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Thesis

A view to obtaining the diploma of

Master of 2° cycle (LMD) in Mathematics
Option: ***Applied Functional Analysis***

The existence and uniqueness of solutions for a second-order iterative differential equation with two point boundary conditions

Presented by:

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CONTENTS

Abstract in English

The existence and uniqueness of solutions for a second-order iterative differential equation with two point boundary conditions

The main objective of this work is to study the existence and uniqueness of solutions of an iterative second-order differential equation with boundary conditions using Schauder's fixed point theorem. We converte the given equation into an integral one in order to create an operator that satisfies the conditions of Schauder's theorem. After that, we define a convex, closed and bounded set of the Banach space of continuous functions over compact in order to prove that the operator associated to the integral equation has a fixed point, which is a solution of the initial equation.

Keywords:Existence, fixed point theorem, iterative differential equation, uniqueness

Abstract in French

L'existence et l'unicité des solutions pour une équation différentielle itérative du second ordre avec des conditions aux limites

L'objectif principal de ce travail est d'étudier l'existence et l'unicité des solutions d'une équation différentielle itérative du second ordre avec des conditions aux limites utilisant le théorème du point fixe de Schauder. Nous convertissons l'équation donnée en une équation intégrale afin de créer un opérateur qui satisfait aux conditions du théorème de Schauder. Ensuite nous définissons un ensemble convexe, fermé et borné de l'espace de Banach des fonctions continues sur un compact afin de prouver que l'opérateur associé à l'équation intégrale a un point fixe, qui est une solution de l'équation initiale.

Mots-clés: Existence, théorème du point fixe, équation différentielle itérative, unicité

Abstract in Arabic

وجود و وحدانية الحلول لمعادلة تفاضلية تكرارية من الدرجة الثانية بشروط حدية

الهدف الرئيسي من هذا العمل هو دراسة وجود و وحدانية الحلول لمعادلة تفاضلية تكرارية من الدرجة الثانية بشروط حدية باستخدام نظرية النقطة الثابتة لشودر، حيث قمنا بتحويل المعادلة المعطاة إلى معادلة تكاملية قصد إنشاء مؤثر يفي بشروط نظرية شودر، بعدها عرفنا مجموعة محدبة، مغلقة ومحدودة من الفضاء البناخي للدوال المستمرة على متراص قصد إثبات أن المؤثر المرفق بالمعادلة التكاملية يملك نقطة ثابتة والتي هي في الأصل حل للمعادلة الابتدائية.

كلمات مفتاحية: الوجود، الوحدانية، نظرية النقطة الثابتة، المعادلة التفاضلية التكرارية.

Dedication

With all love and gratitude, I dedicate this work to those who have been shining stars in the sky of my life and have contributed to the realization of this achievement.

To my dear mother, Aicha, who has been a warm haven for me, and accompanied me with her hope and prayers. Every kind word and tender touch from you has been a light that illuminated my path.

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To my dear brothers: Hicham and AbdErrahim

To my dear sisters: Noura, Saida, Fatima, and Chahra

To my nieces and nephews: Salah Eddin,Akram, Sundus, AbdElghani, HoossamEddin ,Ritaje, Nihad, AlaaErrahmane, Adam, Ranim, Aya,Assil, AbdErrahmane, Sajida, Layane, Rawan,Arij, and Taj Eddin.

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Acronyms

Abbreviation	**Meaning**
DDE	Delay differential equation
FDE	Functional differential equation
SDDE	State-dependent delay differential equation
IDE	Iterative differential equation

List of symbols

Sets and numbers

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers

$\mathbb{R}_+^* = (0, +\infty)$: the set of all non-zero positive real numbers

$\mathbb{R}^n$: n-dimensional real Euclidean space

$[a, b]$: the interval of numbers between a and b, including a and b

(a, b) : an open interval

$[a, +\infty)$: left-closed and right–unbounded interval

$\mathcal{C}\left(\mathbb{R}^{n+1}, (0, +\infty)\right)$ is the space of continuous functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

$\mathcal{C}\left([a, b], (0, +\infty)\right)$ is the space of continuous functions from $[a, b]$ into $(0, +\infty)$

$\mathcal{C}^1\left(\mathbb{R}^{n+1}, (0, +\infty)\right)$: space of continuously differentiable functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

Functions

$|\cdot|$: absolute value

$\|.\|_{\mathbb{Y}}$: a norm on $\mathbb{Y}$

$\|f\|_{\infty}$: the uniform norm defined by $\sup|f(M)|$

$x^{[n]}(t)$: the composition of the function $x(t)$ with itself n times or the n^{th} iterate

of the function $x(t)$

$\sum\limits_{i=1}^{n}$: the summation from index $i=1$ to $i=n$

$\lim\limits_{x\longrightarrow xM_0}$: limit as x approaches x_0

$x'(t) := x^{(1)}(t) := \dfrac{dx(t)}{dt}$: the first derivative of the function $M(t)$ with respect to t

$x''(t) := x^{(2)}(t) := \dfrac{d^2x(t)}{dt^2}$: the second derivative of the function $M(t)$ with respect to t

sup : the supremum

max : the maximum

min : the minimum

Other notations will be clarified upon their initial occurrence.

General introduction

Iterative functional differential equations (IDEs) have gained considerable interest by many scholars due to their ability to create more realistic models for many processes that depend on the composition of the state with itself as many times as needed. They arise in various real-world applications, for example, in biology, more specifically in blood cell production, ecology, more specifically in population growth, in epidemiology like the infection disease transmission, in classical electrodynamics like the description of the motion of charged particles with retarded interactions, and so on. The general form of the iterative differential equations of first order can be written as follows:

$$\frac{d}{dt}x(t) = f\left(x(t), x^{[2]}\left(t\right), ..., x^{[n]}\left(t\right)\right).$$

To the best of our knowledge, first iterative problems are studied by Babbage

[2], Schröder [20] and Abel [1] in the early of the last century. This kind of equation which relates an unknown function, its derivatives and its iterates can be deemed as a particular type of the class of time and state-dependent delay differential equations (SDDEs) of the following form:

$$x'(t) = f\left(t, x(t), x\left(t - \tau_1(t, x(t))\right), \ldots, x\left(t - \tau_n(t, x(t))\right)\right),$$

or advanced differential equations with state-dependent deviating arguments of the following form:

$$x'(t) = f\left(t, x(t), x\left(t + \tau_1(t, x(t))\right), \ldots, x\left(t + \tau_n(t, x(t))\right)\right).$$

Indeed, in many cases the iterates can be yielded from time and state dependent deviating arguments, whether delayed or advanced ones which in turn result from many factors such as the competition for food and habitat during larval stages in insect population growth and the dependence on the history of the mature cell populations in hematopoiesis models and so on.

