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Thesis

A view to obtaining the diploma of

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Option: dynamic systems

Existence, uniqueness and asymptotic behavior of solution for a nonlinear partial differential equation

Presented by:

Boughamsa Wissem

Publicly discussed:

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CONTENTS

ABSTRACT IN ENGLISH

In this thesis, we study the various problems of nonlinear wave equations with source terms and variable coefficients, sometimes with damping and sometimes with viscoelastic terms, under suitable assumptions on variable coefficients. At the beginning, we presented a series of summaries of some previous works by several researchers and the results they have obtained.

We have studied several results. In the first problem, we proved the existence and uniqueness of the solution, and then we proved that the solution blows up in finite time. To verify our theoretical results, we conducted some numerical tests in the form of figures, and their results matched the theoretical analytical study. As for the second problem, we proved the existence and uniqueness of the solution and showed its global existence in the presence of positive initial energy, also demonstrated energy decay when time is sufficiently large, we relied on the Nehari space. The third problem, we proved blow-up in finite time in the both the analytical and numerical results of the solution. In the forth problem, we proved the existence of a local solution and also proved that the local solution is global. Finally, stability of the solution was obtained using a very important result known as the Komornik's condition, and a numerical example was given to illustrate the result, which also matched the analytical results. In fact, Galerkin-Faedo approximations and fixed point theory were used to establish the existence and uniqueness of the solution. At the end of the thesis, we provided a conclusion and perspectives.

Keywords: Viscoelastic term, Variable exponent, Source term, Faedo-Galerkin ap-

proximation, Local solution, Blow up, General decay, Global existence.

ABSTRACT IN FRENCH

Dans cette thèse, nous étudions des différents problèmes des équations d'ondes non linéaires avec des coefficients variables et des termes sources, parfois avec amortissement et parfois avec des termes viscoélastiques, sous des hypothèses appropriées sur les coefficients variables. Au début, nous avons présenté une série de résumés de certains travaux antérieurs réalisés par plusieurs chercheurs et les résultats qu'ils ont obtenus. Nous avons étudié plusieurs problèmes. Dans le premier problème, nous avons prouvé l'existence et l'unicité de la solution, puis nous avons démontré que la solution explosée en temps fini. Pour vérifier nos résultats théoriques, nous avons réalisé quelques tests numériques sous forme de figures, et leurs résultats ont concordé avec l'étude analytique théorique. En ce qui concerne le deuxième problème, nous avons prouvé l'existence et l'unicité de la solution et montré son existence globale en présence d'une énergie initiale positive, démontrant également la décroissance de l'énergie lorsque le temps est suffisamment grand, en nous appuyant sur l'espace de Nehari. Pour le troisième problème, nous avons prouvé l'explosion en temps fini dans les résultats analytiques et numériques de la solution. Dans le quatrième problème, nous avons démontré l'existence d'une solution locale et également prouvé que la solution locale est globale. Enfin, la stabilité de la solution a été obtenue en utilisant un résultat très important connu sous le nom Lemme de Komornik, et un exemple numérique a été donné pour illustrer le résultat, qui correspondait également aux résultats analytiques. En fait, des approximations de Galerkin-Faedo et la théorie des points fixes ont été utilisées pour établir l'existence et l'unicité de la solution. À la fin de la thèse, nous avons fourni une conclusion et perspectives.

Mots-clés: terme viscoélastique, exponent variable, terme source, Approximation de Faedo-Galerkin, Solution locale, Explosion, d´ecroissance générale, l'existence globale.

ملخص

في هذه الرسالة، نقوم بدراسة عدة مسائل مختلفة لمعادلات الموجات غير الخطية مع حد مصدر وذات أسس متغيرة، أحيانًا مع حد التخميد وأحيانًا مع حد مرن - لزج، تحت افتراضات مناسبة حول الأسس المتغيرة. في البداية، قدمنا سلسلة من ملخصات بعض الأعمال السابقة التي قام بها عدة باحثين والنتائج التي حصلوا عليها. لقد درسنا العديد من النتائج. في المسألة الأولى، قمنا بإثبات وجود ووحدانية الحل، ثم أثبتنا أن الحل يتفجر في زمن منتهي. للتحقق من النتائج النظرية لدينا، أجرينا بعض الاختبارات العددية في شكل رسومات، وكانت نتائجها متطابقة مع الدراسة التحليلية النظرية. أما بالنسبة للمسألة الثانية، فقد أثبتنا وجود ووحدانية الحل وأظهرنا وجوده العالمي و في وجود طاقة ابتدائية موجبة، كما قمنا بإظهار اضمحلال الطاقة عندما يكون الزمن كبيرًا بالقدر الكافي، حيث اعتمدنا على فضاء نهاري. بالنسبة للمسألة الثالثة، قمنا بإثبات الانفجار الحل في زمن محدود في النتائج التحليلية والعددية. في المسألة الرابعة، قمنا بإثبات وجود حل محلي وأيضًا أثبتنا أن الحل المحلي هو عالمي. وأخيرًا، تم الحصول على استقرار الحل باستخدام نتيجة هامة جدًا تعرف باسم توطئة كومورنيك، وتم إعطاء مثال عددي لتوضيح النتيجة، والذي كان متطابقًا أيضًا مع النتائج التحليلية. في الواقع، تم استخدام التقريبات فايد غاليركين ونظرية النقطة الثابتة لتأسيس وجود ووحدانية الحل. في نهاية الأطروحة، قدمنا خاتمة وعمل مستقبلي.

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LIST OF SYMBOLS

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers

$\mathbb{R}^n$: n-dimensional real Euclidean space

$[a, b]$: the interval of numbers between a and b, including a and b

(a, b) : an open interval

$[a, +\infty)$: left-closed and right–unbounded interval.

Ω : is open set in $\mathbb{R}^n$.

$\partial\Omega$: the boundary of the domain Ω.

$L^p(\Omega)$: Lebesgue space.

$W^{k,p}(\Omega)$: sobolev space with the norm $\| .\|_{k,p}$.

$L^{p(\cdot)}(\Omega)$: Lebesgue space with variable exponent.

$W^{k,p(\cdot)}(\Omega)$: sobolev space with variable exponent.

LIST OF FIGURES

INTRODUCTION

Literature Review

The natural phenomena are often translated into simple or complex differential equations, as well as linear or nonlinear ones. Many researchers have been interested in this problem due to its significant importance in understanding the behavior of a certain phenomenon, thereby modeling the physical phenomenon into differential mathematical equations of first, second, and higher orders. The differential equations can be classified from second-order equations to elliptic differential equations, hyperbolic differential equations, and parabolic equations. This thesis focuses on hyperbolic differential equations of higher orders due to their various applications in many fields of science and technology especially in physics, specifically in the science of electromagnetism and acoustics.

Considerable effort has been devoted to the study of the wave equation in the case of constant exponent as well as the variable exponent nonlinearities.

Case of constant exponent.

The following equation, with initial and Dirichlet-boundary conditions,

$$\rho_{tt} - \Delta\rho + a\rho_t \left|\rho_t\right|^{r-1} = b \left|\rho\right|^{p-2} \rho, \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{1}$$

where $a, b > 0, p, r > 2$, and Ω is a bounded domain of $R^n, (n \geqslant 1)$ with a smooth boundary $\partial\Omega$. has been studied by many researchers. For $a = 0$, the source term $b\left|\rho\right|^{p-2}\rho$ causes finite time blow-up of solutions with negative initial energy (see [7],

[24], [28], [29]). When $b = 0$, it is known that the damping term $a\rho_t |\rho_t|^{r-1}$ ensures global existence for arbitrary initial data (see [22], [26]).

The interaction between the damping and the source terms was considered for the first time by Levine [28], [29] in the case of linear damping ($r = 2$). His results show that solutions with negative initial energy blow up in finite time. Recently, Georgiev and Todorova [19] have extended the result of Levine to the non-linear case ($r > 2$). In their work the authors introduced a different method and determined appropriate relations between r and p, for which there is global existence or, alternatively, finite time blow-up. Precisely: If $r > p$ they showed that solutions with negative energy have global existence in time and blow up in finite time if $p > r$. Recently, this result has been extended to an abstract setting and to unbounded domains by Levine and Serrin [27] and Levine, Park and Serrin [30]. In these papers, the authors showed that there is no solution with negative energy that can be extended to on $[0, \infty)$ if $p > r$ and proved several noncontinuity theorems. This generalisation allowed them to apply their result also to quasilinear situations, of which problem (1.1) is a special case.

By combining the arguments from [19] and [27], Vitillaro [58] extended these results to situations where the damping is nonlinear and the solution has positive initial energy. with only negative initial energy Messaoudi [35] extended the blow-up result of [19].

Li et al. [31] considered the Petrovsky equation with strong damping term as follows

$$\rho_{tt} + \Delta^2\rho - \Delta\rho_t + \rho_t |\rho_t|^{r-1} = |\rho|^{p-2} \rho, \quad (x, t) \in \Omega \times \mathbb{R}^+. \tag{2}$$

The authors established the blow-up of solutions, existence and decay of the problem (2). Then the global existence and decay of solutions were proved by Polat and Piskin [49].

In [36], Messaoudi studied the Petrovsky equation as follows

$$\rho_{tt} + \Delta^2\rho + a\rho_t |\rho_t|^{r-2} = |\rho|^{p-2} \rho, \quad (x, t) \in \Omega \times \mathbb{R}^+. \tag{3}$$

He investigated the blow-up result in finite time for $p > r$. and the global existence for $r \geq p$

In [61] Tsai and Wu have shown that the solution for equation (3) is global. They also established the blow-up result in finite time for the non-negative initial energy.

In the case when $r=2$ following Petrovesky equation with memory and nonlinear source term

$$\rho_{tt}-\Delta^2\rho-\int_0^t g\left(t-s\right)\Delta^2\rho\left(x,s\right)ds=\left|\rho\right|^{p-1}\rho,\qquad (x,t)\in\Omega\times\mathbb{R}^+. \tag{4}$$

Was investigated by Tahamatani et al [56]. They showed the existence of weak solutions with initial value conditions and proved that there are solution under certain conditions on initial data which blow up in finite time with non-positive initial energy as well as positive initial energy and gave the lifespan estimates of solutions. In the absence of a nonlinear source term, Rivera et al. in [53] considered in a bounded domain $\Omega\subset\mathbb{R}^n$, and showed that the energy of solution decays exponentially, the relaxation function g provided decays exponentially.

Ouaoua et all in [42] considered a nonlinear Timoshenko equation:

$$\rho_{tt}+\Delta^2\rho-\Delta\rho+\rho_t\left|\rho_t\right|^{r-1}=\left|\rho\right|^{p-2}\rho,\quad (x,t)\in\Omega\times\mathbb{R}^+ \tag{5}$$

they proved the local existence, then under suitabel assumption with positive initial energy, proved that solution is global in time. By using komornik's inequality they established the stability result.

Recently, Algarabli [1], investigated the stability of the solutions to a viscoelastic plate equation with a logarithmic nonlinearity of the form

$$\begin{cases}\rho_{tt}-\Delta^2\rho+\rho-\int_0^t g\left(t-s\right)\Delta^2\rho\left(x,s\right)ds=k\rho\ln\left|\rho\right|, & \text{in } \Omega_t\\ \rho\left(x,0\right)=\rho_0\left(x\right),\ \rho_t\left(x,0\right)=\rho_1\left(x\right), & \text{in } \Omega\\ \rho\left(x,\ t\right)=0,\ \frac{\partial\rho}{\partial v}=0, & \text{on } \Gamma_t.\end{cases} \tag{6}$$

Gorka in [20] inquired the following initial boundary value problem

$$\begin{cases}\rho_{tt}-\Delta\rho+\rho=\varepsilon\rho\log\left|\rho\right|^2, & x\in\Omega,\ t\in(0,\ T)\\ \rho\left(x,0\right)=\rho_0\left(x\right),\ \rho_t\left(x,0\right)=\rho_1\left(x\right), & x\in\ \Omega\\ \rho\left(x,t\right)=0, & x\in\partial\Omega,\ t\in\left(0,\ T\right).\end{cases} \tag{7}$$

He proved many different results like as global existence and infinite time blow up in the different cases of initial energy. The absence of the term ρ and remplaced the source term by $\rho\log\left|\rho\right|^k$, equation of (7) becomes

$$\rho_{tt}-\Delta\rho=\rho\log\left|\rho\right|^k. \tag{8}$$

Cazenave and Haraux established in [13] the existence and uniqueness of a solution to the Cauchy problem for the equation (8) in $\mathbb{R}^3$. In [8], Bortkowski and Gorka obtained the existence of classical solutions and studied weak solution for the related initial value problem of the equation (8) in one dimension case. Lian et al. in [33] proved the global existence and infinite time blow of the solition at low energy level $E(0) \leq d$ by using potential well, and with additional some conditions, studied the blow up of a solution with arbitrary positive initial energy.

Concerning the higher-order wave equation with constant exponents:

$$\rho_{tt} + (-\Delta)^m \rho + a\rho_t |\rho_t|^{r-1} = b|\rho|^{p-2}\rho, \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{9}$$

where $m \geq 1$, we can cite the works of Brenner et al. [11] and Pecher [48].

Yaojun Ye [62], studied the following initial boundary value problem

$$\rho_{tt} + (-\Delta)^m \rho - \int_0^t g(t-s)(-\Delta)^m \rho(x,s)\,ds = |\rho|^{p-2}\rho, \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{10}$$

case of $m \geq 1$, the existence of global weak solutions to this equation is established by using the Galerkin method, under suitable conditions on the relaxation function g and the positive initial energy as well as non-positive initial energy, it is proved that the solution blows up in finite time.

In [23] Kafini studied the equation:

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int_0^t g(t-s)(-\Delta)^m \rho(x,s)\,ds + \rho_t = \rho|\rho|^{p-2}\ln|\rho|^k, & x \in \mathbb{R}^n,\ t > 0 \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{11}$$

where $m \geq 1$, $p > 2$ is real number, under certain conditions on the initial data with negative initial energy and with certain classes of relaxation functions, he proved a finite time blow-up result.

For more related results about the existence, blow-up and asymptotic properties of solutions, the readers are also referred to [2, 12, 40, 44, 46, 45, 53, 59, 63]

Case of Variable exponent

In recent years much attention has been given to the study of nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents, of nonlinearity. For example, some models of physical phenomena such as flows of electro-rheological

fluids or fluids with temperature dependent viscosity, filtration processes in porous media, nonlinear viscoelasticity and image processing. More details on this topic can be found in [6] and [5].
For hyperbolic problems of the type nonlinearities, only a few papers have appeared. For example, Antontsev [3] considered the equation

$$\rho_{tt} - div\left(a(x,t)\left|\nabla\rho\right|^{q(x,t)-2}\nabla\rho\right) + \alpha\Delta\rho_t = b(x,t)\left|\rho\right|^{p(x,t)-2}\rho, \text{ in } \Omega\times(0,T). \qquad (12)$$

in a bounded domain $\Omega \subset R^n$ where $\alpha > 0$ is a constant and a, b, q, p are given functions. For certain conditions on a, b, q, p, he proved some blow-up results, for certain solutions with negative initial energy. He also discussed the case when $\alpha = 0$ and proved a blow-up result. Later, Antontsev [4] discussed the same equaton and proved a local and a global existence of some weak solutions under certain hypotheses on the functions a, b, q, p. He also established blow-up results for certain solutions with non-positive initial energy.

Guo and Gao [21] studied the same problem as [6] and obtained several blow-up results for certain solutions associated with negative initial energy. Specifically, they took $p(x,t) = p > 2$, a constant, and obtained a result for the blow-up in a finite amount of time. For the case $p(x,t) = p(x)$, they claimed the same blow up result, but no proof was given. This work is considered to be an improvement on that of [3].

In [55], Sun et al. looked into the following equation

$$\rho_{tt} - div\left(a(x,t)\nabla\rho\right) + c(x,t)\rho_t|\rho_t|^{r(x,t)-1} = b(x,t)\left|\rho\right|^{p(x,t)-1}\rho, \text{ in } \Omega\times(0,T) \qquad (13)$$

Ω bounded domain, with Dirichlet boundary conditions, and established a blow-up result for solutions with positive initial energy. They also gave lower and upper bounds on the blowup time and provided a numerical example of their result.

Messaoudi and Talahmeh [37] studied

$$\rho_{tt} - div\left(|\nabla\rho|^{q(x)-2}\right) + \mu\rho_t = |\rho|^{p(x)-2}\rho, \text{ in } \Omega\times(0,T) \qquad (14)$$

The same authors in [38] extended the letter results to an equation of the form

$$\rho_{tt} - div\left(|\nabla\rho|^{q(x)-2}\right) + a\rho_t|\rho_t|^{r(x)-2} = b\left|\rho\right|^{p(x)-2}\rho, \text{ in } \Omega\times(0,T) \qquad (15)$$

where $a, b > 0$ are constants and the exponents of the non-linearity q, r and p are given functions. They proved that the solutions blow-up in finite-time with negative initial energy as well as with positive energy.

Most recent Messaoudi et al. [39] studied

$$\rho_{tt} - \Delta\rho + a\rho_t|\rho_t|^{r(x)-2} = b\,|\rho|^{p(x)-2}\,\rho, \text{ in } \Omega \times (0,T) \tag{16}$$

they established the existence of a unique weak solution using the Faedo-Galerkin method. under suitable assumptions on the variable exponents r and p. They also proved the finite-time blow up of solutions and gave a two-dimension numerical example to illustrate the blowup result.

Ouaoua et all.[43] considered following p-Kirchhoff type hyperbolic equation with variable exponents

$$\rho_{tt} - M\left(\int_\Omega |\nabla\rho\;|^p\;dx\right)\Delta_p\rho + \rho_t|\rho_t|^{r(x)-2} = |\rho|^{q(x)-2}\rho, \text{ in } \Omega \times (0,T) \tag{17}$$

where $M(s) = a + bs$ with positive parameters a, b, $\Delta_p\rho = div(|\nabla\rho\;|^{p-2}\;\nabla\rho)$, the authore proved the global existence and stability results.

Yunzhu Gao and Wenjie Gao [17] studied a nonlinear viscoelastic equation with variable exponents

$$\rho_{tt} - \Delta\rho - \Delta\rho_{tt} + \int_0^t g\,(t-s)\,\Delta\rho\,(x,s)\,ds + |\rho_t|^{r(x)-2}\rho_t = |\rho|^{p(x)-2}\rho, \text{ in } \Omega \times (0,T) \tag{18}$$

They proved the existence of weak solutions using the Faedo-Galerkin method under suitable assumptions

some other authors investigate hyperbolic type equation with variable exponents (see [18, 32, 44, 50, 54, 15, 16].

Organization of the thesis

Apart from the introduction, this dissertation is divided into fourth chapters.

In chapter 1: we recall some notations and we review some mathematical concepts that will be used throughout this thesis.

In chapter 2: We investigate boundary value problem:

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int\limits_0^t g(t-s)(-\Delta)^m \rho(s)\,ds + \rho_t = |\rho|^{p(x)-2}\rho, & \text{in } \Omega_t, \\ (x,t) = 0,\ \frac{\partial^i \rho}{\partial v^i} = 0,\ i = 1,\ 2, ..., m-1, & \text{on } \Gamma_t \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{19}$$

where $m \geq 1$ is a natural number, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\partial\Omega$ is smooth boundary of Ω, $\Omega_t = \Omega \times \mathbb{R}^+$, $\Gamma_t = \partial\Omega \times \mathbb{R}^+$, g is the relaxation function satisfying some condition. $p(.)$ is given measurable functions on Ω, Using a technique that combines the Faedo-Galarkin and Banach fixed point theorems, we first demonstrate the existence and uniqueness of the local solution. We also demonstrate that the solution blows up in a finite amount of time. In conclusion, we present a numerical example in two dimensions to illustrate the blow-up result.

