

$$|\pounds(t)| \leq \sum_{i=1}^{10} N_i \left| I_i(t) \right|.$$

Exploiting Young's, Cauchy-Schwartz inequalities, we obtain

$$
\begin{aligned}
&|\pounds(t)| \\
\leq\ & \frac{\rho_2 N_1}{2}\int_0^1 \psi_t^2 dx + \frac{\rho_2 N_1}{2}\int_0^1 p_x^2 dx + \frac{\rho_2 l^2 N_1}{2}\int_0^1 \psi_t^2 dx + \frac{\rho_1 l N_1}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1 N_1}{2}\int_0^1 \varphi_t^2 dx \\
&+\frac{\rho_2 l^2 N_1}{2}\int_0^1 \left(\int_0^x p(y)dy\right)^2 dx + \frac{\rho_1 l N_1}{2}\int_0^1 \left(\int_0^x p(y)dy\right)^2 dx + \frac{\rho_1 N_1}{2}\int_0^1 p^2 dx \\
&+\frac{\rho_2 N_2}{2}\int_0^1 \psi_t^2 dx + \frac{\rho_2 N_2}{2}\int_0^1 \left(\int_0^{+\infty} g_1(s)\eta_1(s)ds\right)^2 dx + \frac{\rho_3 N_3}{2}\int_0^1 \theta_{1t}^2 dx \\
&+\frac{\rho_3 N_3}{2}\int_0^1 \left(\int_0^{+\infty} g_2(s)\eta_2(s)ds\right)^2 dx + \frac{\rho_4 N_4}{2}\int_0^1 \theta_{2t}^2 dx + \frac{\rho_4 N_4}{2}\int_0^1 \left(\int_0^{+\infty} g_3(s)\eta_3(s)ds\right)^2 dx \\
&+\frac{\rho_1 N_5}{2}\int_0^1 \varphi^2 dx + \frac{\rho_1 N_5}{2}\int_0^1 \varphi_t^2 dx + \frac{\rho_2 N_5}{2}\int_0^1 \psi^2 dx + \frac{\rho_2 N_5}{2}\int_0^1 \psi_t^2 dx + \frac{\rho_1 N_5}{2}\int_0^1 \omega^2 dx \\
&+\frac{\rho_1 N_5}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1 N_6}{2}\int_0^1 (\omega_x - l\varphi)^2 dx + \frac{\rho_1 N_6}{2}\int_0^1 \left(\int_0^x \omega_t(y)dy\right)^2 dx + \frac{\rho_1 N_6}{2}\int_0^1 \varphi_t^2 dx \\
&+\frac{\rho_1 N_6}{2}\int_0^1 \left(\int_0^x (\varphi_x + \psi + l\omega)(y)dy\right)^2 d + \frac{\rho_1 N_7}{2}\int_0^1 \varphi_t^2 dx + \frac{\rho_1 N_7}{2}\int_0^1 (\omega_x - l\varphi)^2 dx \\
&+\frac{\rho_1 N_7}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1 N_7}{2}\int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \frac{\rho_2 N_8}{2}\int_0^1 (\varphi_x + \psi + l\omega)^2 dx \\
&+\frac{\rho_2 N_8}{2}\int_0^1 \psi_t^2 dx + \frac{\rho_1 b_1 N_8}{2k}\int_0^1 \varphi_t^2 dx + \frac{\rho_1 b_1 N_8}{2k}\int_0^1 \psi_x^2 dx + \frac{\rho_1 \beta_1 N_8}{2k}\int_0^1 \varphi_t^2 dx \\
&+\frac{\rho_1 \beta_1 N_8}{2k}\int_0^1 \left(\int_0^{+\infty} g_1(s)\psi_x(t-s)ds\right)^2 dx + \frac{\rho_3 N_9}{2}\int_0^1 \theta_1^2 dx + \frac{\rho_3 N_9}{2}\int_0^1 \theta_{1t}^2 dx \\
&+\frac{\rho_4 N_9}{2}\int_0^1 \theta_2^2 dx + \frac{\rho_4 N_9}{2}\int_0^1 \theta_{2t}^2 dx + \frac{\rho_2 \rho_4 N_{10}}{2}\int_0^1 \psi_x^2 dx + \frac{\rho_2 \rho_4 N_{10}}{2}\int_0^1 \left(\int_0^x \psi_t(y)dy\right)^2 dx,
\end{aligned}
$$

with (2.18), (1.10), (2.30) *and Poincaré's inequality, we see that*

$$
\int_0^1 \left(\int_0^x \omega_t(y)dy\right)^2 dx \leq \int_0^1 \omega_t^2 dx, \quad \int_0^1 \left(\int_0^x \psi_t(y)dy\right)^2 dx \leq \int_0^1 \psi_t^2 dx,
$$

$$\int_0^1 \left(\int_0^x (\varphi_x+\psi+l\omega)(y)dy\right)^2 dx \le \int_0^1 (\varphi_x+\psi+l\omega)^2\, dx,$$

$$\int_0^1 \left(\int_0^{+\infty} g_1(s)\psi_x(t-s)ds\right)^2 dx \le c\left[\int_0^1 \left(\int_0^{+\infty} g_1(s)\eta_{1x}(s)ds\right)^2 dx + \int_0^1 \psi_x^2 dx\right],$$

$$\int_0^1 \left(\int_0^x p(y)dy\right)^2 dx \le \int_0^1 p^2 dx \le \int_0^1 \psi_x^2 dx,$$

and

$$\begin{aligned}
\int_0^1 \varphi^2 dx &\le \int_0^1 \varphi_x^2 dx \le c\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx,\\
\int_0^1 \omega^2 dx &\le \int_0^1 \omega_x^2 dx \le c\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx,
\end{aligned}$$

so

$$\begin{aligned}
|\pounds(t)| \le\ & \frac{\rho_1}{2}\left[N_1+N_5+N_6+N_7+\rho_1\left(\frac{b}{k}+\frac{\beta_1}{k}\right)N_8\right]\int_0^1 \varphi_t^2 dx + \frac{\rho_3}{2}\left[N_3+N_9\right]\int_0^1 \theta_{1t}^2 dx\\
&+\frac{\rho_4}{2}\left[N_4+N_9\right]\int_0^1 \theta_{2t}^2 dx + \frac{\rho_2}{2}\left[N_1(1+l^2)+N_2+N_5+N_8+N_{10}\right]\int_0^1 \psi_t^2 dx\\
&+\frac{\rho_1}{2}\left[lN_1+N_6+N_7+N_8\right]\int_0^1 \omega_t^2 dx + \frac{N_9}{2}\int_0^1 \theta_{1x}^2 dx + \frac{N_9}{2}\int_0^1 \theta_{2x}^2 dx\\
&+\frac{1}{2}\left[2\rho_1 N_5+\rho_1 N_6+\rho_1 N_7+\rho_2 N_8\right]\int_0^1 (\varphi_x+\psi+l\omega)^2 dx + \frac{\rho_1}{2}\left[2cN_4+2N_5\right]\\
&\int_0^1 (\omega_x-l\varphi)^2 dx + \frac{\rho_2 N_2}{2}\int_0^1 \left(\int_0^{+\infty} g_1(s)\eta_1(s)ds\right)^2 dx + \frac{cN_3}{2}\int_0^1 \left(\int_0^{+\infty} g_2(s)\eta_2(s)ds\right)^2 dx\\
&+\frac{cN_4}{2}\int_0^1 \left(\int_0^{+\infty} g_3(s)\eta_3(s)ds\right)^2 dx + \frac{1}{2}\left[\left(\rho_2+\rho_2 l^2+\rho_1+\rho_1 l\right)N_1 + (2\rho_1+\rho_2)N_5\right.\\
&\left.+\rho_1\left(\frac{b}{k}+\frac{\beta_1}{k}\right)N_8+\rho_2 N_{10}\right]\int_0^1 \psi_x^2 dx\\
\le\ & ME(t),
\end{aligned}$$

2.4. Stability result

where

$$\begin{aligned} M &= \max\Big\{\rho_1\left(N_1+N_5+N_6+N_7+\rho_1\left(\frac{b}{k}+\frac{\beta_1}{k}\right)N_8\right), \rho_3\left(N_3+N_9\right), \rho_4\left(N_4+N_9\right), \\ &\quad \rho_1\left(lN_1+N_6+N_7+N_8\right), \rho_2\left(N_1(1+l^2)+N_2+N_5+N_8+N_{10}\right), , N_3, N_4, N_9, \\ &\quad 2\rho_1N_5+\rho_1N_6+\rho_1N_7+\rho_2N_8, \rho_2N_2, \rho_1\left(2cN_4+2N_5\right), \\ &\quad \left(\left(\rho_2+\rho_2l^2+\rho_1+\rho_1l\right)N_1+2\rho_1N_5+\rho_2N_5+\rho_1\left(\frac{b}{k}+\frac{\beta_1}{k}\right)N_8+\rho_2N_{10}\right)\Big\}. \end{aligned}$$

Consequently$|L(t)-NE(t)|\leq ME(t)$,*which yields*

$$(N-M)E(t)\leq L(t)\leq (M+N)E(t),$$

choosing N *large enough, such that* $N-M>0$. ■

Theorem 2.2 *Let* $U_0\in H$. *Assume that* $(H1)$ *and* $(H2)$ *hold and that, for some* $m_0>0$

$$\max\int_0^1\left\{\psi_{0x}^2(x,s), \psi_{0xs}^2(x,s), \theta_{10x}^2(x,s), \theta_{10xs}^2(x,s), \theta_{20x}^2(x,s), \theta_{20xs}^2(x,s)\right\}dx\leq m_0,\ \ s>0, \tag{2.104}$$

there exists a constant $\delta_0>0$, *such that, for all* $t\in\mathbb{R}^+$,

$$E(t)\leq\frac{\delta_0\left(1+\int_0^t\zeta(s)\int_s^{+\infty}\left(g_1(\tau)+g_2(\tau)+g_3(\tau)\right)d\tau ds\right)}{\int_0^t\zeta(s)ds}, \tag{2.105}$$

in the cases of $\frac{k}{\rho_1}=\frac{b_1}{\rho_2}$ *and* $\frac{k}{\rho_1}\neq\frac{b_1}{\rho_2}$ *respectively.*

Proof. *We have*

$$
\begin{aligned}
& L^{'}(t) \\
\leq\; & -\left[\rho_2(g_1^0N_2+N_5+\frac{m_2}{2}N_{11})-C\left(\varepsilon_1\right)N_1-\frac{\rho_1}{4l}N_6-\frac{\rho_1}{2l}N_7-\rho_2(1+\frac{l}{4\varepsilon_3})N_8-\frac{m_2^2}{4l}N_9\right] \\
& \int_0^1\psi_t^2dx-\left[\lambda_1\left(1-3l^2\right)N_1+\rho_2g_1^0N_2+\left(\lambda_1-\varepsilon_4(k+\beta_1+\gamma)\right)N_{10}-\left(\lambda_1+2l+2cl\right)N_5\right. \\
& \left.-\frac{\lambda_1^2l}{2\varepsilon_3}\left(1+\gamma^2\right)N_8\right]\int_0^1\psi_x^2dx-\left[kN_8-(k+2cl)N_5-(k+l^2\varepsilon_2)N_6-klN_7-\frac{k}{4\varepsilon_4}N_{10}\right] \\
& \int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx-\left[(k_0-\varepsilon_2)N_6+l(k_0-1)N_7-(k_0+2cl)N_5-l\varepsilon_3N_8\right] \\
& \int_0^1\left(\omega_x-l\varphi\right)^2dx-\left[\rho_1N_5+\rho_1(1-l)N_6-\varepsilon_1N_1-\rho_1lN_7-\varepsilon_3N_8-\frac{lm_1^2}{4}N_9\right]\int_0^1\varphi_t^2dx \\
& -\left[\rho_1\left(N_5+\frac{l}{2}N_7\right)-\varepsilon_1N_1-\rho_1N_6-\rho_2l\varepsilon_3N_8-\frac{m_1^2}{4l}N_9\right]\int_0^1\omega_t^2dx \\
& -\left[\rho_3g_2^0N_3-\frac{\gamma^2(1+l^2)}{4l}N_5-\frac{\gamma^2}{4\varepsilon_2}N_6-\frac{l\gamma^2}{4}N_7-\frac{l\varepsilon_3}{2k^2}N_8-\rho_3N_9\right]\int_0^1\theta_{1t}^2dx \\
& -\left[\rho_4g_3^0N_4-\frac{\gamma^2}{2\lambda_1l^2}(1+l^4)N_1-\frac{\gamma^2}{4l}N_5-\rho_4N_9-\frac{\gamma}{4\varepsilon_4}N_{10}\right]\int_0^1\theta_{2t}^2dx-\left[\left(\lambda_2-3l\right)N_9\right] \\
& \int_0^1\theta_{1x}^2dx-\left[\left(\lambda_3-2l\right)N_9\right]\int_0^1\theta_{2x}^2dx+\left[\frac{\beta_1^2}{2l^2}\left(1+l^4\right)N_1+C_1(\delta)N_2+\frac{\beta_1^2}{4l}N_5+\frac{l\beta_1^2}{\varepsilon_3}N_8\right. \\
& \left.+\frac{\beta_1}{4\varepsilon_4}N_{10}\right]g_1^0\int_0^1\int_0^{+\infty}g_1(s)\eta_{1x}^2(s)dsdx+\left[C_2(\delta)N_3+\frac{\beta_2^2}{4l}N_9\right]g_2^0\int_0^1\int_0^{+\infty}g_2(s)\eta_{2x}^2(s)dsdx \\
& +\left[C_3(\delta)N_4+\frac{\beta_3^2}{4l}N_9\right]g_3^0\int_0^1\int_0^{+\infty}g_3(s)\eta_{3x}^2(s)dsdx+\left[\frac{\beta_1N}{2}-\frac{\rho_1^2\beta_1^2}{4k\varepsilon_3}g_1(0)N_8-\frac{\rho_2g_1(0)N_2}{4\delta}\right] \\
& \int_0^1\int_0^{+\infty}g_1'(s)\eta_{1x}^2(s)dsdx+\left[\frac{\gamma\beta_2N}{2m_1}-\frac{\rho_3g_2(0)N_3}{4\delta}\right]\int_0^1\int_0^{+\infty}g_2'(s)\eta_{2x}^2(s)dsdx \\
& +\left[\frac{\gamma\beta_3N}{2m_2}-\frac{\rho_4g_3(0)N_4}{4\delta}\right]\int_0^1\int_0^{+\infty}g_3'(s)\eta_{3x}^2(s)dsdx+\delta\int_0^1\left[(\rho_2N_2+N_4)\psi_t^2+\rho_2N_2\psi_x^2+lN_3\varphi_t^2\right. \\
& \left.+N_3\omega_t^2+\rho_3N_3\theta_{1t}^2+\left(N_2+\rho_4N_4\right)\theta_{2t}^2+N_2\left(\varphi_x+\psi+l\omega\right)^2+N_3\theta_{1x}^2+N_4\theta_{2x}^2\right]dx \\
& +\gamma N_7\int_0^1\theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx-\gamma N_8\int_0^1\theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx+\left(\rho_2-\frac{b_1\rho_1}{k}\right)N_8\int_0^1\psi_t\varphi_{tx}dx.
\end{aligned}
$$

2.4. Stability result

A combination of the estimates of the previous lemmas, and by setting $N_9 = 1,\ N_5 = N_6$, $N_7 = \frac{1}{l} N_5,\ N_{10} = (1-l)\, N_5$ *and* $\varepsilon_4 = l N_{10}$ *we arrive at*

$$
\begin{aligned}
L^{'}(t) \ \leq\ & -\left[\rho_2 g_1^0 N_2 + \left(l\rho_2 - \frac{\rho_1}{4l} - \frac{\rho_1}{2l^2}\right) N_5 - C\left(\varepsilon_1\right) N_1 - \rho_2 (1 + \frac{l}{4\varepsilon_3}) N_8 - \frac{m_2^2}{4l}\right] \\
& \int_0^1 \psi_t^2 dx - \left[\lambda_1 \left(1 - 3l^2\right) N_1 + \rho_2 g_1^0 N_2 - \frac{\lambda_1^2 l}{2\varepsilon_3}\left(1+\gamma^2\right) N_8 - \right. \\
& \left. l\left(\lambda_1 + (1-l)^2 (k + \beta_1 + \gamma) N_5 + 2\left(1 + c\right)\right) N_5\right] \int_0^1 \psi_x^2 dx \\
& - \left[k N_8 - (3k + 2cl + l^2 \varepsilon_2) N_5 - \frac{k}{4l}\right] \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx \\
& - \left[(k_0 - \varepsilon_2 - 2cl - 1) N_5 - l\varepsilon_3 N_8\right] \int_0^1 \left(\omega_x - l\varphi\right)^2 dx \\
& - \left[\rho_1 (1-l) N_5 - \varepsilon_1 N_1 - \varepsilon_3 N_8 - \frac{l m_1^2}{4}\right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{\rho_1}{2} N_5 - \varepsilon_1 N_1 - \rho_2 l \varepsilon_3 N_8 - \frac{m_1^2}{4}\right] \int_0^1 \omega_t^2 dx \\
& - \left[\rho_3 g_2^0 N_3 - \frac{\gamma^2}{4}\left(\frac{1 + l^2}{l} + \frac{1}{\varepsilon_2} + 1\right) N_5 - \frac{l\varepsilon_3}{2k^2} N_8 - \rho_3\right] \int_0^1 \theta_{1t}^2 dx \\
& - \left[\rho_4 g_3^0 N_4 - \frac{\gamma^2}{2\lambda_1 l^2} (1 + l^4) N_1 - \frac{\gamma^2}{4l} N_5 - \frac{\gamma}{4l} - \rho_4\right] \int_0^1 \theta_{2t}^2 dx \\
& - \left[(\lambda_2 - 3l) N_9\right] \int_0^1 \theta_{1x}^2 dx - \left[(\lambda_3 - 2l) N_9\right] \int_0^1 \theta_{2x}^2 dx \\
& + \left[\frac{\beta_1^2}{2l^2}\left(1 + l^4\right) N_1 + C_1(\delta) N_2 + \frac{\beta_1^2}{4l} N_5 + \frac{l\beta_1^2}{\varepsilon_3} N_8 + \frac{\beta_1}{4\varepsilon_4} N_{10}\right] \\
& g_1^0 \int\limits_0^1 \int\limits_0^{+\infty} g_1(s) \eta_{1x}^2(s) ds dx + \left[C_2(\delta) N_3 + \frac{\beta_2^2}{4l} N_9\right] g_2^0 \int\limits_0^1 \int\limits_0^{+\infty} g_2(s) \eta_{2x}^2(s) ds dx \\
& + \left[C_3(\delta) N_4 + \frac{\beta_3^2}{4l} N_9\right] g_3^0 \int\limits_0^1 \int\limits_0^{+\infty} g_3(s) \eta_{3x}^2(s) ds dx + \delta C_{N_2, N_3, N_4, l} E(t) \\
& + \left[\frac{\beta_1 N}{2} - \frac{\rho_1^2 \beta_1^2}{4k\varepsilon_3} g_1(0) N_8 - \frac{\rho_2 g_1(0) N_2}{4\delta}\right] \int\limits_0^1 \int\limits_0^{+\infty} g_1'(s) \eta_{1x}^2(s) ds dx \\
& + \left[\frac{\gamma\beta_2 N}{2m_1} - \frac{\rho_3 g_2(0) N_3}{4\delta}\right] \int\limits_0^1 \int\limits_0^{+\infty} g_2'(s) \eta_{2x}^2(s) ds dx \\
& + \left[\frac{\gamma\beta_3 N}{2m_2} - \frac{\rho_4 g_3(0) N_4}{4\delta}\right] \int\limits_0^1 \int\limits_0^{+\infty} g_3'(s) \eta_{3x}^2(s) ds dx \\
& + \gamma N_7 \int\limits_0^1 \theta_{1tx} \left(\varphi_x + \psi + l\omega\right) dx - \gamma N_8 \int\limits_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx \\
& + \left(\rho_2 - \frac{b_1 \rho_1}{k}\right) N_8 \int\limits_0^1 \psi_t \varphi_{tx} dx.
\end{aligned}
$$