During the recent years, many authors have paid a lot of attention to studying this kind of equations. As a result, a growing number of models that involve iterations of the state, have been proposed in order to gain insight into the dynamics of many phenomena. Unfortunately, IDEs of second order are generally not easy to be studied and they have been investigated only by a few researchers. This thesis attempts to overcome the possible and unexpected hurdles by virtue of the fixed point theory that allows us to establish the existence and uniqueness of the solutions for a second order iterative differential equation with boundary conditions.

Problem statement

The theory of iterative delay differential equations is a young and slow-growing area comparing with the theory of functional differential equations (FDEs). Indeed, such equations still deserve particular attention since despite the multiplicity of works carried out on this direction, the theory still remains emergent and not yet well developed.

Questions that will be addressed include the following:

(i) What are the criteria that guarantee the existence of solutions?

(ii) Does the problem have a unique solution?

Objectives

This work focuses on studying a second order iterative differential equation with two point boundary conditions. Its key aim resides in the application of the fixed point theory with the help of some functional analysis tools to derive a set of sufficient conditions that ensure the existence and uniqueness of solutions for the problem at hand.

Methodology

The study of iterative differential equations poses challenging questions for scholars and practitioners alike, especially for those of order greater than one. Their theory which is not yet well developed, was and still is deemed unpopular area among researchers. Perhaps this is due to the distinctive characteristics of such equations and also their iterative terms which usually hamper the employment of usual methods.

Our principal purposes in the present thesis is to prove the existence and uniqueness of solutions for two-point boundary value problem that involves an iterative source term. The approach used here is based on the fixed point theory that has significant applications in diverse fields. In order to reach the desired target, it is necessary to go through the following steps: firstly, driven by the desire to make the problem as easy to handle as possible, the steps of the proof begins by constructing the cornerstones which are an appropriate Banach space and a subset of it that will facilitate our task whether in controlling the iterative terms or in applying the used fixed point theorems. Secondly, we transform the problem at a fixed point one by converting it into an integral equation before defining an integral operator that fulfil the requirements of the fixed point theorems. Thirdly, we apply the Schauder fixed point theorem under which the existence of solutions is guaranteed. Finally, under an additional condition, the Banach fixed point theorem ensures the existence of a unique solution.

Layout of the thesis

The thesis contains a general introduction, three chapters, a conclusion, and a bibliography. It begins with a brief overview of the topic, the research objectives, the studied problem, the methodology to be used, the steps to be gone through. The rest of the manuscript is furnished as follows: In the first chapter, some basic definitions and necessary results are given such as the Ascoli-Arzela theorem and the fixed point theorems used to establish the main findings. The second chapter provides the requisite mathematical background necessary to grasp the topic being covered. It includes a brief overview on iterative differential equations and some iterative models. In the third chapter, the existence and uniqueness of solutions for the following problem:

$$\begin{cases} x''(t) = f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \ t \in [a,b], \\ x(a) = x_1, \ x(b) = x_2, \end{cases}$$

where $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t)), ..., x^{[n]}(t) = x^{[n-1]}(x(t))$ denote the iterates of the function $x(t)$, are established using the Schauder and the Banach fixed-point theorems. Finally, we end with a general conclusion that recaps the steps of the approach used in this thesis.

CHAPTER 1

Preliminary notions

Contents

The aim of this chapter is to provide the necessary background for ready reference and certain concepts that will be used through the thesis, including necessary functional analysis tools and notions, two fixed point theorems, the notion of *Green*'s function, and so on.

This chapter is mainly going to focus on providing some background definitions and the basic knowledge necessary for a better understanding of the chapters which follow without concentrating too much on the proofs of some results. We strive to introduce the necessary notions, some well-known results and tools from functional analysis that will be crucial in our arguments. Next, we introduce two fixed point theorems that will subsequently be used in our proofs.

1.1 Notation and preliminaries

In this section we collect the useful definitions and preliminary notions and tools.

1.1.1 Convex subset in a vector space

Definition 1.1 [10] Let $\mathbb{X}$ be a vector space over $\mathbb{F}$. A convex subset of $\mathbb{X}$ is a subset $\mathbb{M} \subseteq \mathbb{X}$ such that for every pair of points $x, y \in \mathbb{M}$ and for every $\alpha \in [0,1]$ we have that

$$\alpha x + (1-\alpha)\, y \in \mathbb{M}.$$

Elements of the form $\alpha x + (1-\alpha)\, y$ are called *Convex Combinations* of x and y.

1.1.2 Bounded, closed and compact subset in a normed vector space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$.

Definition 1.2 [18] A subset $\mathbb{M}$ of $\mathbb{X}$ is said to be bounded if there exists $C > 0$ such that

$$\|x\|_{\mathbb{X}} \leq C, \forall x \in \mathbb{M}.$$

Theorem 1.1 *[18] A set $\mathbb{M} \subseteq \mathbb{X}$ is closed if and only if, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{M}$ which converges to an element $x \in \mathbb{X}$, then $x \in \mathbb{M}$.*

Definition 1.3 [18]A set $\mathbb{M} \subseteq \mathbb{X}$ is called compact if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{M}$.

Definition 1.4 [18] The closure of a set $\mathbb{M} \subseteq \mathbb{X}$ (denoted by $\overline{\mathbb{M}}$) is the smallest closed set that contains $\mathbb{M}$.

Definition 1.5 [18] A set $\mathbb{M} \subseteq \mathbb{X}$ is called relatively compact if its closure $\overline{\mathbb{M}}$ is compact.

Corollary 1.1 *[18] A set $\mathbb{M} \subseteq \mathbb{X}$ is relatively compact if and only if every sequence in $\mathbb{M}$ has a subsequence that converges to a point in $\mathbb{X}$.*

1.1.3 Continuous, Lipschitz continuous and compact operators

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two normed vector spaces over the same field $\mathbb{F}$.