In chapter 3: We consider the following boundary value problem:

$$\begin{cases} \rho_{tt} - \Delta\rho - \Delta\rho_{tt} + \int\limits_0^t g(t-s)\Delta\rho(s)\,ds + |\rho_t|^{m(x)-2}\rho_t = |\rho|^{p(x)-2}\rho, & \text{in } Q, \\ \rho(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{20}$$

where $Q = \Omega \times (0,T)$ and Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$ with smooth boundary $\partial\Omega$. $p(.)$ and $m(.)$ are given measurable functions on Ω First, we prove that the problem has a unique local solution for a suitable conditions by using Faedo Galerkin methods, and we also prove that the local solution is global in time. Finally, we demonstrate that the solution with positive initial energy decays exponentially.

In chapter 4: we study a value problem related to the following higher-order wave equation:

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int\limits_0^t g(t-s)(-\Delta)^m \rho(s)\,ds + \rho_t = |\rho|^{p(x)-2}\rho \ln|\rho|^k, & \text{in } \Omega_t, \\ \rho(x,t) = 0,\ \frac{\partial^i u}{\partial v^i} = 0,\ i = 1,\ 2, ..., m-1, & \text{on } \Gamma_t \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{21}$$

where $m \geq 1$ is a natural number, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, $\partial\Omega$ is smooth bouandary of Ω, $\Omega_t = \Omega \times \mathbb{R}^+$, $\Gamma_t = \partial\Omega \times \mathbb{R}^+$, g is the relaxation function and k is positive constant. $p(.)$ is given measurable functions on Ω, We demonstrate that the solution blows up in finite time under suitable conditions for the relaxation functions g and p(.). We also provide a numerical example in two dimensions to demonstrate the blow-up result.

In chapter 5: We focus on a class of fourth-order nonlinear wave equations:

$$\begin{cases} \rho_{tt} + \Delta^2 \rho - \Delta \rho + |\rho_t|^{m(x)-2} \rho_t = |\rho|^{r(x)-2} \rho, & (x,t) \in \ \Omega \times (0,T), \\ \rho(x,t) = 0, & (x,t) \in \ \partial\Omega \times (0,\ T), \\ \rho(x,0) = \rho_0(x), \ \rho_t(x,0) = \rho_1(x), & x \in \Omega, \end{cases} \tag{22}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$. $m(.)$ and $r(.)$ are given measurable functions on Ω. Under suitabel conditions on the variable exponents, we proved the existence of a local solution, then we proved that it is global in time. We also demonstrated the stability result both theoretically and numerically.

CHAPTER 1

PRELIMINARIES

The topic of this chapter is variable exponent Lebesgue and Sobolev spaces, which differentiate from classical spaces L^p and W^p by the fact that exponent p is a function from Ω rather than a constant. In this chapter, we review some mathematical concepts a that will be used throughout this thesis.

1.1 Functional spaces with variable exponents

1.1.1 Lebesgue space with variable exponents

Definition 1.1 [14] Let (ω, Σ, μ) be a σ-finite, complete measure space. We define $P(\omega, \mu)$ to be the set of all $\mu-$ measurable functions. $p : \omega \to [1, \ +\infty]$ be a measurable function. We refer to The function $p \in P(\omega, \mu)$ as variable exponents on ω.
We define

$$1 \leq p_1 := ess\inf_{x \in \omega} p\left(x\right) \leq p\left(x\right) \leq p_2 := ess\sup_{x \in \omega} p\left(x\right) < \infty,$$

if $p_2 < \infty$ then p is said to be a bounded variable exponent.

If $p \in P(\omega, \mu)$, then we define $p^{'}$ by

$$\frac{1}{p(x)} + \frac{1}{p^{'}(x)} = 1, \text{where } \frac{1}{\infty} = 0$$

The function $p^{'}$ is called the dual variable exponent of p.

Definition 1.2 [14] We define the Lebesgue space with a variable-exponent p by

$$L^{(\cdot)}(\Omega) := \left\{\rho : \Omega \to \mathbb{R} : \text{ measurable in } \Omega,\ \varrho_{p(.)}(\lambda\rho) < +\infty,\ \text{ for some } \lambda > 0\right\},$$

where

$$\varrho_{p(.)}(\rho) = \int_{\Omega} |\rho(x)|^{p(x)}\, dx.$$

The set $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg's norm

$$\|\rho\|_{p(.)} := \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{\rho(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space [14].

Lemma 1.1 *[14] If $p(x) = p$, where p is a constant. Then*

$$\| \rho \|_{p(.)} = \| \rho \|_{p} = \lambda_0 = (\int_{\Omega} | \rho^p |)^{\frac{1}{p}}$$

Definition 1.3 [14] we say that a function $q : \Omega \longrightarrow R$ is log Hölder continuous. If there exists $A > 0$ and $0 < \zeta < 1$ such that

$$| q(x_1) - q(x_2)| \leq -\frac{A}{\log | x_1 - x_2 |}, \quad \text{for all} \quad x_1, x_2 \in \Omega, \text{with } | x_1 - x_2| < \zeta.$$

The following lemma gives the relation between the function, called modular function, and the norm

Lemma 1.2 *[14] If ξ is a mesurable function on Ω satisfying:*

$$1 \leq \xi_1 := ess\inf_{x\in\Omega} \xi(x) \leq \xi(x) \leq \xi_2 := ess\sup_{x\in\Omega} \xi(x) < \infty,$$

then we have

$$\min\left\{\|\rho\|_{\xi(.)}^{\xi_1},\ \|\rho\|_{\xi(.)}^{\xi_2}\right\} \leq \varrho_{\xi(.)}(\rho) \leq \max\left\{\|\rho\|_{\xi(.)}^{\xi_1},\ \|\rho\|_{\xi(.)}^{\xi_2}\right\},$$

for any $\rho \in L^{q(\cdot)}(\Omega)$.

As in the case of the constant exponent, we have the following Young's and Hölder's inequalities inequalities.

Lemma 1.3 *[14](**Young's Inequality**)*

Let $p, q, s \geq 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \textit{for a.e} \quad y \in \Omega$$

then for all $a, b \geq 0$,

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{p(.)}}{p(.)} + \frac{(b)^{q(.)}}{q(.)}$$

Given $s = 1$ and $1 < p, q < \infty$, for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \forall a, b \geq 0$$

where

$$C_\varepsilon = 1/q(\varepsilon p)^{\frac{q}{p}}$$

with $p = q = 2$, we have the following:

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

Lemma 1.4 *[14]**Hölder's inequality** Let p, q, s be measurable functions defined on Ω which satisfy the following conditions*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \textit{for a.e} \quad y \in \Omega$$

if $f \in L^{p(.)}(\Omega) and g \in L^{q(.)}(\Omega) then fg \in L^{s(.)}(\Omega)$ and

$$\| fg \|_{s(.)} \leq 2 \| f\|_{p(.)} \| g\|_{q(.)}.$$

*We have the **Cauchy-Schwartz inequality** by taking $p = q = 2$.*

1.1.2 Sobolev space with variable exponents

Definition 1.4 [14] Let k be $\in N$. We define the Sobolev space $W^{k,p(.)}(\Omega)$ of the variable exponent as follows:

$$W^{k,p(.)}(\Omega) = \left\{\rho \in L^{p(.)}(\Omega) \quad such that \quad \partial^{|\alpha|}\rho \in L^{p(.)}(\Omega) \quad with \quad |\alpha| \leq k\right\}$$

equipped with the norm

$$\| \rho\|_{W^{k,p(.)}(\Omega)} = \inf\left\{\lambda > 0 : \varrho_{W^{k,p(.)}(\Omega)}(\frac{\rho}{\lambda}) \leqslant 1\right\} = \sum_{0 \leqslant |\alpha| \leqslant k} \| \partial_\alpha \rho\|_{p(.)},$$

where

$$\varrho_{W^{k,p(.)}(\Omega)}(\rho) = \sum_{0 \leqslant |\alpha| \leqslant k} \varrho_{L^{k,p(.)}(\Omega)}(\partial_\alpha \rho)$$

Clearly

$$W^{0,p(.)}(\Omega) = L^{p(.)}(\Omega)$$

and

$$W^{1,p(.)}(\Omega) = \{\rho \in L^{p(.)}(\Omega) \quad \text{such that} \quad \nabla\rho \quad \text{exists and} \mid \nabla\rho| \in L^{p(.)}(\Omega)\}$$

equipped with the following norm

$$\| \rho\|_{W^{k,p(.)}(\Omega)} = \| \rho\|_{p(.)} + \| \nabla\rho\|_{p(.)}.$$

Theorem 1.1 *[14] Let $p \in P(\Omega)$. The space $W^{k,p(.)}$ is a Banach space which is:*

- *Separable if p is bounded*
- *Reflexive if $1 < p_1 \leq p_2 \leq +\infty$*

Definition 1.5 [14] Let $p \in P(\Omega)$and$k \in \mathbb{N}$. The zero-boundary Sobolev space $W_0^{k,p(.)}$ is the closure of the set of $W^{k,p(.)}$ functions with compact support, i.e.

$$W_0^{k,p(.)}(\Omega) = \{\rho \in W^{k,p(.)}(\Omega) : \rho_{\chi K} \text{for a compact} K \subset \Omega\}$$

Furtheremore,

- $C_0^\infty(\Omega)$ in the space $W_0^{k,p(.)}(\Omega)$ is denoted by $H_0^{k,p(.)}(\Omega)$
- the dual space $W_0^{1,p(.)}(\Omega)$ is indicated as $W_0^{-1,p'(.)}(\Omega)$ as well as usual Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'} = 1$

Remark 1.1 1. $H_0^{k,p(.)}(\Omega) \subset W_0^{k,p(.)}(\Omega)$

2. $W_0^{k,p(.)}(\Omega) = H_0^{k,p(.)}(\Omega)$, if p is log-Hölder continuous on Ω

The variable exponent version of the Poincaré inequality is given in the following theorem

Theorem 1.2 *[14](**Poincare's Inequality**) Let $\Omega \subset \mathbb{R}^n$, p satisfies the log-Hölder inequality on Ω then*

$$\| \rho\|_{p(.)} \leq C \| \nabla\rho\|_{p(.)} \quad \textit{for all} \quad \rho \in W_0^{1,p(.)}(\Omega)$$

where the constant $C \geqslant 0$ depends on Ω and $p(.)$ only. The equivalent norm of the space $W_0^{1,p(.)}(\Omega)$ is given bylog-Hölder inequality on Ω then

$$\| \rho\|_{W_0^{1,p(.)}} =\| \nabla\rho\|_{p(.)}$$

Lemma 1.5 *[14] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ Assume that $p : \Omega \longrightarrow]0, \infty[$ such that*

$$1 < p_1 \leqslant p(x) \leqslant p_2 < \infty \quad \textit{for a.e} \quad x \in \Omega$$

if $p(x) < q^*(x)$ with

$$q^*(x) = \begin{cases} \frac{nq(x)}{n-q(x)}, & \textit{if} \quad q^+ < n \\ +\infty \quad \textit{if} \quad q^+ \geq n \end{cases}$$

Then, the embedding $W^{1,q(.)} \hookrightarrow L^{q(.)}$ is continuous and compact.

Corollary 1.1 *Let $\Omega \subset \mathbb{R}^{\ltimes}$ be a bounded domain with smooth boundary $\partial\Omega$Assume that $p : \overline{\Omega} \longrightarrow]0, \infty[$ is a continuous function such that*

$$2 \leqslant p_1 \leqslant p(x) \leqslant p_2 < \frac{2n}{n-2}, n \geqslant 3$$

Then, the embedding $H_0^1 \hookrightarrow L^{p(.)}$ is continuous and compact.

CHAPTER 2

LOCAL WELL-POSEDNESS AND BLOW-UP OF SOLUTION FOR A HIGHER-ORDER WAVE EQUATION WITH VISCOELASTIC TERM AND VARIABLE-EXPONENT

The main aim of this work is to study the existence and uniqueness of the local solution under suitable conditions for the relaxation function g and variable exponent $p(.)$, using a method which is a mixture of the Faedo-Galarkin and Banach fixed point theorem, and also prove that the solution blows up in finite time. Finally, to illustrate the blow-up result, we give a two-dimensional numerical example.
The results presented in this chapter have been published in[10].

2.1 Setting the problem

We are concerned with the following initial-boundary-value problem:

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int\limits_0^t g(t-s)(-\Delta)^m \rho(s)\,ds + \rho_t = |\rho|^{p(x)-2}\rho, & \text{in } \Omega_t, \\ \rho(x,t) = 0,\ \frac{\partial^i \rho}{\partial v^i} = 0,\ i = 1,\ 2, ..., m-1, & \text{on } \Gamma_t, \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $m \geq 1$ is a natural number, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\partial\Omega$ is smooth boundary of Ω, $\Omega_t = \Omega \times \mathbb{R}^+$, $\Gamma_t = \partial\Omega \times \mathbb{R}^+$.

Assumption:

g : the relaxation function satisfying:

$$1 - \int_0^t g(s)\,ds = \beta > 0,\ g'(t) \leq 0 \text{ for } t \geq 0. \tag{2.2}$$

$p(.)$: is measurable functions on Ω, satisfying

$$\begin{cases} 2 < p_1 \leq p(x) \leq p_2 \leq p^* \text{ for } n \leq 2m, \\ 2 < p_1 \leq p(x) \leq p_2 \leq p^* \text{ for } n > 2m, \end{cases} \tag{2.3}$$

with

$$p^* = \begin{cases} \infty, \text{ if } n \leq 2m, \\ \frac{2n}{n-2m}, \text{ if } n > 2m, \end{cases}$$

where

$$p_1 := ess_{x\in\Omega} \inf(p(x)),\ p_2 := ess_{x\in\Omega} \sup(p(x))$$

We also assume that $p(x)$ satisfy the log-Hölder condition (definition (1.3)).

Befor to state and proof our result we need to define the functional energy of poroblem 2.1 as follow

$$E(t) = \frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|D^m\rho(t)\|_2^2 + \frac{1}{2}(g \circ D^m\rho)(t) - \int\limits_\Omega \frac{1}{p(x)}|\rho(t)|^{p(x)}\,dx, \tag{2.4}$$

where $(g \circ D^m v)(t) = \int_0^t g(t-s) \|D^m v(t) - D^m v(s)\|^2 ds$.

Now, we show that the energy functional is a nonincreasing function along the solution of (2.1) in the next lemma.

Lemma 2.1 *Suppose* (2.2)*and* (2.3) *hold. Then* $E(t)$ *decreases, which*

$$E'(t) = \frac{1}{2}(g' \circ D^m \rho)(t) - \frac{1}{2} g(t) \|D^m \rho(t)\|_2^2 - \|\rho_t\|_2^2 \leq 0,$$

furthermore,

$$E(t) - E(0) \leq 0, \quad t \geq 0. \tag{2.5}$$

Proof. Multiplying the first equation in (2.1) by ρ_t and integrating over Ω, we get

$$\int_\Omega \rho_t \rho_{tt} dx + \int_\Omega \rho_t (-\Delta)^m \rho dx - \int_\Omega \rho_t \int_0^t g(t-s)(-\Delta)^m \rho(x,s)\, dx ds + \int_\Omega \rho_t^2 dx$$
$$= \int_\Omega \rho_t \rho |\rho|^{p(x)-2} dx,$$

then use integration par parts, yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\rho_t|^2 dx + \int_\Omega |D^m \rho|^2 dx\right) - \int_0^t g(t-s) \int_\Omega D^m \rho_t(t) D^m \rho(s)\, dx ds$$
$$+ \int_\Omega \rho_t^2 dx = \frac{d}{dt}\left(\int_\Omega \frac{1}{p(x)} |\rho|^{p(x)} dx\right). \tag{2.6}$$

The third term in (2.6) can be estimated as:

$$\int_0^t g(t-s)\int_\Omega D^m\rho_t(t)\, D^m\rho(s)\, dxds$$

$$= \int_0^t g(t-s)\int_\Omega D^m\rho_t(t)\,.\left(D^m\rho(s) - D^m\rho(t)\right) dxds$$

$$+ \int_0^t g(t-s)\int_\Omega D^m\rho_t(t)\,.D^m\rho(t)\, dxds$$

$$= -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega \left|D^m\rho(s) - D^m\rho(t)\right|^2 dxds$$

$$+ \int_0^t g(t-s)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega \left|D^m\rho(t)\right|^2 dx\right) ds$$

$$= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega \left|D^m\rho(s) - D^m\rho(t)\right|^2 dxds\right]$$

$$+ \frac{1}{2}\int_0^t g'(t-s)\int_\Omega \left|D^m\rho(s) - D^m\rho(t)\right|^2 dxds$$

$$+ \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\, ds\int_\Omega \left|D^m\rho(t)\right|^2 dxds\right] - \frac{1}{2}g(t)\int_\Omega \left|D^m\rho(t)\right|^2 dx. \quad (2.7)$$

Insert (2.7) in (2.6) to get

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |\rho_t|^2 dx + \frac{1}{2}\int_\Omega |D^m\rho|^2 dx + \int_\Omega \frac{1}{p(x)}|\rho|^{p(x)} dx\right.$$

$$\left.+\frac{1}{2}\int_0^t g(t-s)\int_\Omega \left|D^m\rho(s) - D^m\rho(t)\right|^2 dxds - \frac{1}{2}\int_0^t g(s)\, ds\int_\Omega \left|D^m\rho(t)\right|^2 dx\right\}$$

$$= \frac{1}{2}\int_0^t g'(t-s)\int_\Omega \left|D^m\rho(s) - D^m\rho(t)\right|^2 dxds - \frac{1}{2}g(t)\int_\Omega \left|D^m\rho(t)\right|^2 dx - \int_\Omega \rho_t^2 dx,$$

hence, using (2.4), we obtain

$$E'(t) = \frac{1}{2}(g' \circ D^m \rho) - \frac{1}{2} g(t) \|D^m \rho\|_2^2 - \|\rho_t\|^2 \leq 0.$$

Using integration of last inequality, we get

$$E(t) \leq E(0). \tag{2.8}$$

Lemma 2.2 *[38] Suppose that*(2.3)*, and the log-Hölder continuity condition (definition* (1.3)*) hold and* $E(0) < 0$*.Then the solution of* (2.1) *satisfies*

$$\int_{\Omega} |\rho|^{p(x)} dx \geq c \|\rho\|_{p_1}^{p_1}. \tag{2.9}$$

2.2 Existence of weak solutions

In this section, we will prove the nonexistence global solution of (2.1), we state the following lemma witch can be obtained by using the Faedo-Galerkin method by combining the argument of [39, 47, 60, 62].

Lemma 2.3 *: Assume that* (2.3) *and the log Hölder continuity condition (definition* (1.3)*) hold and* $(\rho_0, \rho_1) \in \left(H_0^m(\Omega), L^2(\Omega)\right)$ *and give* $f(t, x)$ *a fixed function on* $\Omega \times (0, t)$*. Then there existe a unique local solution* ρ *of*

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int_0^t g(t-s)(-\Delta)^m \rho(s)\, ds + \rho_t = f(t,x), & \text{in } \Omega_t, \\ \rho(x,t) = 0,\ \frac{\partial^i \rho}{\partial v^i} = 0,\ i = 1,\ 2, ..., m-1, & \text{on } \Gamma_t \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{2.10}$$

Satisfying $\rho \in L^\infty((0, T), H_0^m(\Omega)), u_t \in L^\infty((0, T), L^2(\Omega)) \cap L^2(\Omega \times (0, T))$, where $f \in L^2(\Omega \times (0, T))$.