2.4. Stability result

Now, let us take $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_5}$, $\varepsilon_3 = \frac{1}{N_8}$, *to get*

$$\begin{aligned}
L^{'}(t) \leq\; & -\left[\rho_2 g_1^0 N_2 + \left(l\rho_2 - \frac{\rho_1}{4l} - \frac{\rho_1}{2l^2}\right) N_5 - C\left(\varepsilon_1\right) N_1 - \rho_2 (1 + \frac{lN_8}{4}) N_8 - \frac{m_2^2}{4l}\right] \\
& \int_0^1 \psi_t^2 dx - \left[\lambda_1 \left(1 - 3l^2\right) N_1 + \rho_2 g_1^0 N_2 - \frac{\lambda_1^2 l}{2}\left(1+\gamma^2\right) N_8^2 \right. \\
& \left. - l\left(\lambda_1 + (1-l)^2 (k + \beta_1 + \gamma) N_5 + 2\left(1+c\right)\right) N_5\right] \int_0^1 \psi_x^2 dx \\
& - \left[kN_8 - (3k + 2cl)N_5 - l^2 - \frac{k}{4l}\right] \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx \\
& - \left[(k_0 - 2cl - 1)N_5 - l - 1\right] \int_0^1 \left(\omega_x - l\varphi\right)^2 dx \\
& - \left[\rho_1 (1-l) N_5 - 2 - \frac{lm_1^2}{4}\right] \int_0^1 \varphi_t^2 dx - \left[\frac{\rho_1}{2} N_5 - 1 - \rho_2 l - \frac{m_1^2}{4}\right] \int_0^1 \omega_t^2 dx \\
& - \left[\rho_3 g_2^0 N_3 - \frac{\gamma^2}{4}\left(\frac{1+l^2}{l} + N_5 + 1\right) N_5 - \frac{l}{2k^2} - \rho_3\right] \int_0^1 \theta_{1t}^2 dx \\
& - \left[\rho_4 g_3^0 N_4 - \frac{\gamma^2}{2\lambda_1 l^2}(1 + l^4) N_1 - \frac{\gamma^2}{4l} N_5 - \frac{\gamma}{4l} - \rho_4\right] \int_0^1 \theta_{2t}^2 dx \\
& - \left[(\lambda_2 - 3l)N_9\right] \int_0^1 \theta_{1x}^2 dx - \left[(\lambda_3 - 2l)N_9\right] \int_0^1 \theta_{2x}^2 dx + \alpha_1 \int_0^1 \int_0^{+\infty} g_1(s) \eta_{1x}^2(s) ds dx \\
& + \alpha_2 \int_0^1 \int_0^{+\infty} g_2(s) \eta_{2x}^2(s) ds dx + \alpha_3 \int_0^1 \int_0^{+\infty} g_3(s) \eta_{3x}^2(s) ds dx + \delta C_{N_2,N_3,N_4,l} E(t) \\
& + \left[\frac{\beta_1 N}{2} - \frac{\rho_1^2 \beta_1^2}{4k\varepsilon_3} g_1(0) N_8 - \frac{\rho_2 g_1(0) N_2}{4\delta}\right] \int_0^1 \int_0^{+\infty} g_1'(s) \eta_{1x}^2(s) ds dx \\
& + \left[\frac{\gamma\beta_2 N}{2m_1} - \frac{\rho_3 g_2(0) N_3}{4\delta}\right] \int_0^1 \int_0^{+\infty} g_2'(s) \eta_{2x}^2(s) ds dx + \left[\frac{\gamma\beta_3 N}{2m_2} - \frac{\rho_4 g_3(0) N_4}{4\delta}\right] \\
& \int_0^1 \int_0^{+\infty} g_3'(s) \eta_{3x}^2(s) ds dx + \gamma N_7 \int_0^1 \theta_{1tx}\left(\varphi_x + \psi + l\omega\right) dx - \gamma N_8 \int_0^1 \theta_{2tx}\left(\varphi_x + \psi + l\omega\right) dx \\
& + \left(\rho_2 - \frac{b_1 \rho_1}{k}\right) N_8 \int_0^1 \psi_t \varphi_{tx} dx.
\end{aligned} \tag{2.106}$$

where α_1, α_2 *and* α_3 *are positive constants.*

First, we choose $l < \min\left\{1, \frac{1}{\sqrt{3}}, \frac{k_0 - 1}{2c}, \frac{\lambda_2}{3}, \frac{\lambda_3}{2}\right\}$.

Next, we select N_5 *large enough such that*

$$N_5 > \max\left\{\frac{l+\gamma}{k_0 - 2cl}, \frac{1+l}{\rho_1\left(1-l\right)}, \frac{4 + 2l\rho_2}{\rho_1}\right\}.$$

then, we choose N_1, N_4, N_2 *sufficiently large so that*

$$N_4 > \frac{\gamma^2 (1 + l^4)}{2\lambda_1 l^2 \rho_4 g_3^0} N_1 + \frac{\gamma^2}{4l\rho_4 g_3^0} N_5 + \frac{\gamma}{4l\rho_4 g_3^0} + \frac{1}{g_3^0}.$$

2.4. Stability result

$$\lambda_1\left(1-3l^2\right)N_1>-\rho_2 g_1^0 N_2+l\left(\lambda_1+(1-l)^2(k+\beta_1+\gamma)N_5+2(1+c)\right)N_5+\frac{\lambda_1^2 l}{2}\left(1+\gamma^2\right)N_8^2.$$

$$\rho_2 g_1^0 N_2>\left(\frac{\rho_1}{2l^2}+\frac{\rho_1}{4l}-l\rho_2\right)N_5+C\left(N_1\right)N_1+\rho_2(1+\frac{lN_8}{4})N_8+\frac{m_2^2}{4l}.$$

After that, we pick N_3, N_8 *very large so that*

$$N_3>\frac{\gamma^2}{4\rho_3 g_2^0}\left(\frac{1+l^2}{l}+N_5+1\right)N_5+\frac{l}{2k^2\rho_3 g_2^0}+\frac{1}{g_2^0}.$$

$$N_8>(3+\frac{2cl}{k})N_5+\frac{l^2}{k}+\frac{1}{4l}.$$

Finally, we choose N *sufficiently large so that* (2.103) *remains valid and*

$$\frac{\beta_1 N}{2}-\frac{\rho_1^2\beta_1^2}{4k\varepsilon_3}g_1(0)N_8-\frac{\rho_2 g_1(0)N_2}{4\delta}>0.$$

$$\frac{\gamma\beta_2 N}{2m_1}-\frac{\rho_3 g_2(0)N_3}{4\delta}>0.$$

$$\frac{\gamma\beta_3 N}{2m_2}-\frac{\rho_4 g_3(0)N_4}{4\delta}>0.$$

Therefor, (2.106) *takes the form*

$$\begin{aligned}L^{'}(t) \;\leq\; & -(\lambda_0-\delta C_{N_2,N_3,N_4,l})E(t)+\alpha_1\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx+\alpha_2\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx\\ & +\alpha_3\int_0^1\int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx+\gamma N_7\int_0^1\theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx\\ & -\gamma N_8\int_0^1\theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx+\left(\rho_2-\frac{b_1\rho_1}{k}\right)N_8\int_0^1\psi_t\varphi_{tx}dx, \qquad (2.107)\end{aligned}$$

for some positive constant λ_0. *Using* (2.5) *and* (2.23), *we obtain that for all* $t\in\mathbb{R}^+$.

$$\begin{aligned}\zeta(t)\int_0^1\int_0^t g_1(s)\eta_{1x}^2(s)dsdx \;&\leq\; \int_0^1\int_0^t\zeta(s)g_1(s)\eta_{1x}^2(s)dsdx\leq-\int_0^1\int_0^t g_1'(s)\eta_{1x}^2(s)dsdx\\ &\leq\; -\int_0^1\int_0^{+\infty} g_1'(s)\eta_{1x}^2(s)dsdx\leq-\frac{2}{\beta_1}E'(t).\end{aligned}$$

On the other hand, using (2.22) *and the fact that* $E(t)$ *is nonincreasing, we ask for* $t,s\in\mathbb{R}^+$,

$$\begin{aligned}\int_0^1\eta_{1x}^2(s)dx \;&=\; \int_0^1\left(\psi_x(x,t)-\psi_x(x,t-s)\right)^2dx\leq 2\int_0^1\psi_x^2(x,t)dx+2\int_0^1\psi_x^2(x,t-s)dx\\ &\leq\; \frac{8}{\lambda_1}E(0)+2m_{0.}\end{aligned}$$

Then, we obtain

$$\zeta(t)\int_0^1\int_t^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx \leq \left(\frac{8}{\lambda_1}E(0)+2m_0\right)\zeta(t)\int_t^{+\infty} g_1(s)ds.$$

Then, we deduce that, for all $t \in \mathbb{R}^+$,

$$\zeta(t)\int_0^1\int_0^{+\infty} g_1(s)\eta_{1x}^2(s)dsdx \leq -\frac{2}{\beta_1}E'(t)+\left(\frac{8}{\lambda_1}E(0)+2m_0\right)\zeta(t)\int_t^{+\infty} g_1(s)ds. \tag{2.108}$$

Similarly, we have

$$\zeta(t)\int_0^1\int_0^{+\infty} g_2(s)\eta_{2x}^2(s)dsdx \leq -\frac{2m_1}{\gamma\beta_2}E'(t)+\left(\frac{8m_1}{\gamma\lambda_2}E(0)+2m_0\right)\zeta(t)\int_t^{+\infty} g_2(s)ds. \tag{2.109}$$

$$\zeta(t)\int_0^1\int_0^{+\infty} g_3(s)\eta_{3x}^2(s)dsdx \leq -\frac{2m_2}{\gamma\beta_3}E'(t)+\left(\frac{8m_2}{\gamma\lambda_3}E(0)+2m_0\right)\zeta(t)\int_t^{+\infty} g_3(s)ds. \tag{2.110}$$

At this point, multiplying (2.107) *by* $\zeta(t)$ *and we take* $\delta < \frac{\lambda_0}{C}.$ *, we obtain*

$$\begin{aligned}
&\zeta(t)L'(t)+\left(\frac{2\alpha_1}{\beta_1}+\frac{2m_1\alpha_2}{\gamma\beta_2}+\frac{2m_2\alpha_3}{\gamma\beta_3}\right)E'(t)\\
\leq\;& -c_1\zeta(t)E(t)+\alpha_1\left(\frac{8}{\lambda_1}E(0)+2m_0\right)r_1(t)+\alpha_2\left(\frac{8m_1}{\gamma\lambda_2}E(0)+2m_0\right)r_2(t)\\
&+\alpha_3\left(\frac{8m_2}{\gamma\lambda_3}E(0)+2m_0\right)r_3(t)+\gamma N_7\zeta(t)\int_0^1\theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx\\
&-\gamma N_8\zeta(t)\int_0^1\theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx+\left(\rho_2-\frac{b_1\rho_1}{k}\right)N_8\zeta(t)\int_0^1\psi_t\varphi_{tx}dx,
\end{aligned} \tag{2.111}$$

for some $c_1=\lambda_0-\delta C>0$ *and* $r_i(t)=\zeta(t)\int_t^{+\infty} g_i(s)ds$ *for* $i=1,2,3.$

It is clear that

$$L_1(t)=\zeta(t)L(t)+c_2E(t)\sim E(t),$$

that is, exist positive constants δ_1 *and* δ_2*,such that*

$$\delta_1E(t)\leq L_1(t)\leq \delta_2E(t).$$

So

$$\begin{aligned}
L_1'(t)\;\leq\;& -c_3\zeta(t)E(t)+\alpha_1\gamma_1r_1(t)+\alpha_2\gamma_2r_2(t)+\alpha_3\gamma_3r_3(t)\\
&+\gamma N_7\zeta(t)\int_0^1\theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx-\gamma N_8\zeta(t)\int_0^1\theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx\\
&+\left(\rho_2-\frac{b_1\rho_1}{k}\right)N_8\zeta(t)\int_0^1\psi_t\varphi_{tx}dx,
\end{aligned}$$

where $c_3 = \frac{c_1}{\delta_2}, \gamma_1 = \frac{8}{\lambda_1} E(0) + 2m_0, \gamma_2 = \frac{8m_1}{\gamma\lambda_2} E(0) + 2m_0$ *and* $\gamma_3 = \frac{8m_2}{\gamma\lambda_3} E(0) + 2m_0$.

By using (2.95) *and* (2.96)*, we get*

$$\begin{aligned} L_1^{'}(t) &\leq -c_3\zeta(t)E(t) + \alpha_1\gamma_1 r_1(t) + \alpha_2\gamma_2 r_2(t) + \alpha_3\gamma_3 r_3(t) + 2\varepsilon\zeta(t)E(t) \\ &\quad + c_\varepsilon \zeta(t) \int_0^1 \left[\int_0^{+\infty} g_2(s)\eta_{2tx}^2(s)ds + \int_0^{+\infty} g_3(s)\eta_{3tx}^2(s)ds \right] dx \\ &\quad - \left(\frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \zeta(t)E^{'}(t) + \left(\rho_2 - \frac{b_1\rho_1}{k} \right) N_8 \zeta(t) \int_0^1 \psi_t \varphi_{tx} dx, \end{aligned}$$

we have

$$\begin{aligned} \zeta(t)\int_0^1 \int_0^{+\infty} g_2(s)\eta_{2tx}^2(s)dsdx &\leq -\frac{2m_1}{\gamma\beta_2}\tilde{E}^{'}(t) + \left(\frac{8m_1}{\gamma\lambda_2}\tilde{E}(0) + 2m_0 \right) r_2(t) \\ \zeta(t)\int_0^1 \int_0^{+\infty} g_3(s)\eta_{3tx}^2(s)dsdx &\leq -\frac{2m_2}{\gamma\beta_3}\tilde{E}^{'}(t) + \left(\frac{8m_2}{\gamma\lambda_3}\tilde{E}(0) + 2m_0 \right) r_2(t) \end{aligned}$$

So

$$\begin{aligned} &\frac{d}{dt}\left[L_1(t) + \left(\frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \left(\tilde{E}(t) + \zeta(t)E(t) \right) \right] \\ \leq\ & -(c_3 - 2\varepsilon)\zeta(t)E(t) + \alpha_1\gamma_1 r_1(t) + (\alpha_2 + c_\varepsilon)\gamma_2 r_2(t) + (\alpha_3 + c_\varepsilon)\gamma_3 r_3(t) \\ & + \left(\rho_2 - \frac{b_1\rho_1}{k} \right) N_8 \zeta(t) \int_0^1 \psi_t \varphi_{tx} dx + \left(\frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \zeta^{'}(t)E(t). \end{aligned} \tag{2.112}$$

Because is nonincreasing, the last term of (2.112) *is nonpositive, therefore, by integration on* $[0, T]$ *and using the fact* $E(t)$ *is nonincreasing, we obtain*

$$\begin{aligned} &(c_3 - 2\varepsilon) E(T) \int_0^T \zeta(t)dt \leq L_1(0) + \left(\frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \left(\tilde{E}(0) + \zeta(0)E(0) \right) \\ &+\alpha_1\gamma_1 \int_0^T r_1(t)dt + (\alpha_2 + c_\varepsilon)\gamma_2 \int_0^T r_2(t)dt + (\alpha_3 + c_\varepsilon)\gamma_3 \int_0^T r_3(t)dt \\ &+ \left(\rho_2 - \frac{b_1\rho_1}{k} \right) N_8 \zeta(t) \int_0^1 \psi_t \varphi_{tx} dx. \end{aligned} \tag{2.113}$$

1. *If* $\frac{k}{\rho_1} = \frac{b_1}{\rho_2}$ *holds, relation* (2.113) *gives* (2.105) *with*

$$\begin{aligned} \delta_0 &= \frac{1}{c_3 - 2\varepsilon} \max \left\{ L_1(0) + \left(\frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \left(\tilde{E}(0) + \zeta(0)E(0) \right), \right. \\ &\quad \left. \alpha_1\gamma_1, (\alpha_2 + c_\varepsilon)\gamma_2, (\alpha_3 + c_\varepsilon)\gamma_3 \right\}. \end{aligned}$$

2. *If* $\frac{k}{\rho_1} \neq \frac{b_1}{\rho_2}$*,the relation* (2.113) *becomes as follows*

$$\begin{aligned} &(c_3 - 3\varepsilon) E(T) \int_0^T \zeta(t)dt \leq L_1(0) + \left(\frac{2c_\varepsilon}{\beta_1} + \frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3} \right) \left(\tilde{E}(0) + \zeta(0)E(0) \right) \\ &+(\alpha_1 + c_\varepsilon)\gamma_1 \int_0^T r_1(t)dt + (\alpha_2 + c_\varepsilon)\gamma_2 \int_0^T r_2(t)dt + (\alpha_3 + c_\varepsilon)\gamma_3 \int_0^T r_3(t)dt, \end{aligned} \tag{2.114}$$

by using (2.94) *and*

$$\zeta(t)\int_0^1\int_0^{+\infty} g_1(s)\eta_{1tx}^2(s)dsdx \leq -\frac{2}{\beta_1}\tilde{E}'(t) + \left(\frac{8}{\lambda_1}\tilde{E}(0) + 2m_0\right) r_1(t),$$

therefore the relation (2.114) *gives* (2.105) *with*

$$\begin{aligned}\delta_0 \quad = \quad & \frac{1}{c_3 - 3\varepsilon}\max\left\{L_1(0) + \left(\frac{2c_\varepsilon}{\beta_1} + \frac{2c_\varepsilon m_1}{\gamma\beta_2} + \frac{2c_\varepsilon m_2}{\gamma\beta_3}\right)\left(\tilde{E}(0) + \zeta(0)E(0)\right),\right.\\ & \left.(\alpha_1 + c_\varepsilon)\gamma_1, (\alpha_2 + c_\varepsilon)\gamma_2, (\alpha_3 + c_\varepsilon)\gamma_3\right\}.\end{aligned}$$

This completes the proof. ■

2.4. Stability result

CHAPTER 3

Energy decay of the Bresse system by two thermo-viscoelastic dampings

3.1 Introduction

In the present chapter, we consider the following thermo-viscoelastic Bresse system of type III [10]

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + l\gamma\theta_{1t} = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_{2tx} = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_{1tx} = 0, \\ \theta_{1tt} - k_1\theta_{1xx} + \int\limits_0^t g\left(t-s\right)\theta_{1xx}\left(x,s\right)ds + m_1\left(\omega_x - l\varphi\right)_t = 0, \\ \theta_{2tt} - k_2\theta_{2xx} + \int\limits_0^t g\left(t-s\right)\theta_{2xx}\left(x,s\right)ds + m_2\psi_{xt} = 0, \end{cases} \tag{3.1}$$

with the following initial data conditions

$$\left(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_{1t}, \theta_2, \theta_{2t}\right)(x,0) = \left(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{11}, \theta_{20}, \theta_{21}\right), \tag{3.2}$$

and the homogeneous Dirichlet boundary conditions

$$\omega(x,t) = \varphi(x,t) = \psi(x,t) = \theta_1(x,t) = \theta_2(x,t) = 0, \text{ for } x = 0, 1, \tag{3.3}$$

where $t \in (0, +\infty)$ represents the time variable and $x \in (0,1)$ is the space variable. Here φ, ω and ψ are, respectively, the vertical, longitudinal and shear angle displacements of the beam and θ_1, θ_2 represent the temperature difference. g is a positive function satisfying some conditions to be specified later, and the coeffitions $\rho_1, \rho_2, k, k_0, k_1, k_2, l, b, \gamma, m_1, m_2$ are positive constants.

The isothermal system [6] which was first considered by Bresse in 1859, is more general than the well-known system of Timoshenko where the longitudinal displacement ω

is not considered ($l = 0$).There are many publications concerning the stabilization of Timoshenko system with different kinds of damping to be added to the equations or to the boundary (see [12, 15, 24, 26, 36, 37].).

Ammar-Khodja et al. [2] concerned with the following system of Timoshenko type with memory:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, & in\ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + g*\psi_{xx} + k\left(\varphi_x + \psi\right) = 0, & in\ (0,L)\times(0,+\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & for\quad t\geq 0, \\ \varphi(.,0) = \varphi_0, \varphi_t(.,0) = \varphi_1, \psi(.,0) = \psi_0, \psi_t(.,0) = \psi_1, & in\ (0,L), \end{cases}$$

where $g*\psi_{xx}(x,t) = \int\limits_0^t g\left(t-s\right)\psi_{xx}\left(x,s\right)ds$ represents the memory effect with a real-valued C^2-function g. Also, Guesmia and Messaoudi also, considered, in [23] the same previous system and established the same resulta of stability of [2] with weaker conditions on g. Although they used the same method and adopted almost all the multipliers used in [2], the use of a functional similar to the one in [8, 9] made the difference and played an essential role in weakening the requirements on g.