Definition 1.6 [12] An operator $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be continuous at a point $x_0 \in \mathbb{X}$ if

$$\lim_{x \longrightarrow x_0} \mathcal{S}x = \mathcal{S}x_0.$$

The continuity at $x_0 \in \mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{X}, \quad \left(\|x - x_0\|_{\mathbb{X}} < \delta\right) \Longrightarrow \left(\|\mathcal{S}x - \mathcal{S}x_0\|_{\mathbb{Y}} < \varepsilon\right).$$

If $\mathcal{S}$ is continuous at every point of $\mathbb{X}$, then $\mathcal{S}$ is said to be continuous on $\mathbb{X}$. The continuity on $\mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \forall x \in \mathbb{X}, \exists \delta > 0, \forall y \in \mathbb{X}, \left(\|x - y\|_{\mathbb{X}} < \delta\right) \Longrightarrow \left(\|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} < \varepsilon\right).$$

Definition 1.7 [19] A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is called *Lipschitz continuous* if there is a positive constant C such that

$$\forall x, y \in \mathbb{X} : \|\mathcal{S}x - \mathcal{S}y\|_{\mathbb{Y}} \leq C \|x - y\|_{\mathbb{X}}.$$

If $C \in [0, 1[$, $\mathcal{S}$ is called a contraction mapping.

Remark 1.1 If $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ then

$$\mathcal{S} \text{ is a contraction} \Longrightarrow \mathcal{S} \text{ is Lipschitz continuous} \Longrightarrow \mathcal{S} \text{ is continuous on } \mathbb{X}.$$

Theorem 1.2 *[18] A continuous function on a closed bounded interval is bounded and attains its bounds.*

Remark 1.2 The above theorem is hidden in the proof of many theorems and lemmas in the rest of this thesis where we integrate a continuous function over a compact interval.

Definition 1.8 A map $\mathcal{S} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be compact if and only if $\mathcal{S}$ maps bounded sets into relatively compact sets, i.e.,

$$[\mathcal{S} \text{ compact}] \Longleftrightarrow \left[\forall M \subset E, \ \ (M \text{ bounded}) \Longrightarrow \left(\overline{\mathcal{S}(M)} \text{ compact}\right)\right].$$

Equivalently, $\mathcal{S}$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(\mathcal{S}x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

1.1.4 The Banach space of continuous real-valued function on $[a, b]$

Theorem 1.3 *Let $[a, b]$ be a closed bounded interval. Let $\mathcal{C}[a, b]$ be the normed space of continuous real-valued function on $[a, b]$, equipped with the sup norm. Then $\mathcal{C}[a, b]$ is a Banach space.*

1.1.5 Arzelà-Ascoli theorem

Let $\mathbb{X}$ be a compact subset of a normed vector space over $\mathbb{F}$ and let $\mathcal{C}(\mathbb{X})$ be the normed vector space of real valued continuous functions on $\mathbb{X}$ with the *sup*-norm

$$\|f\|_{\infty} = \sup_{x\in\mathbb{X}} |f(x)|.$$

Let $\mathcal{F}$ be a collection of functions in $\mathcal{C}(\mathbb{X})$.

Definition 1.9 [4] The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and $x, y \in \mathbb{X}$ with $\|x-y\|_{\mathbb{X}} < \delta$ we have $|f(x) - f(y)| < \varepsilon$, i.e.,

$$\forall \varepsilon > 0,\ \forall x \in \mathbb{X}, \exists \delta > 0,\ \forall y \in \mathbb{X}, [\|x-y\|_{\mathbb{X}} < \delta] \Longrightarrow [\forall f \in \mathcal{F},\ \ |f(x) - f(y)| < \varepsilon].$$

Definition 1.10 [4] The collection $\mathcal{F}$ is said to be uniformly bounded if there is an $M \geq 0$ so that $\|f\|_{\infty} = \sup_{x\in\mathbb{X}} |f(x)| \leq M$ for all $f \in \mathcal{F}$, i.e.,

$$\exists M \geq 0 : \|f\|_{\infty} = \sup_{x\in\mathbb{X}} |f(x)| \leq M,\ \forall f \in \mathcal{F}.$$

Theorem 1.4 [4] *If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $\mathcal{C}(\mathbb{X})$, then $\mathcal{F}$ is relatively compact in $\mathcal{C}(\mathbb{X})$.*

1.2 Fixed point theorems

Fixed point theory is designed to be used in studying different types of equations arising in various fields.

Definition 1.11 [19] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$. A fixed point of a mapping $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ of $\mathbb{X}$ into itself is an $x \in \mathbb{X}$ which is mapped onto itself, that is

$$\mathcal{S}(x) = x.$$

1.2.1 Schauder's fixed point theorem

The *Schauder* fixed point theorem relies on the compactness of the operator in *Banach* spaces.

Theorem 1.5 *[19] Let* $\mathbb{M}$ *be a non-empty bounded closed convex subset of a Banach space* $(\mathbb{X}, \|.\|)$ *and let* $\mathcal{S} : \mathbb{M} \longrightarrow \mathbb{M}$ *be a compact and continuous mapping. Then* $\mathcal{S}$ *has a fixed point in* $\mathbb{M}$*.*

An alternative version of the *Schauder* fixed point theorem can be stated as follows:

Theorem 1.6 *[19] Let* $\mathbb{M}$ *be a non-empty compact convex subset of a Banach space* $(\mathbb{X}, \|.\|)$ *and let* $\mathcal{S} : \mathbb{M} \longrightarrow \mathbb{M}$ *be a continuous mapping. Then* $\mathcal{S}$ *has a fixed point in* $\mathbb{M}$*.*

1.2.2 Banach's fixed point theorem

One of the very helpful tools which is broadly applicable in proving the existence and uniqueness of solutions, is the well-known *Banach* fixed point theorem (also known as the contraction mapping theorem or contractive mapping theorem).

Theorem 1.7 *[19] Let* $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *be a Banach space and let* $\mathcal{S} : \mathbb{X} \rightarrow \mathbb{X}$ *be a contraction on* $\mathbb{X}$*. Then* $\mathcal{S}$ *has a unique fixed point* $x \in \mathbb{X}$ *such that*

$$\mathcal{S}(x) = x.$$

Theorem 1.8 *[14] If* $\mathbb{M}$ *is a closed subset of a Banach space* $\mathbb{X}$ *and* $\mathcal{S} : \mathbb{M} \rightarrow \mathbb{M}$ *is a contraction, then* $\mathcal{S}$ *has a unique fixed point in* $\mathbb{M}$*.*

CHAPTER 2

Iterative Functional Differential Equations

Contents

T

2.1 Notion of Iterations

Definition 2.1 The composition $x \circ y$ of the function x with the function y is

$$(x \circ y)(t) = x(y(t)),$$

The domain of $x \circ y$ is the set of all t in the domain of y such that $y(t)$ is in the domain of x.