Now, we prove the local existence of (2.1) by using the method of Banach fixed point theorem.

Theorem 2.1 : *Suppose that* (2.3) *holds. Suppose further that*

$$2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2(n-m)}{n-2m}, \quad (n > 2m) \tag{2.11}$$

and $(\rho_0, \rho_1) \in \left(H_0^m(\Omega), L^2(\Omega)\right)$. *Then there existe* $T > 0$, *such that* (2.1) *has unique local solution*

$$\rho \in L^\infty((0, T), H_0^m(\Omega)), \rho_t \in L^\infty((0, T), L^2(\Omega)) \cap L^2(\Omega \times (0, T)).$$

Proof. Let $v \in L^\infty((0,T), H_0^m(\Omega))$ and $f(v) = |v|^{p(x)-2} v$, we have

$$\|f(v)\|_{L^2}^2 = \int_\Omega |v|^{2p(x)-2} dx \leq \int_\Omega |v|^{2p_2-2} dx + \int_\Omega |v|^{2p_1-2} dx < \infty,$$

Since

$$2p_1 - 2 \leq 2p_2 - 2 \leq \frac{2n}{n-2m}, \quad (n > 2m).$$

So,

$$f(v) \in L^\infty((0,t), H_0^m(\Omega)) \subset L^2(\Omega \times (0, T)).$$

Therefore, for each $v \in L^\infty((0, T), H_0^m(\Omega))$, there existe a unique

$$\rho \in L^\infty((0, T), H_0^m(\Omega)), \ \rho_t \in L^\infty((0,T), L^2(\Omega)) \cap L^2(\Omega \times (0, T)).$$

Satisfying the following problem

$$\begin{cases} \rho_{tt} + (-\Delta)^m \rho - \int_0^t g(t-s)(-\Delta)^m \rho(s)\, ds + \rho_t = f(v), & \text{in } \Omega_t, \\ \rho(x, t) = 0, \ \frac{\partial^i \rho}{\partial v^i} = 0, \ i = 1, 2, ..., m-1, & \text{on } \Gamma_t, \\ \rho(x, 0) = \rho_0(x), \ \rho_t(x, 0) = \rho_1(x), & \text{in } \Omega. \end{cases} \tag{2.12}$$

Let a map $G : X_T \longrightarrow X_T$, by $G(v) = \rho$, where

$$X_T = \left\{ w \in L^{\infty}\left((0,T),\ H_0^m(\Omega)\right), w_t \in L^{\infty}((0,T),\ L^2(\Omega)) \right\}.$$

X_T is Banach space with respect to the norm

$$\|w\|_{X_T} = \frac{1}{2}\sup_{(0,T)} \|w_t\|_2^2 + \frac{1}{2} l \sup_{(0,T)} \|D^m w\|_2^2 .$$

Multiplying the equation (2.12) by ρ_t and integrating over $\Omega \times (0,t)$, to get

$$\begin{aligned}
&\frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|D^m\rho(t)\|_2^2 + \frac{1}{2}(g \circ D^m\rho)(t) \\
&- \int_0^t \left[\frac{1}{2}(g' \circ D^m\rho)(s) - \frac{1}{2}g(s)\|D^m\rho(s)\|_2^2\right] ds + \int_0^t \int_\Omega \rho_t^2\,dx\,ds \\
&\frac{1}{2}\|\rho_1\|_2^2 + \frac{1}{2}\|D^m\rho_0\|_2^2 + \int_0^t \int_\Omega |v|^{2p(x)-2} v\rho_t\,dx\,ds.
\end{aligned} \tag{2.13}$$

Using the Young and the Sobolev-Poincare inequalities, we obtain

$$\begin{aligned}
\left|\int_\Omega |v|^{p(x)-2} v\rho_t\,dx\right| &\le \frac{\delta}{4}\|\rho_t(t)\|_2^2 + \frac{4}{\delta}\int_\Omega |v|^{2p(x)-2}\,dx \\
&\le \frac{\delta}{4}\|\rho_t(t)\|_2^2 + \frac{4}{\delta}\left(\int_\Omega |v|^{2p_2-2}\,dx + \int_\Omega |v|^{2p_1-2}\,dx\right) \\
&\le \frac{\delta}{4}\|\rho_t(t)\|_2^2 + \frac{4c_*}{\delta}\left(\|D^m v\|_2^{2p_2-2} + \|D^m v\|_2^{2p_1-2}\right).
\end{aligned} \tag{2.14}$$

Thus by (2.13) and (2.14), we get

$$\begin{aligned}
\frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{1}{2}l\|D^m\rho(t)\|_2^2 &\le k_0 + \frac{\delta T}{4}\sup_{(0,\,T)}\|\rho_t(t)\|_2^2 \\
&\quad + \frac{4c_*}{\delta}\left(\int_0^T \|D^m v\|_2^{2p_2-2} + \int_0^T \|D^m v\|_2^{2p_1-2}\right) ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\sup_{(0,\,T)}\|\rho_t(t)\|_2^2 + \frac{1}{2}l\sup_{(0,\,T)}\|D^m\rho(t)\|_2^2 &\le k_0 + \frac{\delta T}{4}\sup_{(0,\,T)}\|\rho_t(t)\|_2^2 \\
&\quad + \frac{4c_*T}{\delta l^{2p_2-2}}\left[\int_0^T \|v\|_{X_T}^{p_2-1} + \int_0^T \|v\|_{X_T}^{p_1-1}\right].
\end{aligned}$$

Where $k_0 = \frac{1}{2}\|\rho_1\|_2^2 + \frac{1}{2}\|D^m\rho_0\|_2^2$ and c_* is the embedding constant.

Taking $\delta T = 1$, we get

$$\frac{1}{2}\sup_{(0,\ T)}\|\rho_t(t)\|_2^2 + \frac{1}{2}l\sup_{(0,\ T)}\|D^m\rho(t)\|_2^2 \leq 2k_0 + \frac{8c_*T}{\delta l^{2p_2-2}}\left[\|v\|_{X_T}^{p_2-1} + \|v\|_{X_T}^{p_1-1}\right].$$

Then

$$\|\rho\|_{X_T} \leq K + T\alpha\left[\|v\|_{X_T}^{p_2-1} + \|v\|_{X_T}^{p_1-1}\right].$$

Choosing M_0 large enough and T Sufficiently small such that

$$\|\rho\|_{X_T} \leq K + 2T\alpha M_0^{2p_2-2} \leq M_0.$$

If $K \leq M_0^2$ and $T \leq T_0 < \frac{M_0^2-K}{2\alpha M_0^{2P_2-2}}$.

Thus, we have $G : \Lambda \longrightarrow \Lambda$, where

$$\Lambda = \left\{w \in X_T,\ \|w\|_{X_T} \leq M_0^2\right\}.$$

Next, we show that G is contraction. For this purpose, let $\rho_1 = G(v_1)$ and $\rho_2 = G(v_2)$ and set $\rho = \rho_1 - \rho_2$ satisfyies

$$\begin{cases} \rho_{tt} + (-\Delta)^m\rho - \int\limits_0^t g(t-s)(-\Delta)^m\rho(s)\,ds + (\rho_{1t} - \rho_{2t}) & \\ \qquad = |v_1|^{p(x)-2}v_1 - |v_2|^{p(x)-2}v_2 & \text{in } \Omega\times(0,t). \\ \rho(x,t) = 0,\ \frac{\partial^i u}{\partial v^i} = 0,\ i = 1,\ 2,...,m-1, & \text{on } \Gamma_t, \\ \rho(x,\ 0) = \rho_t(x,\ 0) = 0, & \text{in } \Omega. \end{cases} \tag{2.15}$$

Multiplying (2.15) by $\rho_t = \rho_{1t} - \rho_{2t}$ and integrate on $\Omega\times(0,t)$, we get

$$\frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|D^m\rho(t)\|_2^2 + \frac{1}{2}(g\circ D^m\rho)(t)$$
$$-\int_0^t\left[\frac{1}{2}(g'\circ D^m\rho)(s) - \frac{1}{2}g(s)\|D^m\rho(s)\|_2^2\right]ds + \int_0^t\|\rho_{1t} - \rho_{2t}\|^2\,dx$$
$$+\int_0^t\int_\Omega\left(|v_1|^{p(x)-2}v_1 - |v_2|^{p(x)-2}v_2\right)\rho_t ds.$$

Then, we have

$$\frac{1}{2}\left\|\rho_t(t)\right\|_2^2+\frac{1}{2}l\left\|D^m\rho(t)\right\|_2^2\leq\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\rho_t dxds, \qquad (2.16)$$

where $h(s)=|s|^{p(x)-2}s$.

Now, we estimate

$$I=\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\rho_t dxds.$$

We get

$$\begin{aligned}I&\leq\left|\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\rho_t dxds\right|\\&\leq\int_\Omega\left|h'(\xi)\right||v|\left\|\rho_t\right\|dx,\end{aligned}$$

where $v=v_1-v_2$ and $\xi=\alpha v_1+(1-\alpha)v_2,\ \ 1\geq\alpha\geq 0$. By the Young inequality implies

$$\begin{aligned}I&\leq\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+\frac{2}{\delta}\int_\Omega\left|h'(\xi)\right|^2|v|^2dx\\&\leq\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+\frac{2\left(p_2-1\right)^2}{\delta}\int_\Omega\left|\alpha v_1+(1-\alpha)v_2\right|^{2(p(x)-2)}|v|^2dx\\&\leq\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+\frac{2\left(p_2-1\right)}{\delta}\left(\int_\Omega|v|^{\frac{2n}{n-2m}}dx\right)^{\frac{n-2m}{n}}\left[\int_\Omega\left|\alpha v_1+(1-\alpha)v_2\right|^{\frac{n}{m}(p(x)-2)}\right]^{\frac{2m}{n}}\\&\leq\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+c_\delta\left(\int_\Omega|v|^{\frac{2n}{n-2m}}dx\right)^{\frac{n-2m}{n}}\left[\int_\Omega\left|\alpha v_1+(1-\alpha)v_2\right|^{\frac{n}{m}(p_2-2)}dx+\right.\\&\left.\int_\Omega\left|\alpha v_1+(1-\alpha)v_2\right|^{\frac{n}{m}(p_2-2)}dx\right]^{\frac{2m}{n}}. \qquad (2.17)\end{aligned}$$

Since $2\leq p_1\leq p(x)\leq p_2\leq\frac{2(n-m)}{n-2m},\ (n>2)$, we get

$$\begin{aligned}I\leq&\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+c_\delta c_*\left\|D^m v\right\|_2^2\left(\left\|D^m v_1\right\|_2^{2(p_2-2)}+\left\|D^m v_1\right\|_2^{2(p_1-2)}\right.\\&\left.+\left\|D^m v_2\right\|_2^{2(p_2-2)}+\left\|D^m v_2\right\|_2^{2(p_1-2)}\right)\\\leq&\frac{\delta}{2}\left\|\rho_t(t)\right\|_2^2+4c_{\delta,l}c_*M_0^{2(p_2-2)}\left\|D^m v\right\|_2^2.\end{aligned}$$

Therefore, (2.16) takes the form

$$\begin{aligned}&\frac{1}{2}\sup_{(0,\,T)}\|\rho_t(t)\|_2^2+\frac{1}{2}l\sup_{(0,T)}\|D^m\rho(t)\|_2^2\\&\leq\frac{\delta}{2}T_0\frac{1}{2}\sup_{(0,\,T)}\|\rho_t(t)\|_2^2+4c_{\delta,l}c_*T_0M_0^{2(p_2-2)}\sup_{(0,\,T)}\|D^m v\|_2^2.\end{aligned}$$

Then, we arrive at

$$\|\rho\|_{X_T}\leq\delta T_0\|\rho\|_{X_T}+8c_{\delta,l}c_*T_0M_0^{2(p_2-2)}\|v\|_{X_T}.$$

We take δ small sufficient, we obtain

$$\|\rho\|_{X_T}\leq\lambda T_0\|v\|_{X_T}.$$

There existe T_0 small for that $\lambda T_0<1$, then we get

$$\|\rho\|_{X_T}\leq\theta\|v\|_{X_T},\quad 0<\theta<1.$$

Then, G is a contraction maping. Thus, implies that the unique solution $\rho\in\Lambda$ satisfied $G(\rho)=\rho$. Thus, ρ is nonglobal solution of (2.1).

Uniqueness: All that remains is to prove its uniqueness.

Let ρ^1, ρ^2 be two solutions in the class described in the statement of this theorem, and $w=\rho^1-\rho^2$. Then w satisfies

$$w_{tt}+(-\Delta)^m w-\int_0^t g(t-s)(-\Delta)^m w(s)\,ds+w_t=\left|\rho^1\right|^{p(x)-2}\rho^1-\left|\rho^2\right|^{p(x)-2}\rho^2 \tag{2.18}$$

and

$$w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x)$$

Multiplying (2.18) by w_t, then integrating with respect to x, we get

$$\frac{1}{2}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|D^m w\|_2^2$$

$$+\frac{1}{2}(g\circ D^m w)(t) - \frac{1}{2}\int_0^t \left(g'\circ D^m w\right)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|D^m w\|_2^2\,ds + \int_\Omega w_t^2$$

$$=\int_0^t\int_\Omega \left(|\rho^1|^{p(x)-2}\rho^1 - |\rho^2|^{p(x)-2}\rho^2\right) w_t dxds$$

This implies

$$\begin{aligned}&\frac{1}{2}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|D^m w\|_2^2\\ &\leq\; C\int_0^t\int_\Omega \left(|\rho^1|^{p(x)-2}\rho^1 - |\rho^2|^{p(x)-2}\rho^2\right) w_t dxds. \end{aligned}\tag{2.19}$$

By repeating the estimate as in [39], and the fact $\|\nabla w\|_2^2 \leq c \| D^m w\|_2^2$, we arive at

$$\int_\Omega |w_t|^2\,dx + \|D^m w\|_2^2 \leq C\int_0^t\left(\int_\Omega |w_t|^2\,dx + \|D^m w\|_2^2\right)ds. \tag{2.20}$$

Gronwall's inequality, yields

$$\int_\Omega |w_t|^2\,dx + \|D^m w\|_2^2 = 0.$$

Thus, $w = 0$. The shows the uniqueness.

2.3 Blow up of solution

The main result of this section addresses the blow up at finite time of weak solutions.

We start by establishing several lemma needed for the proof of our main result

Lemma 2.4 *[65] Assme* $\rho(t)$ *is a twice continuously differentiable satisfying*

$$\begin{cases} \Phi''(t) + \Phi'(t) \geq C\Phi^{1+\alpha}(t), & t,\ C,\ \alpha > 0 \\ \Phi(0) > 0,\ \Phi'(0) \geq 0. \end{cases} \tag{2.21}$$

Then, $\Phi(t)$ *blows up in finite time.*

Now, we state our main result.

Theorem 2.2 *Suppose that* 2.2 *holds. Assume further*

$$\int_0^t g(s)\,ds < \frac{p_1(p_1-2)}{(p_1-1)^2},\ \forall t \geq 0, \tag{2.22}$$

and the initial condition

$$(\rho_0,\ \rho_1) \in H_0^m(\Omega) \times L^2(\Omega),$$

satisfying

$$E(0) < 0 \text{ and } \rho_0\rho_1 > 0.$$

Then the solution of (2.1) blows up in finite time.

Proof. To apply the Lemma 2.4, the following is defined:

$$\Phi(t) = \frac{1}{2}\int_\Omega |\rho(x,t)|^2\,dx.$$

Therefore,

$$\Phi'(t) = \int_\Omega \rho\rho_t dx,\ \Phi''(t) = \int_\Omega \left(\rho\rho_{tt} + |\rho_t|^2\right) dx. \tag{2.23}$$

By using the first equation of (2.1), the second equation of (2.23) becomes

$$\begin{aligned}
\Phi''(t) &= \int_\Omega \left(\rho\rho_{tt} + |\rho_t|^2\right) dx = \int_\Omega \rho\rho_{tt} dx + \int_\Omega |\rho_t|^2 dx \\
&= \int_\Omega |\rho_t|^2 dx + \int_\Omega \rho(t) \left(-(-\Delta)^m \rho(t) + \int_0^t g(t-s)(-\Delta)^m \rho(s)\, ds \right. \\
&\left. -\rho_t + |\rho(t)|^{p(x)-2} \rho(t) \right) dx \\
&= \int_\Omega |\rho_t|^2 dx - \int_\Omega |D^m \rho(t)|^2 dx - \int_\Omega \rho(t)\rho_t dx + \int_\Omega |\rho(t)|^{p(x)} dx \\
&+ \int_0^t g(t-s) \int_\Omega D^m\rho(t) . D^m \rho(s)\, dx ds.
\end{aligned}$$

We add and substract the term $\int_0^t g(t-s) \int_\Omega D^m\rho(t) . D^m\rho(t)\, dxds$, and we take account that

$$\int_0^t g(t-s) \int_\Omega D^m\rho(t) . D^m\rho(t)\, dxds = \int_0^t g(t-s)\, ds \int_\Omega D^m\rho(t) . D^m\rho(t)\, dx,$$

we obtain

$$\begin{aligned}
\Phi''(t) = &\int_\Omega |\rho_t|^2 dx - \int_\Omega |D^m\rho|^2 dx - \int_\Omega \rho\rho_t dx + \int_\Omega |\rho|^{p(x)} dx \\
&+ \int_0^t g(t-s) \int_\Omega D^m\rho(t) . D^m\rho(t)\, dxds \\
&- \int_0^t g(t-s) \int_\Omega D^m\rho(t) . (D^m\rho(t) - D^m\rho(s))\, dxds.
\end{aligned}$$

We recall that $\int\limits_0^t g(t-s)\,ds = \int\limits_0^t g(s)\,ds$, then

$$\Phi''(t) = \int_\Omega |\rho_t|^2\,dx - \int\limits_\Omega |D^m\rho|^2\,dx - \int_\Omega \rho\rho_t dx + \int_\Omega |\rho|^{p(x)}\,dx$$
$$+ \int\limits_0^t g(s)\,ds \int\limits_\Omega D^m\rho(t)\,.D^m\rho(t)\,dx$$
$$- \int\limits_0^t g(t-s) \int\limits_\Omega D^m\rho(t)\,.\left(D^m\rho(t) - D^m\rho(s)\right)dxds$$

So that,

$$\Phi''(t) = -\left(1 - \int\limits_0^t g(s)\,ds\right)\int\limits_\Omega |D^m\rho(t)|^2\,dx$$
$$- \int\limits_0^t g(t-s) \int\limits_\Omega D^m\rho(t)\,.\left(D^m\rho(t) - D^m\rho(s)\right)dxds$$
$$+ \int_\Omega |\rho|^{p(x)}\,dx + \int_\Omega |\rho_t|^2\,dx - \int_\Omega \rho\rho_t dx. \tag{2.24}$$

Using the following Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta}b^2,$$

for $a, b \in \mathbb{R}$, and $\delta > 0$, we estimate

$$\int\limits_0^t g(t-s) \int\limits_\Omega D^m\rho(t)\,.\left(D^m\rho(t) - D^m\rho(s)\right)dxds$$
$$= \int\limits_\Omega D^m\rho(t) \int\limits_0^t g(t-s)\left(D^m\rho(t) - D^m\rho(s)\right)dsdx$$
$$\leq \int\limits_\Omega \delta\,|D^m\rho(t)|^2\,dx + \int\limits_\Omega \frac{1}{4\delta}\left(\int\limits_0^t g(t-s)\left(D^m\rho(t) - D^m\rho(s)\right)ds\right)^2 dx$$