As for the Timoshenko type III thermoelasticity system, there are some publications that have been interested in the study of the stabilization of this type of systems, among these publications we can mention [11] where Djebabla and Tatar studied the following system

$$\begin{cases} \rho_1\phi_{tt} - k\left(\phi_x + \psi\right)_x = 0, & in\ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\phi_x + \psi\right) + \gamma\theta_x = 0, & in\ (0,L)\times(0,+\infty), \\ \rho_3\theta_{tt} - l\theta_{xx} + \beta\int\limits_0^t g\left(t-s\right)\theta_{xx}\left(x,s\right)ds + \gamma\psi_{ttx} = 0, & in\ (0,L)\times(0,+\infty), \end{cases}$$

and they proved, under Dirichlet-Dirichlet-Neumann boundary conditions, suitable conditions on its coefficients and for g decaying exponentially, that the energy function also decays exponentially.

Kafini [29] improved the result of Djebabla and Tatar [11] with more general relaxation functions. He proved, under the same conditions on the coefficients, a general decay result, from which the usual exponential and polynomial decays are only special cases.

Our objective in this chapter is to study the effect of type III thermoelasticity on the Bresse system with the presence of two memory type dampers. More precisely, we prove the decay of solutions and establish the energy decay rate when

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}, k = k_0. \tag{3.4}$$

In the next section, we present some materials needed in the proof of our results and in the second we present the statement and the proof of the result with the different functionals by which we modify the classical energy to obtain an equivalent.

3.2 Preliminaries

In this section, we present the mathematical bases to be used later for the proof of our stability result. We will use the following assumptions related to the function g

$(G1)$ $g:\mathbb{R}_+\to\mathbb{R}_+$ is a differentiable function such that

$$g(0)>0,\qquad k_1-\int_0^\infty g(s)ds=\lambda_1>0, and\quad k_2-\int_0^\infty g(s)ds=\lambda_2>0. \tag{3.5}$$

$(G2)$ There exists a non-increasing differentiable function $\zeta:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$g'(t)\leq-\zeta(t)g(t),\qquad t\geq 0\quad\text{and}\quad\int_0^\infty\zeta(t)dt=+\infty. \tag{3.6}$$

For completeness, we state, without proof, the following global existence and uniqueness result which can be proved by using the Faedo-Galerkin method.

Theorem 3.1 *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1),(\omega_0,\omega_1),(\theta_{10},\theta_{11}),(\theta_{20},\theta_{21})\in H_0^1(0,1)\times L^2(0,1)$. *Assume that* $(G1)$ *and* $(G2)$ *are satisfied. Then, the problem* (3.1)-(3.3) *has a unique weak solution* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *of such that*

$$\varphi,\psi,\omega,\theta_1,\theta_2\in C\left(\mathbb{R}_+;H_0^1(0,1)\right)\cap C^1\left(\mathbb{R}_+;L^2(0,1)\right).$$

3.3 Technical Lemmas

In this section, we start with establish several lemmas needed for our work then proof our main result.

Let us first prove that the energy function E is decreasing, we have

Lemma 3.1 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be the solution of* (3.1)-(3.3). *Then the energy functional, defined by*

$$\begin{aligned}E(t)\;=\;&\frac{1}{2}\int_0^1\Big[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\frac{\gamma}{m_1}\theta_{1t}^2+\frac{\gamma}{m_2}\theta_{2t}^2+b\psi_x^2\\&+k\left(\varphi_x+\psi+l\omega\right)^2+k_0\left(\omega_x-l\varphi\right)^2+\frac{\gamma}{m_1}\left(k_1-\int_0^t g(s)ds\right)\theta_{1x}^2\\&+\frac{\gamma}{m_2}\left(k_2-\int_0^t g(s)ds\right)\theta_{2x}^2+\frac{\gamma}{m_1}(g\circ\theta_{1x})+\frac{\gamma}{m_2}(g\circ\theta_{2x})\Big]dx,\end{aligned} \tag{3.7}$$

satisfies

$$E^{'}(t)=\frac{\gamma}{2m_1}\left(\int_0^1(g^{'}\circ\theta_{1x})dx-g(t)\int_0^1\theta_{1x}^2dx\right)+\frac{\gamma}{2m_2}\left(\int_0^1(g^{'}\circ\theta_{2x})dx-g(t)\int_0^1\theta_{2x}^2dx\right)\leq 0. \tag{3.8}$$

Proof. *Multiplying Equation* $(3.1)_1$ *by* φ_t , $(3.1)_2$ *by* ψ_t, $(3.1)_3$ *by* ω_t, $(3.1)_4$ *by* $\frac{\gamma}{m_1}\theta_{1t}$ *and* $(3.1)_5$ *by* $\frac{\gamma}{m_2}\theta_{2t}$ *and integrating over* $(0,1)$*, we get after summing up*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Big[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\frac{\gamma}{m_1}\theta_{1t}^2+\frac{\gamma}{m_2}\theta_{2t}^2+b\psi_x^2+k\left(\varphi_x+\psi+l\omega\right)^2\\&+k_0\left(\omega_x-l\varphi\right)^2+\frac{\gamma k_1}{m_1}\theta_{1x}^2+\frac{\gamma k_2}{m_2}\theta_{2x}^2\Big]dx+\frac{\gamma}{m_1}\int_0^1\left(\int_0^t g\left(t-s\right)\theta_{1xx}\left(x,s\right)ds\right)\theta_{1t}dx\\&+\frac{\gamma}{m_2}\int_0^1\left(\int_0^t g\left(t-s\right)\theta_{2xx}\left(x,s\right)ds\right)\theta_{2t}dx\\=\;&0.\end{aligned}\tag{3.9}$$

By using (1.6)*, we obtain*

$$\begin{aligned}\int_0^1\left(\int_0^t g\left(t-s\right)\theta_{ixx}\left(x,s\right)ds\right)\theta_{it}dx\;=\;&\frac{1}{2}\frac{d}{dt}(g\circ\theta_{ix})(t)-\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_0^1\theta_{ix}^2dx\right)\\&-\frac{1}{2}\int_0^1(g'\circ\theta_{ix})(t)dx+\frac{1}{2}g(t)\int_0^1\theta_{ix}^2dx,\quad i=1,2.\end{aligned}$$

We obtain the result by substituting this last equality in (3.9). ■

Now, we introduce the multiplier p given by the solution of the Dirichlet problem

$$-p_{xx}=\psi_x,\quad p(0)=p(1)=0,$$

then we can obtain the following inequality

$$\int_0^1 p_t^2dx\leq\int_0^1 p_{tx}^2dx\leq\int_0^1\psi_t^2dx,\tag{3.10}$$

$$\int_0^1 p^2dx\leq\int_0^1 p_x^2dx\leq\int_0^1\psi^2dx\leq\int_0^1\psi_x^2dx,\tag{3.11}$$

and we define the functional

$$I_1\left(t\right)=-\rho_2\int_0^1\psi_t p_x dx-l\int_0^1\left(\rho_2 l\psi_t-\rho_1\omega_t\right)\int_0^x p(y)dydx+\rho_1\int_0^1\varphi_t p dx.\tag{3.12}$$

So, we have the following lemma.

Lemma 3.2 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be a solution of* (3.1)-(3.3)*. Then we have for any* $\varepsilon_1>0$,

$$\begin{aligned}I_1'\left(t\right)\;\leq\;&-b\left(1-2l^2\right)\int_0^1\psi_x^2dx+\frac{\gamma^2}{2bl^2}(1+l^4)\int_0^1\theta_{2t}^2dx\\&+C\left(\varepsilon_1\right)\int_0^1\psi_t^2dx+\varepsilon_1\int_0^1\omega_t^2dx+\varepsilon_1\int_0^1\varphi_t^2dx,\end{aligned}\tag{3.13}$$

where $C\left(\varepsilon_1\right)=\rho_2+\frac{\rho_1^2}{4\varepsilon_1}+\rho_2l^2+\frac{\rho_1^2l^2}{4\varepsilon_1}$.

Proof. By differentiating the expression of I_1, using equations in (3.1) and integrating by parts, we obtain

$$\begin{aligned} I_1^{'} &= -\rho_2\int_0^1 \psi_t p_{tx}dx + b\int_0^1 \psi_x p_{xx}dx + \rho_1\int_0^1 \varphi_t p_t dx + bl^2\int_0^1 \psi_x p dx \\ &\quad -\gamma\int_0^1 \theta_{2t}(p_{xx}+l^2p)dx - l\int_0^1 (\rho_2 l\psi_t - \rho_1\omega_t)\int_0^x p_t(y)dydx. \end{aligned}$$

Using the fact that $p_{xx} = -\psi_x$,we get

$$\begin{aligned} I_1^{'}(t) &= -\rho_2\int_0^1 \psi_t p_{tx}dx - b\int_0^1 \psi_x^2 dx + \rho_1\int_0^1 \varphi_t p_t dx \\ &\quad +bl^2\int_0^1 p_x^2 dx - \gamma l^2\int_0^1 \theta_{2t} p dx + \gamma\int_0^1 \theta_{2t}\psi_x dx \\ &\quad -\rho_2 l^2\int_0^1 \psi_t\int_0^x p_t(y)dydx + \rho_1 l\int_0^1 \omega_t\int_0^x p_t(y)dydx, \end{aligned} \tag{3.14}$$

by using Young's inequality, Poincaré's inequality, (3.10) and (3.11), Using the fact that $p_{xx} = -\psi_x$,we get

$$\rho_1\int_0^1 \varphi_t p_t dx \leq \varepsilon_1\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_1}\int_0^1 \psi_t^2 dx,$$

$$-\gamma l^2\int_0^1 \theta_{2t} p dx \leq \frac{bl^2}{2}\int_0^1 \psi_x^2 dx + \frac{\gamma^2 l^2}{2b}\int_0^1 \theta_{2t}^2 dx,$$

$$\gamma\int_0^1 \theta_{2t}\psi_x dx \leq \frac{bl^2}{2}\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{2bl^2}\int_0^1 \theta_{2t}^2 dx,$$

$$-\rho_2 l^2\int_0^1 \psi_t\int_0^x p_t(y)dydx \leq \rho_2 l^2\int_0^1 \psi_t^2 dx,$$

and

$$\rho_1 l\int_0^1 \omega_t\int_0^x p_t(y)dydx \leq \varepsilon_1\int_0^1 \omega_t^2 dx + \frac{\rho_1^2 l^2}{4\varepsilon_1}\int_0^1 \psi_t^2 dx.$$

We obtain the result by substituting these last inequalities in (3.14). ■

Lemma 3.3 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be a solution of* (3.1)-(3.3). *Then the functional*

$$I_2(t) = -\int_0^1 \theta_{1t}\left(\int_0^t g(t-s)(\theta_1(t)-\theta_1(s))ds\right)dx \tag{3.15}$$

satisfies for any $\varepsilon_2 > 0$ *and* $0 < \delta < 1$

$$\begin{aligned} I_2^{'}(t) &\leq -\left(\int_0^t g(s)\,ds - \delta\right)\int_0^1 \theta_{1t}^2 dx + l\varepsilon_2\int_0^1 \varphi_t^2 dx + \varepsilon_2\int_0^1 \omega_t^2 dx \\ &\quad +\frac{1}{4\delta}\left(k_1 - \int_0^t g(s)\,ds\right)^2\int_0^1 \theta_{1x}^2 dx - \frac{g(0)}{4\delta}\int_0^1 (g' \circ \theta_{1x})\,dx \\ &\quad +(C(\varepsilon_2) + \bar{g}\delta)\int_0^1 (g\circ\theta_{1x})\,dx, \end{aligned} \tag{3.16}$$

where $C(\varepsilon_2) = \bar{g}\left(1 + \frac{m_1^2(1+l)}{4\varepsilon_2}\right)$.

Proof. *Taking the derivative of* I_2*, using the fourth equation in* (3.1)*, we obtain*

$$\begin{aligned} I_2'(t) &= \left(k_1 - \int_0^t g(s)\,ds\right)\int_0^1 (g \diamond \theta_{1x})\,\theta_{1x}dx - m_1\int_0^1 (g \diamond \theta_{1x})\,\omega_t dx \\ &\quad - m_1 l \int_0^1 (g \diamond \theta_1)\,\varphi_t dx - \int_0^1 \theta_{1t}\,(g' \diamond \theta_1)\,dx \\ &\quad - \left(\int_0^t g(s)\,ds\right)\int_0^1 \theta_{1t}^2 dx + \int_0^1 (g \diamond \theta_{1x})^2\,dx. \end{aligned} \tag{3.17}$$

By using Young's inequality, (1.7) *and* (1.8)*, we get, for any* $\varepsilon_2 > 0$ *and* $0 < \delta < 1$

$$-\int_0^1 \theta_{1t}\,(g' \diamond \theta_1)\,dx \le \delta\int_0^1 \theta_{1t}^2 dx - \frac{g(0)}{4\delta}\int_0^1 (g' \circ \theta_{1x})\,dx, \tag{3.18}$$

$$\begin{aligned} \left(k_1 - \int_0^t g(s)\,ds\right)\int_0^1 (g \diamond \theta_{1x})\,\theta_{1x}dx &\le \bar{g}\delta\int_0^1 (g \circ \theta_{1x})\,dx \\ &\quad + \frac{1}{4\delta}\left(k_1 - \int_0^t g(s)\,ds\right)^2 \int_0^1 \theta_{1x}^2 dx, \end{aligned} \tag{3.19}$$

$$-m_1\int_0^1 (g \diamond \theta_{1x})\,\omega_t dx \le \varepsilon_2\int_0^1 \omega_t^2 dx + \frac{\bar{g}m_1^2}{4\varepsilon_2}\int_0^1 (g \circ \theta_{1x})\,dx, \tag{3.20}$$

and

$$-m_1 l\int_0^1 (g \diamond \theta_1)\,\varphi_t dx \le l\varepsilon_2\int_0^1 \varphi_t^2 dx + \frac{\bar{g}lm_1^2}{4\varepsilon_2}\int_0^1 (g \circ \theta_{1x})\,dx. \tag{3.21}$$

Inserting (3.18)-(3.21) *into* (3.17)*, we complete the proof of* (3.16). ■

Lemma 3.4 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be a solution of* (3.1)-(3.3)*. Then the functional*

$$I_3(t) = -\int_0^1 \theta_{2t}\left(\int_0^t g(t-s)(\theta_2(t) - \theta_2(s))ds\right)dx, \tag{3.22}$$

satisfies for any $\varepsilon_3 > 0$ *and* $0 < \delta < 1$

$$\begin{aligned} I_3'(t) &\le -\left(\int_0^t g(s)\,ds - \delta\right)\int_0^1 \theta_{2t}^2 dx \\ &\quad + \frac{1}{4\delta}\left(k_2 - \int_0^t g(s)\,ds\right)^2\int_0^1 \theta_{2x}^2 dx + \varepsilon_3\int_0^1 \psi_t^2 dx \\ &\quad - \frac{g(0)}{4\delta}\int_0^1 (g' \circ \theta_{2x})\,dx + (C(\varepsilon_3) + \bar{g}\delta)\int_0^1 (g \circ \theta_{2x})\,dx, \end{aligned} \tag{3.23}$$

where $C(\varepsilon_3) = \bar{g}\left(1 + \frac{m_2^2}{4\varepsilon_3}\right)$.

Proof. *Taking the derivative of I_3, using the fifth equation in (3.1), we obtain*

$$\begin{aligned} I_3'(t) &= \left(k_2 - \int_0^t g(s)\,ds\right)\int_0^1 (g \diamond \theta_{2x})\,\theta_{2x}dx \\ &\quad -m_2\int_0^1 (g \diamond \theta_{2x})\,\psi_t dx - \int_0^1 \theta_{2t}\,(g' \diamond \theta_2)\,dx \\ &\quad -\left(\int_0^t g(s)\,ds\right)\int_0^1 \theta_{2t}^2 dx + \int_0^1 (g \diamond \theta_{2x})^2\,dx. \end{aligned} \tag{3.24}$$

By using Young's inequality, (1.7) and (1.8), we get, for any $\varepsilon_3 > 0$ and $0 < \delta < 1$

$$-\int_0^1 \theta_{2t}\,(g' \diamond \theta_2)\,dx \leq \delta\int_0^1 \theta_{2t}^2 dx - \frac{g(0)}{4\delta}\int_0^1 (g' \circ \theta_{2x})\,dx, \tag{3.25}$$

$$\begin{aligned} \left(k_2 - \int_0^t g(s)\,ds\right)\int_0^1 (g \diamond \theta_{2x})\,\theta_{2x}dx &\leq \delta\bar{g}\int_0^1 (g \circ \theta_{2x})\,dx \\ &\quad +\frac{1}{4\delta}\left(k_2 - \int_0^t g(s)\,ds\right)^2\int_0^1 \theta_{2x}^2 dx, \end{aligned} \tag{3.26}$$

and

$$-m_2\int_0^1 (g \diamond \theta_{2x})\,\psi_t dx \leq \varepsilon_3\int_0^1 \psi_t^2 dx + \frac{\bar{g}m_2^2}{4\varepsilon_3}\int_0^1 (g \circ \theta_{2x})\,dx. \tag{3.27}$$

Combining (3.24)-(3.27), we obtain the desired result. ■

Lemma 3.5 *Let $(\varphi, \psi, \omega, \theta_1, \theta_2)$ be a solution of (3.1)-(3.3). Then the functional*

$$I_4(t) = -\int_0^1 (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t)\,dx,$$

satisfies

$$\begin{aligned} I_4'(t) &\leq -\rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx - \rho_1\int_0^1 \omega_t^2 dx + (b + l + 2cl)\int_0^1 \psi_x^2 dx \\ &\quad +(k + 2cl)\int_0^1 (\varphi_x + \psi + l\omega)^2\,dx + (k_0 + 2cl)\int_0^1 (\omega_x - l\varphi)^2\,dx \\ &\quad +\frac{\gamma^2}{4l}(1 + l^2)\int_0^1 \theta_{1t}^2 dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{2t}^2 dx. \end{aligned} \tag{3.28}$$

Proof. Taking the derivative of I_4, by using equations in (3.1), we get

$$\begin{aligned} I_4'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx - \rho_1\int_0^1 \omega_t^2 dx + b\int_0^1 \psi_x^2 dx \\ &\quad +k\int_0^1 (\varphi_x + \psi + l\omega)^2\,dx + k_0\int_0^1 (\omega_x - l\varphi)^2\,dx \\ &\quad +l\gamma\int_0^1 \theta_{1t}\varphi dx - \gamma\int_0^1 \theta_{2t}\psi_x dx - \gamma\int_0^1 \theta_{1t}\omega_x dx, \end{aligned} \tag{3.29}$$

We obtain the result by using (1.10) and Young's inequality to estimate the last three terms on the right-hand side of (3.29), as follows

$$\begin{aligned} l\gamma\int_0^1 \theta_{1t}\varphi dx &\leq l\int_0^1 \varphi_x^2 dx + \frac{\gamma^2 l}{4}\int_0^1 \theta_{1t}^2 dx \\ &\leq cl\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx + \frac{\gamma^2 l}{4}\int_0^1 \theta_{1t}^2 dx, \end{aligned} \tag{3.30}$$

$$\begin{aligned} -\gamma\int_0^1 \theta_{1t}\omega_x dx &\leq l\int_0^1 \omega_x^2 dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{1t}^2 dx \\ &\leq cl\int_0^1 \left[(\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{1t}^2 dx, \end{aligned} \tag{3.31}$$

$$-\gamma\int_0^1 \theta_{2t}\psi_x dx \leq l\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{4l}\int_0^1 \theta_{2t}^2 dx. \tag{3.32}$$

■

Lemma 3.6 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be a solution of (3.1)-(3.3), and let $k=k_0$. The functional I_5 defined by*

$$I_5(t) = -\rho_1\int_0^1 (\omega_x - l\varphi)\int_0^x \omega_t(y)dydx - \rho_1\int_0^1 \varphi_t\int_0^x (\varphi_x+\psi+l\omega)(y)dydx, \tag{3.33}$$

satisfies, for any $\varepsilon_4 > 0$

$$\begin{aligned} I_5^{'}(t) &\leq \rho_1\int_0^1 \omega_t^2 dx + \rho_1(l-1)\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4l}\int_0^1 \psi_t^2 dx \\ &\quad + \frac{\gamma}{2\varepsilon_4}\int_0^1 \theta_{1t}^2 dx + (\varepsilon_4\gamma - k_0)\int_0^1 (\omega_x - l\varphi)^2 dx \\ &\quad + (k+\gamma l^2\varepsilon_4)\int_0^1 (\varphi_x+\psi+l\omega)^2 dx. \end{aligned} \tag{3.34}$$