Definition 2.2 For $x : E \longrightarrow E$, the n^{th} iterate of function x, denoted by $x^{[n]}$ for some nonnegative integer n, is defined recursively by

$$x^{[0]} = Id_E,$$

and

$$x^{[n+1]} = x \circ x^{[n]},$$

where Id_E is the identity map on E.

Definition 2.3 [7] Let $x^{[0]} = t$, $x^{[1]} = x(t)$, $x^{[2]} = x(x(t))$,..., $x^{[n]} = x\left(x^{[n-1]}(t)\right)$ be the iterates of the function $x(t)$. When the derivatives and the iterates of an unknown function appear in a well defined functional relation, we are then dealing with an iterative functional differential equation.

Now, we present a useful estimate that will help us later.

Lemma 2.1 *Let*

$$\begin{aligned}\Omega \ &: = \{x \in \mathcal{C}([a,b],[a,b]) : x(a) = x_1,\ x(b) = x_2, \\ &\qquad\qquad |x(t_1) - x(t_2)| \leq M|t_1 - t_2|, t_1, t_2 \in [a,b]\},\end{aligned}$$

(1) Ω is a convex, closed, and compact subset of $\mathcal{C}([a,b],\mathbb{R})$.

(2) If $x, y \in \Omega$, then

$$\left\|x^{[k]} - y^{[k]}\right\| \leq \sum_{j=0}^{k-1} M^j \|x - y\|,\ \ k = 1, 2, ...$$

where $x^{[k]} = x \circ x \circ ... \circ x$ k times.

Proof.

(1) Ω is a convex: Let $\alpha \in [0,1]$ and $x, y \in \Omega$. We have

$$(\alpha x + (1-\alpha)y)(x_1) = \alpha x(x_1) + (1-\alpha)y(x_1) = \alpha x(x_1) + y(x_1) - \alpha y(x_1),$$

and

$$(\alpha x + (1-\alpha)y)(x_2) = \alpha x(x_2) + (1-\alpha)y(x_2) = \alpha x(x_2) + y(x_2) - \alpha y(x_2).$$

Since $x, y \in \Omega$ then $x(x_1) = x_1$, $y(x_1) = x_1$, $x(x_2) = x_2$, and $y(x_2) = x_2$. So

$$(\alpha x + (1-\alpha)y)(x_1) = \alpha x_1 + x_1 - \alpha x_1 = x_1,$$

and

$$(\alpha x + (1-\alpha)y)(x_2) = \alpha x_2 + x_2 - \alpha x_2 = x_2.$$

On the other hand

$$\left|\left(\alpha x+(1-\alpha)y\right)(t_1)-\left(\alpha x+(1-\alpha)y\right)(t_2)\right|$$

$$=\left|(\alpha x)(t_1)+(1-\alpha)y(t_1)-(\alpha x)(t_2)+(1-\alpha)y(t_2)\right|$$

$$\leq \alpha\left|x(t_1)-x(t_2)\right|+(1-\alpha)\left|y(t_1)-y(t_2)\right|.$$

Since $x,y\in\Omega$ then

$$\left|x(t_1)-x(t_2)\right|\leq M\left|t_1-t_2\right|,\text{ and }\left|y(t_1)-y(t_2)\right|\leq M\left|t_1-t_2\right|.$$

So

$$\begin{aligned}\left|\left(\alpha x+(1-\alpha)y\right)(t_1)-\left(\alpha x+(1-\alpha)y\right)(t_2)\right| &\leq \alpha M\left|t_1-t_2\right|+(1-\alpha)M\left|t_1-t_2\right|\\ &= M\left|t_1-t_2\right|.\end{aligned}$$

Thus, Ω is a convex.

<u>Ω is a closed:</u> Let $(x_n)_{n\in\mathbb{N}}\subset\Omega$ be a converge sequence to $x\in\mathcal{C}([a,b],\mathbb{R})$, then

$$x_n(x_1)=x_1\text{ and }x_n(x_2)=x_2,$$

so

$$\lim_{n\to\infty}x_n(x_1)=x(x_1)\text{ and }\lim_{n\to\infty}x_n(x_2)=x(x_2).$$

But

$$\lim_{n\to\infty}x_n(x_1)=x_1\text{ and }\lim_{n\to\infty}x_n(x_2)=x_2,$$

so

$$x\left(x_1\right)=x_1, \text{ and } x\left(x_2\right)=x_2.$$

On the other hand

$$\begin{aligned}
\left|x\left(t_2\right)-x\left(t_1\right)\right| &= \left|x\left(t_2\right)-x_n\left(t_2\right)+x_n\left(t_2\right)-x_n\left(t_1\right)+x_n\left(t_1\right)-x\left(t_1\right)\right| \\
&\leq \left|x\left(t_2\right)-x_n\left(t_2\right)\right|+\left|x_n\left(t_2\right)-x_n\left(t_1\right)\right|+\left|x_n\left(t_1\right)-x\left(t_1\right)\right| \\
&\leq \left|x_n\left(t_2\right)-x_n\left(t_1\right)\right| \\
&\leq M\left|t_2-t_1\right|.
\end{aligned}$$

So $x \in \Omega$ and Ω is closed.

Ω is a compact: In view that

$$\|x\| \leq \max\{|a|,|b|\} \text{ and } \left|x\left(t_1\right)-x\left(t_2\right)\right| \leq M\left|t_1-t_2\right|,$$

Ω is uniformly bounded and equicontinuous and is relatively compact by the Arzelà-Ascoli theorem. Therefore, Ω is a closed, convex and relatively compact subset of the Banach space $\mathcal{C}([a,b],\mathbb{R})$.