Using the Hölder inequality, we get

$$\int_0^t g(t-s)\int_\Omega D^m\rho(t).\left(D^m\rho(t)-D^m\rho(s)\right)dxds \leq \delta\int_\Omega |D^m\rho(t)|^2\,dx$$
$$+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\,ds\right)\left(\int_0^t g(t-s)\,|D^m\rho(t)-D^m\rho(s)|^2\,ds\right)dx$$
$$=\delta\int_\Omega |D^m\rho(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(t-s)\,ds\right)\int_0^t g(t-s)\,\|D^m\rho(t)-D^m\rho(s)\|_2^2\,ds$$
$$=\delta\int_\Omega |D^m\rho(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(t-s)\,ds\right)(g\circ D^m\rho)(t)$$
$$=\delta\int_\Omega |D^m\rho(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\rho)(t).$$

We deduce that

$$-\int_0^t g(t-s)\int_\Omega D^m\rho(t).\left(D^m\rho(t)-D^m\rho(s)\right)dxds$$
$$\geq -\delta\int_\Omega |D^m\rho(t)|^2\,dx-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\rho)(t). \tag{2.25}$$

By combining (2.24) and (2.25), we get

$$\Phi''(t)\geq -\left(1+\delta-\int_0^t g(s)\,ds\right)\|D^m\rho\|_2^2-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\rho)(t)$$
$$+\int_\Omega |\rho|^{p(x)}\,dx+\int_\Omega |\rho_t|^2\,dx-\int_\Omega \rho\rho_t dx.$$

Now, we exploit (2.4) to substitute for $\|D^m\rho\|_2^2$, Therefore,

$$\begin{aligned}\Phi''(t)+\Phi'(t) \geq & -\frac{2}{\beta}\left(1+\delta-\int_0^t g(s)\,ds\right)E(t)\\ & +\left(1+\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}\right)\int_\Omega |\rho_t|^2\,dx\\ & +\left(\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right)(g\circ D^m\rho)\\ & +\left(1-2\frac{1+\delta-\int_0^t g(s)\,ds}{\beta p_1}\right)\int_\Omega |\rho|^{p(x)}\,dx.\end{aligned}$$

Using Lemma 2.4, we get

$$\begin{aligned}\Phi''(t)+\Phi'(t) \geq & -\frac{2}{\beta}\left(1+\delta-\int_0^t g(s)\,ds\right)E(t)\\ & +\left(1+\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}\right)\int_\Omega |\rho_t|^2\,dx\\ & +\left(\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right)(g\circ D^m\rho)\\ & +c\left(1-2\frac{1+\delta-\int_0^t g(s)\,ds}{\beta p_1}\right)\|\rho\|_{p_1}^{p_1}. \end{aligned} \tag{2.26}$$

At this point $\delta > 0$ is chosen so that:

$$\frac{1+\delta-\int\limits_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\geq 0$$

$$c\left(1-2\frac{1+\delta-\int\limits_0^t g(s)\,ds}{\beta p_1}\right)>0.$$

This is, of course, possible by (2.22). Thus by using (2.5) and the negative initial energy, (2.26) becomes:

$$\Phi''(t)+\Phi'(t)\geq\gamma\left\|\rho\right\|_{p1}^{p_1}, \tag{2.27}$$

where $\gamma = c\left(1-2\frac{1+\delta-\int\limits_0^t g(s)ds}{\beta p_1}\right)$.

Now, we use Hölder's inequality to estimate

$$\int_\Omega |\rho|^2\,dx\leq\left(\int_\Omega |\rho|^{p1}\,dx\right)^{\frac{2}{p_1}}\left(\int_\Omega 1dx\right)^{\frac{p_1-2}{p1}}.$$

Where $|\Omega|$ is measure of the doman Ω, then

$$\left(\int_\Omega |\rho|^{p_1}\,dx\right)^{\frac{2}{p_1}}\geq\left(\int_\Omega |\rho|^2\,dx\right)|\Omega|^{\frac{2-p_1}{p_1}}.$$

So,

$$\int_\Omega |\rho|^{p_1}\,dx\geq\left(\int_\Omega |\rho|^2\,dx\right)^{\frac{p_1}{2}}|\Omega|^{\frac{2-p_1}{2}}. \tag{2.28}$$

From the expression of $\Phi(t)=\frac{1}{2}\int_\Omega |\rho(x,t)|^2\,dx$,we get

$$2\Phi(t)=\int_\Omega |\rho(x,t)|^2\,dx.$$

Then

$$(2\Phi(t))^{\frac{p_1}{2}}=\left(\int_\Omega |\rho(x,t)|^2\,dx\right)^{\frac{p_1}{2}}. \tag{2.29}$$

Combining (2.28), (2.29),and (2.27) yield

$$\Phi''(t) + \Phi'(t) \geq 2^{\frac{p_1}{2}} \gamma (\Phi(t))^{\frac{p_1}{2}} |\Omega|^{\frac{2-p_1}{2}}$$

We simplify the last inequality, we arrive at

$$\Phi''(t) + \Phi'(t) \geq \varpi \Phi^{1+\alpha}(t) \tag{2.30}$$

where

$$\varpi = 2^{\frac{p_1}{2}} \gamma |\Omega|^{\frac{2-p_1}{2}} > 0, \qquad \alpha = \frac{p_1 - 2}{2}$$

Therefore $\Phi(t)$ blows up in the finite time.

2.4 Numerical example

Now, we present an example to illustrate numerically the result of Theorem 2.2. For solve problem (2.1), we consider $m = 1$, $n = 2$ where the domain is taken to be $\Omega = [-1,\ 1]^2$. We chosen $g(t) = \lambda e^{-t}$, $(0 < \lambda < 1)$, $\rho_0(x_1, x_2) = \rho_1(x_1, x_2) = 3(2 - x_1^2 + x_2^2)$, such that $E(0) < 0$, $\rho_0 \rho_1 > 0$, and we take $p(x_1, x_2) = 4.8$, which satisfy condition (2.3).

2.4.1 Numerical method

We first choose a suitable numerical scheme to discretize (2.1) using finite differences for the time variable t and the space variable $x = (x_1, x_2)$. Comprehensive details about the finite difference methods, see in [52, 51]. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $\quad t_n = n\ \delta t, \quad n = 1, 2, ..., N+1$, where δt is the time step.

Let $\rho^n(x) = \rho(x_1, x_2,\ t_n)$, and use the finite-difference formulas:

$$\partial_t \rho^n(x) = \frac{\rho^n(x) - \rho^{n-1}(x)}{\delta t},$$

and

$$\partial_{tt}\rho^n(x) = \frac{\rho^{n+1}(x) - 2\rho^n(x) + \rho^{n-1}(x)}{(\delta t)^2}.$$

Then the discrete problem of (2.1) reads: Let ρ_0 and ρ_1, calculate $\{\rho^2,\ \rho^3, ..., \rho^{n+1}\}$ such that

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - \frac{\rho^n - \rho^{n-1}}{\delta t} & \\ -\int_0^{t_{n+1}} g(t_{n+1} - s)\,\Delta\rho^n(s)\,ds + |\rho^n|^{p(x_1,x_2)-2}\rho^n, & \text{in } \Omega, \\ \rho^{n+1} = 0, & \text{on } \partial\Omega, \\ \rho^0 = \rho_0, \quad \rho^1 = \rho^0 + (\delta t)\,\rho_1, & \text{in } \Omega. \end{cases} \tag{2.31}$$

Problem (2.31) is solved iteratively by using the history data ρ^n and ρ^{n-1} in the second side of the equation, satisfies the boundary-value problem:

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = F(\rho^n,\ \rho^{n-1}), & \text{in } \Omega_h, \\ \rho^{n+1} = 0, & \text{on } \partial\Omega_h, \end{cases} \tag{2.32}$$

where:

$$F(\rho^n,\ \rho^{n-1}) = \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - \frac{\rho^n - \rho^{n-1}}{\delta t} - \int_0^{t_{n+1}} g(t_{n+1} - s)\,\Delta\rho^n(s)\,ds + |\rho^n|^{p(x_1,x_2)-2}\rho^n.$$

2.4.2 Numerical results

Now, we present the results of the numerical scheme(2.31). The numerical results are obtained using the Matlab codes. The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 100×100;
- Time step is: $\delta t = 0.01$;
- The spatial discretisation step $h \simeq 0.01$;

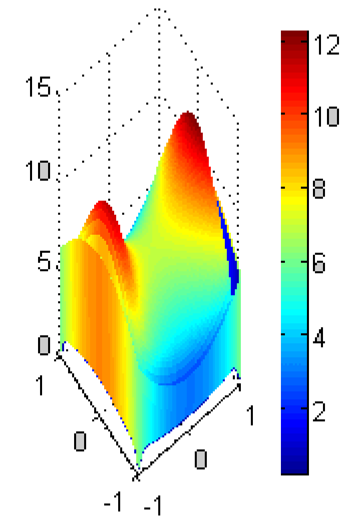

Figure 2.1: ρ^{16}

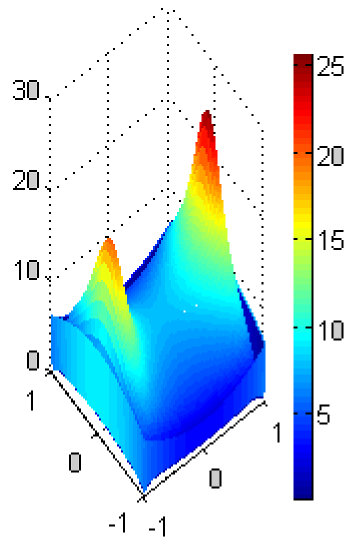

Figure 2.2: ρ^{26}

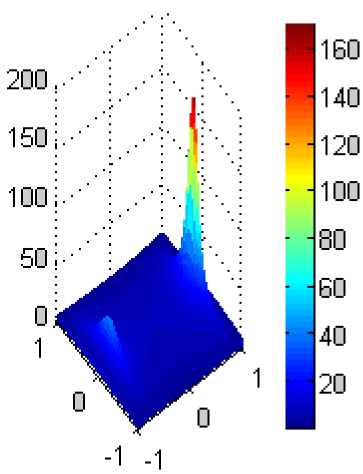

Figure 2.3: ρ^{29}

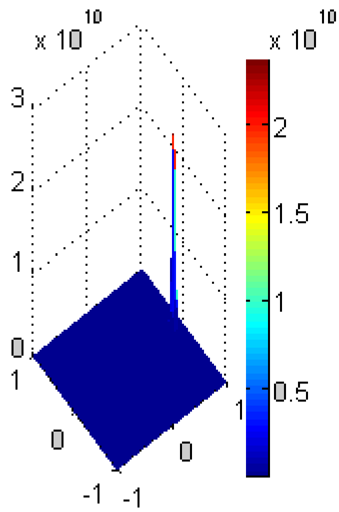

Figure 2.4: ρ^{30}

- $\lambda = 10^{-3}$.

Figures. 1, 2, 3 and 4 present ρ^n for iterations $n = 16$ $(t = 0.16)$, $n = 26$ $(t = 0.26)$, $n = 29$ $(t = 0.29)$ and $n = 30$ $(t = 0.30)$ respectively.

Figure. 4 present ρ^n for iteration $n = 30$ $(t = 0.30)$, which the blowup.

In conclusion, the previous numerical example verifies and agrees with the results of Theorem 2.2.

CHAPTER 3

GLOBAL EXISTENCE AND GENERAL DECAY OF SOLUTION FOR A NONLENEAR WAVE EQUATION WITH VARIABLE EXPONENTS AND MEMORY TERM

This work deals with viscoelastic wave equation with damping and source terms involving variable exponents nonlinearity.
The main prupose is to prove with suitable assumptions on the variable exponent the local solution by using Faedo-Galerkin methods, and also prove that the local solution is global in time. Then, we demonstrate that solution with positive initial energy decay exponentially. The outcomes that appear in this context have been published in [9].

3.1 Setting the problem

We consider the following boundary value problem:

$$\begin{cases} \rho_{tt} - \Delta\rho - \Delta\rho_{tt} + \int_0^t g(t-s)\,\Delta\rho(s)\,ds + |\rho_t|^{m(x)-2}\rho_t = |\rho|^{p(x)-2}\rho, & \text{in } Q, \\ \rho(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ \rho(x,0) = \rho_0(x), \; \rho_t(x,0) = \rho_1(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $Q = \Omega \times (0,T)$ and Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$ with smooth boundary $\partial\Omega$. $p(.)$ and $m(.)$ are given measurable functions on Ω satisfying

$$2 \leq \theta^- \leq \theta(x) \leq \theta^+ \leq \theta^* \tag{3.2}$$

$$\theta^- := ess\inf_{x\in\Omega}\theta(x), \qquad \theta^+ := ess\sup_{x\in\Omega}\theta(x)$$

and

$$\theta^* = \begin{cases} \infty, & \text{if } n = 2, \\ \frac{2n}{n-2}, & \text{if } n \geq 3. \end{cases} \tag{3.3}$$

We also assume that $p(.)$ and $m(.)$ satisfy the log-Hölder continuity condition (definition 1.3).

Equation (3.1) can be viewed as a generalization of the evolutional equation

$$\rho_{tt} - \Delta\rho - \Delta\rho_{tt} + \int_0^t g(t-s)\,\Delta\rho(s)\,ds + \omega\,|\rho_t|^{m-2}\rho_t = b\,|\rho|^{r-2}\rho, \text{ in } \Omega\times(0,T)$$

with the constant exponent of nonlinearity, m, $r \in (2, \infty)$, which appears in various physical contexts.

We denote the total energy related to problem (3.1)

$$E(t) = \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\|\nabla\rho_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho\|_2^2 + \frac{1}{2}(g\circ\nabla\rho)(t) - \int_\Omega \frac{1}{p(x)}|\rho|^{p(x)}\,dx, \tag{3.4}$$

where

$$(g\circ\nabla\rho)(t) = \int_0^t\int_\Omega g(t-s)\,|\nabla\rho(t)-\nabla\rho(s)|^2\,dxds.$$

We also introduce the following functionals:

$$\widetilde{E}(t) = \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\|\nabla\rho_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho\|_2^2 + \frac{1}{2}(g\circ\nabla\rho)(t) - \frac{1}{p^-}\int_\Omega |\rho|^{p(x)}\,dx, \tag{3.5}$$

$$\widetilde{\widetilde{E}}(t) = \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\|\nabla\rho_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho\|_2^2 + \frac{1}{2}(g\circ\nabla\rho)(t) - \frac{1}{p^+}\int_\Omega |\rho|^{p(x)}\,dx, \tag{3.6}$$

$$I(t) = \left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho\|_2^2 + (g\circ\nabla\rho)(t) - \int_\Omega |\rho|^{p(x)}\,dx, \tag{3.7}$$

and

$$J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho\|_2^2 + \frac{1}{2}(g\circ\nabla\rho)(t) - \frac{1}{p^-}\int_\Omega |\rho|^{p(x)}\,dx, \tag{3.8}$$

we show that

$$\widetilde{E}(t) \le E(t) \le \widetilde{\widetilde{E}}(t). \tag{3.9}$$

Assumptions:

(A_1) : $g : \mathbb{R}^+ \to \mathbb{R}^+_*$ is a bounded C^1 function satisfying

$$1 - \int_0^\infty g(s)\, ds = l > 0 \text{ and } g'(t) \leq -g(t). \tag{3.10}$$

(A_2) : Assume that

$$I(0) > 0, \tag{3.11}$$

and

$$Max\left(\frac{B^{p^-}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^- -2}{2}}, \frac{B^{p^+}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^+ -2}{2}}\right) = \lambda < 1. \tag{3.12}$$

Lemma 3.1 *[14] If p is a measurable function on Ω satisfying (3.2) and (3.3), then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.*

From the Lemma 3.1, there exists the positive constant B satisfying

$$\|\rho\|_{p(.)} \leq B\|\nabla \rho\|_2, \quad \text{for } \rho \in H_0^1(\Omega).$$

Theorem 3.1 *Suppose that $m(.)$, $p(.) \in C\left(\overline{\Omega}\right)$ and verify the log-Hölder continuity condition (definition (1.3)) with*

$$2 \leq p^- \leq p(x) \leq p^+ \leq 2\frac{n-1}{n-2}, \quad \text{if } n \geq 3 \tag{3.13}$$
$$p(x) \geq 2 \qquad \text{if } n = 2,$$

and

$$2 \leq m^- \leq m(x) \leq m^+ \leq 2\frac{n-1}{n-2}, \quad \text{if } n \geq 3 \tag{3.14}$$
$$m(x) \geq 2 \qquad \text{if } n = 2.$$

Then for any $(\rho_0,\ \rho_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *problem* (3.1) *has a unique weak local solution*

$$\begin{aligned} \rho &\in L^{\infty}\left([0,\ T);\ H_0^1(\Omega)\right), \\ \rho_t &\in L^{\infty}\left([0,\ T);\ H_0^1(\Omega)\right) \cap L^{m(.)}(\Omega \times [0,T)), \\ \rho_{tt} &\in L^{2}\left([0,\ T);\ H_0^1(\Omega)\right). \end{aligned}$$

3.2 Existence of weak solutions

In this section we are going to obtain the existence of weak solutions to the problem (3.1). We will use Faedo- Galerkin's method.

Existence: To prove that a local solution exists, we take several steps.

- Step 1: Approximate problem.

Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^1(\Omega)$ wich constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = span\{v_1,\ v_2, ..., v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. By the normalization, we have $\|v_l\| = 1$, for any given integer k, we consider the approximation solution

$$\rho_k(t) = \sum_{l=1}^{k} \rho_{lk}(t)\, v_l,$$

where ρ_k is the solutions to the following Cauchy problem

$$\left(\rho_k''(t),\ v_l\right) - \left(\Delta\rho_k(t),\ v_l\right) - \left(\Delta\rho_k''(t),\ v_l\right) - \int_0^t g(t-s)\left(\Delta\rho_k(s),\ v_l\right) ds$$

$$+\left(\left|\rho_k'(t)\right|^{m(x)-2} \rho_k'(t),\ v_l\right) = \left(\left|\rho_k(t)\right|^{p(x)-2} \rho_k(t),\ v_l\right), \quad l = 1,\ 2, ..., k, \tag{3.15}$$

$$\rho_k(0) = \rho_{0k} = \sum_{i=1}^{k} (\rho_k(0),\ v_i)\, v_i \to \rho_0 \text{ in } H_0^1(\Omega) \tag{3.16}$$

$$\rho_k^{'}(0) = \rho_{1k} = \sum_{l=1}^{k} \left(\rho_k^{'}(0),\ v_l\right) v_l \to \rho_1 \text{ in } H_0^1(\Omega)\,. \tag{3.17}$$

Note that, we can solve the system (3.15)-(3.17) by a Picard's iteration method in ordinary differential equations. Hence, there exists a solution in $[0,\ T_*)$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0,\ T]$ for any given $T > 0$ by making use of the priori estimates below.