Proof. By differentiating I_5, then exploiting the first and third equations in (3.1), under the condition $k=k_0$ we get

$$\begin{aligned} I_5^{'}(t) &= \rho_1\int_0^1 \omega_t^2 dx - \rho_1\int_0^1 \varphi_t^2 dx - k_0\int_0^1 (\omega_x - l\varphi)^2 dx \\ &\quad + k\int_0^1 (\varphi_x+\psi+l\omega)^2 dx - \rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y)dydx \\ &\quad + \gamma\int_0^1 (\omega_x - l\varphi)\theta_{1t} dx + \gamma l\int_0^1 \theta_{1t}\int_0^x (\varphi_x+\psi+l\omega)(y)dydx. \end{aligned} \tag{3.35}$$

By using Young's and Poincaré's inequalities, we obtain

$$-\rho_1\int_0^1 \varphi_t\int_0^x \psi_t(y)dydx \leq \rho_1 l\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4l}\int_0^1 \psi_t^2 dx, \tag{3.36}$$

$$\gamma\int_0^1 \left(\omega_x - l\varphi\right)\theta_{1t}dx \leq \gamma\varepsilon_4\int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \frac{\gamma}{4\varepsilon_4}\int_0^1 \theta_{1t}^2 dx, \tag{3.37}$$

and

$$\gamma l\int_0^1 \theta_{1t}\int_0^x \left(\varphi_x + \psi + l\omega\right)(y)dydx \leq \gamma l^2\varepsilon_4\int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \frac{\gamma}{4\varepsilon_4}\int_0^1 \theta_{1t}^2 dx. \tag{3.38}$$

By substituting the last three inequalities in relation (3.35), we obtain the desired result. ■

Lemma 3.7 *Let* $(\varphi,\psi,\omega,\theta_1,\theta_2)$ *be the solution of* (3.1)-(3.3) *and let* $k = k_0$*. Then, the functional*

$$I_6(t) = -\rho_1\int_0^1 \varphi_t\left(\omega_x - l\varphi\right)dx - \rho_1\int_0^1 \omega_t\left(\varphi_x + \psi + l\omega\right)dx, \tag{3.39}$$

satisfies the estimate

$$\begin{aligned} I_6'(t) \quad \leq \quad & -l(k_0 - l)\int_0^1 \left(\omega_x - l\varphi\right)^2 dx - \frac{\rho_1 l}{2}\int_0^1 \omega_t^2 dx \\ & + kl\int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_1 l\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{2l}\int_0^1 \psi_t^2 dx \\ & + \frac{\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx + \gamma\int_0^1 \theta_{1tx}\left(\varphi_x + \psi + l\omega\right)dx. \end{aligned} \tag{3.40}$$

Proof. A simple differentiation of I_6, using the first and third equations in (3.1), under the condition $k = k_0$ leads to

$$\begin{aligned} I_6'(t) \quad = \quad & -k_0 l\int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \rho_1 l\int_0^1 \varphi_t^2 dx - \rho_1 l\int_0^1 \omega_t^2 dx \\ & + kl\int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \gamma\int_0^1 \theta_{1tx}\left(\varphi_x + \psi + l\omega\right)dx \\ & - \rho_1\int_0^1 \omega_t\psi_t dx + l\gamma\int_0^1 \theta_{1t}\left(\omega_x - l\varphi\right)dx. \end{aligned} \tag{3.41}$$

Use of Young's inequality for the last two terms in the right-hand side of (3.41) , we get

$$l\gamma\int_0^1 \theta_{1t}\left(\omega_x - l\varphi\right)dx \leq l^2\int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \frac{\gamma^2}{4}\int_0^1 \theta_{1t}^2 dx,$$

$$-\rho_1\int_0^1 \omega_t\psi_t dx \leq \frac{\rho_1 l}{2}\int_0^1 \omega_t^2 dx + \frac{\rho_1}{2l}\int_0^1 \psi_t^2 dx.$$

We obtain the result by substituting these last inequalities in (3.41). ■

3.3. Technical Lemmas

Lemma 3.8 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be the solution of* (3.1)-(3.3), *and let* $k = k_0, \frac{k}{\rho_1} = \frac{b}{\rho_2}$. *Then the functional*

$$I_7(t) = \rho_2 \int_0^1 \psi_t \left(\varphi_x + \psi + l\omega\right) dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx, \tag{3.42}$$

satisfies, for any $\varepsilon_5 > 0$, *the estimate*

$$\begin{aligned} I_7'(t) \leq & -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_2 l \varepsilon_5 \int_0^1 \omega_t^2 dx + l\varepsilon_5 \int_0^1 \left(\omega_x - l\varphi\right)^2 dx \\ & + \frac{b^2 l}{4\varepsilon_5} \left(1 + \frac{\gamma^2}{k^2}\right) \int_0^1 \psi_x^2 dx + \rho_2 \left(1 + \frac{l}{4\varepsilon_5}\right) \int_0^1 \psi_t^2 dx \\ & + l\varepsilon_5 \int_0^1 \theta_{1t}^2 dx - \gamma \int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx. \end{aligned} \tag{3.43}$$

Proof. By differentiating I_7, then exploiting the first and second equations in (3.1), we get

$$\begin{aligned} I_7'(t) = & -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + bl \int_0^1 \psi_x \left(\omega_x - l\varphi\right) dx \\ & + \rho_2 l \int_0^1 \psi_t \omega_t dx - \frac{bl\gamma}{k} \int_0^1 \theta_{1t} \psi_x dx - \gamma \int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx \\ & + \left(\rho_2 - \frac{b\rho_1}{k}\right) \int_0^1 \psi_t \varphi_{tx} dx. \end{aligned}$$

Under the condition $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ we get

$$\begin{aligned} I_7'(t) = & -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + bl \int_0^1 \psi_x \left(\omega_x - l\varphi\right) dx \\ & + \rho_2 l \int_0^1 \psi_t \omega_t dx - \frac{bl\gamma}{k} \int_0^1 \theta_{1t} \psi_x dx - \gamma \int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx. \end{aligned} \tag{3.44}$$

By using Young's inequality, we obtain

$$\rho_2 l \int_0^1 \psi_t \omega_t dx \leq \rho_2 l \varepsilon_5 \int_0^1 \omega_t^2 dx + \frac{\rho_2 l}{4\varepsilon_5} \int_0^1 \psi_t^2 dx,$$

$$bl \int_0^1 \psi_x \left(\omega_x - l\varphi\right) dx \leq l\varepsilon_5 \int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \frac{b^2 l}{4\varepsilon_5} \int_0^1 \psi_x^2 dx,$$

and

$$-\frac{bl\gamma}{k} \int_0^1 \theta_{1t} \psi_x dx \leq l\varepsilon_5 \int_0^1 \theta_{1t}^2 dx + \frac{b^2 l \gamma^2}{4k^2 \varepsilon_5} \int_0^1 \psi_x^2 dx.$$

By substituting the last three inequalities in (3.44), we obtain the desired result. ■

3.3. Technical Lemmas

Lemma 3.9 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be the solution of* (3.1)-(3.3)*. Then the functional*

$$I_8(t)=\int_0^1 (\theta_1\theta_{1t}+\theta_2\theta_{2t})\,dx, \tag{3.45}$$

satisfies, for any $0<\delta<1$, the estimate

$$\begin{aligned} I_8'(t) \leq & -\left[\lambda_1-\frac{1}{4\delta}(1+m_1^2+l^2m_1^2)\right]\int_0^1 \theta_{1x}^2dx+\int_0^1 \theta_{1t}^2dx+\delta\int_0^1 \varphi_t^2dx \\ & -\left[\lambda_2-\frac{1}{4\delta}(1+m_2^2)\right]\int_0^1 \theta_{2x}^2dx+\int_0^1 \theta_{2t}^2dx+\delta\int_0^1 \psi_t^2dx \\ & +\delta\int_0^1 \omega_t^2dx+\delta\bar{g}\int_0^1 (g\circ\theta_{1x})\,dx+\delta\bar{g}\int_0^1 (g\circ\theta_{2x})\,dx. \end{aligned} \tag{3.46}$$

Proof. Using the fourth and fifth equations of (3.1) and repeating calculations similar to the one above, we arrive at

$$\begin{aligned} I_8'(t) = & -\left(k_1-\int_0^t g(s)\,ds\right)\int_0^1 \theta_{1x}^2dx+\int_0^1 \theta_{1t}^2dx-\left(k_2-\int_0^t g(s)\,ds\right)\int_0^1 \theta_{2x}^2dx \\ & +\int_0^1 \theta_{2t}^2dx+m_1\int_0^1 \theta_{1x}\omega_tdx+lm_1\int_0^1 \theta_1\varphi_tdx+m_2\int_0^1 \theta_{2x}\psi_tdx \\ & -\int_0^1 (g\diamond\theta_{1x})\,\theta_{1x}dx-\int_0^1 (g\diamond\theta_{2x})\,\theta_{2x}dx. \end{aligned} \tag{3.47}$$

By using (1.7), Young's and Poincaré's inequalities, we get

$$-\int_0^1 (g\diamond\theta_{ix})\,\theta_{1x}dx \leq \delta\bar{g}\int_0^1 (g\circ\theta_{ix})\,dx+\frac{1}{4\delta}\int_0^1 \theta_{ix}^2dx, \text{ for } i=1,2, \tag{3.48}$$

$$m_1\int_0^1 \theta_{1x}\omega_tdx \leq \delta\int_0^1 \omega_t^2dx+\frac{m_1^2}{4\delta}\int_0^1 \theta_{1x}^2dx, \tag{3.49}$$

$$lm_1\int_0^1 \theta_1\varphi_tdx \leq \delta\int_0^1 \varphi_t^2dx+\frac{l^2m_1^2}{4\delta}\int_0^1 \theta_{1x}^2dx, \tag{3.50}$$

and

$$m_2\int_0^1 \theta_{2x}\psi_tdx \leq \delta\int_0^1 \psi_t^2dx+\frac{m_2^2}{4\delta}\int_0^1 \theta_{2x}^2dx. \tag{3.51}$$

Combining (3.47)-(3.51), gives the desired result. ■

Lemma 3.10 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be the solution of* (3.1)-(3.3)*. Then the functional*

$$I_9(t)=-\rho_2\int_0^1 \psi_x\int_0^x \psi_t(y,t)dydx, \tag{3.52}$$

satisfies, for any $\varepsilon_6 > 0$, *the estimate*

$$\begin{aligned} I_9^{'}(t) &\leq -(b-k\varepsilon_6-\gamma\varepsilon_6)\int_0^1 \psi_x^2 dx \rho_2 \int_0^1 \psi_t^2 dx \\ &+\frac{k}{4\varepsilon_6}\int_0^1 (\varphi_x+\psi+l\omega)^2 dx + \frac{\gamma}{4\varepsilon_6}\int_0^1 \theta_{2t}^2 dx. \end{aligned} \tag{3.53}$$

Proof. By differentiating I_9, then exploiting the second equation in (3.1), we get

$$\begin{aligned} I_9^{'}(t) &= \rho_2 \int_0^1 \psi_t^2 dx - b\int_0^1 \psi_x^2 dx + \gamma \int_0^1 \psi_x \theta_{2t} dx \\ &-k\int_0^1 \psi(\varphi_x+\psi+l\omega)\, dx. \end{aligned} \tag{3.54}$$

Use of Young's and Poincaré's inequalities for the last two terms in the right-hand side of (3.54) , we get

$$-k\int_0^1 \psi(\varphi_x+\psi+l\omega)\, dx \leq k\varepsilon_6 \int_0^1 \psi_x^2 dx + \frac{k}{4\varepsilon_6}\int_0^1 (\varphi_x+\psi+l\omega)^2 dx, \tag{3.55}$$

and

$$\gamma\int_0^1 \psi_x \theta_{2t} dx \leq \gamma\varepsilon_6 \int_0^1 \psi_x^2 dx + \frac{\gamma}{4\varepsilon_6}\int_0^1 \theta_{2t}^2 dx. \tag{3.56}$$

Combining (3.54)-(3.56), gives the desired result. ■

Lemma 3.11 *Let* $(\varphi, \psi, \omega, \theta_1, \theta_2)$ *be the solution of* (3.1)-(3.3). *Then the functional*

$$I_{10} = -\rho_2 \int_0^1 \theta_{2t} \int_0^x \psi_t(y,t) dy dx. \tag{3.57}$$

satisfies, for any $\varepsilon_7 > 0$ *and* $0 < \delta < 1$, *the estimate*

$$\begin{aligned} I_{10}^{'}(t) &\leq -\frac{\rho_2 m_2}{2}\int_0^1 \psi_t^2 dx + b\varepsilon_7 \int_0^1 \psi_x^2 dx \\ &+k\varepsilon_7 \int_0^1 (\varphi_x+\psi+l\omega)^2 dx + \left(\gamma + \frac{b+k}{4\varepsilon_7}\right)\int_0^1 \theta_{2t}^2 dx \\ &+\frac{(\rho_2 k_2+\bar{g})^2}{\rho_2 m_2}\int_0^1 \theta_{2x}^2 dx + \frac{\rho_2 \bar{g}}{m_2}\int_0^1 (g\circ\theta_{2x})\, dx. \end{aligned} \tag{3.58}$$

Proof. Using the second and fifth quations of (3.1) and repeating similar computations as above, we arrive at

$$\begin{aligned} I_{10}^{'}(t) &= -\rho_2 m_2 \int_0^1 \psi_t^2 dx + \left(\rho_2 k_2 - \int_0^t g(s)\, ds\right)\int_0^1 \theta_{2x}\psi_t dx \\ &+\rho_2 \int_0^1 (g\diamond\theta_{2x})\, \psi_t dx - b\int_0^1 \theta_{2t}\psi_x dx + \gamma\int_0^1 \theta_{2t}^2 dx \\ &+k\int_0^1 \theta_{2t}\int_0^x (\varphi_x+\psi+l\omega)(y,t) dy dx. \end{aligned} \tag{3.59}$$

By using (1.7), Young's and Poincaré's inequalities, we get

$$\left(\rho_2 k_2 - \int_0^t g\left(s\right) ds\right) \int_0^1 \theta_{2x} \psi_t dx \leq \frac{\rho_2 m_2}{4} \int_0^1 \psi_t^2 dx + \frac{\left(\rho_2 k_2 + \bar{g}\right)^2}{\rho_2 m_2} \int_0^1 \theta_{2x}^2 dx,$$

$$\rho_2 \int_0^1 \left(g \diamond \theta_{2x}\right) \psi_t dx \leq \rho_2 \frac{m_2}{4} \int_0^1 \psi_t^2 dx + \frac{\rho_2 \bar{g}}{m_2} \int_0^1 \left(g \circ \theta_{2x}\right) dx,$$

$$-b \int_0^1 \theta_{2t} \psi_x dx \leq b\varepsilon_7 \int_0^1 \psi_x^2 dx + \frac{b}{4\varepsilon_7} \int_0^1 \theta_{2t}^2 dx,$$

and

$$k \int_0^1 \theta_{2t} \int_0^x \left(\varphi_x + \psi + l\omega\right)(y,t) dy dx \leq k\varepsilon_7 \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \frac{k}{4\varepsilon_7} \int_0^1 \theta_{2t}^2 dx.$$

By substituting the last three inequalities in (3.59), we obtain the desired result. ■

Firstly, to estimate the terms $\gamma \int_0^1 \theta_{1tx} \left(\varphi_x + \psi + l\omega\right) dx$ and $-\gamma \int_0^1 \theta_{2tx} \left(\varphi_x + \psi + l\omega\right) dx$ appeared in (3.40),(3.43) respectively, we use the system obtained by differentiating (3.1)-(3.3) with respect to time, defined by

$$\begin{cases} \rho_1 \varphi_{ttt} - k\left(\varphi_x + \psi + l\omega\right)_{tx} - k_0 l \left(\omega_x - l\varphi\right)_t + l\gamma\theta_{1tt} = 0, \\ \rho_2 \psi_{ttt} - b\psi_{xxt} + k\left(\varphi_x + \psi + l\omega\right)_t + \gamma\theta_{2ttx} = 0, \\ \rho_1 \omega_{ttt} - k_0 \left(\omega_x - l\varphi\right)_{xt} + kl\left(\varphi_x + \psi + l\omega\right)_t + \gamma\theta_{1ttx} = 0, \\ \theta_{1ttt} - k_1 \theta_{1xxt} + \frac{\partial}{\partial t} \int_0^t g\left(t-s\right) \theta_{1xx}\left(x,s\right) ds + m_1 \left(\omega_x - l\varphi\right)_{tt} = 0 \\ \theta_{2ttt} - k_2 \theta_{2xxt} + \frac{\partial}{\partial t} \int_0^t g\left(t-s\right) \theta_{2xx}\left(x,s\right) ds + m_2 \psi_{xtt} = 0 \\ \omega_t(x,t) = \varphi_t(x,t) = \psi_t(x,t) = \theta_{1t}(x,t) = \theta_{2t}(x,t) = 0, \text{ for } x = 0, 1, \end{cases} \tag{3.60}$$

and we introduce the second-order energy functional

$$\begin{aligned} \tilde{E}(t) &= \frac{1}{2} \int_0^1 \Bigg[\rho_1 \varphi_{tt}^2 + \rho_2 \psi_{tt}^2 + \rho_1 \omega_{tt}^2 + \frac{\gamma}{m_1} \theta_{1tt}^2 + \frac{\gamma}{m_2} \theta_{2tt}^2 + b\psi_{xt}^2 \\ &\quad + k\left(\varphi_x + \psi + l\omega\right)_t^2 + k_0 \left(\omega_x - l\varphi\right)_t^2 + \frac{\gamma}{m_1}(g \circ \theta_{1tx}) + \frac{\gamma}{m_2}(g \circ \theta_{2tx}) \\ &\quad + \frac{\gamma}{m_1} \left(k_1 - \int_0^t g(s) ds\right) \theta_{1tx}^2 + \frac{\gamma}{m_2} \left(k_2 - \int_0^t g(s) ds\right) \theta_{2tx}^2 \Bigg] dx. \end{aligned} \tag{3.61}$$

At this point, we need the following technical lemmas:

Lemma 3.12 *The second-order energy $\tilde{E}(t)$ satisfies, for all $t \geq 0$*

$$\begin{aligned}\tilde{E}'(t) &= \frac{\gamma}{2m_1}\left(\int_0^1 (g' \circ \theta_{1tx})dx - g(t)\int_0^1 \theta_{1tx}^2 dx\right) \\ &+\frac{\gamma}{2m_2}\left(\int_0^1 (g' \circ \theta_{2tx})dx - g(t)\int_0^1 \theta_{2tx}^2 dx\right) \\ &-\frac{\gamma}{m_1}g(t)\int_0^1 \theta_{10xx}\theta_{1tt}dx - \frac{\gamma}{m_2}g(t)\int_0^1 \theta_{20xx}\theta_{2tt}dx, \end{aligned} \tag{3.62}$$

and

$$\tilde{E}(t) \leq c\left(\tilde{E}(0) + \int_0^1 (\theta_{10xx}^2 + \theta_{20xx}^2)dx\right). \tag{3.63}$$

Proof. *Differentiating $\tilde{E}(t)$ with respect to t, using system (3.60) and noting that the following fact*

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t g(t-s)\theta_{ixx}(x,s)ds &= \frac{\partial}{\partial t}\int_0^t g(s)\theta_{ixx}(t-s)ds \\ &= \int_0^t g(s)\theta_{ixxt}(t-s)ds + g(t)\theta_{ixx}(0) \\ &= \int_0^t g(t-s)\theta_{ixxt}(s)ds + g(t)\theta_{i0xx}, \quad \text{for } i=1,2,\end{aligned}$$

we obtain (3.62). Then, by using Young's inequality, we find that

$$\begin{aligned}\tilde{E}'(t) &\leq -\frac{\gamma}{m_1}g(t)\int_0^1 \theta_{10xx}\theta_{1tt}dx - \frac{\gamma}{m_2}g(t)\int_0^1 \theta_{20xx}\theta_{2tt}dx \\ &\leq \frac{\gamma}{m_1}g(t)\int_0^1\left(\theta_{10xx}^2 + \frac{1}{4}\theta_{1tt}^2\right)dx + \frac{\gamma}{m_2}g(t)\int_0^1\left(\theta_{20xx}^2 + \frac{1}{4}\theta_{2tt}^2\right)dx \\ &\leq g(t)\tilde{E}(t) + \gamma g(t)\int_0^1\left(\frac{\theta_{10xx}^2}{m_1} + \frac{\theta_{20xx}^2}{m_2}\right)dx,\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tilde{E}(t)e^{-\int_0^t g(s)ds}\right) &= \tilde{E}'(t)e^{-\int_0^t g(s)ds} - g(t)\tilde{E}(t)e^{-\int_0^t g(s)ds}\\ &\leq g(t)\tilde{E}(t)e^{-\int_0^t g(s)ds} + \gamma g(t)e^{-\int_0^t g(s)ds}\int_0^1\left(\frac{\theta_{10xx}^2}{m_1}+\frac{\theta_{20xx}^2}{m_2}\right)dx\\ &\quad -g(t)\tilde{E}(t)e^{-\int_0^t g(s)ds}\\ &\leq \gamma g(t)\int_0^1\left(\frac{\theta_{10xx}^2}{m_1}+\frac{\theta_{20xx}^2}{m_2}\right)dx,\end{aligned}$$

which gives us

$$\begin{aligned}\tilde{E}(t)e^{-\int_0^{+\infty} g(s)ds} &\leq \tilde{E}(t)e^{-\int_0^t g(s)ds}\\ &\leq \tilde{E}(0)+\gamma\left(\int_0^t g(s)ds\right)\int_0^1\left(\frac{\theta_{10xx}^2}{m_1}+\frac{\theta_{20xx}^2}{m_2}\right)dx\\ &\leq \tilde{E}(0)+\gamma\left(\int_0^{+\infty} g(s)ds\right)\int_0^1\left(\frac{\theta_{10xx}^2}{m_1}+\frac{\theta_{20xx}^2}{m_2}\right)dx.\end{aligned}$$

Consequently, (3.63) *follows.* ■

Lemma 3.13 *There holds for any* $0<\delta<1$ *and* $t\geq t_0>0$*, we have the following inequalities:*

$$\begin{aligned}\pm\gamma\int_0^1\theta_{itx}\left(\varphi_x+\psi+l\omega\right)dx &\leq \delta\int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx+\frac{c}{\delta}g(t)E(0)\\ &\quad +\frac{c}{\delta}\int_0^1\left[(g\circ\theta_{ixt})-(g'\circ\theta_{ix})\right]dx, \qquad (3.64)\end{aligned}$$

for $i=1,2$.