(2) We will prove this inequality by induction. So, the proof will now proceed in two steps:

(2) We will prove this inequality by induction. So, the proof will now proceed in two steps:

The basis step: For $k=1$, we have

$$\|x-y\| \leq \|x-y\|.$$

then, the inequality holds for $k = 1$

The inductive step: Now, we assume that the inequality holds for a given $k = m$ and we want to show that it also holds for $k = m + 1$. Suppose that

$$\left\|x^{[m]} - y^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \left\|x - y\right\|,$$

then

$$\begin{aligned}
\left|x^{[m+1]}(t) - y^{[m+1]}(t)\right| &\leq \left|x\left(x^{[m]}(t)\right) - x\left(y^{[m]}(t)\right)\right| + \left|x\left(y^{[m]}(t)\right) - y\left(y^{[m]}(t)\right)\right| \\
&\leq M\left|x^{[m]}(t) - y^{[m]}(t)\right| + \left|x\left(y^{[m]}(t)\right) - y\left(y^{[m]}(t)\right)\right|,
\end{aligned}$$

so

$$\begin{aligned}
\left\|x^{[m+1]} - y^{[m+1]}\right\| &\leq M\left\|x^{[m]} - y^{[m]}\right\| + \left\|x - y\right\| \\
&\leq M\sum_{j=0}^{m-1} M^j \left\|x - y\right\| + \left\|x - y\right\| \\
&\leq \left(\sum_{j=0}^{m-1} M^{j+1} + 1\right)\left\|x - y\right\| \\
&\leq \sum_{j=0}^{m} M^j \left\|x - y\right\|.
\end{aligned}$$

By induction we deduce that

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \left\|\varphi - \psi\right\| \qquad \forall m \in \mathbb{N},$$

which finishes the proof. ■

2.2 Some Examples of Iterative Functional Differential Equation

Example 2.1 [11] Let consider the the following second order iterative differential equation:

$$x'' + f\left(t, x\left(t\right), x^{[2]}\left(t\right), ..., x^{[n]}\left(t\right)\right) = 0,\ \ 0 < t < b,$$
$$x\left(0\right) = 0, \alpha \int_0^{\eta} x\left(s\right) ds = x\left(b\right)\ \text{ with } \eta \in \left(0, b\right),\ \ \alpha \in \mathbb{R}^* ,$$

where $x^{[2]}\left(t\right) = x\left(x\left(t\right)\right)$ and $f\left[a, b\right] \times \mathbb{R}^n \longrightarrow [0, \infty)$ is a continuous function with respect to its arguments. This equation describes diffusion phenomena with a source or a reaction term. For instance, in thermal conduction, it can be interpreted as the one-dimensional heat conduction equation which models the steady-states of a heated bar of length b with a controller at $x = b$ that adds or removes heat according to a sensor, while the left endpoint is maintained at $0°C$ and f is the distributed temperature source function depending on delayed temperatures. We refer the interested reader to [5, 6, 16] and the references therein for more details.

Example 2.2 [8] Let consider the the following third order iterative differential equation:

$$x''' + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) = 0,\ \ 0 < t < b,$$

$$x(0) = x''(0) = 0, \alpha \int_0^{\eta} x(s)\, ds = x(b)\ \text{ with } \eta \in (0, b),\ \ \alpha \in \mathbb{R}^*\ ,$$

where f is a continuous function with respect to its arguments.This problem includes many important models. For instance, it arises in the modeling of draining or coating fluid flow problems, electromagnetic waves, thin film flow and gravity-driven flows (see [21, 22]).

CHAPTER 3

Existence and uniqueness results of a second order iterative differential equation

Contents

In this chapter we prove the existence and uniqueness of solutions of a second order iterative differential equation under general boundary value conditions by the fixed point theorems of Schauder and Banach.

3.1 Introduction

Iterative differential equations which is a relationship between a function, its derivatives and its iterates can be considered as special type of the class of differential equations with state-dependent deviating arguments that occur in many real phenomena such as infectious disease transmission models and the two-body problem of classical electrodynamics.

The study of iterative differential equations can be traced back to the works of *Babage* [2], Eder [13] and Petuhov [17]. At that time, most studies were concerned with first-order iterative equations, which were later developed to include second-order and higher-order iterative equations. There has been a special of interest in these equations especially those of second order such as the following equation:

$$x'' = f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

or its particular cases with solutions satisfying some boundary conditions, where $x^{[2]}(t) = x(x(t))$ and $x^{[n]}(t) = x\left(x^{[n-1]}(t)\right)$.

In 2018 and by using Green's function, E. R. Kaufmann [15] investigated the second order equation involving 2-th iterates

$$x'' = f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \; t \in [a, b]$$

associated with the boundary value conditions

$$x\left(a\right)=a,\ \ x\left(b\right)=b\text{ or }x\left(a\right)=b,\ \ x\left(b\right)=a,$$

He gave the sufficient conditions for the existence and uniqueness of solutions by fixed point theorems. In 2020, A. Bouakkaz and R. Khemis [3] They studied the existence of positive periodic solutions for the following class of second-order iterative differential equations

$$\begin{aligned}\frac{d^2}{dt^2}x\left(t\right)+p\left(t\right)\frac{d}{dt}x\left(t\right)+q\left(t\right)x\left(t\right) &= \frac{d}{dt}f\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right)\\ &\quad+\sum_{i=1}^{n}c_i\left(t\right)x^{[i]}\left(t\right),\end{aligned}$$

where p and q are positive continuous real-valued functions and the functions $f:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ is continuous with respect to its arguments. The main results of this work are obtained by virtue of a Krasnoselskii fixed point theorem and some useful properties of a Green's function. In 2022, S. Chouaf et al [11], the authors they used the Schauder fixed point theorem to study the existence, uniqueness and dependance continuous of solutions of the following nonlinear second order differential equation with iterative source term and integral boundary conditions

$$\begin{aligned}&x''+f\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right)=0,\ \ 0<t<b,\\ &x\left(0\right)=0,\alpha\int_0^{\eta}x\left(s\right)ds=x\left(b\right)\ \text{ with }\eta\in\left(0,b\right),\ \ \alpha\in\mathbb{R}^*.\end{aligned}$$

In [9], the authors they investigated the following second-order iterative differential equation with boundary conditions

$$x'' = h\left(t, x\left(t\right), x^{[2]}\left(t\right), ..., x^{[n]}\left(t\right)\right), \quad -b \leq t \leq b,$$

associated with the boundary value conditions

$$x\left(-b\right) = \eta_1, \; x\left(b\right) = \eta_2.$$

The results of existence and uniqueness of solutions are obtained by the principle of contraction mappings and the Schauder's fixed point theorem.