- Step 2: Apriori estimates

Multiplying equation (3.15) by $\rho_{lk}^{'}(t)$ and summing over l from 1 to k,

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\rho_k^{'}\right\|_2^2 + \frac{1}{2}\left\|\nabla\rho_k^{'}\right\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2 + \frac{1}{2}(g \circ \nabla\rho_k)(t)\right.$$
$$\left. - \int_\Omega \frac{1}{p(x)}|\rho_k|^{p(x)}\,dx\right) = -\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dx + \frac{1}{2}\left(g^{'} \circ \nabla\rho_k\right)(t) - \frac{1}{2}g(t)\|\nabla\rho_k\|_2^2\,. \tag{3.18}$$

Then by virtue of (3.4), assumption (A_1) and definition of the expression $\left(g^{'} \circ \nabla\rho_k\right)(t)$, we will have

$$E^{'}(\rho_k(t)) = -\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dx + \frac{1}{2}\left(g^{'} \circ \nabla\rho_k\right)(t) - \frac{1}{2}g(t)\|\nabla\rho_k\|_2^2 \le 0.$$

Integrating (3.18) over $(0,\ t)$, we obtain the estimate

$$\frac{1}{2}\left\|\rho_k^{'}\right\|_2^2+\frac{1}{2}\left\|\nabla\rho_k^{'}\right\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2+\frac{1}{2}(g\circ\nabla\rho_k)(t)$$
$$-\int_\Omega\frac{1}{p(x)}|\rho_k|^{p(x)}dx+\int_0^t\int_\Omega\left|\rho_k^{'}\right|^{m(x)}dxds-\frac{1}{2}\int_0^t\left(g^{'}\circ\nabla\rho_k\right)(s)\,ds$$
$$+\frac{1}{2}\int_0^t g(s)\|\nabla\rho_k\|_2^2\,ds\leq E(0). \tag{3.19}$$

Since $I(0)>0$, then there exists $T_*<T$ by continuity such that $I(t)\geq 0$, for all $t\in[0,\ T_*]$. We get from (3.6) and (3.7) that

$$J(\rho_k(t))=\frac{p^--2}{2p^-}\left(\left(1-\int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2+(g\circ\nabla\rho_k)(t)\right)+\frac{1}{p^-}I(t). \tag{3.20}$$

Then

$$J(\rho_k(t))\geq\frac{p^--2}{2p^-}\left(\left(1-\int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2+(g\circ\nabla\rho_k)(t)\right). \tag{3.21}$$

Hence, we have

$$\left(1-\int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2\leq\frac{2p^-}{p^--2}J(\rho_k(t)). \tag{3.22}$$

From (3.4), (3.5) and (3.7), we obviosly have

$$J(\rho_k(t))\leq\widetilde{E}(\rho_k(t))\leq E(\rho_k(t))\leq E(0)\quad\forall t\in[0,\ T_*].$$

Thus, we obtain

$$\left(1-\int_0^t g(s)\,ds\right)\|\nabla\rho_k\|_2^2\leq\frac{2p^-}{p^--2}E(0). \tag{3.23}$$

Before the rest of the proof, we need

Lemma 3.2 *Suppose that* (3.2) *and assumptions* (A_1), (A_2) *hold, then*

$$\varrho_{p(.)}(\rho_k) \leq l \ \|\nabla \rho_k\|_2^2, \tag{3.24}$$

where l *is defined in* (3.10).

Proof. By the Lemmas 1.2 and 3.1, we have

$$\begin{aligned}\varrho_{p(.)}(\rho_k) &\leq \max\left\{\|\rho_k\|_{p(.)}^{p^-}, \ \|\rho_k\|_{p(.)}^{p^+}\right\} \\ &\leq \max\left\{B^{p^-}\|\nabla \rho_k\|_2^{p^-}, \ B^{p^+}\|\nabla \rho_k\|_2^{p^+}\right\},\end{aligned}$$

and from assumptions (A_1), (A_2) and (3.23), we get

$$\begin{aligned}\varrho_{p(.)}(\rho_k) &\leq \max\left\{B^{p^-}\|\nabla \rho_k\|_2^2 \times \|\nabla \rho_k\|_2^{p^--2}, \ B^{p^+}\|\nabla \rho_k\|_2^2 \times \|\nabla \rho_k\|_2^{p^+-2}\right\} \\ &\leq \max\left(l\|\nabla \rho_k\|_2^2 \times \frac{B^{p^-}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^--2}{2}}, \right. \\ &\left. l\|\nabla \rho_k\|_2^2 \times \frac{B^{p^+}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^+-2}{2}}\right) \leq l\|\nabla \rho_k\|_2^2 .\end{aligned}$$

From (3.24), the inequality (3.19) becames

$$\frac{1}{2}\left\|\rho_k'\right\|_2^2 + \frac{1}{2}\left\|\nabla \rho_k'\right\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla \rho_k\|_2^2 + \frac{1}{2}(g \circ \nabla \rho_k)(t)$$

$$+ \int_0^t \int_\Omega \left|\rho_k'\right|^{m(x)} dxds - \frac{1}{2}\int_0^t \left(g' \circ \nabla \rho_k\right)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla \rho_k\|_2^2\,ds \leq E(0)$$

$$\frac{1}{2}\sup_{t\in(0,\,T_*)}\left\|\rho_k'\right\|_2^2 + \frac{1}{2}\sup_{t\in(0,\,T_*)}\left\|\nabla \rho_k'\right\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\left(1 - \int_0^t g(s)\,ds\right)\sup_{t\in(0,\,T_*)}\|\nabla \rho_k\|_2^2$$

$$+ \frac{1}{2}(g \circ \nabla \rho_k)(t) + \int_0^t \int_\Omega \left|\rho_k'\right|^{m(x)} dxds - \frac{1}{2}\int_0^t \left(g' \circ \nabla \rho_k\right)(s)\,ds$$

$$+ \frac{1}{2}\int_0^t g(s)\|\nabla \rho_k\|_2^2\,ds \leq E(0). \tag{3.25}$$

From (3.25), we conclude that

$$\begin{cases} \rho_k \text{ is uniformly bounded in } L^\infty\left([0,\ T),\ H_0^1(\Omega)\right), \\ \rho_k^{'} \text{ is uniformly bounded in } L^\infty\left([0,\ T),\ H_0^1(\Omega)\right) \cap L^{m(.)}(\Omega \times [0,\ T)). \end{cases} \tag{3.26}$$

Furthemore, we have from Lemma (3.1) and (3.26) that

$$\begin{aligned} &\left\{|\rho_k|^{p(x)-2}\rho_k\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T),\ L^2(\Omega)\right), \\ &\left\{\left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{'}\right\} \text{ is uniformly bounded in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times [0,\ T)). \end{aligned} \tag{3.27}$$

By (3.26) and (3.27), we infer that there exists a subsequence ρ_n of ρ_k and a function ρ such that

$$\begin{cases} \rho_k \rightharpoonup \rho \text{ weakly star in } L^\infty\left([0,\ T),\ H_0^1(\Omega)\right), \\ \rho_k^{'} \rightharpoonup \rho^{'} \text{ weakly star in } L^\infty\left([0,\ T),\ H_0^1(\Omega)\right), \\ \left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{'} \rightharpoonup \psi \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times [0,\ T)). \end{cases} \tag{3.28}$$

By the Aubin-Lions compactness Lemma [34], we conclude from (3.28) that

$$\rho_k \rightharpoonup \rho \text{ strongly in } C\left([0,\ T),\ H_0^1(\Omega)\right).$$

Which implies

$$\rho_k \rightharpoonup \rho \text{ everywhere in } [0,T] \times \Omega. \tag{3.29}$$

It follow from (3.28) and (3.29) that

$$\begin{cases} |\rho_k|^{p(x)-2}\rho_k \rightharpoonup |\rho|^{p(x)-2}\rho \text{ weakly in } L^\infty\left([0,\ T),\ L^2(\Omega)\right), \\ \left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{'} \rightharpoonup \left|\rho^{'}\right|^{m(x)-2}\rho^{'} \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times [0,\ T)). \end{cases} \tag{3.30}$$

Next, multiplying the equation (3.15) by $\rho_{lk}^{''}(t)$ and summing over l from 1 to k, we get

$$\left\|\rho_k^{''}\right\|_2^2 + \left\|\nabla\rho_k^{''}\right\|_2^2 + \frac{d}{dt}\left(\int_\Omega \frac{1}{m(x)}\left|\rho_k^{'}\right|^{m(x)} dx\right) = -\int_\Omega \nabla\rho_k \nabla\rho_k^{''} dx$$

$$+\int_0^t g(t-\tau)\int_\Omega \nabla\rho_k(\tau)\nabla\rho_k''(t)\,dxd\tau+\int_\Omega |\rho_k|^{p(x)-2}\rho_k\rho_k''dx. \tag{3.31}$$

From the Young's inequality, we have

$$\left|-\int_\Omega \nabla\rho_k\nabla\rho_k''dx\right|\le \delta\left\|\nabla\rho_k''\right\|_2^2+\frac{1}{4\delta}\|\nabla\rho_k\|_2^2, \tag{3.32}$$

$$\begin{aligned}
&\left|\int_0^t g(t-\tau)\int_\Omega \nabla\rho_k(\tau)\nabla\rho_k''(t)\,dxd\tau\right|\\
&\le \delta\left\|\nabla\rho_k''\right\|_2^2+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)\nabla\rho_k(\tau)\,d\tau\right)^2dx\\
&\le \delta\left\|\nabla\rho_k''\right\|_2^2+\frac{1}{4\delta}\int_0^t g(s)\,ds\int_0^t g(t-\tau)\int_\Omega|\nabla\rho_k(\tau)|^2\,dxd\tau\\
&\le \delta\left\|\nabla\rho_k''\right\|_2^2+\frac{(1-l)\,g(0)}{4\delta}\int_0^t\|\nabla\rho_k(\tau)\|^2\,d\tau,
\end{aligned} \tag{3.33}$$

and

$$\left|\int_\Omega |\rho_k|^{p(x)-2}\rho_k\rho_k''dx\right|\le\delta\left\|\rho_k''\right\|_2^2+\frac{1}{4\delta}\int_\Omega|\rho_k|^{2p(x)-2}\,dx \tag{3.34}$$

From (3.31)-(3.34), the inequality (3.31) becames

$$\begin{aligned}
&(1-\delta)\left\|\rho_k''\right\|_2^2+(1-2\delta)\left\|\nabla\rho_k''\right\|_2^2+\frac{d}{dt}\left(\int_\Omega\frac{1}{m(x)}\left|\rho_k'\right|^{m(x)}dx\right)\\
&\le\frac{1}{4\delta}\|\nabla\rho_k\|_2^2+\frac{(1-l)\,g(0)}{4\delta}\int_0^t\|\nabla\rho_k(\tau)\|^2\,d\tau+\frac{1}{4\delta}\int_\Omega|\rho_k|^{2(p(x)-1)}\,dx
\end{aligned} \tag{3.35}$$

We have $\rho_k \in L^{\infty}\left([0,\ T),\ H_0^1(\Omega)\right)$, then

$$\int\limits_{\Omega} |\rho_k|^{2p(x)-2}\, dx \leq \int\limits_{\Omega} |\rho_k|^{2p^- -2}\, dx + \int\limits_{\Omega} |\rho_k|^{2p^+ -2}\, dx < +\infty,$$

since, $2(p^- - 1) \leq 2(p(x) - 1) \leq 2(p^+ - 1) \leq \frac{2n}{n-2}$. We chosen δ small enough for that we find λ constant positive such that

$$\int\limits_0^t \left\|\rho_k^{''}\right\|_2^2 ds + \lambda \int\limits_0^t \left\|\nabla \rho_k^{''}\right\|_2^2 ds + \int\limits_{\Omega} \frac{1}{m(x)} \left|\rho_k^{'}\right|^{m(x)} dx \leq C. \tag{3.36}$$

Then

$$\rho_k^{''} \text{ is bounded in } L^2\left([0,\ T),\ H_0^1(\Omega)\right). \tag{3.37}$$

Similarly, we have

$$\rho_k^{''} \rightharpoonup \rho^{''} \text{ weakly star in } L^2\left([0,\ T),\ H_0^1(\Omega)\right). \tag{3.38}$$

- Step 3: Passing to the limit

Setting up $k \longrightarrow \infty$ and passing to the limit in (3.15), we obtain

$$\left(\rho^{''}(t),\ v_l\right) - \left(\Delta\rho(t),\ v_l\right) - \left(\Delta\rho^{''}(t),\ v_l\right) - \int\limits_0^t g(t-s)\left(\Delta\rho(s),\ v_l\right) ds$$

$$+ \left(\left|\rho^{'}(t)\right|^{m(x)-2} \rho^{'}(t),\ v_l\right) = \left(|\rho(t)|^{p(x)-2} \rho(t),\ v_l\right), \quad l = 1,\ 2, ..., k. \tag{3.39}$$

Since $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^1(\Omega)$, we deduce that ρ satisfies the equation of 3.1. From 3.28, 3.30, 3.38 and Lemma 3.1.7 in [64] with $B = H_0^1(\Omega)$ in the both cases, we infer that

$$\begin{cases} \rho_k(0) \rightharpoonup \rho(0) \text{ weakly in } H_0^1(\Omega), \\ \rho_k^{'}(0) \rightharpoonup \rho^{'}(0) \text{ weakly in } H_0^1(\Omega). \end{cases} \tag{3.40}$$

We get from (3.16) and (3.40) that $\rho(0) = \rho_0$, $\rho'(0) = \rho_1$. Thus, the proof of existence is complete.

Uniqueness: Now it remains to prove uniqueness.

Let ρ^1, ρ^2 be two solutions in the class described in the statement of this theorem, and $w = \rho^1 - \rho^2$. Then w satisfies

$$w_{tt} - \Delta w - \Delta w_{tt} + \int_0^t g(t-s)\Delta w(s)\,ds + \omega\left(\left|\rho_t^1\right|^{m(x)-2}\rho_t^1 - \left|\rho_t^2\right|^{m(x)-2}\rho_t^2\right)$$
$$= \left|\rho^1\right|^{p(x)-2}\rho^1 - \left|\rho^2\right|^{p(x)-2}\rho^2 \tag{3.41}$$

and

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x)$$

Multiplying (3.41) by w_t, then integrating with respect to x, we get

$$\frac{1}{2}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\int_\Omega |\nabla w_t|^2\,dx + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla w\|_2^2$$
$$+\frac{1}{2}(g\circ\nabla w)(t) - \frac{1}{2}\int_0^t\left(g'\circ\nabla w\right)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla w\|_2^2\,ds$$
$$+\omega\int_0^t\int_\Omega\left(\left|\rho_t^1\right|^{m(x)-2}\rho_t^1 - \left|\rho_t^2\right|^{m(x)-2}\rho_t^2\right)w_t\,dx\,ds$$
$$= \int_0^t\int_\Omega\left(\left|\rho^1\right|^{p(x)-2}\rho^1 - \left|\rho^2\right|^{p(x)-2}\rho^2\right)w_t\,dx\,ds$$

By using the inequality

$$\left(|a|^{m(x)-2}a - |b|^{m(x)-2}b\right)(a-b) \geq 0$$

for all $a, b \in \mathbb{R}$ and a.e. $x \in \Omega$. This implies

$$\frac{1}{2}\int_{\Omega} |w_t|^2 \, dx + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) \|\nabla w\|_2^2$$
$$\leq C \int_0^t \int_{\Omega} \left(|\rho^1|^{p(x)-2} \rho^1 - |\rho^2|^{p(x)-2} \rho^2 \right) w_t dx ds. \tag{3.42}$$

By repeating the estimate as in [39], we arive at

$$\int_{\Omega} |w_t|^2 \, dx + \|\nabla w\|_2^2 \leq C \int_0^t \left(\int_{\Omega} |w_t|^2 \, dx + \|\nabla w\|_2^2 \right) ds. \tag{3.43}$$

Gronwall's inequality, yields

$$\int_{\Omega} |w_t|^2 \, dx + \|\nabla w\|_2^2 = 0.$$

Thus, $w = 0$. The shows the uniqueness.

3.3 Global Existence

In this section, we state and prove a global existence result, we shall use the following theorem.

Theorem 3.2 *Suppose that assumptions of theorem 3.1 hold,* (A_1) *and* (A_2). *If* $(\rho_0, \ \rho_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *then the solution of* (3.1) *is bounded and global in time.*

Proof. It is sufficient to show that $\|\nabla \rho(t)\|_2^2 + \|\rho_t(t)\|_2^2$ is bounded independently of t.

To obtain this, we observe that

$$\begin{aligned} E(0) \geq E(t) \geq \widetilde{E}(t) &= \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\|\nabla \rho_t\|_2^2 \\ &+ \frac{p^- - 2}{2p^-}\left(\left(1 - \int_0^t g(s)\,ds\right)\|\nabla \rho\|_2^2 + (g \circ \nabla \rho)(t)\right) + \frac{1}{p^-} I(t) \\ &\geq \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\|\nabla \rho_t\|_2^2 + \frac{p^- - 2}{2p^-}\Big(l\,\|\nabla \rho\|_2^2 + (g \circ \nabla \rho)(t)\Big), \end{aligned} \tag{3.44}$$

since $I(t) > 0$, $(g \circ \nabla \rho)(t)$ are positives. Therefore,

$$\|\nabla \rho\|_2^2 + \|\rho_t\|_2^2 \leq CE(0),$$

where C is positive constant, depends only on p^- and l and is independent of t. This infer that the solution of (3.1) is bounded and global in time.

3.4 Energy Decay

In this section, we prove the energy decay result by constructing a suitable functional.

We define

$$G(t) = M\ E(t) + \epsilon\ \Theta(t) + \Psi(t), \tag{3.45}$$

where M and ϵ are positive constants which specified later and

$$\Theta(t) = \int_\Omega \rho_t \rho dx + \int_\Omega \nabla \rho_t(t)\nabla \rho(t)\,dx, \tag{3.46}$$

$$\Psi(t) = \int_\Omega (\Delta \rho_t - \rho_t) \int_0^t g(t-s)(\rho(t) - \rho(s))\,dsdx. \tag{3.47}$$

We need the following technical Lemmas in the course of our investigation.

Lemma 3.3 *Under the assumptions of Theorem* (3.1), *we have*

$$\int_\Omega |\rho|^{2p(x)-2}\, dx \le c\,\|\nabla\rho\|_2^2\,. \tag{3.48}$$

$$\int_\Omega |\rho_t|^{2m(x)-2}\, dx \le c\,\|\nabla\rho_t\|_2^2\,. \tag{3.49}$$

Proof. *By Lemma 1.2, we have*

$$\int_\Omega |\rho|^{2(p(x)-1)}\, dx \le \max\left\{\|\rho\|_{2(p(.)-1)}^{2(p^--1)},\ \|\rho\|_{2(p(.)-1)}^{2(p^+-1)}\right\}.$$

On the other hand, by Lemma 3.1, we have

$$\begin{aligned}\int_\Omega |\rho|^{2(p(x)-1)}\, dx &\le \max\left\{B^{2(p^--1)}\,\|\nabla\rho\|_2^{2(p^--1)},\ B^{2(p^+-1)}\,\|\nabla\rho\|_2^{2(p^+-1)}\right\}\\ &\le \max\left\{B^{2(p^--1)}\,\|\nabla\rho\|_2^{2(p^--2)},\ B^{2(p^+-1)}\,\|\nabla\rho\|_2^{2(p^+-2)}\right\}\|\nabla\rho\|_2^2,\end{aligned}$$

since,

$$2\left(p^- - 1\right) \le 2\left(p\left(x\right) - 1\right) \le 2\left(p^+ - 1\right) \le \frac{2n}{n-2}.$$

Using (3.44), *we obtain*

$$\int_\Omega |\rho|^{2(p(x)-1)}\, dx \le$$

$$\max\left\{B^{2(p^--1)}\left(\frac{2p^-}{l\left(p^--2\right)}E\left(0\right)\right)^{p^--2}, B^{2(p^+-1)}\left(\frac{2p^-}{l\left(p^--2\right)}E\left(0\right)\right)^{p^+-2}\right\}\|\nabla\rho\|_2^2$$

$$\le c\,\|\nabla\rho\|_2^2\,.$$

Similarly, we get

$$\int_\Omega |\rho_t|^{2m(x)-2}\, dx \le c\,\|\nabla\rho_t\|_2^2\,.$$

Lemma 3.4 *Let* $\rho \in L^{\infty}\left([0,\ T);\ H_0^1(\Omega)\right)$, *then, we have*

$$\int_{\Omega}\left(\int_0^t g(t-s)(\rho(t)-\rho(s))\,ds\right)^2 dx \le (1-l)\,c^2\,(g\circ\nabla\rho)(t), \quad (3.50)$$

where c is Sobolev-Poincaré constant.