Proof. We have, for $i=1,2$

$$\begin{aligned}\pm\gamma\int_0^1\theta_{itx}\left(\varphi_x+\psi+l\omega\right)dx &= \pm\frac{\gamma}{\int_0^t g(s)ds}\int_0^1\left(\varphi_x+\psi+l\omega\right)(g\diamond\theta_{itx})dx\\ &\quad \pm\frac{\gamma}{\int_0^t g(s)ds}\int_0^1\left(\varphi_x+\psi+l\omega\right)\int_0^t g(t-s)\theta_{itx}(s)dsdx.\end{aligned} \qquad (3.65)$$

Recall that $\frac{1}{\int_0^t g(s)ds} \le \frac{1}{\int_0^{t_0} g(s)ds}$, for all $t \ge t_0 > 0$, which together with Young's inequality yields for any $0 < \delta < 1$,

$$\pm \frac{\gamma}{\int_0^t g(s)ds}\int_0^1 (\varphi_x+\psi+l\omega)\,(g\diamond\theta_{itx})dx \le \frac{\delta}{2}\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \frac{c}{\delta}\int_0^1 (g\circ\theta_{ixt})dx, \tag{3.66}$$

and

$$\begin{aligned}
&\pm \frac{\gamma}{\int_0^t g(s)ds}\int_0^1 (\varphi_x+\psi+l\omega)\left(\int_0^t g(t-s)\theta_{itx}(s)ds\right)dx\\
&= \pm\frac{\gamma}{\int_0^t g(s)ds}\int_0^1 (\varphi_x+\psi+l\omega)\left(g(0)\theta_{ix} - g(t)\theta_{i0x} + \int_0^t g^{'}(t-s)\theta_{ix}(s)ds\right)dx\\
&= \pm\frac{\gamma}{\int_0^t g(s)ds}\int_0^1 (\varphi_x+\psi+l\omega)\left[g(t)(\theta_{ix}-\theta_{i0x}) - (g^{'}\diamond\theta_{ix})\right]dx\\
&\le \frac{\delta}{2}\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \frac{c}{\delta}g(t)\int_0^1 (\theta_{ix}^2+\theta_{i0x}^2)dx - \frac{c}{\delta}\int_0^1 (g^{'}\circ\theta_{ix})dx\\
&\le \frac{\delta}{2}\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx + \frac{c}{\delta}g(t)E(0) - \frac{c}{\delta}\int_0^1 (g^{'}\circ\theta_{ix})dx.
\end{aligned} \tag{3.67}$$

Inserting (3.66)-(3.67) into (3.65), we can get (3.64). The proof is done. ■

Lemma 3.14 *For any* $t \ge t_0 > 0$, *we have the following inequalities:*

$$-\frac{\gamma}{m_1}g(t)\int_0^1 \theta_{10xx}\theta_{1tt}dx - \frac{\gamma}{m_2}g(t)\int_0^1 \theta_{20xx}\theta_{2tt}dx \le cg(t)\left(\tilde{E}(0) + \int_0^1 (\theta_{10xx}^2+\theta_{20xx}^2)dx\right). \tag{3.68}$$

Proof. by using Young's inequality and (3.63), we find that

$$\begin{aligned}
&-\frac{\gamma}{m_1}g(t)\int_0^1 \theta_{10xx}\theta_{1tt}dx - \frac{\gamma}{m_2}g(t)\int_0^1 \theta_{20xx}\theta_{2tt}dx\\
\le\ & \frac{\gamma g(t)}{2m_1}\int_0^1 (\theta_{10xx}^2+\theta_{1tt}^2)dx + \frac{\gamma g(t)}{2m_2}\int_0^1 (\theta_{20xx}^2+\theta_{2tt}^2)dx\\
\le\ & cg(t)\left(\tilde{E}(0) + \int_0^1 \left(\theta_{10xx}^2+\theta_{20xx}^2\right)dx\right).
\end{aligned}$$

■

3.4 Stability result

Now, we are ready to state and prove the main result of this section.

First, we must prove the equivalence between $E(t)+\tilde{E}(t)$ and $L(t)$.

Lemma 3.15 *We define a Lyapunov functional L as follows*

$$L(t)=N\left(E(t)+\tilde{E}(t)\right)+\sum_{i=1}^{10}N_iI_i(t), \tag{3.69}$$

satisfies, for N_i, $i=1,2,...,10$ are positive constants to be proprerly chosen later, with sufficiently large N,

$$\alpha_1\left(E(t)+\tilde{E}(t)\right)\leq L(t)\leq\alpha_2\left(E(t)+\tilde{E}(t)\right), \qquad \forall t\geq 0, \tag{3.70}$$

where α_1 and α_2 are positive constants.

Theorem 3.2 *Let $(\varphi,\psi,\omega,\theta_1,\theta_2)$ be the solution of (3.1)-(3.3) and assume that $(G1)$, $(G2)$ and (3.4) hold. Then there exists a constant $C>0$ such that the energy functional (3.7) satisfies,*

$$E(t)\leq C\left(\frac{1+\int\limits_0^t g(s)ds}{\int\limits_0^t\zeta(s)ds}\right), \qquad \forall t\geq 0, \tag{3.71}$$

where

$$C=c\left(E(0)+\tilde{E}(0)+\int\limits_0^1\left(\theta_{10xx}^2+\theta_{20xx}^2\right)dx\right).$$

Proof. *A combination of the estimates of the previous lemmas gives*

$$\begin{aligned}
L^{'}(t) \;\leq\; & -\Bigg[\rho_2\left(N_4-N_9\right)-C\left(\varepsilon_1\right)N_1-\varepsilon_3N_3-\frac{\rho_1}{4l}N_5-\frac{\rho_1}{2l}N_6-\rho_2\left(1+\frac{l}{4\varepsilon_5}\right)N_7 \\
& +\frac{\rho_2m_2}{2}N_{10}\Bigg]\int_0^1\psi_t^2dx-\Big[b\left(1-2l^2\right)N_1-\left(b+l+2cl\right)N_4 \\
& -\frac{b^2l}{4\varepsilon_5}\left(1+\frac{\gamma^2}{k^2}\right)N_7-\left(k\varepsilon_6+\gamma\varepsilon_6-b\right)N_9-b\varepsilon_7N_{10}\Bigg]\int_0^1\psi_x^2dx \\
& -\left[kN_7-(k+2cl)N_4-(k+\gamma l^2\varepsilon_4)N_5-klN_6-\frac{k}{4\varepsilon_6}N_9-k\varepsilon_7N_{10}\right] \\
& \int_0^1\left(\varphi_x+\psi+l\omega\right)^2dx-\left[(k_0-\gamma\varepsilon_4)N_5+l(k_0-l)N_6-(k_0+2cl)N_4-l\varepsilon_5N_7\right] \\
& \int_0^1\left(\omega_x-l\varphi\right)^2dx-\left[\rho_1N_4+\rho_1(1-l)N_5-\varepsilon_1N_1-l\varepsilon_2N_2-\rho_1lN_6\right]\int_0^1\varphi_t^2dx \\
& -\left[\rho_1N_4+\frac{\rho_1l}{2}N_6-\varepsilon_1N_1-\varepsilon_2N_2-\rho_1N_5-\rho_2l\varepsilon_5N_7\right]\int_0^1\omega_t^2dx \\
& -\left[\left(\int_0^tg\left(s\right)ds\right)N_2-\frac{\gamma^2(1+l^2)}{4l}N_4-\frac{\gamma}{2\varepsilon_4}N_5-\frac{\gamma^2}{4}N_6-l\varepsilon_5N_7-N_8\right]\int_0^1\theta_{1t}^2dx \\
& -\Bigg[\left(\int_0^tg\left(s\right)ds\right)N_3-\frac{\gamma^2}{2bl^2}(1+l^4)N_1-\frac{\gamma^2}{4\varepsilon_4}N_4-N_8-\frac{\gamma}{4\varepsilon_6}N_9 \\
& -\left(\gamma+\frac{b+k}{4\varepsilon_7}\right)N_{10}\Bigg]\int_0^1\theta_{2t}^2dx-\left[\frac{\gamma}{2m_1}g(t)N+\lambda_1N_8\right]\int_0^1\theta_{1x}^2dx \\
& -\left[\frac{\gamma}{2m_2}g(t)N+\lambda_2N_8-\frac{\left(\rho_2k_2+\bar{g}\right)^2}{\rho_2m_2}N_{10}\right]\int_0^1\theta_{2x}^2dx+C\left(\varepsilon_2\right)N_2\int_0^1\left(g\circ\theta_{1x}\right)dx \\
& +\left[C\left(\varepsilon_3\right)N_3+\frac{\rho_2\bar{g}}{m_2}N_{10}\right]\int_0^1\left(g\circ\theta_{2x}\right)dx+\frac{\gamma N}{2m_1}\int_0^1\left(g'\circ\theta_{1x}\right)dx \\
& +\frac{\gamma N}{2m_2}\int_0^1\left(g'\circ\theta_{2x}\right)dx+\delta\int_0^1\Big[N_8(\varphi_t^2+\omega_t^2)+N_8\psi_t^2+N_2\theta_{1t}^2 \\
& +N_3\theta_{2t}^2+\bar{g}(N_2+N_8)\left(g\circ\theta_{1x}\right)+\bar{g}(N_3+N_8)\left(g\circ\theta_{2x}\right)\Big]dx \\
& +\frac{1}{4\delta}\int_0^1\Bigg[\left((k_1-\int_0^tg\left(s\right)ds)N_2+(1+m_1^2+l^2m_1^2)N_8\right)\theta_{1x}^2 \\
& +\left((k_2-\int_0^tg\left(s\right)ds)N_3+(1+m_2^2)N_8\right)\theta_{2x}^2-g(0)N_2\left(g'\circ\theta_{1x}\right) \\
& -g(0)N_3\left(g'\circ\theta_{2x}\right)\Bigg]dx+\frac{N\gamma}{2m_1}\int_0^1(g^{'}\circ\theta_{1tx})dx-\frac{N\gamma}{2m_1}g(t)\int_0^1\theta_{1tx}^2dx \\
& -\frac{N\gamma}{2m_2}g(t)\int_0^1\theta_{2tx}^2dx+\frac{N\gamma}{2m_2}\int_0^1(g^{'}\circ\theta_{2tx})dx-\frac{\gamma N}{m_1}g(t)\int_0^1\theta_{10xx}\theta_{1tt}dx \\
& -\frac{N\gamma}{m_2}g(t)\int_0^1\theta_{20xx}\theta_{2tt}dx+\gamma N_6\int_0^1\theta_{1tx}\left(\varphi_x+\psi+l\omega\right)dx-\gamma N_7\int_0^1\theta_{2tx}\left(\varphi_x+\psi+l\omega\right)dx.
\end{aligned}$$

3.4. Stability result

By setting $N_8 = 1,\ N_5 = N_4,\ N_6 = \frac{1}{l}N_4,\ N_9 = (1-l)\,N_4,\ \varepsilon_6 = lN_9$ *and by using* (3.64),(3.68)*, we arrive at*

$$
\begin{aligned}
L^{'}(t) \leq\ & -\Big[\rho_2\left(m_2-\varepsilon_7\right)N_{10} + \Big(l - \frac{\rho_1}{4l} - \frac{\rho_1}{2l}\Big)N_4 - C_1\left(\varepsilon_1\right)N_1 - \varepsilon_3 N_3 \\
& -\rho_2\left(1+\frac{l}{4\varepsilon_5}\right)N_7\Big]\int\limits_0^1 \psi_t^2 dx - \Big[b\left(1-2l^2\right)N_1 - \frac{b^2 l}{4\varepsilon_5}\left(1+\frac{\gamma^2}{k^2}\right)N_7 \\
& -l\left((1-l)^2(k+\gamma)N_4 + 2c + b + 1\right)N_4 - b\varepsilon_7 N_{10}\Big]\int\limits_0^1 \psi_x^2 dx \\
& -\left[kN_7 - \left(3k + 2lc + \gamma l^2\varepsilon_4\right)N_4 - \frac{k}{4l} - k\varepsilon_7 N_{10}\right]\int\limits_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx \\
& -\left[\left(k_0 - 2cl - \gamma\varepsilon_4\right)N_4 - l\varepsilon_5 N_7\right]\int\limits_0^1 \left(\omega_x - l\varphi\right)^2 dx \\
& -\left[\rho_1(1-l)N_4 - \varepsilon_1 N_1 - l\varepsilon_2 N_2\right]\int\limits_0^1 \varphi_t^2 dx \\
& -\left[\frac{\rho_1}{2}N_4 - \varepsilon_1 N_1 - \varepsilon_2 N_2 - \rho_2 l\varepsilon_5 N_7\right]\int\limits_0^1 \omega_t^2 dx \\
& -\left[\left(\int\limits_0^t g\left(s\right)ds\right)N_2 - \left(\frac{\gamma^2(1+l^2)}{4l} + \frac{\gamma}{2\varepsilon_4} + \frac{\gamma^2}{4l}\right)N_4 - l\varepsilon_5 N_7 - 1\right]\int\limits_0^1 \theta_{1t}^2 dx \\
& -\Big[\left(\int\limits_0^t g\left(s\right)ds\right)N_3 - \frac{\gamma^2(1+l^4)}{2bl^2}N_1 - \frac{\gamma^2}{4l}N_4 - \left(\gamma + \frac{b+k}{4\varepsilon_7}\right)N_{10} \\
& -\frac{\gamma}{4l} - 1\Big]\int\limits_0^1 \theta_{2t}^2 dx - \left[\frac{\gamma}{2m_1}g(t)N + \lambda_1\right]\int\limits_0^1 \theta_{1x}^2 dx + C\left(\varepsilon_2\right)N_2\int\limits_0^1 \left(g\circ\theta_{1x}\right)dx \\
& -\left[\frac{\gamma}{2m_2}g(t)N + \lambda_2\right]\int\limits_0^1 \theta_{2x}^2 dx + \left[C\left(\varepsilon_3\right)N_3 + \frac{\rho_2\bar{g}}{4\varepsilon_7}N_{10}\right]\int\limits_0^1 \left(g\circ\theta_{2x}\right)dx \\
& +\frac{\gamma N}{2m_1}\int\limits_0^1 \left(g'\circ\theta_{1x}\right)dx + \frac{\gamma N}{2m_2}\int\limits_0^1 \left(g'\circ\theta_{2x}\right)dx + \delta C_{N_2,N_3,N_8,N_{10}}E(t) \\
& -\frac{1}{\delta}C_{N_2,N_3,N_8,N_{10}}E^{'}(t) + cNg(t)\left[\tilde{E}(0) + \int\limits_0^1 (\theta_{10xx}^2 + \theta_{20xx}^2)dx\right] \\
& -\delta\left(\frac{N_4}{l} + N_7\right)\int\limits_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \frac{c}{\delta}(N_4 + N_7)g(t)E(0) \\
& +\frac{c}{\delta}N_4\int\limits_0^1 \left[\left(g\circ\theta_{1xt}\right) - \left(g^{'}\circ\theta_{1xt}\right)\right]dx + \frac{c}{\delta}N_7\int\limits_0^1 \left[\left(g\circ\theta_{2xt}\right) - \left(g^{'}\circ\theta_{2xt}\right)\right]dx.
\end{aligned}
$$

Now, let us take $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_2}$, $\varepsilon_3 = \frac{1}{N_3}$, $\varepsilon_4 = \frac{1}{N_4}$, $\varepsilon_5 = \frac{1}{N_7}$, $\varepsilon_7 = \frac{1}{N_{10}}$, *to get*

$$\begin{aligned}
L^{'}(t) \leq\; & -\Bigg[\rho_2 m_2 N_{10} + \left(l - \frac{\rho_1}{4l} - \frac{\rho_1}{2l}\right) N_4 - \left(N_1\rho_2 + \frac{N_1^2\rho_1^2}{4}\right)(1+l^2) \\
& -\rho_2\left(1 + \frac{lN_7}{4\varepsilon_5}\right) N_7 - 2\Bigg] \int\limits_0^1 \psi_t^2 dx - \big[b\left(1-2l^2\right) N_1 - b \\
& -l\left((1-l)^2(k+\gamma)N_4 + 2c + b + 1\right) N_4 - \frac{b^2 l}{4}\left(1 + \frac{\gamma^2}{k^2}\right)\Bigg] \int\limits_0^1 \psi_x^2 dx \\
& -\left[kN_7 - (3k+2lc)\, N_4 - \gamma l^2 - \frac{k}{4l} - k\varepsilon\right] \int\limits_0^1 (\varphi_x + \psi + l\omega)^2\, dx \qquad (3.72) \\
& -\left[(k_0 - 2cl)\, N_4 - l - \gamma\right] \int\limits_0^1 (\omega_x - l\varphi)^2\, dx - \left[\rho_1(1-l)N_4 - 1 - l\right] \int\limits_0^1 \varphi_t^2 dx \\
& -\left[\frac{\rho_1}{2} N_4 - 2 - \rho_2 l\right] \int\limits_0^1 \omega_t^2 dx \\
& -\left[\left(\int\limits_0^t g(s)\, ds\right) N_2 - \left(\frac{\gamma^2(1+l^2)}{4l} + \frac{\gamma N_4}{2} + \frac{\gamma^2}{4l}\right) N_4 - l - 1\right] \int\limits_0^1 \theta_{1t}^2 dx \\
& -\Bigg[\left(\int\limits_0^t g(s)\, ds\right) N_3 - \frac{\gamma^2(1+l^4)}{2bl^2} N_1 - \frac{\gamma^2}{4l} N_4 \\
& -\left(\gamma + N_{10}\frac{b+k}{4}\right) N_{10} - \frac{\gamma}{4l} - 1\Bigg] \int\limits_0^1 \theta_{2t}^2 dx \\
& -\left[\frac{\gamma}{2m_1} g(t) N + \lambda_1\right] \int\limits_0^1 \theta_{1x}^2 dx + \bar{g}\left(1 + N_2 \frac{m_1^2(1+l)}{4}\right) N_2 \int\limits_0^1 (g \circ \theta_{1x})\, dx \\
& -\left[\frac{\gamma}{2m_2} g(t) N + \lambda_2\right] \int\limits_0^1 \theta_{2x}^2 dx + \bar{g}\left[\left(1 + \frac{m_2^2 N_3}{4}\right) N_3 + \frac{\rho_2}{4} N_{10}^2\right] \int\limits_0^1 (g \circ \theta_{2x})\, dx \\
& + \frac{\gamma N}{2m_1} \int\limits_0^1 (g' \circ \theta_{1x})\, dx + \frac{\gamma N}{2m_2} \int\limits_0^1 (g' \circ \theta_{2x})\, dx + \delta C_{N_2,N_3,N_4,N_7,N_8,N_{10}} E(t) \\
& - \frac{1}{\delta} C_{N_2,N_3,N_8,N_{10}} E^{'}(t) + cNg(t)\left[\tilde{E}(0) + \int\limits_0^1 (\theta_{10xx}^2 + \theta_{20xx}^2) dx\right] \\
& + \frac{c}{\delta}(N_4 + N_7) g(t) E(0) + \frac{c}{\delta} N_4 \int\limits_0^1 \left[(g \circ \theta_{1xt}) - \left(g^{'} \circ \theta_{1xt}\right)\right] dx \\
& + \frac{c}{\delta} N_7 \int\limits_0^1 \left[(g \circ \theta_{2xt}) - \left(g^{'} \circ \theta_{2xt}\right)\right] dx.
\end{aligned}$$