In this chapter we study the equation involving $n-th$ iterates, together with the general boundary value condition, that is, the two-point boundary value problems

$$\begin{cases} x''\left(t\right) = f\left(t, x\left(t\right), x^{[2]}\left(t\right), ..., x^{[n]}\left(t\right)\right), \; t \in \left[a, b\right], \\ \qquad x\left(a\right) = x_1, \; x\left(b\right) = x_2. \end{cases} \tag{3.1}$$

is discussed, where the different points $x_1, x_2 \in [a, b]$. Using an auxiliary integral equation without the help of Green's functions, we prove the existence and uniqueness of solutions by the fixed point theorems of Schauder and Banach, respectively. Our theorems generalize and revise the related result

3.2 Integral equation

Lemma 3.1 *The boundary value problem (3.1) is equivalent to the $\mathcal{C}^0$ solution of the integral equation*

$$\begin{aligned} x(t) &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}.\int_a^b (b-s)\, f\left(s, x(s), x^{[2]}(s),\ \dots,\ x^{[n]}(s)\right) ds \\ &\quad + \frac{x_2 - x_1}{b-a}.t - \frac{t}{b-a}.\int_a^b (b-s)\, f\left(s, x(s), x^{[2]}(s),\ \dots,\ x^{[n]}(s)\right) ds \\ &\quad + \int_a^t (t-s)\, f\left(s, x(s), x^{[2]}(s), \dots,\ x^{[n]}(s)\right) ds. \qquad (3.2) \end{aligned}$$

Proof. An integration of equation (3.1) from a to t leads to

$$\int_a^t x''(s)\, ds = \int_a^t f\left(s, x(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds.$$

Hence

$$x'(t) = x'(a) + \int_a^t f\left(s, x(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds.$$

Integrating again from a to t, we get

$$\int_a^t x'(s)\, ds = \int_a^t x'(a)\, ds + \int_a^t \left[\int_a^\zeta f\left(s, x(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds\right] d\zeta,$$

which implies that

$$\begin{aligned} x(t) &= x(a) + x'(a)(t-a) \\ &\quad + \int_a^t (t-s)\, f\left(s, x(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds. \qquad (3.3) \end{aligned}$$

Since $x(a) = x_1$ and $x(b) = x_2$ we obtain

$$x(t) = x_1 + x'(a)(t-a) + \int_a^t (t-s)\, f\left(s, x(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds,$$

and

$$\begin{aligned} x(b) &= x_2 = x_1 + x'(a)(b-a) \\ &\quad + \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \end{aligned} \tag{3.4}$$

Thus

$$x'(a) = \frac{x_2 - x_1}{b-a} - \frac{1}{b-a} \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds.$$

The substitution of (3.4) in (3.3) gives

$$\begin{aligned} x(t) &= x_1 + (t-a)\frac{x_2 - x_1}{b-a} - \frac{t-a}{b-a} \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad + \int_a^t (t-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a} \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad + t\frac{x_2 - x_1}{b-a} - \frac{t}{b-a} \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &\quad + \int_a^t (t-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \end{aligned}$$

Conversely, from (3.2) we get

$$\begin{aligned} x(a) &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}. \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s),\ ...,\ x^{[n]}(s)\right) ds \\ &\quad + \frac{x_2 - x_1}{b-a}.a - \frac{a}{b-a}. \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s),\ ...,\ x^{[n]}(s)\right) ds \\ &\quad + \int_a^a (a-s) f\left(s, x(s), x^{[2]}(s), ...,\ x^{[n]}(s)\right) ds \\ &= \frac{bx_1 - ax_2}{b-a} + \frac{x_2 - x_1}{b-a} a = x_1. \end{aligned}$$

and

$$\begin{aligned}
x\left(b\right) &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}\int_a^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&\quad + b\frac{x_2 - x_1}{b-a} - \frac{b}{b-a}\int_a^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&\quad + \int_a^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&= \frac{bx_1 - ax_2}{b-a} + b\frac{x_2 - x_1}{b-a} \\
&\quad + \left(\frac{a}{b-a} - \frac{b}{b-a} + 1\right)\int_a^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&= x_2.
\end{aligned}$$

The derivation

$$\begin{aligned}
x'\left(t\right) &= \frac{x_2 - x_1}{b-a} - \frac{1}{b-a}.\int_a^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&\quad + \int_a^t f\left(s, x\left(s\right), x^{[2]}\left(s\right), \ ..., x^{[n]}\left(s\right)\right) ds \\
&= \frac{x_2 - x_1}{b-a} + \frac{1}{b-a}.\int_a^t \left(s-a\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds \\
&\quad - \frac{1}{b-a}\int_t^b \left(b-s\right) f\left(s, x\left(s\right), x^{[2]}\left(s\right), ..., x^{[n]}\left(s\right)\right) ds,
\end{aligned}$$

then

$$x''\left(t\right) = f\left(t, x\left(t\right), x^{[2]}\left(t\right), ..., x^{[n]}\left(t\right)\right).$$

and the proof is completed. ■

3.3 Existence results

In this section, we apply the Schauder's fixed point theorem to prove the existence of a $\mathcal{C}^0$ solution of (3.1). In order to meet the requirements of Schauder's fixed point theorem we consider the subset Ω of the Banach space $\mathcal{C}([a,b],\mathbb{R})$ defined by

$$\begin{aligned}\Omega \ &:= \{x \in \mathcal{C}([a,b],[a,b]) : x(a) = x_1,\ x(b) = x_2, \\ &\qquad |x(t_1) - x(t_2)| \leq M|t_1 - t_2|, t_1, t_2 \in [a,b]\}, \qquad (3.5)\end{aligned}$$

where

$$M := \frac{|x_2 - x_1|}{b-a} + 2L.(b-a). \qquad (3.6)$$

and we defined an operator $T : [a,b] \to \mathbb{R}$ by

$$\begin{aligned}(Tx)(t) \ &:= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \frac{x_2 - x_1}{b-a}.t - \frac{t}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \int_a^t (t-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds. \qquad (3.7)\end{aligned}$$

Theorem 3.1 *Suppose that $f : [a,b]^{n+1} \to \mathbb{R}$ is a $\mathcal{C}^0$ function and satisfies*

$$|f(t, u_1, ..., u_n) - f(t, v,\ ..., v_n)| \leq \sum_{i=1}^{n} L_i |u_i - v_i|, \qquad (3.8)$$

for some constants $L_i \geq 0$. *If*

$$L < \frac{|x_2 - x_1|}{(b-a)^2}, \tag{3.9}$$

where

$$L := \|f\|_{[a,b]^{n+1}} = \max_{(u_0,u_1,...,u_n)\in[a,b]^{n+1}} |f(u_0, u_1, ..., u_n)|, \tag{3.10}$$

then the boundary value problem (*3.1*) *has at least a solution* x *on* Ω.

Proof. The proof will be made in two steps.