Proof. By Hölder inequality, we get

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t g(t-s)(\rho(t)-\rho(s))\,ds\right)^2 dx \\
&\le \int_{\Omega}\left(\int_0^t g(t-s)\,ds\right)\left(\int_0^t g(t-s)\left|\rho(t)-\rho(s)\right|^2 ds\right) dx \\
&\le (1-l)\,c^2\int_0^t g(t-s)\left\|\nabla\rho(t)-\nabla\rho(s)\right\|_2^2 ds \\
&\le (1-l)\,c^2\,(g\circ\nabla\rho)(t).
\end{aligned}$$

Lemma 3.5 *Let* ρ *be solution of* (3.1), *then there exists two positive constants* B_1 *and* B_2 *such that*

$$B_1E(t) \le G(t) \le B_2E(t). \quad (3.51)$$

Proof. By Young's inequality, we have

$$\begin{aligned}
\left|\int_{\Omega}\rho_t\rho dx\right| &\le \delta\left\|\rho_t\right\|_2^2 + \frac{1}{4\delta}\left\|\rho\right\|_2^2 \\
&\le \delta\left\|\rho_t\right\|_2^2 + \frac{c}{4\delta}\left\|\nabla\rho\right\|_2^2, \quad (3.52)
\end{aligned}$$

and

$$\left|\int_{\Omega}\nabla\rho_t\nabla\rho dx\right| \le \delta\left\|\nabla\rho_t\right\|_2^2 + \frac{1}{4\delta}\left\|\nabla\rho\right\|_2^2. \quad (3.53)$$

It following from (3.47) that

$$\Psi(t) = -\int_{\Omega} \nabla\rho_t \int_0^t g(t-s)(\nabla\rho(t) - \nabla\rho(s))\,dsdx$$
$$-\int_{\Omega} \rho_t \int_0^t g(t-s)(\rho(t) - \rho(s))\,dsdx. \tag{3.54}$$

By Young′s inequality and Hölder inequality, the first term on the right-hand side of (3.54) can be estimated as

$$\left| -\int_{\Omega} \nabla\rho_t \int_0^t g(t-s)(\nabla\rho(t) - \nabla\rho(s))\,dsdx \right|$$
$$\leq \frac{1}{2}\|\nabla\rho_t\|_2^2 + \frac{1}{2}\int_{\Omega}\left(\int_0^t g(t-s)(\nabla\rho(t) - \nabla\rho(s))\,ds\right)^2 dx$$
$$\leq \frac{1}{2}\|\nabla\rho_t\|_2^2 + \frac{1-l}{2}(g\circ\nabla\rho)(t). \tag{3.55}$$

Appliying similar arguments as in deriving (3.55) and then using Lemma (3.4), we have

$$\left| -\int_{\Omega} \rho_t \int_0^t g(t-s)(\rho(t) - \rho(s))\,dsdx \right|$$
$$\leq \frac{1}{2}\|\rho_t\|_2^2 + \frac{1}{2}\int_{\Omega}\left(\int_0^t g(t-s)(\rho(t) - \rho(s))\,ds\right)^2 dx$$
$$\leq \frac{1}{2}\|\rho_t\|_2^2 + \frac{1-l}{2}c^2(g\circ\nabla\rho)(t). \tag{3.56}$$

Hence, by using (3.52)-(3.56), we have the following inequalities from (3.45)

$$\begin{aligned} G(t) &\leq M\ E(t) + \epsilon\Theta(t) + \Psi(t) \\ &\leq M\ E(t) + \lambda_1 \|\rho_t\|_2^2 + \lambda_2 \|\nabla\rho_t\|_2^2 + \lambda_3 \|\nabla\rho\|_2^2 + \lambda_4 (g \circ \nabla\rho)(t) \\ &\leq M\ E(t) + \lambda_5 \left(\|\rho_t\|_2^2 + \|\nabla\rho_t\|_2^2 + \|\nabla\rho\|_2^2 + (g \circ \nabla\rho)(t)\right) \end{aligned} \tag{3.57}$$

where $\lambda_1 = \frac{1}{2} + \epsilon\delta$, $\lambda_2 = \frac{1}{2} + \epsilon\delta$, $\lambda_3 = \frac{1+c}{4\delta}$, $\lambda_4 = \frac{1-l}{2}\left(1 + c^2\right)$. On the other hand, we have

$$G(t) \geq M\ E(t) - \lambda_5 \left(\|\rho_t\|_2^2 + \|\nabla\rho_t\|_2^2 + \|\nabla\rho\|_2^2 + (g \circ \nabla\rho)(t)\right), \tag{3.58}$$

where $\lambda_5 = \max(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)$.

Thus, from the definition of $E(t)$ and (3.44), we can chosing M sufficiently large and ϵ small enough, there exist two positive constants B_1 and B_2 such that

$$B_1 E(t) \leq G(t) \leq B_2 E(t).$$

Theorem 3.3 *Let* $(\rho_0,\ \rho_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be given. Suppose that* (A_1) *and* (A_2) *hold. Then for* $t \geq t_0$ *the solution energy of* (3.1) *satisfies*

$$E(t) \leq k\ e^{-\xi(t-t_0)}, \quad t \geq t_0 \tag{3.59}$$

where ζ *is a positive constant.*

Proof. In order to obtain the decay result of $E(t)$. To this end, we need to estimate the derivative of $G(t)$. It follows from (3.46) and first equation of (3.1) that

$$\Theta'(t) = \|\rho_t\|_2^2 + \|\nabla\rho_t\|_2^2 - \|\nabla\rho\|_2^2 - \int_\Omega |\rho_t|^{m(x)-2}\rho_t\rho dx + \int_\Omega |\rho|^{p(x)} dx$$
$$+ \int_\Omega \nabla\rho \int_0^t g(t-s)\nabla\rho(s)\, dsdx. \tag{3.60}$$

The last term on the right-hand side of (3.60) can be estimated as

$$\begin{aligned}
&\left|\int_\Omega \nabla\rho \int_0^t g(t-s)\nabla\rho(s)\, dsdx\right| \\
&\le \int_\Omega \left(\int_0^t g(t-s)\,|\nabla\rho(s) - \nabla\rho(t)|\, ds\right) dx + \int_0^t g(s)\, ds\, \|\nabla\rho\|_2^2 \\
&\le (1+\eta)\int_0^t g(s)\, ds\, \|\nabla\rho\|_2^2 + \frac{1}{4\eta}(g\circ\nabla\rho)(t) \\
&\le (1+\eta)(1-l)\|\nabla\rho\|_2^2 + \frac{1}{4\eta}(g\circ\nabla\rho)(t), \quad \text{for } \eta > 0.
\end{aligned} \tag{3.61}$$

Also, by Hölder inequality, Young inequality, we get

$$\left|\int_\Omega |\rho_t|^{m(x)-2}\rho_t\rho dx\right| \le \eta\|\rho\|_2^2 + \frac{1}{4\eta}\int_\Omega |\rho_t|^{2m(x)-2} dx \tag{3.62}$$

Substitution of (3.61) and (3.62) into (3.60), yields

$$\Theta'(t) \le \|\rho_t\|_2^2 + \|\nabla\rho_t\|_2^2 - \|\nabla\rho\|_2^2 + (1+\eta)(1-l)\|\nabla\rho\|_2^2 + \frac{1}{4\eta}(g\circ\nabla\rho)(t)$$
$$+\eta\|\rho\|_2^2 + \frac{1}{4\eta}\int_\Omega |\rho_t|^{2m(x)-2} + \int_\Omega |\rho|^{p(x)} dx. \tag{3.63}$$

Next, we would like to estimate $\Psi'(t)$. Taking the derivative of $\Psi(t)$

in (3.47) and using the first equation of (3.1), we get

$$\begin{aligned}
\Psi'(t) &= \int_\Omega \nabla\rho(t) \int_0^t g(t-s)(\nabla\rho(t) - \nabla\rho(s))\,dsdx \\
&- \int_\Omega \left(\int_0^t g(t-s)\nabla\rho(s)\,ds\right)\left(\int_0^t g(t-s)(\nabla\rho(t)-\nabla\rho(s))\,ds\right)dx \\
&+ \int_\Omega |\rho_t|^{m(x)-2}\rho_t \int_0^t g(t-s)(\rho(t)-\rho(s))\,dsdx \\
&- \int_\Omega |\rho|^{p(x)-2}\rho \int_0^t g(t-s)(\rho(t)-\rho(s))\,dsdx \\
&- \int_\Omega \rho_t \int_0^t g'(t-s)(\rho(t)-\rho(s))\,dsdx \\
&- \left(\int_0^t g(s)\,ds\right)\|\nabla\rho_t\|_2^2 - \left(\int_0^t g(s)\,ds\right)\|\rho_t\|_2^2 \\
&- \int_\Omega \nabla\rho_t \int_0^t g'(t-s)(\nabla\rho(t)-\nabla\rho(s))\,dsdx. \qquad (3.64)
\end{aligned}$$

Similar to (3.63), in what follows we will estimate the right-hand side of (3.64)

$$\begin{aligned}
&\left|\int_\Omega \nabla\rho \int_0^t g(t-s)(\nabla\rho(t)-\nabla\rho(s))\,dsdx\right| \\
&\leq \delta\|\nabla\rho\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g(t-s)(\nabla\rho(t)-\nabla\rho(s))\,ds\right)^2 dx \\
&\leq \delta\|\nabla\rho\|_2^2 + \frac{1-l}{4\delta}(g\circ\nabla\rho)(t). \qquad (3.65)
\end{aligned}$$

and

$$\left| \int_\Omega \left(\int_0^t g(t-s) \nabla \rho(s) \, ds \right) \left(\int_0^t g(t-s) (\nabla \rho(t) - \nabla \rho(s)) \, ds \right) dx \right|$$
$$\leq \delta I_1 + \frac{1}{4\delta} I_2, \tag{3.66}$$

where

$$I_1 = \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(s)| \, ds \right)^2 dx$$

and

$$I_2 = \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(t) - \nabla \rho(s)| \, ds \right)^2 dx$$

By Hölder inequality, Young$^{'}$s inequality, for $\eta > 0$, we obtain

$$I_1 \leq \int_\Omega \left(\int_0^t g(t-s) (|\nabla \rho(s) - \nabla \rho(t)| + |\nabla \rho(t)|) \, ds \right)^2 dx$$
$$\leq \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(s) - \nabla \rho(t)| \, ds \right)^2 dx + \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(t)| \, ds \right)^2 dx$$
$$+ 2 \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(s) - \nabla \rho(t)| \, ds \right) \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(t)| \, ds \right) dx$$
$$\leq \left(\int_0^t g(s) \, ds \right)^2 \|\nabla \rho\|_2^2 + \int_\Omega \left(\int_0^t g(t-s) \, ds \right) \left(\int_0^t g(t-s) |\nabla \rho(s) - \nabla \rho(t)|^2 \, ds \right.$$
$$+ \eta \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(t)| \, ds \right)^2 dx + \frac{1}{\eta} \int_\Omega \left(\int_0^t g(t-s) |\nabla \rho(s) - \nabla \rho(t)| \, ds \right)^2 dx$$
$$\leq (1+\eta)(1-l)^2 \|\nabla \rho\|_2^2 + \left(1 + \frac{1}{\eta}\right)(1-l)(g \circ \nabla \rho)(t). \tag{3.67}$$

and

$$I_2 = \int_\Omega \left(\int_0^t g(t-s) \left| \nabla \rho(t) - \nabla \rho(s) \right| ds \right)^2 dx \leq (1-l)(g \circ \nabla \rho)(t). \tag{3.68}$$

Taking $\eta = \frac{l}{1-l}$ in (3.67) and using (3.68), we then get from (3.66)

$$\left| -\int_\Omega \left(\int_0^t g(t-s) \nabla \rho(s)\, ds \right) \left(\int_0^t g(t-s) (\nabla \rho(t) - \nabla \rho(s))\, ds \right) dx \right|$$
$$\leq (1-l) \left(\delta \|\nabla \rho\|_2^2 + \left(\frac{\delta}{l} + \frac{1}{4\delta} \right) (1-l)(g \circ \nabla \rho)(t) \right). \tag{3.69}$$

By the Hölder inequality, Young's inequality and Poincaré's inequality, we have

$$\left| \int_\Omega |\rho_t|^{m(x)-2} \rho_t \int_0^t g(t-s)(\rho(t) - \rho(s))\, ds dx \right|$$
$$\leq \delta \int_\Omega |\rho_t|^{2m(x)-2} dx + \frac{(1-l)c^2}{4\delta} (g \circ \nabla \rho)(t), \tag{3.70}$$

and

$$\left| \int_\Omega |\rho|^{p(x)-2} \rho \int_0^t g(t-s)(\rho(t) - \rho(s))\, ds dx \right|$$
$$\leq \delta \int_\Omega |\rho|^{2p(x)-2} dx + \frac{(1-l)c^2}{4\delta} (g \circ \nabla \rho)(t). \tag{3.71}$$

Using the Young's inequality and (A_1) to deal with the last term of (3.64), we have

$$\left| -\int_\Omega \nabla \rho_t \int_0^t g'(t-s)(\nabla \rho(t) - \nabla \rho(s))\, ds dx \right|$$
$$\leq \delta \|\nabla \rho_t\|_2^2 - \frac{g(0)}{4\delta} \left(g' \circ \nabla \rho \right)(t). \tag{3.72}$$

Exploiting again Young's inequality and (A_1) to estimate the fiveth term, we get

$$\begin{aligned}
&\left| -\int_{\Omega} \rho_t \int_0^t g'(t-s)\left(\rho(t)-\rho(s)\right) dsdx \right| \\
&\leq \delta \|\rho_t\|_2^2 + \frac{1}{4\delta} \int_{\Omega}\int_0^t g'(t-s)\left|\rho(t)-\rho(s)\right|^2 dsdx \\
&\leq \delta \|\rho_t\|_2^2 - \frac{g(0)c^2}{4\delta}\left(g' \circ \nabla\rho\right)(t). \qquad (3.73)
\end{aligned}$$

Then, combinig these estimates (3.65)-(3.73), (3.64) becomes

$$\begin{aligned}
\Psi'(t) \leq\ & \delta \|\rho_t\|_2^2 + \delta \|\nabla\rho_t\|_2^2 + (1-l)\delta \|\nabla\rho\|_2^2 \\
&+ \delta \int_{\Omega} |\rho_t|^{2m(x)-2} dx + \delta \int_{\Omega} |\rho|^{2p(x)-2} dx \\
&+ \delta \|\nabla\rho\|_2^2 + \frac{(1-l)}{4\delta}(g\circ\nabla\rho)(t) \\
&+ \left(\frac{\delta}{l} + \frac{1}{4\delta}\right)(1-l)^2 (g\circ\nabla\rho)(t) \\
&+ \frac{(1-l)}{4\delta}c^2 (g\circ\nabla\rho)(t) + \frac{(1-l)}{4\delta}c^2 (g\circ\nabla\rho)(t) \\
&- \frac{g(0)}{4\delta}\left(g'\circ\nabla\rho\right)(t) - \frac{g(0)c^2}{4\delta}\left(g'\circ\nabla\rho\right)(t) \\
&- \left(\int_0^t g(s)\,ds\right)\|\nabla\rho_t\|_2^2 - \left(\int_0^t g(s)\,ds\right)\|\rho_t\|_2^2. \qquad (3.74)
\end{aligned}$$

By (3.74) and Lemma 3.3, we obtain

$$\begin{aligned}
\Psi'(t) \leq\ & c_1 \|\rho_t\|_2^2 + c_2 \|\nabla\rho_t\|_2^2 + c_3 \|\nabla\rho\|_2^2 \\
&+ c_4 (g\circ\nabla\rho)(t) - c_5 \left(g'\circ\nabla\rho\right)(t), \qquad (3.75)
\end{aligned}$$

where

$c_1 = \left(\delta - \int_0^t g(s)\,ds\right)$, $c_2 = \left(\delta + c\delta - \int_0^t g(s)\,ds\right)$, $c_3 = \left((1-l)\delta + \delta + c\delta\right)$,

$c_4 = \left(\left(\frac{\delta}{l} + \frac{1}{4\delta} \right) (1-l)^2 + \frac{(1-l)}{4\delta} + \frac{2(1-l)}{4\delta} c^2 \right)$ and $c_5 = \left(\frac{g(0)}{4\delta} + \frac{g(0)c^2}{4\delta} \right)$.

Since $g(t)$ is positive and continuous, then for any $t_0 > 0$, there exists g_1, g_0 for that

$$g(t) \geq g_1 \text{ and} \int_0^t g(s)\, ds \geq \int_0^{t_0} g(s)\, ds = g_0, \quad \forall t \geq t_0. \tag{3.76}$$

Hence, we conclude from (3.45), (3.63), (3.75) and (3.76) that for any $t \geq t_0 > 0$,

$$\begin{aligned} G'(t) &= M\ E'(t) + \epsilon \Theta'(t) + \Psi'(t) \\ &\leq \left(\frac{M}{2} - c_5 \right) \left(g' \circ \nabla \rho \right)(t) + (\epsilon + c_1) \|\rho_t\|_2^2 + \left(\epsilon + c_2 + \frac{\epsilon c}{4\eta} \right) \|\nabla \rho_t\|_2^2 \\ &+ \left(-\frac{M}{2} g_1 + c_3 - \epsilon + \epsilon c \eta + (1-\eta)(1-l) \right) \|\nabla u\|_2^2 \\ &+ \left(c_4 + \frac{\epsilon}{4\eta} \right) (g \circ \nabla \rho)(t) + \epsilon \int_\Omega |\rho|^{p(x)}\, dx. \end{aligned}$$

However, $g'(t) \leq -g(t)$ by (A_1), thus, we see that

$$\begin{aligned} G'(t) \leq &- (-\epsilon - c_1) \|\rho_t\|_2^2 - \left(-\epsilon - c_2 - \frac{\epsilon c}{4\eta} \right) \|\nabla \rho_t\|_2^2 \\ &- \left(\frac{M}{2} g_1 - c_3 + \epsilon - \epsilon c \eta - (1-\eta)(1-l) \right) \|\nabla \rho\|_2^2 \\ &- \left(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta} \right) (g \circ \nabla \rho)(t) + \epsilon \int_\Omega |\rho|^{p(x)}\, dx. \end{aligned}$$

At this point, we take $\delta = \epsilon$, $\eta = \sqrt{\delta}$ and we choose ϵ small enough for that $g_0 > (c+2)\epsilon + c\sqrt{\epsilon}$.

One ϵ fixed, we pick M sufficiently large so that $\left(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta} \right) > 0$ and $\left(\frac{M}{2} g_1 - c_3 + \epsilon - \epsilon c \eta - (1-\eta)(1-l) \right) > 0$.

Therefore, for any $t \geq t_0$, we have

$$G^{'}(t) \leq -(c_6 \|\rho_t\|_2^2 + c_7 \|\nabla \rho_t\|_2^2 + c_8 \|\nabla \rho\|_2^2 + c_9 (g \circ \nabla \rho)(t) - \epsilon \int_{\Omega} |\rho|^{p(x)} dx), \tag{3.77}$$

where

$$c_6 = (-\epsilon - c_1), c_7 = \left(-\epsilon - c_2 - \frac{\epsilon c}{4\eta}\right),$$

$$c_8 = \left(\frac{M}{2} g_1 - c_3 + \epsilon - \epsilon c\eta - (1-\eta)(1-l)\right), \text{and} c_9 = \left(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta}\right).$$

Combine Lemma 3.5 with (3.44) and (3.9) to get

$$G^{'}(t) \leq -c_{10} E(t) \leq -\frac{c_{10}}{B_2} G(t), \tag{3.78}$$

for some positive constant $c_{10} > 0$. The integration of (3.78) over $(t_0,\ t)$, gives

$$G(t) \leq G(t_0)\, e^{-\frac{c_{10}}{B_2}(t-t_0)}, \quad t \geq t_0.$$

Again, by virtue of Lemma 3.5,

$$E(t) \leq \frac{G(t_0)}{B_1} e^{-\frac{c_{10}}{B_2}(t-t_0)}, \quad t \geq t_0.$$

This, completes the proof.