First, we choose $l < \min\left\{\frac{1}{\sqrt{2}}, \frac{k_0}{2c}, \frac{\sqrt{3\rho_1}}{2}\right\}$.
Next, we select N_4 *large enough such that*

$$N_4 > \max\left\{\frac{l+\gamma}{k_0 - 2cl}, \frac{1+l}{\rho_1(1-l)}, \frac{4 + 2l\rho_2}{\rho_1}\right\},$$

then, we choose N_1, N_7, N_{10} *sufficiently large so that*

$$N_7 > \left(3 + \frac{2cl}{k}\right) N_4 + \frac{\gamma l^2}{k} + \frac{1}{4l} + 1,$$

$$b\left(1 - 2l^2\right) N_1 > l\left((1-l)^2(k+\gamma)N_4 + 2c + b + 1\right) N_4 + \frac{b^2 l}{4}\left(1 + \frac{\gamma^2}{k^2}\right) + b,$$

$$\rho_2 m_2 N_{10} > \left(\frac{\rho_1}{4l} + \frac{\rho_1}{2l} - l\right) N_4 + \left(N_1\rho_2 + \frac{N_1^2\rho_1^2}{4}\right)(1+l^2) + \rho_2\left(N_7 + \frac{lN_7^2}{4\varepsilon_5}\right) + 2.$$

After that, we pick N_2, N_3 *very large so that*

$$\left(\int_0^t g(s)\,ds\right) N_2 > \left(\frac{\gamma^2(1+l^2)}{4l} + \frac{\gamma N_4}{2} + \frac{\gamma^2}{4l}\right) N_4 + l + 1,$$

$$\left(\int_0^t g(s)\,ds\right) N_3 > \frac{\gamma^2(1+l^4)}{2bl^2} N_1 + \frac{\gamma^2}{4l} N_4 + \left(\gamma + N_{10}\frac{b+k}{4}\right) N_{10} + \frac{\gamma}{4l} + 1.$$

Finally, we choose N *sufficiently large so that* (3.70) *remains valid and*

$$\frac{\gamma}{2m_1} g(t) N + \lambda_1 > 0,$$

$$\frac{\gamma}{2m_2} g(t) N + \lambda_2 > 0.$$

Therefor, (3.72) *takes the form*

$$\begin{aligned} L'(t) \leq\ & -(\lambda_0 - C_{N_2,N_3,N_4,N_7,N_8,N_{10}}\delta)E(t) \\ & + c\int_0^1 \left[(g \circ \theta_{1x}) + (g \circ \theta_{2x}) + (g \circ \theta_{1tx}) + (g \circ \theta_{2tx})\right] dx \\ & + cg(t)\left[E(0) + \tilde{E}(0) + \int_0^1 (\theta_{10xx}^2 + \theta_{20xx}^2)dx\right] - \frac{\lambda_1}{\delta} E'(t), \end{aligned}$$

for some positive constant λ_0, λ_1, C. *At this point, we take* $\delta < \frac{\lambda_0}{C}$. *So for some* $c > 0$, *we obtain*

$$\begin{aligned} L'(t) + cE'(t) \leq\ & -cE(t) + c\int_0^1 \left[(g \circ \theta_{1x}) + (g \circ \theta_{2x}) + (g \circ \theta_{1tx}) + (g \circ \theta_{2tx})\right] dx \\ & + cg(t)\left[E(0) + \tilde{E}(0) + \int_0^1 (\theta_{10xx}^2 + \theta_{20xx}^2)dx\right]. \end{aligned} \tag{3.73}$$

It is clear that

$$L_1(t) = L(t) + cE(t) \sim \left(E(t) + \tilde{E}(t)\right).$$

Multiplying (3.73) *by* $\zeta(t)$ *, and by using the assumption* $(G2)$, *we have*

$$\zeta(t)(g \circ \theta_{ix}) \leq -(g' \circ \theta_{ix}), \qquad \zeta(t)(g \circ \theta_{itx}) \leq -(g' \circ \theta_{itx}), \quad \text{for } i = 1, 2,$$

we deduce

$$\begin{aligned}\zeta(t)E(t) &\leq -c\zeta(t)L_1^{'}(t)-c\int\limits_0^1\left[(g^{'}\circ\theta_{1x})+(g^{'}\circ\theta_{2x})+(g^{'}\circ\theta_{1tx})+(g^{'}\circ\theta_{2tx})\right]dx\\ &\quad +c\left[E(0)+\tilde{E}(0)+\int\limits_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\zeta(t)g(t). \qquad (3.74)\end{aligned}$$

By (3.8),(3.62)*, we get*

$$-c\int\limits_0^1\left[(g^{'}\circ\theta_{1x})+(g^{'}\circ\theta_{2x})\right]dx\leq -cE^{'}(t), \qquad (3.75)$$

and

$$-c\int\limits_0^1\left[(g^{'}\circ\theta_{1tx})+(g^{'}\circ\theta_{2tx})\right]dx\leq -c\tilde{E}^{'}(t)-cg(t)\int\limits_0^1\left[\theta_{10xx}\theta_{1tt}+\theta_{20xx}\theta_{2tt}\right]dx.$$

By using (3.68)*, we obtain*

$$-c\int\limits_0^1\left[(g^{'}\circ\theta_{1tx})+(g^{'}\circ\theta_{2tx})\right]dx\leq -c\tilde{E}^{'}(t)+cg(t)\left(\tilde{E}(0)+\int\limits_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right). \qquad (3.76)$$

Noting that $E(t)$ *and* $\zeta(t)$ *are non-increasing, integrating* (3.74) *over* $(0,t)$ *and using* (3.75)-(3.76)*, we obtain for any* $t>t_0>0$.

$$\begin{aligned}E(t)\int\limits_0^t\zeta(s)ds &\leq \int\limits_0^t E(s)\zeta(s)ds=\int\limits_0^{t_0}E(s)\zeta(s)ds+\int\limits_{t_0}^t E(s)\zeta(s)ds\\ &\leq t_0E(0)\zeta(0)+c\left[\zeta(t_0)L_1(t_0)-\zeta(t)L_1(t)+\int\limits_{t_0}^t\zeta^{'}(s)L_1(s)ds\right]\\ &\quad +c\left[E(t_0)-E(t)+\tilde{E}(t_0)-\tilde{E}(t)\right]\\ &\quad +c\left[\tilde{E}(0)+\int\limits_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\int\limits_{t_0}^t g(s)ds\\ &\quad +c\left[E(0)+\tilde{E}(0)+\int\limits_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\int\limits_{t_0}^t\zeta(s)g(s)ds. \qquad (3.77)\end{aligned}$$

Noting the fact $0\leq L_1(t)\leq c\left(E(t)+\tilde{E}(t)\right)$ *and* $\zeta(t)$ *is non-increasing, we can get*

$$L_1(t_0)\leq c(E(t_0)+\tilde{E}(t_0)),\qquad -\zeta(t)L_1(t)+\int\limits_{t_0}^t\zeta^{'}(s)L_1(s)ds\leq 0,$$

and

$$\int\limits_{t_0}^t\zeta(s)g(s)ds\leq\zeta(0)\int\limits_0^t g(s)ds,$$

which, together with (3.77), gives us

$$\begin{aligned}
E(t)\int_0^t \zeta(s)ds &\leq c\left[E(0)+E(t_0)\right]+c\left[E(t_0)+\tilde{E}(t_0)\right] \\
&\quad +c\left[\tilde{E}(0)+\int_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\int_0^t g(s)ds \\
&\quad +c\left[E(0)+\tilde{E}(0)+\int_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\int_0^t g(s)ds \\
&\leq c\left[E(0)+\tilde{E}(0)+\int_0^1(\theta_{10xx}^2+\theta_{20xx}^2)dx\right]\left(1+\int_0^t g(s)ds\right),
\end{aligned}$$

which yields (3.71). This completes the proof of Theorem 3.2. ■

CHAPTER 4

Energy decay of Bresse system with a viscoelastic term and constant delay

4.1 Introduction

In the present chapter, we are concerned with the thermoelastic Bresse system of type III with constant delay and viscoelastic term, which has the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \beta\int\limits_0^t g\left(t-s\right)\psi_{xx}\left(s\right)ds = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_{tx} = 0, \\ \rho_3\theta_{tt} - l\theta_{xx} + \gamma\omega_{tx} - \mu_1\theta_{txx}\left(x,t\right) - \mu_2\theta_{txx}\left(x,t-\tau\right) = 0, \end{cases} \tag{4.1}$$

where $(x,t) \in (0,1) \times \mathbb{R}_+$, with the following initial data and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,1), \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in (0,1), \\ \theta_{tx}\left(x,t-\tau\right) = f_0\left(x,t-\tau\right), & (x,t) \in (0,1) \times (0,\tau), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0, & t \in (0,+\infty), \\ \omega_x(0,t) = \omega_x\left(1,t\right) = \theta\left(0,t\right) = \theta\left(1,t\right) = 0, & t \in (0,+\infty), \end{cases} \tag{4.2}$$

$\rho_1, \rho_2, \rho_3, k, k_0, l, b, \beta, \gamma, \mu_1$ are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay and g is a positive function satisfying some conditions to be specified later.

The issue of existence and stability of Bresse system has attracted a great deal of attention in the last decades.

In [35], the authors considered the Bresse system with finite memory of the following form

$$\begin{cases} \rho_1\varphi_{tt} - k_1\left(\varphi_x + \psi + l\omega\right)_x - k_3 l\left(\omega_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x + \psi + l\omega\right) + \int\limits_0^t g\left(t-s\right)\psi_{xx}\left(s\right)ds = 0, \\ \rho_1\omega_{tt} - k_3\left(\omega_x - l\varphi\right)_x + k_1 l\left(\varphi_x + \psi + l\omega\right) = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = \omega_x(0,t) = \omega_x(L,t) = 0, \\ \varphi(.,0) = \varphi_0, \varphi_t(.,0) = \varphi_1, \\ \psi(.,0) = \psi_0, \psi_t(.,0) = \psi_1, \\ \omega(.,0) = \omega_0, \omega_t(.,0) = \omega_1, \end{cases}$$

in $(0, L) \times (0, +\infty)$, where g is a positive strictly increasing function satisfying, for some nonnegative functions ξ and H,

$$g'(t) \leq -\xi(t)H\left(g(t)\right), \qquad \forall t \geq 0.$$

They proved some new decay results of energy decay associated with this system in the case of equal and non-equal speed of wave propagation, under appropriate conditions on ξ and H.

Li et al. [32] were interested in Bresse system, but this time with second sound and delay term, as shown in the following system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu\varphi_t(x, t-\tau_0) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\theta_t + \beta q + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases}$$

They proved the result of the existence and uniqueness of the solutions by using the semi-group method, and under a similar condition on the previous parameters, i.e.

$$\zeta = \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\gamma^2\rho_1}{bk\rho_3} = 0 \quad and \quad k = k_0,$$

they showed that the dissipation induced by the heat is strong enough to exponentially stabilize the system in the presence of a "small" delay when the stable number is zero.

In the same context, Houasni et al. [28] studied the same type of Bresse system with finite memory and two constant delay terms positioned within the system as follows

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau_1) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \delta\int\limits_0^t g\left(t-s\right)\psi_{xx}\left(s\right)ds + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \lambda_1\omega_t + \lambda_2\omega_t(x, t-\tau_2) = 0, \\ \rho_3\theta_t + \beta q + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases}$$

where they proved the energy decay result regardless of the stable number ζ.

Motivated by the works mentioned above, we show that even if we use the thermoelasticity type III instead of the thermoelasticity by second sound and in the presence of one delay term we can establish an energy decay result. We prove our result by using the energy method together with some hypotheses on the weights of the delay and the frictional damping as well the relaxation function g.

The chapter is organized as follows: In Section 2, we introduce some preliminary results. In Section 3, we use the Lyapunov functional to establish an stability result of the energy when $\frac{k}{\rho_1} = \frac{b}{\rho_2}, k = k_0$, and l is small enough.

4.2 Preliminaries

In this section, we present the mathematical bases to be used later for the proof of our stability result. We will use the following assumptions related to the function g

$(G1)$ $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad b - \beta \int_0^\infty g(s)ds = \lambda > 0, \tag{4.3}$$

$(G2)$ There exists a non-increasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\zeta(t)g(t), \qquad t \geq 0 \quad \text{and} \quad \int_0^\infty \zeta(t)dt = +\infty. \tag{4.4}$$

We introduce the new variable as in [39]

$$z(x,\rho,t) = \theta_{tx}(x, t - \tau\rho), \qquad x \in (0,1), \rho \in (0,1), t > 0. \tag{4.5}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad x \in (0,1), \rho \in (0,1), t > 0.$$

Hence, problem (4.1)-(4.2) is equivalent to the following system, where $(x,\rho,t) \in (0,1) \times (0,1) \times \mathbb{R}_+$

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \beta\int_0^t g\left(t-s\right)\psi_{xx}\left(s\right)ds = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_{tx} = 0, \\ \rho_3\theta_{tt} - l\theta_{xx} + \gamma\omega_{tx} - \mu_1\theta_{txx}\left(x,t\right) - \mu_2 z_x\left(x,1,t\right) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{cases} \tag{4.6}$$

with the following initial data and boundary conditions

$$\begin{cases} \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right), & x\in(0,1), \\ \psi\left(x,0\right)=\psi_0\left(x\right),\psi_t\left(x,0\right)=\psi_1\left(x\right), & x\in(0,1), \\ \omega\left(x,0\right)=\omega_0\left(x\right),\ \omega_t\left(x,0\right)=\omega_1\left(x\right), & x\in(0,1), \\ \theta\left(x,0\right)=\theta_0\left(x\right),\ \theta_t\left(x,0\right)=\theta_1\left(x\right), & x\in(0,1), \\ z(x,\rho,0)=f_0(x,-\rho\tau), & (x,\rho)\in(0,1)\times(0,1) \\ z(x,0,t)=\theta_{tx}(x,t), & (x,t)\in(0,1)\times(0,+\infty), \\ \varphi\left(0,t\right)=\varphi\left(1,t\right)=\psi_x\left(0,t\right)=\psi_x\left(1,t\right)=0, & t\in(0,+\infty), \\ \omega_x\left(0,t\right)=\omega_x\left(1,t\right)=\theta\left(0,t\right)=\theta\left(1,t\right)=0, & t\in(0,+\infty). \end{cases} \tag{4.7}$$

In order to be able to use Poincaré's inequality for ψ and ω, integrating on $[0,1]$ the second and third equations in (4.6), and using the boundary conditions, we verify that

$$\frac{d^2}{dt^2}\int_0^1\psi dx+\frac{k}{\rho_2}\int_0^1\psi dx+\frac{kl}{\rho_2}\int_0^1\omega dx=0, \tag{4.8}$$

and

$$\frac{d^2}{dt^2}\int_0^1\omega dx+\frac{kl^2}{\rho_1}\int_0^1\omega dx+\frac{kl}{\rho_1}\int_0^1\psi dx=0. \tag{4.9}$$

According to (4.8) we have

$$\int_0^1\omega dx=-\frac{\rho_2}{kl}\frac{d^2}{dt^2}\int_0^1\psi dx-\frac{1}{l}\int_0^1\psi dx. \tag{4.10}$$

Substituting (4.10) into (4.9), we get

$$\frac{d^4}{dt^4}\int_0^1\psi dx+\left(\frac{k}{\rho_2}+\frac{kl^2}{\rho_1}\right)\frac{d^2}{dt^2}\int_0^1\psi dx=0. \tag{4.11}$$

Let $l_0=\sqrt{\frac{k}{\rho_2}+\frac{kl^2}{\rho_1}}$. Then, solving the ODE (4.11), we find

$$\int_0^1\psi dx=a_1\cos(l_0t)+a_2\sin(l_0t)+a_3t+a_4, \tag{4.12}$$

where $a_1,...,a_4$ are real constants. By combining (4.10) and (4.12), we get

$$\int_0^1\omega dx=a_1\left(\frac{\rho_2l_0^2}{kl}-\frac{1}{l}\right)\cos(l_0t)+a_2\left(\frac{\rho_2l_0^2}{kl}-\frac{1}{l}\right)\sin(l_0t)-\frac{a_3}{l}t-\frac{a_4}{l}. \tag{4.13}$$

Using the initial data of ψ and ω in (4.6), we deduce

$$\begin{cases} a_1=\frac{k}{\rho_2l_0^2}\int_0^1\psi_0dx+\frac{lk}{\rho_2l_0^2}\int_0^1\omega_0dx, \\ a_2=\frac{k}{\rho_2l_0^3}\int_0^1\psi_1dx+\frac{lk}{\rho_2l_0^3}\int_0^1\omega_1dx, \\ a_3=\left(1-\frac{k}{\rho_2l_0^2}\right)\int_0^1\psi_1dx-\frac{lk}{\rho_2l_0^2}\int_0^1\omega_1dx, \\ a_4=\left(1-\frac{k}{\rho_2l_0^2}\right)\int_0^1\psi_0dx-\frac{lk}{\rho_2l_0^2}\int_0^1\omega_0dx. \end{cases}$$

Therefore, the new variables are defined as follows

$$\begin{aligned}\tilde{\psi} &= \psi - \left(a_1\cos(l_0 t) + a_2\sin(l_0 t) + a_3 t + a_4\right),\\ \tilde{\omega} &= \omega - \left(a_1\left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right)\cos(l_0 t) + a_2\left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right)\sin(l_0 t) - \frac{a_3}{l}t - \frac{a_4}{l}\right),\end{aligned}$$

to check that

$$\int_0^1 \tilde{\psi}(x,t)dx = \int_0^1 \tilde{\omega}(x,t)dx = 0, \qquad \forall t \geq 0.$$

In this case, Poincaré's inequality is applicable for $\tilde{\psi}$ and $\tilde{\omega}$ in the Sobolev space $H_*^1(0,1)$, where

$$\begin{aligned}L_*^2(0,1) &= \left\{v \in L^2(0,1) : \int_0^1 v(x)dx = 0\right\},\\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1),\\ H_*^2(0,1) &= \left\{v \in H^2(0,1) : v_x(0) = v_x(1) = 0\right\},\end{aligned}$$

furthermore, $(\varphi, \tilde{\psi}, \tilde{\omega}, \theta, z)$ satisfies the equations and the boundary conditions in (4.6),(4.7) with the initial data

$$\begin{aligned}\tilde{\psi}_0 &= \psi_0 - (a_1 + a_4), \qquad \tilde{\psi}_1 = \psi_1 - (l_0 a_2 + a_3),\\ \tilde{\omega}_0 &= \omega_0 - \left(a_1\left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right) - \frac{a_4}{l}\right), \quad \tilde{\omega}_1 = \omega_1 - \left(a_2 l_0\left(\frac{\rho_2 l_0^2}{kl} - \frac{1}{l}\right) - \frac{a_3}{l}\right).\end{aligned}$$

In what follows, we work with $\tilde{\psi}$, $\tilde{\omega}$ and, respectively, write ψ, ω for convenience.