Step 1. We first show that $Tx \in \Omega$ for any $x \in \Omega$. We have

$$\begin{aligned}(Tx)(a) \quad : \quad &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \frac{x_2 - x_1}{b-a}.a - \frac{a}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \int_a^a (a-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ = \quad & \frac{bx_1 - ax_2}{b-a} + \frac{x_2 - x_1}{b-a} a = x_1, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}(Tx)(b) \quad : \quad &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \frac{x_2 - x_1}{b-a}.b - \frac{b}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ &+ \int_a^b (t-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ = \quad & \frac{bx_1 - ax_2}{b-a} + \frac{x_2 - x_1}{b-a} b \\ &+ \left(\frac{a}{b-a} - \frac{b}{b-a} + 1\right) \int_a^b (b-s) f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) ds \\ = \quad & x_2. \end{aligned} \tag{3.12}$$

Moreover,

$$\begin{aligned}(Tx)'(t) &= \frac{x_2-x_1}{b-a}+\frac{1}{b-a}.\int_a^t (s-a)\, f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds\\ &\quad -\frac{1}{b-a}\int_t^b (b-s)\, f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds. \end{aligned}\tag{3.13}$$

Since

$$\begin{aligned}(Tx)'(t) &\geq \frac{x_2-x_1}{b-a}-\frac{L}{b-a}.\frac{(b-a)^2}{2}-\frac{L}{b-a}.\frac{(b-a)^2}{2}\\ &= \frac{x_2-x_1}{b-a}+(b-a)\,L>0.\end{aligned}\tag{3.14}$$

for $x_2 > x_1$ and

$$\begin{aligned}(Tx)'(t) &\geq \frac{x_2-x_1}{b-a}-\frac{L}{b-a}.\frac{(b-a)^2}{2}-\frac{L}{b-a}.\frac{(b-a)^2}{2}\\ &= \frac{x_2-x_1}{b-a}+(b-a)\,L<0.\end{aligned}\tag{3.15}$$

for $x_2 < x_1$, it follows from (3.11)-(3.12) that

$$Tx:[a,b]\rightarrow[a,b].$$

Now, for any $t_1, t_2 \in [a, b]$, we have

$$\begin{aligned}
&\left|(Tx)(t_1) - (Ty)(t_2)\right| \\
&= \left| \frac{x_2 - x_1}{b-a}.(t_1 - t_2) \right. \\
&-\frac{t_1 - t_2}{b-a}.\int_a^b (b-s) f\left(s, x(s), x^2(s), ..., x^m(s)\right) ds \\
&+\int_a^{t_1} (t_1 - s) f\left(s, x(s), x^2(s), ..., x^m(s)\right) ds \\
&\left. -\int_a^{t_2} (t_2 - s) f\left(s, x(s), x^2(s), ..., x^m(s)\right) ds \right| \\
&\leq \frac{|x_2 - x_1|}{b-a}.|t_1 - t_2| + \frac{L}{b-a}.\frac{(b-a)^2}{2}.|t_1 - t_2| \\
&+\left|\int_{t_1}^{t_2} (t_1 - s) f\left(s, x(s), x^2(s), \ ..., x^m(s)\right) ds\right| \\
&+\left|\int_{t_2}^{t_1} (t_1 - t_2) f\left(s, x(s), x^2(s), ..., x^m(s)\right) ds\right| \\
&\leq \frac{|x_2 - x_1|}{b-a}.|t_1 - t_2| + L.\frac{b-a}{2}.|t_1 - t_2| \\
&+L.\frac{(t_1 - t_2)^2}{2} + L.|t_1 - t_2|.|t_2 - a| \\
&\leq \frac{|x_2 - x_1|}{b-a}.|t_1 - t_2| + L.\frac{b-a}{2}.|t_1 - t_2| \\
&+L.\frac{b-a}{2}.|t_1 - t_2| + L.(b-a).|t_1 - t_2| \\
&= \left(\frac{|x_2 - x_1|}{b-a} + 2L.(b-a)\right).|t_1 - t_2| \\
&= M.|t_1 - t_2|. \qquad (3.16)
\end{aligned}$$

Those relations imply that $Tx \in \Omega$, i,e.,T is a self-mapping.

Step 2. We show that T is continuous. For any $x, y \in \Omega$, using Lemma 2.1 we have

$$
\begin{aligned}
& \|Tx - Ty\| \\
= \; & \max_{t\in[a,b]} \left| \frac{a-t}{b-a} . \int_a^b (b-s) \, f\left(s, x_1(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \\
& -f\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds \\
& + \int_a^t (t-s) \left(f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right| \\
& -f\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds| \\
\leq \; & \max_{t\in[a,b]} \left| \int_a^b (b-s) \left(f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \right. \\
& -f\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right)\big) ds| \\
& + \max_{t\in[a,b]} \left| \int_a^t (t-s) \left(f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \right. \\
& -f\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right)\big)\big| \, ds \\
\leq \; & (b-a)^2 . \max_{t\in[a,b]} \left| f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \right. \\
& -f\left(t, y(t), y^{[2]}(t), ..., y^{[n]}(t)\right)\big| \\
\leq \; & (b-a)^2 . \sum_{i=1}^{n} L_i \left(\sum_{j=1}^{i=1} M^j \right) \|x-y\|, \qquad (3.17)
\end{aligned}
$$

which implies that$T$ is continuous.

In view of Lemma 2.1, Ω is a closed, convex and relatively compact subset of the Banach space $\mathcal{C}([a,b], \mathbb{R})$ and since T is a continuous operator. By Schauder's fixed point theorem T has a fixed point $x \in \Omega$, which is a solution

$x \in \Omega$ of (3.1) and the proof is completed. ■

Theorem 3.2 *Suppose that all conditions of Theorem 3.1 hold. If*

$$(b-a)^2 . \sum_{i=1}^{n} L_i \left(\sum_{j=1}^{i\,=\,1} M^j \right) < 1, \tag{3.18}$$

where M is defined by (3.6) Then the boundary value problem (3.1) has a unique solution $x(t)$ on $[a, b]$.