CHAPTER 4

A BLOW-UP RESULT FOR A HIGHER-ORDER NONLINEAR WAVE EQUATION WITH LOGARITHMIC SOURCE TERM: ANALYTICAL AND NUMERICAL RESULTS

this chapter is devoted to the study of blows up in finite time of the solution for the higher-order wave equation with logarithmic source term. We gave a two-dimensional numerical example to illustrate the blow-up result. (submission)

4.1 Setting the problem

We consider the following boundary value problem:

$$\begin{cases} \rho_{tt}+(-\Delta)^m\rho-\int\limits_0^t g(t-s)(-\Delta)^m\rho(s)\,ds+\rho_t=|\rho|^{p(x)-2}\rho\ln|\rho|^k, & \text{in } \Omega_t \\ \rho(x,t)=0,\ \frac{\partial^i\rho}{\partial v^i}=0,\ i=1,\ 2,...,m-1, & \text{on } \Gamma_t \\ \rho(x,0)=\rho_0(x),\ \ \rho_t(x,0)=\rho_1(x), & \text{in } \Omega \end{cases} \tag{4.1}$$

where $m\geq 1$ is a natural number, Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$, $\partial\Omega$ is smooth bouandary of Ω, $\Omega_t=\Omega\times\mathbb{R}^+$, $\Gamma_t=\partial\Omega\times\mathbb{R}^+$, g is the relaxation function satisfying some condition to be specified later and k is positive constant. $p(.)$ is given measurable functions on Ω, satisfying

$$\begin{cases} 2<p_1\leq p(x)\leq p_2<\infty \text{ for } n\leq 2m, \\ 2<p_1\leq p(x)\leq p_2<\frac{2(n-m)}{n-2m} \text{ for } n>2m. \end{cases} \tag{4.2}$$

Where

$$p_1:=\ ess_{x\in\Omega}\inf(p(x)),\ p_2:=ess_{x\in\Omega}\sup(p(x)).$$

We also assume that $p(x)$ satisfy the log $-$Hölder continuity condition (definition 1.3). Let us assume that

(H) $g\in\mathbb{C}^1([0,+\infty))$ is non-negative function satisfaying

$$1-\int_0^t g(s)\,ds=l>0,\ g'(t)\leq 0 \text{ for } t\geq 0. \tag{4.3}$$

It necessary to state the local existence theorem for problem 4.1 wich can be established by combining the argument of [57, 52, 65]

Theorem 4.1 *Assume that g, $p(.)$ satisfy (H), (4.2) respectively. Then for any given*

$$(\rho_0,\rho_1)\in H_0^m(\Omega)\cap H^{2m}(\Omega)\times H^m(\Omega),$$

problem (4.1) *has a unique local solution*

$$\rho \in C\left((0,T)\,,\ H_0^{2m}(\Omega)\right)$$

$$\rho_t \in C\left((0,T)\,,\ H^m(\Omega)\right).$$

In the proof of our main result, we define the energy functional by

$$E(t) = \frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|D^m\rho(t)\|_2^2 + \frac{1}{2}(g \circ D^m\rho)(t)$$
$$+ \int_\Omega \frac{k}{p^2(x)}|\rho(t)|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|\rho|^{p(x)}\ln|\rho|^k\,dx. \tag{4.4}$$

Where $(g \circ D^m v)(t) = \int_0^t g(t-s)\|D^m v(t) - D^m v(s)\|^2\,ds, \forall v \in H_0^m(\Omega).$

Lemma 4.1 *Assume that* (H), (4.2) *hold, and let* ρ *be the solution of problem* 4.1. *Then* $E(t)$ *decreases, which*

$$E'(t) = \frac{1}{2}(g' \circ D^m\rho)(t) - \frac{1}{2}g(t)\|D^m\rho(t)\|_2^2 - \|\rho_t\|_2^2 \le 0,$$

furthermore,

$$E(t) \le E(0)\,,\quad \textit{for all } t \ge 0.$$

Proof. By multiplying the equation in (4.1) by ρ_t and integrating over Ω, we get

$$\int_\Omega \rho_t\rho_{tt}dx + \int_\Omega \rho_t(-\Delta)^m\rho dx - \int_\Omega \rho_t\int_0^t g(t-s)(-\Delta)^m\rho(x,s)\,dxds + \int_\Omega \rho_t^2 dx$$
$$= \int_\Omega \rho_t\rho|\rho|^{p(x)-2}\ln|\rho|^k\,dx,$$

then use integration par parts, yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}|\rho_t|^2\,dx+\int_{\Omega}\|D^m\rho\|_2^2\,dx\right)-\int_0^t g\left(t-s\right)\int_{\Omega}D^m\rho_t\left(t\right)D^m\rho\left(s\right)dxds$$
$$+\int_{\Omega}\rho_t^2dx=\frac{d}{dt}\left(-\int_{\Omega}\frac{k}{p^2\left(x\right)}\rho^{p(x)}+\int_{\Omega}\frac{1}{p\left(x\right)}|\rho|^{p(x)}\ln|\rho|^k\,dx\right). \tag{4.5}$$

The third term in the left side of can be estimated as follows.

$$\begin{aligned}
&\int_0^t g\left(t-s\right)\int_{\Omega}D^m\rho_t\left(t\right)D^m\rho\left(s\right)dxds\\
&=\int_0^t g\left(t-s\right)\int_{\Omega}D^m\rho_t\left(t\right).\left(D^m\rho\left(s\right)-D^m\rho\left(t\right)\right)dxds\\
&+\int_0^t g\left(t-s\right)\int_{\Omega}D^m\rho_t\left(t\right).D^m\rho\left(t\right)dxds\\
&=-\frac{1}{2}\int_0^t g\left(t-s\right)\frac{d}{dt}\int_{\Omega}|D^m\rho\left(s\right)-D^m\rho\left(t\right)|^2\,dxds\\
&+\int_0^t g\left(t-s\right)\left(\frac{d}{dt}\frac{1}{2}\int_{\Omega}|D^m\rho\left(t\right)|^2\,dx\right)ds
\end{aligned}$$

$$\begin{aligned}
&=-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g\left(t-s\right)\int_{\Omega}|D^m\rho\left(s\right)-D^m\rho\left(t\right)|^2\,dxds\right]\\
&+\frac{1}{2}\int_0^t g'\left(t-s\right)\int_{\Omega}|D^m\rho\left(s\right)-D^m\rho\left(t\right)|^2\,dxds\\
&+\frac{1}{2}\frac{d}{dt}\left[g\left(s\right)\int_{\Omega}|D^m\rho\left(s\right)|^2\,dxds\right]-\frac{1}{2}g\left(t\right)\int_{\Omega}|D^m\rho\left(t\right)|^2\,dx.
\end{aligned} \tag{4.6}$$

Insert (4.6) in (4.5) to get

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\Omega}|\rho_t|^2\,dx+\frac{1}{2}\int_{\Omega}\|D^m\rho\|_2^2\,dx+\int_{\Omega}\frac{k}{p^2(x)}|\rho|^{p(x)}\,dx\right.$$
$$-\int_{\Omega}\frac{1}{p(x)}|\rho|^{p(x)}\ln|\rho|^{p(x)}\,dx$$
$$\left.+\frac{1}{2}\int_0^t g(t-s)\int_{\Omega}|D^m\rho(s)-D^m\rho(t)|^2\,dxds-\frac{1}{2}\int_0^t g(s)\,ds\int_{\Omega}|D^m\rho(t)|^2\,dxds\right\}$$
$$=\frac{1}{2}\int_0^t g'(t-s)\int_{\Omega}|D^m\rho(s)-D^m\rho(t)|^2\,dxds-\frac{1}{2}g(t)\int_{\Omega}|D^m\rho(t)|^2\,dx-\int_{\Omega}\rho_t^2dx,$$

hence, using (4.4), we obtain

$$E'(t)=\frac{1}{2}(g'\circ D^m\rho)-\frac{1}{2}g(t)\|D^m\rho\|^2-\|\rho_t\|^2\le 0.$$

Using integration of last inequality, we get

$$E(t)\le E(0) \tag{4.7}$$

Lemma 4.2 *[40] Assume that* (4.2)*, the log-Hölder continuity condition (definition* (1.3) *) hold and* $E(0)<0$*. Then the solution of* (4.1)*, satisfies for some* $c>0$,

$$\int_{\Omega}|\rho|^{p(x)}\,dx\ge c\|\rho\|_{p_1}^{p_1}. \tag{4.8}$$

4.2 Blow up of solution

Lemma 4.3 *[59] Suppose that* $\Phi(t)$ *is a twice continuously differentiable satisfying*

$$\begin{cases}\Phi''(t)+\Phi'(t)\ge C_0\Phi^{1+\alpha}(t), & t>0,\ C_0>0,\ \alpha>0\\ \Phi(0)>0,\ \Phi'(0)\ge 0.\end{cases} \tag{4.9}$$

Then, $\Phi(t)$ blows up in finite time.

Our main result as follows

Theorem 4.2 *Assume that (H) holds. Assume further*

$$\int_0^t g(s)\,ds < \frac{p_1(p_1-2)}{(p_1-1)^2},\ \forall t \geq 0, \tag{4.10}$$

and the initial condition

$$(\rho_0, \rho_1) \in H_0^m(\Omega) \times L^2(\Omega),$$

with compact supports, satisfying

$$E(0) < 0 \text{ and } \rho_0\rho_1 > 0.$$

Then any solution of 4.1 blows up in finite time.

Proof. We define the following function

$$\Phi(t) = \frac{1}{2}\int_\Omega |\rho(x,t)|^2\,dx.$$

Therefore,

$$\Phi'(t) = \int_\Omega \rho\rho_t dx,\ \Phi''(t) = \int_\Omega \left(\rho\rho_{tt} + |\rho_t|^2\right) dx. \tag{4.11}$$

Multiplying the equation of (4.1) by $\rho(t)$ and integrating over Ω, gives

$$\int_\Omega \rho\rho_{tt}dx = -\int_\Omega |D^m\rho|^2\,dx - \int_\Omega \rho\rho_t dx + \int_\Omega |\rho|^{p(x)} \ln|\rho|^k\,dx$$
$$+\int_0^t g(t-s)\int_\Omega D^m\rho(t)\,D^m\rho(s)\,dxds.$$

We have

$$\int_0^t g(t-s)\int_\Omega D^m\rho(t)\,D^m\rho(s)\,dxds = \int_0^t g(s)\int_\Omega |D^m\rho|^2\,dx$$
$$+\int_0^t g(t-s)\int_\Omega D^m\rho(t)\left(D^m\rho(s) - D^m\rho(t)\right)dxds.$$

The second equation of (4.11) becomes:

$$\Phi''(t) = -\left(1 - \int_0^t g(s)\right)\int_\Omega |D^m\rho|^2\,dx$$
$$-\int_0^t g(t-s)\int_\Omega D^m\rho(t)\,.\,(D^m\rho(s) - D^m\rho(t))\,dxds$$
$$+\int_\Omega |\rho|^{p(x)}\ln|\rho|^k\,dx + \int_\Omega |\rho_t|^2\,dx - \int_\Omega \rho\rho_t dx. \tag{4.12}$$

Using Young's inequality, we estimate

$$-\int_0^t g(t-s)\int_\Omega D^m\rho(t)\,.\,(D^m\rho(s) - D^m\rho(t))\,dxds$$
$$\geq -\delta\int_\Omega |D^m\rho|^2\,dx - \frac{1}{4\delta}\left(\int_0^t g(s)\right)(g\circ D^m\rho)(t), \qquad \forall\delta > 0. \tag{4.13}$$

By combining (4.12) and (4.13), we get

$$\Phi''(t) = -\left(1 + \delta - \int_0^t g(s)\,ds\right)\|D^m\rho\|^2 - \frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\rho)(t)$$
$$+\int_\Omega |\rho|^{p(x)}\ln|\rho|^k\,dx + \int_\Omega |\rho_t|^2\,dx - \int_\Omega \rho\rho_t dx.$$

Now, we exploit (4.4) to substitute for $\int_\Omega |\rho|^{p(x)}\ln|\rho|^k\,dx$,

$$\int_\Omega |\rho|^{p(x)}\ln|\rho|^k\,dx$$
$$\geq p_1\left[\frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|D^m\rho\|_2^2 + \frac{1}{2}(g\circ D^m\rho)(t) + \frac{k}{p_2^2}\int_\Omega |\rho|^{p(x)}\,dx\right]$$

Using Lemma 4.2, leads to:

$$\Phi''(t)+\Phi'(t)\geq\left(1+\frac{p_1}{2}\right)\|\rho_t\|^2+\left[\frac{p_1}{2}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right](g\circ D^m\rho)(t)$$
$$+\left[\left(\frac{p_1}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right)-\delta\right]\|D^m\rho\|_2^2+\frac{p_1kc}{p_2^2}\|\rho\|_{p_1}^{p_1}. \tag{4.14}$$

At this point, we aim to have

$$\frac{p_1}{2}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)>0 \text{ and } \left(\frac{p_1}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right)-\delta>0. \tag{4.15}$$

So, from the first part of (4.15), we need

$$\delta>\frac{1}{2p_1}\left(\int_0^t g(s)\,ds\right)$$

and from the second part of (4.15) that

$$\delta<\left(\frac{p_1}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right),$$

or

$$\frac{1}{2p_1}\left(\int_0^t g(s)\,ds\right)<\delta<\left(\frac{p_1}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right),$$

this implise that

$$\frac{1}{2p_1}\left(\int_0^t g(s)\,ds\right)<\left(\frac{p_1}{2}-1\right)\left(1-\int_0^t g(s)\,ds\right)$$

or

$$\int_0^t g(s)\,ds<\frac{p_1(p_1-2)}{(p_1-1)^2},\ \forall t\geq 0,$$

where it guaranteed by (4.15). Therefore, the estimate (4.14) becomes

$$\Phi''(t) + \Phi'(t) \geq \gamma \|\rho\|_{p_1}^{p_1}, \tag{4.16}$$

where $\gamma = \frac{p_1 kc}{p_2^2}$. Now, we use Hölder's inequality to estimate

$$\int_\Omega |\rho|^2 dx \leq \left(\int_\Omega |\rho|^{p_1} dx\right)^{\frac{2}{p_1}} \left(\int_\Omega 1 dx\right)^{\frac{p_1-2}{p_1}},$$

where $|\Omega|$ is the measure of the doman Ω, then

$$\left(\int_\Omega |\rho|^{p_1} dx\right)^{\frac{2}{p_1}} \geq \left(\int_\Omega |\rho|^2 dx\right) |\Omega|^{\frac{2-p_1}{p_1}}.$$

So,

$$\int_\Omega |\rho|^{p_1} dx \geq \left(\int_\Omega |\rho|^2 dx\right)^{\frac{p_1}{2}} |\Omega|^{\frac{2-p_1}{2}}. \tag{4.17}$$

From the definition of $\Phi(t)$,we have

$$(2\Phi(t))^{\frac{p_1}{2}} = \left(\int_\Omega |\rho(x,t)|^2 dx\right)^{\frac{p_1}{2}}. \tag{4.18}$$

Combining (4.17),(4.18), and (4.16), yield

$$\Phi''(t) + \Phi'(t) \geq 2^{\frac{p_1}{2}} \gamma (\Phi(t))^{\frac{p_1}{2}} |\Omega|^{\frac{2-p_1}{2}}.$$

It is easy to verify that the requirements of Lemma 4.3 are satisfied by:

$$C_0 = 2^{\frac{p_1}{2}} \gamma |\Omega|^{\frac{2-p_1}{2}} > 0, \qquad \alpha = \frac{p_1 - 2}{2}.$$

Therefore $\Phi(t)$ blows up in the finite time.

4.3 Numerical study

In this section, we present an application to illustrate numerically the blowup result of Theorem 4.2. For this purpose, we numerically solve problem (4.1), for $m = 1$, $n = 2$, $k = 2$, where the domain is taken to be $\Omega = [-1,\ 1]^2$. We chosen $g(t) = \lambda e^{-t}$, $(0 < \lambda < 1)$, $\rho_0(x_1, x_2) = \rho_1(x_1, x_2) = 4 + 5x_1^2 - 3x_2^2$, where will be chosen such that $E(0) < 0$ and $\rho_0\rho_1 > 0$, we take the exponent function $p(x_1, x_2) = \lceil x_1 \rceil^2 + 4.8$, which satisfy condition (4.2), where $\lceil . \rceil$ denotes the greatest integer function.

4.3.1 Numerical method

We first introduce a suitable numerical scheme to discretize (4.1) using finite differences for the time variable $t \in [0, T]$ and the space variable $x = (x_1, x_2) \in \Omega$. Comprehensive details about the finite difference methods are available in [52, 51]. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $\quad t_n = n\ \delta t, \quad n = 1,\ 2, ..., N+1$, where δt is the time step.

Let $\rho^n\ (x_1, x_2) = \rho(x_1, x_2,\ t_n)$, and use the finite-difference formulas: the first-order backward difference for

$$\partial_t \rho^n(x_1, x_2) = \frac{\rho^n(x_1, x_2) - \rho^{n-1}(x_1, x_2)}{\delta t}.$$

and the second-order center difference for

$$\partial_{tt} \rho^n(x_1, x_2) = \frac{\rho^{n+1}(x_1, x_2) - 2\rho^n(x_1, x_2) + \rho^{n-1}(x_1, x_2)}{(\delta t)^2}.$$

Then the time discrete problem of (4.1) reads: Given ρ_0 and ρ_1, find $\left\{\rho^2,\ \rho^3, ..., \rho^{n+1}\right\}$ such that

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - \frac{\rho^n - \rho^{n-1}}{\delta t} \\ -\int_0^{t_{n+1}} g\left(t_{n+1} - s\right)\Delta\rho^n\left(s\right)ds + \left|\rho^n\right|^{p(x_1,x_2)-2}\rho^n \ln\left|\rho^n\right|^2, & \text{in } \Omega \\ \rho^{n+1} = 0, & \text{on } \partial\Omega \\ \rho^0 = \rho_0\left(x_1, x_2\right), \quad \rho^1 = \rho^0 + \left(\delta t\right)\rho_1\left(x_1, x_2\right), & \text{in } \Omega \end{cases} \tag{4.19}$$

Note that the above problem is linear in ρ^{n+1}, which is achieved by using the history data ρ^n and ρ^{n-1} in the second side of the equation. Problem(4.19) is solved iteratively as for given regular ρ^n, the solution ρ^{n+1} satisfies the boundary-value problem:

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = F\left(\rho^n,\ \rho^{n-1}\right), & \text{in } \Omega_h \\ \rho^{n+1} = 0, & \text{on } \partial\Omega_h \end{cases} \tag{4.20}$$

where

$$\begin{aligned} F\left(\rho^n,\ \rho^{n-1}\right) &= \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - \frac{\rho^n - \rho^{n-1}}{\delta t} - \int_0^{t_{n+1}} g\left(t_{n+1} - s\right)\Delta\rho^n\left(s\right)ds \\ &+ \left|\rho^n\right|^{p(x_1,x_2)-2}\rho^n \ln\left|\rho^n\right|^2. \end{aligned}$$

4.3.2 Numerical results

In this subsection, we present and discuss the blow up results of the numerical scheme(4.19). The numerical results are obtained using the Matlab codes.