For completeness, we state, without proof, the following existence and uniqueness result which can be proved by using the Faedo-Galerkin method.

Theorem 4.1 *Let*

$$\begin{aligned}(\varphi_0, \varphi_1), (\theta_0, \theta_1) &\in H_0^1(0,1) \times L_2(0,1),\\ (\psi_0, \psi_1), (\omega_0, \omega_1) &\in H_*^1(0,1) \times L_*^2(0,1),\\ f_0 &\in L^2\left((0,1)\times(0,1)\right).\end{aligned}$$

Assume that $|\mu_2| < \mu_1$, (G1) and (G2) are satisfied. Then there exists a unique weak solution $(\varphi, \psi, \omega, \theta, z)$ of problem (4.6)-(4.7) such that

$$\begin{aligned}\varphi, \theta &\in C\left(\mathbb{R}_+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0,1)\right),\\ \psi, \omega &\in C\left(\mathbb{R}_+; H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L_*^2(0,1)\right),\\ z &\in C\left(\mathbb{R}_+; L^2\left((0,1)\times(0,1)\right)\right).\end{aligned}$$

4.2. Preliminaries

4.3 Energy stability result

In this section, we state and prove our energy decay result for the energy of the solution of system (4.6)-(4.7), using the Lyapunov functional which is equivalent to the energy functional. To achieve our goal, we need the following technical lemmas.

Lemma 4.1 *Let* $(\varphi, \psi, \omega, \theta, z)$ *be the solution of* (4.6)-(4.7)*. Then the energy functional, defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \rho_3\theta_t^2 + \xi\int_0^1 z^2(x,\rho,t)d\rho + l\theta_x^2 \\ &\quad + k\left(\varphi_x + \psi + l\omega\right)^2 + k_0\left(\omega_x - l\varphi\right)^2 + \beta(g\circ\psi_x) + (b - \beta\int_0^t g(s)ds)\psi_x^2\Big]\, dx, \end{aligned} \tag{4.14}$$

satisfies, for some $C_0 \geq 0$,

$$E^{'}(t) \leq \frac{\beta}{2}(g^{'}\circ\psi_x) - \frac{\beta}{2}g(t)\int_0^1 \psi_x^2 dx - C_0\left(\int_0^1 \theta_{tx}^2 dx + \int_0^1 z^2(x,1,t)dx\right) \leq 0,$$

where

$$\tau\left|\mu_2\right| < \xi < \tau(2\mu_1 - |\mu_2|). \tag{4.15}$$

Proof. *Multiplying Equation* $(4.6)_1$ *by* φ_t *,* $(4.6)_2$ *by* ψ_t*,* $(4.6)_3$ *by* ω_t *and*$(4.6)_4$ *by* θ_t *and integrating over* $(0,1)$*and* $(4.6)_5$ *by* $(\xi/\tau)z$ *and integrating over* $(0,1)\times(0,1)$*with respect to* ρ *and* x*, we get after summing up*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \rho_3\theta_t^2 + \xi\int_0^1 z^2(x,\rho,t)d\rho + l\theta_x^2 + b\psi_x^2 \\ &\quad + k\left(\varphi_x + \psi + l\omega\right)^2 + k_0\left(\omega_x - l\varphi\right)^2\Big]\, dx + \mu_1\int_0^1 \theta_{tx}^2 dxx \\ &\quad + \mu_2\int_0^1 z(x,1,t)\theta_{tx}dx + \frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx + \beta\int_0^1 \psi_t\int_0^t g(t-s)\psi_{xx}(s)dsdx \\ &= 0. \end{aligned} \tag{4.16}$$

Now, we estimate the last three terms on the left-hand side of the above equation. By using (1.6)*, we obtain*

$$\begin{aligned} \beta\int_0^1 \psi_t(t)\int_0^t g(t-s)\psi_{xx}(s)dsdx &= \frac{\beta}{2}\frac{d}{dt}(g\circ\psi_x) + \frac{\beta}{2}g(t)\int_0^1 \psi_x^2(t)dx \\ &\quad - \frac{\beta}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_0^1 \psi_x^2(t)dx\right) - \frac{\beta}{2}(g^{'}\circ\psi_x), \end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx &= \frac{\xi}{\tau}\int_0^1\int_0^1 \frac{d}{2d\rho}z^2(x,\rho,t)d\rho dx \\ &= \frac{\xi}{2\tau}\int_0^1[z^2(x,1,t)-z^2(x,0,t)]dx \\ &= \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)dx-\frac{\xi}{2\tau}\int_0^1\theta_{tx}^2dx, \qquad (4.18)\end{aligned}$$

$$-\mu_2\int_0^1 z(x,1,t)\theta_{tx}dx \le \frac{|\mu_2|}{2}\left(\int_0^1\theta_{tx}^2dx+\int_0^1 z^2(x,1,t)dx\right). \qquad (4.19)$$

So, we conclude

$$\begin{aligned}E^{'}(t) \ \le\ & \frac{\beta}{2}(g^{'}\circ\psi_x)-\frac{\beta}{2}g(t)\int_0^1\psi_x^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^1\theta_{tx}^2dx \\ & -\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)dx.\end{aligned}$$

Using (4.15)*, we have, for some* $C_0>0$*,*

$$E^{'}(t)\le\frac{\beta}{2}(g^{'}\circ\psi_x)-\frac{\beta}{2}g(t)\int_0^1\psi_x^2dx-C_0\left(\int_0^1\theta_{tx}^2dx+\int_0^1 z^2(x,1,t)dx\right)\le 0.$$

■

Now, we introduce the multiplier p given by the solution of the Dirichlet problem

$$-p_{xx}=\psi_x, \quad p(0)=p(1)=0,$$

then we can obtain the following inequality

$$\int_0^1 p_t^2dx\le\int_0^1 p_{tx}^2dx\le\int_0^1\psi_t^2dx, \qquad (4.20)$$

$$\int_0^1 p^2dx\le\int_0^1 p_x^2dx\le\int_0^1\psi^2dx\le\int_0^1\psi_x^2dx, \qquad (4.21)$$

and we define the functional

$$I_1(t)=-\rho_2\int_0^1\psi_tp_xdx-l\int_0^1(\rho_2l\psi_t-\rho_1\omega_t)\int_0^x p(y)dydx+\rho_1\int_0^1\varphi_tpdx. \qquad (4.22)$$

So, we have the following lemma.

Lemma 4.2 *Let* $(\varphi,\psi,\omega,\theta)$ *be a solution of* (4.6)-(4.7)*. Then we have for any* $\varepsilon_1>0$*,*

$$\begin{aligned}I_1^{'}(t)\ \le\ & -\left(\lambda-4bl^2\right)\int_0^1\psi_x^2dx+\frac{\gamma^2}{4b}\int_0^1\theta_{tx}^2dx+C_1(\varepsilon_1)\int_0^1\psi_t^2dx \\ & +\varepsilon_1\int_0^1\omega_t^2dx+\varepsilon_1\int_0^1\varphi_t^2dx+C\int_0^1(g\circ\psi_x)dx, \qquad (4.23)\end{aligned}$$

where $C_1(\varepsilon_1)=\left(\rho_2+\frac{\rho_1^2}{4\varepsilon_1}+\rho_2l^2+\frac{\rho_1^2l^2}{4\varepsilon_1}\right).$

Proof. By differentiating the expression of I_1, using the first and second equations of (4.6), and integrating by parts, we obtain

$$\begin{aligned} I_1^{'} = & -\rho_2\int_0^1 \psi_t p_{tx} dx + b\int_0^1 \psi_x p_{xx} dx - \beta\int_0^1 \left(g * \psi_x\right) p_{xx} dx + \rho_1\int_0^1 \varphi_t p_t dx \\ & + bl^2\int_0^1 \psi_x p dx - l\int_0^1 \left(\beta_1 l \left(g * \psi_x\right) - \gamma\theta_t\right) p dx \\ & - l\int_0^1 \left(\rho_2 l\psi_t - \rho_1\omega_t\right)\int_0^x p_t(y) dy dx. \end{aligned}$$

Using the fact that $p_{xx} = -\psi_x$,we get

$$\begin{aligned} I_1^{'}(t) = & -\rho_2\int_0^1 \psi_t p_{tx} dx - b\int_0^1 \psi_x^2 dx + \beta\int_0^1 \left(g * \psi_x\right)\psi_x dx + \rho_1\int_0^1 \varphi_t p_t dx \\ & + bl^2\int_0^1 p_x^2 dx + \gamma l\int_0^1 \theta_t p dx - \beta l^2\int_0^1 \left(g * \psi_x\right) p dx \\ & - \rho_2 l^2\int_0^1 \psi_t\int_0^x p_t(y) dy dx + \rho_1 l\int_0^1 \omega_t\int_0^x p_t(y) dy dx, \end{aligned} \tag{4.24}$$

by using Young's inequality, Poincaré's inequality, (4.20) and (4.21), we obtain

$$\begin{aligned} \beta\int_0^1 \left(g * \psi_x\right)\psi_x dx = & \ \beta\left(\int_0^t g(s) ds\right)\int_0^1 \psi_x^2 dx - \beta\int_0^1 \left(g \diamond \psi_x\right)\psi_x dx \\ \leq & \ \beta\bar{g}\int_0^1 \psi_x^2 dx - \beta_1\int_0^1 \left(g \diamond \psi_x\right)\psi_x dx \\ \leq & \ \beta\bar{g}\int_0^1 \psi_x^2 dx + \delta\int_0^1 \psi_x^2 dx + \frac{\beta_1^2\bar{g}}{4\delta}\int_0^1 \left(g \circ \psi_x\right) dx, \end{aligned} \tag{4.25}$$

$$\begin{aligned} -\beta l^2\int_0^1 \left(g * \psi_x\right) p dx = & \ \beta l^2\int_0^1 \left(g * p_{xx}\right) p dx = -\beta l^2\int_0^1 \left(g * p_x\right) p_x dx \\ = & \ \beta l^2\int_0^1 \left(g \diamond p_x\right) p_x dx - \beta l^2\left(\int_0^t g(s) ds\right)\int_0^1 p_x^2 dx \\ \leq & \ \beta l^2\int_0^1 \left(g \diamond p_x\right) p_x dx \leq \delta\int_0^1 p_x^2 dx + \frac{\beta^2 l^4}{4\delta}\int_0^1 \left(g \diamond p_x\right)^2 dx \\ \leq & \ \delta\int_0^1 \psi_x^2 dx + \frac{c}{\delta}\int_0^1 \left(g \circ \psi_x\right) dx, \end{aligned} \tag{4.26}$$

$$\rho_1\int_0^1 \varphi_t p_t dx \leq \varepsilon_1\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_1}\int_0^1 \psi_t^2 dx, \tag{4.27}$$

$$\gamma l\int_0^1 \theta_t p dx \leq bl^2\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{4b}\int_0^1 \theta_{tx}^2 dx, \tag{4.28}$$

and

$$\rho_1 l \int_0^1 \omega_t \int_0^x p_t(y)dydx \leq \varepsilon_1 \int_0^1 \omega_t^2 dx + \frac{\rho_1^2 l^2}{4\varepsilon_1} \int_0^1 \psi_t^2 dx. \tag{4.29}$$

We obtain the result by substituting these last inequalities in (4.24). ■

Lemma 4.3 *Let* $(\varphi, \psi, \omega, \theta, z)$ *be a solution of* (4.6)-(4.7)*. Then the functional*

$$I_2(t) = -\rho_2 \int_0^1 \psi_t \left(\int_0^t g(t-s)(\psi(t) - \psi(s)ds \right) dx, \tag{4.30}$$

satisfies for any $\varepsilon_2 > 0$ *and* $0 < \delta < 1$

$$\begin{aligned} I_2'(t) \leq & -\left(\rho_2 \int_0^t g(s)\, ds - \delta \right) \int_0^1 \psi_t^2 dx + \varepsilon_2 \left(1 + \bar{g}^2\right) \int_0^1 \psi_x^2 dx \\ & + \varepsilon_2 \int_0^1 (\varphi_x + \psi + l\omega)^2 dx - \frac{\rho_2^2 g(0)}{4\delta} \int_0^1 (g' \circ \psi_x)\, dx \\ & + C(\varepsilon_2) \int_0^1 (g \circ \psi_x)\, dx, \end{aligned} \tag{4.31}$$

where $C(\varepsilon_2) = \frac{\bar{g}}{4\varepsilon_2}\left(b^2 + 2\beta_1^2 + k^2 + 4\varepsilon_2^2\right).$

Proof. *Taking the derivative of* I_2*, using the second equation in* (4.6)*, we obtain*

$$\begin{aligned} I_2'(t) = & -\rho_2 \int_0^1 \psi_t (g' \diamond \psi)\, dx - \rho_2 \left(\int_0^t g(s)\, ds \right) \int_0^1 \psi_t^2 dx \\ & + b \int_0^1 (g \diamond \psi_x)\, \psi_x dx + k \int_0^1 (\varphi_x + \psi + lw)(g \diamond \psi)\, dx \\ & - \beta_1 \int_0^1 (g * \psi_x)(g \diamond \psi_x)\, dx. \end{aligned} \tag{4.32}$$

By using Young's inequality, (1.7) *and* (1.8)*, we get, for any* $\varepsilon_2 > 0$ *and* $0 < \delta < 1$

$$-\int_0^1 \psi_t (g' \diamond \psi)\, dx \leq \delta \int_0^1 \psi_t^2 dx - \frac{\rho_2 g(0)}{4\delta} \int_0^1 (g' \circ \psi_x)\, dx, \tag{4.33}$$

$$k \int_0^1 (\varphi_x + \psi + lw)(g \diamond \psi)\, dx \leq \varepsilon_2 \int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{\bar{g} k^2}{4\varepsilon_2} \int_0^1 (g \circ \psi_x)\, dx, \tag{4.34}$$

$$b \int_0^1 (g \diamond \psi_x)\, \psi_x dx \leq \varepsilon_2 \int_0^1 \psi_x^2 dx + \frac{\bar{g} b^2}{4\varepsilon_2} \int_0^1 (g \circ \psi_x)\, dx, \tag{4.35}$$

and

$$-\beta_1 \int_0^1 (g * \psi_x)(g \diamond \psi_x)\, dx \leq \varepsilon_2 \bar{g}^2 \int_0^1 \psi_x^2 dx + \bar{g} \left(\varepsilon_2 + \frac{\beta_1^2}{2\varepsilon_2} \right) \int_0^1 (g \circ \psi_x)\, dx. \tag{4.36}$$

Combining (4.32)-(4.36)*, we obtain the desired result.* ■

4.3. Energy stability result

Lemma 4.4 *Let* $(\varphi,\psi,\omega,\theta,z)$ *be a solution of* (4.6)-(4.7). *Then the functional*

$$I_3(t) = -\rho_1 \int_0^1 (\varphi\varphi_t + \omega\omega_t)\, dx, \tag{4.37}$$

satisfies, for any $0<\delta<1$, *the estimate*

$$\begin{aligned} I_3'(t) \quad &\leq \quad -\rho_1\int_0^1 \varphi_t^2 dx - \rho_1 \int_0^1 \omega_t^2 dx + (2k + c\delta)\int_0^1 (\varphi_x+\psi+l\omega)^2\, dx \\ &\quad + (k_0 + c\delta)\int_0^1 (\omega_x - l\varphi)^2\, dx + \left(\frac{k}{4}+c\delta\right)\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{4\delta}\int_0^1 \theta_{tx}^2 dx. \end{aligned} \tag{4.38}$$

Proof. Taking the derivative of I_3, by using equations in (4.6), we get

$$\begin{aligned} I_3'(t) \quad &= \quad -\rho_1\int_0^1 \varphi_t^2 dx - \rho_1\int_0^1 \omega_t^2 dx + k\int_0^1 (\varphi_x+\psi+l\omega)^2\, dx \\ &\quad + k_0\int_0^1 (\omega_x - l\varphi)^2\, dx - \gamma\int_0^1 \theta_t\omega_x dx - k\int_0^1 (\varphi_x+\psi+l\omega)\,\psi dx. \end{aligned} \tag{4.39}$$

We obtain the result by using (1.10), Young's and Poincaré's inequalities to estimate the last two terms on the right-hand side of (4.39), as follows

$$-k\int_0^1 (\varphi_x+\psi+l\omega)\,\psi dx \leq k\int_0^1 (\varphi_x+\psi+l\omega)^2\, dx + \frac{k}{4}\int_0^1 \psi_x^2 dx,$$

$$-\gamma\int_0^1 \theta_t\omega_x dx \leq \delta\int_0^1 \omega_x^2 dx + \frac{\gamma^2}{4\delta}\int_0^1 \theta_t^2 dx.$$

■

Lemma 4.5 *Let* $(\varphi,\psi,\omega,\theta,z)$ *be a solution of* (4.6)-(4.7), *and let* $k=k_0$. *The functional* I_4 *defined by*

$$I_4(t) = -\rho_1\int_0^1 (\omega_x - l\varphi)\int_0^x \omega_t(y)dydx - \rho_1\int_0^1 \varphi_t \int_0^x (\varphi_x+\psi+l\omega)(y)dydx, \tag{4.40}$$

satisfies, for any $0<\delta<1$,

$$\begin{aligned} I_4'(t) \quad &\leq \quad \rho_1\int_0^1 \omega_t^2 dx + \rho_1(l-1)\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4l}\int_0^1 \psi_t^2 dx + \frac{\gamma}{4\delta}\int_0^1 \theta_t^2 dx \\ &\quad + (\delta\gamma - k_0)\int_0^1 (\omega_x - l\varphi)^2\, dx + k\int_0^1 (\varphi_x+\psi+l\omega)^2\, dx. \end{aligned} \tag{4.41}$$

Proof. By differentiating I_3, then exploiting the first and third equations in (4.6), under the condition $k = k_0$, we get

$$\begin{aligned} I_4'(t) &= \rho_1 \int_0^1 \omega_t^2 dx - \rho_1 \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (\omega_x - l\varphi)^2 dx + k \int_0^1 (\varphi_x + \psi + l\omega)^2 dx \\ &\quad - \rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y) dy dx + \gamma \int_0^1 (\omega_x - l\varphi) \theta_t dx. \end{aligned} \tag{4.42}$$

By using Young's and Poincaré's inequalities, we obtain

$$-\rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y) dy dx \leq \rho_1 l \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4l} \int_0^1 \psi_t^2 dx, \tag{4.43}$$

and

$$\gamma \int_0^1 (\omega_x - l\varphi) \theta_t dx \leq \delta\gamma \int_0^1 (\omega_x - l\varphi)^2 dx + \frac{\gamma}{4\delta} \int_0^1 \theta_t^2 dx. \tag{4.44}$$

Combining (4.42)-(4.44), we obtain the desired result. ■

Lemma 4.6 *Let $(\varphi, \psi, \omega, \theta, z)$ be the solution of (4.6)-(4.7) and let $k = k_0$. Then, the functional*

$$I_5(t) = -\rho_1 \int_0^1 \varphi_t (\omega_x - l\varphi) dx - \rho_1 \int_0^1 \omega_t (\varphi_x + \psi + l\omega) dx, \tag{4.45}$$

satisfies the estimate

$$\begin{aligned} I_5'(t) &\leq -k_0 l \int_0^1 (\omega_x - l\varphi)^2 dx + \rho_1 l \int_0^1 \varphi_t^2 dx + \frac{\gamma^2}{2kl} \int_0^1 \theta_{tx}^2 dx + \frac{\rho_1}{2l} \int_0^1 \psi_t^2 dx \\ &\quad + \frac{3kl}{2} \int_0^1 (\varphi_x + \psi + l\omega)^2 dx - \frac{\rho_1 l}{2} \int_0^1 \omega_t^2 dx. \end{aligned} \tag{4.46}$$

Proof. A simple differentiation of I_5, using the first and third equations in (4.6), leads to

$$\begin{aligned} I_5'(t) &= -k_0 l \int_0^1 (\omega_x - l\varphi)^2 dx + \rho_1 l \int_0^1 \varphi_t^2 dx + kl \int_0^1 (\varphi_x + \psi + l\omega)^2 dx \\ &\quad - \rho_1 l \int_0^1 \omega_t^2 dx + \gamma \int_0^1 \theta_{tx} (\varphi_x + \psi + l\omega) dx - \rho_1 \int_0^1 \omega_t \psi_t dx. \end{aligned} \tag{4.47}$$

Use of Young's inequality for the last two terms in the right-hand side of (4.47) , we get

$$\gamma \int_0^1 \theta_{tx} (\varphi_x + \psi + l\omega) dx \leq \frac{kl}{2} \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \frac{\gamma^2}{2kl} \int_0^1 \theta_{tx}^2 dx, \tag{4.48}$$

and

$$-\rho_1 \int_0^1 \omega_t \psi_t dx \leq \frac{\rho_1 l}{2} \int_0^1 \omega_t^2 dx + \frac{\rho_1}{2l} \int_0^1 \psi_t^2 dx. \tag{4.49}$$

By combining these last three relations, we obtain the desired result, under the condition $k = k_0$. ■

4.3. Energy stability result

Lemma 4.7 *Let* $(\varphi, \psi, \omega, \theta, z)$ *be the solution of* (4.6)-(4.7)*, and let* $k = k_0, \frac{k}{\rho_1} = \frac{b}{\rho_2}$. *Then the functional*

$$\begin{aligned} I_6(t) &= \rho_2 \int_0^1 \psi_t \left(\varphi_x + \psi + l\omega\right) dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx \\ &\quad - \frac{\beta\rho_1}{k} \int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s) ds dx, \end{aligned} \tag{4.50}$$

satisfies, for any $\varepsilon_3 > 0$ *and* $0 < \delta < 1$, *the estimate*

$$\begin{aligned} I_6'(t) &\leq -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \delta \int_0^1 \varphi_t^2 dx + l\varepsilon_3 \int_0^1 \omega_t^2 dx \\ &\quad + \left(lC(\varepsilon_3) + \frac{c}{\delta} g(0)\right) \int_0^1 \psi_x^2 dx + \left(\rho_2 + \frac{\rho_2^2 l}{4\varepsilon_3}\right) \int_0^1 \psi_t^2 dx \\ &\quad + 2l\varepsilon_3 \int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \frac{l\beta^2 \bar{g}^2}{2\varepsilon_3} \int_0^1 (g \circ \psi_x) dx \\ &\quad + \left(\frac{b\rho_1}{k} - \rho_2\right) \int_0^1 \varphi_t \psi_{tx} dx - \frac{c}{\delta} \int_0^1 (g' \circ \psi_x) dx, \end{aligned} \tag{4.51}$$

where $C(\varepsilon_3) = \frac{b^2}{4\varepsilon_3} + \frac{\beta^2 \bar{g}^2}{2\varepsilon_3}$.