Proof. It is known from (3.17) and (3.18) that T is a contractive operator, and the remainder is same as that of Theorem 3.1. Then, the problem (3.1) has a unique solution $x \in \Omega$ from Banach fixed point theorem and the proof is completed. ■

Remark 3.1 From the assumption

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$$

we can get

$$\begin{aligned}
&\left|f\left(s,x\left(s\right),x^{[2]}\left(s\right)\right)-f\left(s,y\left(s\right),y^{[2]}\left(s\right)\right)\right|\\
&\leq L_1\left|x\left(s\right)-y\left(s\right)\right|+L_2\left|x^{[2]}\left(s\right)-y^{[2]}\left(s\right)\right|\\
&\leq L_1\left|x\left(s\right)-y\left(s\right)\right|+L_2\left(\left|x^{[2]}\left(s\right)-x\circ y\left(s\right)\right|+\left|x\circ y\left(s\right)-y^{[2]}\left(s\right)\right|\right)\\
&\leq L_1\left\|x-y\right\|+L_2\left|x^{[2]}\left(s\right)-x\circ y\left(s\right)\right|+L_2\left\|x-y\right\|\\
&\leq L_1\left\|x-y\right\|+L_2\left|x^{[2]}-x\circ y\right|+L_2\left\|x-y\right\|\\
&\leq L_1\left\|x-y\right\|+L_1L_2\left\|x-y\right\|+L_2\left\|x-y\right\|\\
&=L_1\left\|x-y\right\|+\left(L_1+1\right)L_2\left\|x-y\right\|. \qquad (3.19)
\end{aligned}$$

3.4 Examples

We give two examples to illustrate our main results.

Example 3.1 Consider the problem

$$\begin{cases} x"\left(t\right)=k\cos\left(c_1x\left(t\right)+c_2x^2\left(t\right)+c_3x^3\left(t\right)\right), \\ x\left(0\right)=x_1, x\left(\pi\right)=x_2, \end{cases} \qquad (3.20)$$

where k, c_i $(i=1,2,3)\in\mathbb{R}$ and $x_1,x_2\in[0,\pi]$ and

$$f\left(t,x_1\left(t\right),x^{[2]}\left(t\right),x^{[3]}\left(t\right)\right):=k\cos\left(c_1x\left(t\right)+c_2x^{[2]}\left(t\right)+c_3x^{[3]}\left(t\right)\right).$$

Clearly

$$\left\|f\right\|\leq\left|k\right|,$$

and

$$\begin{aligned}&\left|f\left(t,x_1\left(t\right),x_2\left(s\right),x_3\left(s\right)\right)-f\left(t,y_1\left(t\right),y_2\left(s\right),y_3\left(s\right)\right)\right|\\&\leq\left|c_1k\right|.\left|x_1\left(t\right)-y_1\left(t\right)\right|+\left|c_2k\right|.\left|x_2\left(t\right)-y_2\left(t\right)\right|\\&+\left|c_3k\right|.\left|x_3\left(t\right)-y_3\left(t\right)\right|.\end{aligned}$$

From Theorem 3.1, problem (3.20) has at least a solution $x(t)$ defined on $[0,\pi]$ if

$$k<\frac{\left|x_2-x_1\right|}{\pi^2}.$$

When

$$x_1=0,x_2=\pi,c_1=0,c_2=1,c_3=0, \tag{3.21}$$

then example (3.20) is reduced to Example 3.3 of [15], i.e.,

$$\begin{cases}x"\left(t\right)=k\cos\left(x^2\left(t\right)\right),t\in[0,\pi].\\ \quad x\left(0\right)=0,x\left(\pi\right)=\pi.\end{cases} \tag{3.22}$$

Example 3.2 Consider the problem (3.20) again. We have

$$\begin{aligned}&\left(b-a\right)^2\sum_{i\,=\,1}^{3}L_i\left(\sum_{j=0}^{i-1}M^j\right)\\&=\left(b-a\right)^2\left(L_1+L_2\left(1+M\right)+L_3\left(1+M+M^2\right)\right)\\&=\pi^2\left(\left|c_1k\right|+\left|c_2k\right|\left(1+M\right)+\left|c_3k\right|\left(1+M+M^2\right)\right)\\&=\pi^2\left|k\right|.\sum_{i\,=\,1}^{3}\left|c_i\right|\left(\sum_{j=0}^{i-1}M^j\right),\end{aligned}$$

where

$$\begin{aligned} M \quad : \quad &= \frac{|x_2 - x_1|}{b-a} + 2L.(b-a) \\ &= \frac{|x_2 - x_1|}{\pi} + 2|k|.\pi \\ &= \frac{|x_2 - x_1| + 2|k|\pi^2}{\pi}. \end{aligned}$$

By using Theorem 3.2, we know that the problem (3.20) has a unique solution $x(t)$ on $[0,\pi]$ if

$$\pi^2.|k|.\sum_{i=1}^{3}|c_i|\left(\sum_{j=0}^{i-1}M^j\right) < 1. \tag{3.23}$$

For instance, choose

$$k = -\frac{1}{10},\ c_1 = -c_2 = c_3 = \frac{1}{3\pi^2},\ x_1 = 0, x_2 = \pi,$$

note that

$$\begin{aligned} &\pi^2|k|\sum_{i=1}^{3}|c_i|\left(\sum_{j=0}^{i-1}M^j\right) \\ &= \frac{1}{10}\left[1 + \pi.\frac{2}{3\pi^2}\left(\pi + 2.\frac{1}{10}.\pi^2\right) + \frac{1}{3\pi^2}\left(\pi + 2.\frac{1}{10}.\pi^2\right)^2\right] \\ &= \frac{1}{10}\left(1 + \frac{2}{3}\left(1 + \frac{\pi}{5}\right) + \frac{1}{3}\left(1 + \frac{\pi}{5}\right)^2\right) \\ &= 0.296\,94 \\ &< 1, \end{aligned}$$

so the problem

$$\begin{cases} x''(t) = -\frac{1}{10}\cos\frac{1}{3\pi^2}\left(x(t) - x^2(t) + x^3(t)\right), \\ x(0) = 0, x(\pi) = \pi \end{cases}$$

has a unique solution $x(t)$ on $[0, \pi]$.

Conclusion

Boundary value problems for second-order iterative differential equations can be encountered in the modelling of many phenomena in various scientific and engineering fields.

This thesis dealt with a class of second-order iterative differential equations with two point boundary conditions. The method used here is an attractive technique based on the fixed point theory. The essence of this approach lies in transforming the problem at hand into an equivalent integral equation which helped us in constructing an integral operator before employing the Schauder fixed point theorem and hence proving the existence of at least one solution of the iterative boundary value problem.

Furthermore, besides the previously established conditions we added an extra condition under which the Banach fixed point theorem can be applied to ensure the existence and uniqueness of solutions.

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