Proof. By differentiating I_6, then exploiting the first and second equations in (4.6), and for $k = k_0, \frac{k}{\rho_1} = \frac{b}{\rho_2}$, we get

$$\begin{aligned} I_6'(t) &= -k \int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ &\quad + \rho_2 l \int_0^1 \psi_t \omega_t dx + bl \int_0^1 \psi_x \left(\omega_x - l\varphi\right) dx \\ &\quad - \frac{\beta\rho_1}{k} \int_0^1 \left(g' * \psi_x\right) \varphi_t dx - l\beta \int_0^1 \left(g * \psi_x\right)\left(\omega_x - l\varphi\right) dx. \end{aligned} \tag{4.52}$$

Estimate (4.51) follows thanks to Young's inequality and the fact that

$$\rho_2 l \int_0^1 \psi_t \omega_t dx \leq l\varepsilon_3 \int_0^1 \omega_t^2 dx + \frac{\rho_2^2 l}{4\varepsilon_3} \int_0^1 \psi_t^2 dx, \tag{4.53}$$

$$bl \int_0^1 \psi_x \left(\omega_x - l\varphi\right) dx \leq l\varepsilon_3 \int_0^1 \left(\omega_x - l\varphi\right)^2 dx + \frac{lb^2}{4\varepsilon_3} \int_0^1 \psi_x^2 dx, \tag{4.54}$$

$$\begin{aligned} -\frac{\beta\rho_1}{k} \int_0^1 \left(g' * \psi_x\right) \varphi_t dx &= \frac{\beta_1\rho_1}{k} \int_0^1 \left(g' \diamond \psi_x\right) \varphi_t dx - \frac{\beta_1\rho_1}{k} \int_0^t g'(s) ds \int_0^1 \psi_x \varphi_t dx \\ &\leq \delta \int_0^1 \varphi_t^2 dx + \frac{\beta^2 \rho_1^2 g^2(0)}{2k^2\delta} \int_0^1 \psi_x^2 dx \\ &\quad - \frac{\beta^2 \rho_1^2 g(0)}{2k^2\delta} \int_0^1 \left(g' \circ \psi_x\right) dx, \end{aligned} \tag{4.55}$$

and

$$\begin{aligned}
-l\beta\int_0^1 (g*\psi_x)(\omega_x-l\varphi)\,dx &= l\beta_1\int_0^1 (g\diamond\psi_x)(\omega_x-l\varphi)\,dx \\
&\quad -l\beta_1\int_0^t g(s)ds\int_0^1(\omega_x-l\varphi)\,\psi_x dx \\
&\le l\varepsilon_3\int_0^1(\omega_x-l\varphi)^2\,dx+\frac{\beta^2 l\bar{g}^2}{2\varepsilon_3}\int_0^1\psi_x^2dx \\
&\quad +\frac{\beta^2 l\bar{g}}{2\varepsilon_3}\int_0^1(g\circ\psi_x)\,dx. \qquad (4.56)
\end{aligned}$$

■

Lemma 4.8 *Let* $(\varphi,\psi,\omega,\theta,z)$ *be the solution of* (4.6)-(4.7). *Then the functional*

$$K(t)=\int_0^1\left(\rho_3\theta\theta_t+\frac{\mu_1}{2}\theta_x^2+\gamma\theta\omega_x\right)dx, \qquad (4.57)$$

satisfies, for any $0<\delta<1$, *the estimate*

$$\begin{aligned}
K'(t) &\le \left(\rho_3+\frac{\gamma^2}{4\delta}\right)\int_0^1\theta_t^2dx+(\delta-l)\int_0^1\theta_x^2dx \\
&\quad +\frac{\mu_2^2}{4\delta}\int_0^1 z^2(x,1,t)dx+c\delta\int_0^1(\varphi_x+\psi+l\omega)^2\,dx \\
&\quad +c\delta\int_0^1(\omega_x-l\varphi)^2\,dx+c\delta\int_0^1\psi_x^2dx. \qquad (4.58)
\end{aligned}$$

Proof. Using the fourth equation of (4.6) and repeating similar computations as above, we arrive at

$$K'(t)=\rho_3\int_0^1\theta_t^2dx-l\int_0^1\theta_x^2dx-\mu_2\int_0^1\theta_x z(x,1,t)\,dx+\gamma\int_0^1\theta_t\omega_x dx, \qquad (4.59)$$

by using Young's inequality and (1.10), we get

$$\gamma\int_0^1\theta_t\omega_x dx\le\frac{\gamma^2}{4\delta}\int_0^1\theta_t^2dx+c\delta\int_0^1\left[(\varphi_x+\psi+l\omega)^2+(\omega_x-l\varphi)^2+\psi_x^2\right]dx, \qquad (4.60)$$

and

$$-\mu_2\int_0^1\theta_x z(x,1,t)\,dx\le\delta\int_0^1\theta_x^2dx+\frac{\mu_2^2}{4\delta}\int_0^1 z^2(x,1,t)dx. \qquad (4.61)$$

Combining (4.59)-(4.61), gives the desired result. ■

Lemma 4.9 *Let* $(\varphi,\psi,\omega,\theta,z)$ *be the solution of* (4.6)-(4.7). *Then the functional* R *defined by*

$$R(t)=\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx, \qquad (4.62)$$

satisfies

$$R^{'}(t) \leq -R(t) - \frac{C_1}{\tau}\int_0^1 z^2(x,1,t)dx + \frac{1}{\tau}\int_0^1 \theta_{tx}^2 dx. \tag{4.63}$$

Proof. *By differentiating R, then by using (4.6)$_5$, and integrating by parts, we get*

$$R^{'}(t) = \frac{-1}{\tau}\int_0^1\int_0^1 e^{-\tau\rho}\frac{d}{d\rho}z^2(x,\rho,t)d\rho dx,$$

we have

$$\frac{-1}{\tau}\int_0^1\int_0^1 \frac{d}{d\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx = R(t) - \frac{1}{\tau}\int_0^1\int_0^1 e^{-\tau\rho}\frac{d}{d\rho}z^2(x,\rho,t)d\rho dx,$$

so

$$\begin{aligned} R^{'}(t) &= -R(t) - \frac{1}{\tau}\int_0^1\int_0^1 \frac{d}{d\rho}\left(e^{-\tau\rho}z^2(x,\rho,t)\right)d\rho dx \\ &\leq -R(t) - \frac{C_1}{\tau}\int_0^1 z^2(x,1,t)dx + \frac{1}{\tau}\int_0^1 \theta_{tx}^2 dx, \end{aligned}$$

for $C_1 > 0$. ■

Now, we are ready to state and prove the main result of this section.

Lemma 4.10 *We define a Lyapunov functional L as follows*

$$L(t) = NE(t) + \sum_{i=1}^{6} N_i I_i(t) + k(t) + R(t), \tag{4.64}$$

satisfies, for N_i, $i = 1, 2, ..., 6$ are positive constants to be proprerly chosen later, with sufficiently large N,

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \quad \forall t \geq 0, \tag{4.65}$$

where α_1 and α_2 are positive constants.

Theorem 4.2 *Let $(\varphi, \psi, \omega, \theta, z)$ be the solution of (4.6)-(4.7) and assume that $(G1)$, $(G2)$ (3) and $\mu_1 > |\mu_2|$ hold. Then, the energy functional (4.14) satisfies,*

$$E(t) \leq Ae^{-B\int_{t_0}^{t}\zeta(s)ds}, \quad \forall t \geq t_0, \tag{4.66}$$

where A and B are positive constants.

Proof. *A combination of the estimates of the previous lemmas and using* $\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx$ *gives*

$$\begin{aligned}
L^{'}(t) \leq\; & -\left[N_2\int_0^t g\left(s\right)ds - N_1C_1\left(\varepsilon_1\right) - N_5\frac{\rho_1}{2l} - N_6\left(\rho_2 + \frac{\rho_2^2 l}{4\varepsilon_3}\right) - \frac{\rho_1}{4l}N_4\right]\int_0^1 \psi_t^2 dx \\
& - \left[\left(\lambda - 4bl^2\right)N_1 - \frac{k}{4}N_3 - \varepsilon_2\left(1+\bar{g}^2\right)N_2 - N_6 lC(\varepsilon_3)\right]\int_0^1 \psi_x^2 dx \\
& - \left[kN_6 - 2kN_3 - \frac{3kl}{2}N_5 - N_2\varepsilon_2 - kN_4\right]\int_0^1 \left(\varphi_x + \psi + l\omega\right)^2 dx \\
& - \left[k_0 l N_5 + k_0 N_4 - k_0 N_3 - 2l\varepsilon_3 N_6\right]\int_0^1 \left(\omega_x - l\varphi\right)^2 dx - [l]\int_0^1 \theta_x^2 dx \\
& - \left[\rho_1 N_3 - N_1\varepsilon_1 - N_5\rho_1 l - \rho_1(l-1)N_4\right]\int_0^1 \varphi_t^2 dx \\
& - \left[\rho_1 N_3 - \varepsilon_1 N_1 + N_5\rho_1\frac{l}{2} - l\varepsilon_3 N_6 - \rho_1 N_4\right]\int_0^1 \omega_t^2 dx - R(t) \\
& - \left[N\mu_1 - \frac{\gamma^2}{4b}N_1 - N_5\frac{\gamma^2}{2kl} - \rho_3 - \frac{1}{\tau}\right]\int_0^1 \theta_{tx}^2 dx - \left[NC_0 + \frac{C_1}{\tau}\right]\int_0^1 z^2(x,1,t)dx \\
& + \left(CN_1 + C\left(\varepsilon_2\right)N_2 + \frac{\beta^2 l\bar{g}^2}{2\varepsilon_3}N_6\right)\int_0^1 \left(g\circ\psi_x\right)dx + N\frac{\beta}{2}\int_0^1 \left(g'\circ\psi_x\right)dx \\
& + \delta\int_0^1 \left[c(N_3+1)\left(\psi_x^2 + \left(\varphi_x + \psi + l\omega\right)^2\right) + c(N_3+N_4+1)\left(\omega_x - l\varphi\right)^2\right. \\
& \left. + N_2\psi_t^2 + N_6\varphi_t^2 + \theta_x^2 dx\right] + \frac{1}{\delta}\int_0^1 \left[-c(N_2+N_6)\left(g'\circ\psi_x\right)\right. \\
& \left. + \frac{\gamma^2}{4}(N_3+N_4+1)\theta_{tx}^2 + cg(0)N_6\psi_x^2 + \frac{\mu_2^2}{4}z^2(x,1,t)\right]dx
\end{aligned}$$

4.3. Energy stability result

By setting $N_4 = N_3$ *and* $N_5 = \frac{1}{l}N_3$ *we arrive at*

$$\begin{aligned}
L^{'}(t) \;\leq\; & -\left[N_2\int_0^t g(s)\,ds - N_1C_1(\varepsilon_1) - N_3\rho_1\left(\frac{1}{2l^2}+\frac{1}{4}\right) - N_6\rho_2\left(1+\frac{\rho_2 l}{4\varepsilon_3}\right)\right]\int_0^1 \psi_t^2dx \\
& -\left[\left(\lambda-4bl^2\right)N_1 - \frac{k}{4}N_3 - \varepsilon_2\left(1+\bar{g}^2\right)N_2 - N_6lC(\varepsilon_3)\right]\int_0^1 \psi_x^2dx \\
& -\left[kN_6 - \frac{9}{2}kN_3 - N_2\varepsilon_2\right]\int_0^1 (\varphi_x+\psi+l\omega)^2\,dx - [l]\int_0^1 \theta_x^2dx \\
& -\left[k_0N_3 - 2l\varepsilon_3N_6\right]\int_0^1 (\omega_x - l\varphi)^2\,dx - \left[\rho_1(1-l)N_3 - N_1\varepsilon_1\right]\int_0^1 \varphi_t^2dx \\
& -\left[\frac{1}{2}\rho_1N_3 - \varepsilon_1N_1 - l\varepsilon_3N_6\right]\int_0^1 \omega_t^2dx - \left[NC_0+\frac{C_1}{\tau}\right]\int_0^1 z^2(x,1,t)dx \\
& -\left[N\mu_1 - \frac{\gamma^2}{4b}N_1 - N_3\frac{\gamma^2}{2kl^2} - \rho_3 - \frac{1}{\tau}\right]\int_0^1 \theta_{tx}^2dx - R(t) + N\frac{\beta}{2}\int_0^1 (g'\circ\psi_x)\,dx \\
& +\left(CN_1 + C(\varepsilon_2)N_2 + \frac{\beta^2 l\bar{g}^2}{2\varepsilon_3}N_6\right)\int_0^1 (g\circ\psi_x)\,dx \\
& +\delta C_{N_3,N_2,N_6}E(t) - \frac{1}{\delta}C_{N_2,N_3,N_6}E^{'}(t).
\end{aligned} \tag{4.67}$$

First, let us take $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_2}$, $\varepsilon_3 = \frac{1}{N_6}$, *and choose* $l < \min\left\{1, \sqrt{\frac{\lambda}{4b}}\right\}$. *Next, we select* N_3 *large enough such that*

$$N_3 > \max\left\{\frac{2l}{k}, \frac{1}{\rho_1(1-l)}, \frac{2+2l}{\rho_1}\right\},$$

then, we choose N_1, N_6 *sufficiently large so that*

$$N_6 > \frac{9}{2}N_3 + \frac{1}{k},$$

$$\left(\lambda-4bl^2\right)N_1 > \frac{k}{4}N_3 + 1 + \bar{g}^2 + N_6^2 l\left(\frac{b^2+2\beta^2\bar{g}^2}{4}\right).$$

After that, we pick N_2 *very large so that*

$$N_2\int_0^t g(s)\,ds > \left(N_1\rho_2 + \frac{N_1^2\rho_1^2}{4}\right)(1+l^2) + N_3\rho_1\left(\frac{1}{2l^2}+\frac{1}{4}\right) + N_6\rho_2\left(1+N_6\frac{\rho_2 l}{4}\right).$$

Finally, we choose N *sufficiently large to satisfy*

$$N\mu_1 - \frac{\gamma^2}{4b}N_1 - N_3\frac{\gamma^2}{2kl^2} - \rho_3 - \frac{1}{\tau} > 0,$$

$$NC_0 + \frac{C_1}{\tau} > 0.$$

4.3. Energy stability result

Therefor, (4.67) takes the form

$$L^{'}(t) \leq -(\lambda_0 - C_{N_2,N_3,N_6}\delta)E(t) + \lambda_1 \int_0^1 (g \circ \psi_x)dx - \frac{\lambda_2}{\delta}E^{'}(t),$$

for some positive constant $\lambda_0, \lambda_1, \lambda_2, C$*. At this point, we take* $\delta < \frac{\lambda_0}{C}$ *So for some* $m > 0$*, we obtain*

$$L^{'}(t) \leq -mE(t) + \lambda_1 \int_0^1 (g \circ \psi_x)dx - \frac{\lambda_2}{\delta}E^{'}(t). \tag{4.68}$$

Multiplying (4.68) by $\zeta(t)$ *gives*

$$\zeta(t)L^{'}(t) \leq -m\zeta(t)E(t) + \lambda_1\zeta(t) \int_0^1 (g \circ \psi_x)dx - \frac{\lambda_2}{\delta}\zeta(t)E^{'}(t). \tag{4.69}$$

The second term can be estimated, using $(G2)$*, as follows*

$$\begin{aligned}
\lambda_1\zeta(t)\int_0^1 (g \circ \psi_x)dx &= \lambda_1\zeta(t)\int_0^1\int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right)^2 dsdx \\
&\leq \lambda_1\int_0^1\int_0^t \zeta(t-s)g(t-s)\left(\psi_x(t) - \psi_x(s)\right)^2 dsdx \\
&\leq -\lambda_1\int_0^1\int_0^t g^{'}(t-s)\left(\psi_x(t) - \psi_x(s)\right)^2 dsdx \\
&= -\lambda_1\int_0^1 (g^{'} \circ \psi_x)dx \leq -\frac{2\lambda_1}{\beta}E^{'}(t),
\end{aligned}$$

so for some $\lambda_3 > 0$*, (4.69) becomes as follows*

$$\zeta(t)L^{'}(t) \leq -m\zeta(t)E(t) - \lambda_3E^{'}(t) - \frac{\lambda_2}{\delta}\zeta(t)E^{'}(t). \tag{4.70}$$

It is clear that

$$L_1(t) = \zeta(t)\left(L(t) + \frac{\lambda_2}{\delta}E(t)\right) \sim E(t).$$

Therefore, using (4.70) and the fact that $\zeta^{'}(t) \leq 0$*, we arrive at,*

$$L_1^{'}(t) = \zeta^{'}(t)\left(L(t) + \frac{\lambda_2}{\delta}E(t)\right) + \zeta(t)\left(L^{'}(t) + \frac{\lambda_2}{\delta}E^{'}(t)\right) \leq \zeta(t)\left(L^{'}(t) + \frac{\lambda_2}{\delta}E^{'}(t)\right).$$

So

$$L_1^{'}(t) \leq -m\zeta(t)E(t) - \lambda_3E^{'}(t).$$

Now, we set

$$L_2(t) = L_1(t) + \lambda_3E(t) \sim E(t),$$

gives

$$L_2^{'}(t) = L_1^{'}(t) + \lambda_3E^{'}(t) \leq -m\zeta(t)E(t). \tag{4.71}$$

4.3. Energy stability result

A simple integration of (4.71) *over* (t_0, t) *leads to*

$$L_2(t) \leq L_2(t_0) e^{-m\int_{t_0}^{t} \zeta(s)ds}. \tag{4.72}$$

Recalling (4.65), *estimate* (4.72) *yields the desired result* (4.66). ■

4.3. Energy stability result

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