The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 300×300;

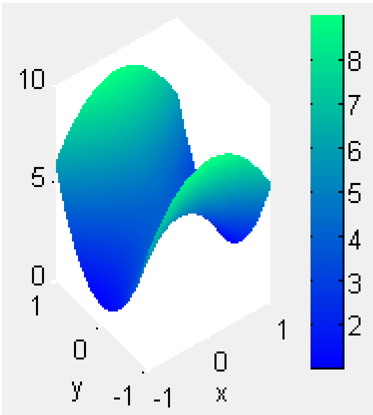

Figure 4.1: ρ^0

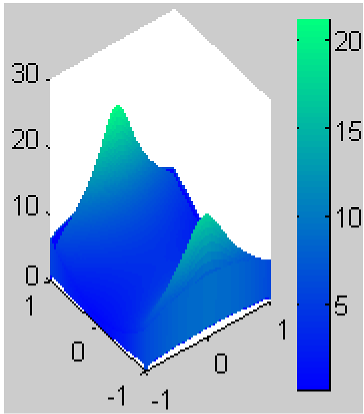

Figure 4.2: ρ^{39}

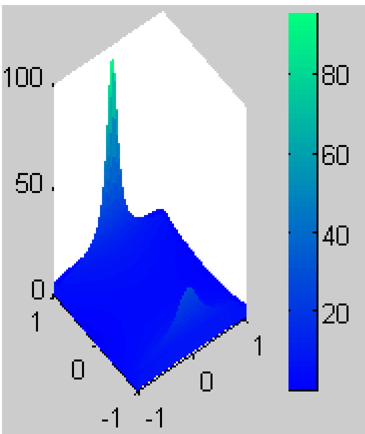

Figure 4.3: ρ^{44}

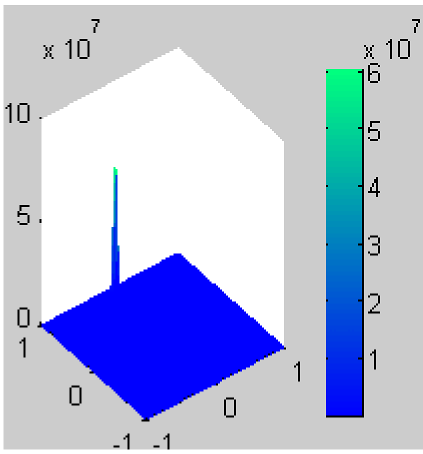

Figure 4.4: ρ^{45}

- Time step is: $\delta t = 0.01$;
- The spatial discretisation step $h = 0.003$;
- $k = \frac{1}{2}$.

Figures. 4.1 shows the graphs of the initial data u_0. Figures. 4.2, 4.3 and 4.4 present the solution of ρ^n from iterations $n = 39$ ($t = 0.39$), $n = 44$ ($t = 0.44$) and $n = 45$ ($t = 0.45$) respectively .

Figure. 4.4 present the solution ρ^n for iteration $n = 45$ ($t = 0.45$), which the blowup.

CHAPTER 5

WELL-POSEDNESS AND STABILITY RESULTS FOR A CLASS OF NONLINEAR FOURTH-ORDER WAVE EQUATION WITH VARIABLE-EXPONENTS

In this chapter, we consider a class of nonlinear wave equations of the fourth order with damping and source terms of the variable exponent type. First, under suitable conditions on the variable exponents $m(.)$ and $r(.)$, we establish local existence. Then we prove that the local solution is global. Finally, using the Komornik inequality to estimate the stability of the solution.

The results, which appear in this context, have been published in[41].

5.1 Setting the problem

We consider the following boundary value problem:

$$\begin{cases} \rho_{tt} + \Delta^2 \rho - \Delta \rho + |\rho_t|^{m(x)-2} \rho_t = |\rho|^{r(x)-2} \rho, & (x,t) \in \Omega \times (0,T), \\ \rho(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\ T), \\ \rho(x,0) = \rho_0(x),\ \rho_t(x,0) = \rho_1(x), & x \in \Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$. $m(.)$ and $r(.)$ are given measurable functions on Ω, satisfying

$$\begin{aligned} &2 < r_1 \leq r(x) \leq r_2 < 2\frac{n-1}{n-2}, && \text{if } n \geq 3, \\ &2 < r(x) < \infty, && \text{if } n = 1,\ 2, \end{aligned}$$

$$\begin{aligned} &2 < m_1 \leq m(x) \leq m_2 < 2\frac{n}{n-2}, && \text{if } n \geq 3, \\ &2 < m(x) < \infty, && \text{if } n = 1,\ 2, \end{aligned}$$

and

$$\begin{aligned} r_1 &:= ess\inf_{x\in\Omega} r(x), \quad & r_2 &:= ess\sup_{x\in\Omega} r(x), \\ m_1 &:= ess\inf_{x\in\Omega} m(x), \quad & m_2 &:= ess\sup_{x\in\Omega} m(x), \end{aligned}$$

We also assume that $m(.)$ and $r(.)$ satisfy the log-Hölder continuity condition (definition (1.3)).

Lemma 5.1 *[14] If p is a measurable function on Ω satisfying (definition (1.3)), then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.*

From the Lemma 5.1, there exists the positive constant c_* satisfying

$$\|\rho\|_{p(.)} \leq c_* \|\nabla \rho\|_2, \quad \text{for } \rho \in H_0^1(\Omega).$$

Lemma 5.2 *[25] Let $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $C > 0$ such that*

$$\int_t^{\infty} G^{\alpha+1}(s)\, ds \leq C G^{\alpha}(0) G(s), \quad \forall t \in \mathbb{R}_+.$$

Then we have

$$G(t) \leq G(0) \left(\frac{C + \alpha t}{C + \alpha C} \right)^{\frac{-1}{\alpha}}, \quad \forall\, t \geq C.$$

Theorem 5.1 *Suppose that $r,\ m \in C\left(\overline{\Omega}\right)$ and satisfies the log-Hölder continuity condition (definition (1.3)). Then, for any $(\rho_0,\ \rho_1) \in H^2(\Omega) \cap H^4(\Omega) \times L^2(\Omega)$, problem (5.1) has a unique weak local solution*

$$\begin{aligned} &\rho \in L^{\infty}\left((0,T),\ H^2(\Omega)\right), \\ &\rho_t \in L^{\infty}\left((0,T),\ L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0,\ T)), \\ &\rho_{tt} \in L^{\infty}\left((0,T),\ L^2(\Omega)\right). \end{aligned}$$

In the order to state and prove our result, we define the potential energy functional and Nehari's functional, by the following

$$E(t) = E(\rho(t)) = \frac{1}{2}\left(\|\rho_t(t)\|_2^2 + \|\nabla \rho(t)\|_2^2 + \|\Delta \rho(t)\|_2^2 \right) - \int_{\Omega} \frac{1}{r(x)} |\rho(t)|^{r(x)}\, dx. \tag{5.2}$$

$$I(t) = I(\rho(t)) = \|\nabla \rho(t)\|_2^2 + \|\Delta \rho(t)\|_2^2 - \int_{\Omega} |\rho(t)|^{r(x)}\, dx. \tag{5.3}$$

$$J(t) = J(\rho(t)) = \frac{1}{2}\left(\|\nabla \rho(t)\|_2^2 + \|\Delta \rho(t)\|_2^2 \right) - \int_{\Omega} \frac{1}{r(x)} |\rho(t)|^{r(x)}\, dx. \tag{5.4}$$

Lemma 5.3 *Under the assumptions of theorem 5.1, we have*

$$E^{'}(t) = -\int_{\Omega} |\rho_t(t)|^{m(x)} dx \leq 0, \quad t \in [0,\ T].$$

and

$$E(t) \leq E(0).$$

Proof. We multiply the first equation of (5.1) by ρ_t and integrating over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\left(\|\rho_t(t)\|_2^2 + \|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2\right) - \int_{\Omega}\frac{1}{r(x)}|\rho(t)|^{r(x)} dx\right)$$
$$= -\int_{\Omega} |\rho_t(t)|^{m(x)} dx,$$

then

$$E^{'}(t) = -\int_{\Omega} |\rho_t(t)|^{m(x)} dx \leq 0. \tag{5.5}$$

Integratying (5.5) over $(0,\ t)$, we obtain

$$E(t) \leq E(0).$$

Lemma 5.4 *Assume that the assumptions of theorem 5.1 and $E(0) > 0$ hold,*

$$I(0) > 0,$$

and

$$\theta_1 + \theta_2 < 1, \tag{5.6}$$

where

$$\theta_1 := \alpha \max\left\{ c_{1,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{1,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}} \right\},$$

$$\theta_2 := (1-\alpha)\max\left\{ c_{2,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{2,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}} \right\},$$

with $0 < \alpha < 1$, $c_{1,*}$ *and* $c_{2,*}$ *are the bests embedding constants of* $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ *and* $H_0^2(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ *respectively, then* $I(t) > 0$, *for all* $t \in [0,\ T]$.

Proof. By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{5.7}$$

Now, we have for all $t \in [0,\ T_*]$:

$$\begin{aligned} J(t) = J(\rho(t)) &= \frac{1}{2}\left(\|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|\rho(t)|^{r(x)}dx \\ &\geq \frac{1}{2}\|\nabla\rho(t)\|_2^2 + \frac{1}{2}\|\Delta\rho(t)\|_2^2 - \frac{1}{r_1}\left(\|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2 - I(t)\right) \\ &\geq \frac{r_1-2}{2r_1}\left(\|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2\right) + \frac{1}{r_1}I(t), \end{aligned}$$

using (5.7), we obtain

$$\|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2 \leq \frac{2r_1}{r_1-2}J(t), \qquad \text{for all } t \in [0,\ T_*]. \tag{5.8}$$

By Lemma 5.3, we get

$$\|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2 \leq \frac{2r_1}{r_1-2}E(t) \leq \frac{2r_1}{r_1-2}E(0) \tag{5.9}$$

On the other hand, by Lemma 1.2, we have

$$\int_\Omega |\rho(t)|^{r(x)}\,dx \leq \ Max\left\{\|\rho(t)\|_{r(.)}^{r_1},\ \|\rho(t)\|_{r(.)}^{r_2}\right\}$$
$$= \alpha\ Max\left\{\|\rho(t)\|_{r(.)}^{r_1},\ \|\rho(t)\|_{r(.)}^{r_2}\right\}$$
$$+(1-\alpha)\,Max\left\{\|\rho(t)\|_{r(.)}^{r_1},\ \|\rho(t)\|_{r(.)}^{r_2}\right\}.$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, we obtain

$$\int_\Omega |\rho(t)|^{r(x)}\,dx \leq \alpha\ Max\left\{c_{1,*}^{r_1}\|\nabla\rho(t)\|_2^{r_1},\ c_{1,*}^{r_2}\|\nabla\rho(t)\|_2^{r_2}\right\}$$
$$+(1-\alpha)\,Max\left\{c_{2,*}^{r_1}\|\Delta\rho(t)\|_2^{r_1},\ c_{2,*}^{r_2}\|\Delta\rho(t)\|_2^{r_2}\right\}$$
$$\leq \alpha\ Max\left\{c_{1,*}^{r_1}\|\nabla\rho(t)\|_2^{r_1-2},\ c_{1,*}^{r_2}\|\nabla\rho(t)\|_2^{r_2-2}\right\}\times\|\nabla\rho(t)\|_2^2$$
$$+(1-\alpha)\,Max\left\{c_{2,*}^{r_1}\|\Delta\rho(t)\|_2^{r_1-2},\ c_{2,*}^{r_2}\|\Delta\rho(t)\|_2^{r_2-2}\right\}\times\|\Delta\rho(t)\|_2^2$$
$$\leq \alpha\ Max\left\{c_{1,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{1,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\}$$
$$\times\|\nabla\rho(t)\|_2^2$$
$$+(1-\alpha)\,Max\left\{c_{2,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{2,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\}$$
$$\times\|\Delta\rho(t)\|_2^2$$

Then, we get

$$\int_\Omega |\rho(t)|^{r(x)}\,dx \leq \theta_1\|\nabla\rho(t)\|_2^2+\theta_2\|\Delta\rho(t)\|_2^2,\qquad \text{for all } t\in[0,\ T_*]. \tag{5.10}$$

Since $\theta_1+\theta_2<1$, then

$$\int_\Omega |\rho(t)|^{r(x)}\,dx < \|\nabla\rho(t)\|_2^2+\|\Delta\rho(t)\|_2^2,\qquad \text{for all } t\in[0,\ T_*]. \tag{5.11}$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*].$$

By repeating the above procedure, we can extend T_* to T.

5.2 Existence of weak solution

In this section we are going to obtain the existence of local solution to the problem (5.1). We will use the Faedo- Galerkin's method.

Existence:

We take several steps to prove the existence of a local solution to problem (5.1).

- Step 1: Approximate problem.

Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$ wich constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = span\{v_1,\ v_2, ..., v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. By the normalization, we have $\|v_l\| = 1$, for any given integer k, we consider the approximation solution

$$\rho_k(t) = \sum_{l=1}^{k} \rho_{lk}(t)\, v_l,$$

where ρ_k is the solutions to the following Cauchy problem

$$\left(\rho_k''(t),\ v_l\right) + \left(\Delta^2 \rho_k(t),\ v_l\right) - \left(\Delta \rho_k(t),\ v_l\right) + \left(\left|\rho_k'(t)\right|^{m(x)-2} \rho_k'(t),\ v_l\right)$$
$$= \left(|\rho_k(t)|^{r(x)-2} \rho_k(t),\ v_l\right), \quad l = 1,\ 2, ..., k, \tag{5.12}$$

$$\rho_k(0) = \rho_{0k} = \sum_{i=1}^{k} (\rho_k(0), \ v_l) v_l \to \rho_0 \text{ in } H_0^2(\Omega) \cap H^4(\Omega), \tag{5.13}$$

$$\rho_k^{'}(0) = \rho_{1k} = \sum_{l=1}^{k} \left(\rho_k^{'}(0), \ v_l\right) v_l \to \rho_1 \text{ in } L^2(\Omega). \tag{5.14}$$

Note that, we can solve the system (5.12)-(5.14) by a Picard's iteration method in ordinary differential equations. Hence, there exists a solution in $[0, \ T_*)$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0, \ T]$ for any given $T > 0$ by making use of the priori estimates below.

- Step 2: A priori Estimates.

The first estimate. Multiplying equation (5.12) by $u_{lk}^{'}(t)$ and summing over l from 1 to k,

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\rho_k^{'}\right\|_2^2 + \frac{1}{2}\|\nabla\rho_k\|_2^2 + \frac{1}{2}\|\Delta\rho_k\|_2^2 - \int_\Omega \frac{1}{r(x)}|\rho_k|^{r(x)}\right) dx = -\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dx. \tag{5.15}$$

Then

$$E^{'}(\rho_k(t)) = -\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dx \leq 0.$$

Integrating (5.15) over $(0, \ t)$, we obtain the estimate

$$\begin{aligned} &\frac{1}{2}\left\|\rho_k^{'}\right\|_2^2 + \frac{1}{2}\|\nabla\rho_k\|_2^2 + \frac{1}{2}\|\Delta\rho_k\|_2^2 \\ &- \int_\Omega \frac{1}{r(x)}|\rho_k|^{r(x)} dx + \int_0^t\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dxds \leq E(0). \end{aligned} \tag{5.16}$$

Then, from (5.11), the inequality (5.16) becomes

$$\frac{1}{2}\sup_{t\in(0,\ T)}\left\|\rho_k^{'}\right\|_2^2 + \frac{r_1-2}{2r_1}\sup_{t\in(0,\ T)}\|\nabla\rho_k\|_2^2 + \frac{r_1-2}{2r_1}\sup_{t\in(0,\ T)}\|\Delta\rho_k\|_2^2$$

$$+\int_0^t\int_\Omega \left|\rho_k^{'}\right|^{m(x)} dxds \leq E(0). \tag{5.17}$$

From (5.17), we conclude that

$$\begin{cases} \{\rho_k\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ \{\rho_k^{'}\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ L^2(\Omega)\right) \cap L^{m(x)}(\Omega\times[0,\ T]). \end{cases} \tag{5.18}$$

Since $\{\rho_k^{'}\}$ is uniformly bounded in $L^{m(x)}(\Omega\times[0,\ T])$,
then $\left\{\left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{'}\right\}$ is bounded in $L^{\frac{m(x)}{m(x)-1}}(\Omega\times[0,\ T])$
hence, up to a subsequence, $\left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{'} \rightharpoonup \Phi$ weakly in $L^{\frac{m(x)}{m(x)-1}}(\Omega\times[0,\ T])$.
As in [39], we have to show that $\Phi = \left|\rho^{'}\right|^{m(x)-2}\rho'$. Furthermore, we have from Lemma 5.1 and (5.18) that

$$\left\{|\rho_k|^{r(x)-2}\rho_k\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \tag{5.19}$$

By (5.18) and (5.19), we infer that there exists a subsequence of ρ_k (denote still by the same symbol) and a function ρ such that

$$\begin{cases} \rho_k \rightharpoonup \rho \text{ weakly star in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ \rho_k^{'} \rightharpoonup \rho^{'} \text{ weakly star in } L^\infty\left([0,\ T],\ L^2(\Omega)\right) \text{ and weakly in } L^{m(x)}(\Omega\times[0,\ T]) \\ |\rho_k|^{r(x)-2}\rho_k \rightharpoonup \Psi \text{ weakly in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \end{cases} \tag{5.20}$$

By the Aubin-Lions compactness Lemma [34], we conclude from (5.20) that

$$\rho_k \rightharpoonup \rho \text{ strongly in } C\left([0,\ T],\ H_0^2(\Omega)\right),$$

which implies

$$\rho_k \rightharpoonup \rho \text{ everywhere in } \Omega\times[0, T]. \tag{5.21}$$

It follow from (5.20) and (5.21) that

$$|\rho_k|^{r(x)-2}\rho_k \rightharpoonup |\rho|^{r(x)-2}\rho \text{ weakly in } L^{\infty}\left([0,\ T],\ L^2(\Omega)\right). \tag{5.22}$$

The second estimate. Now, we would like to get more estimates. In doing so, differentiating (5.12) with respect to t, we get

$$\begin{aligned}
&\left(\rho_k^{'''}(t),\ v_l\right) + \left(\Delta^2\rho_k^{'}(t),\ v_l\right) - \left(\Delta\rho_k^{'}(t),\ v_l\right) \\
&\quad + \left((m(x)-1)\left|\rho_k^{'}(t)\right|^{m(x)-2}\rho_k^{''}(t),\ v_l\right) \\
&= \left((r(x)-1)\left|\rho_k(t)\right|^{r(x)-2}\rho_k^{'}(t),\ v_l\right), \quad l=1,\ 2,...,k,
\end{aligned} \tag{5.23}$$

Next, multiplying the equation (5.23) by $\rho_{lk}^{''}(t)$ and summing over l from 1 to k, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}\left|\rho_k^{''}\right|^2 dx + \int_{\Omega}\left|\Delta\rho_k^{'}\right|^2 dx + \int_{\Omega}\left|\nabla\rho_k^{'}\right|^2 dx\right) \\
&+ \int_{\Omega}(m(x)-1)\left|\rho_k^{'}\right|^{m(x)-2}\rho_k^{''2}dx \\
&= \int_{\Omega}(r(x)-1)\left|\rho_k\right|^{r(x)-2}\rho_k^{'}\rho_k^{''}dx
\end{aligned} \tag{5.24}$$

We have from Hölder's inequality that

$$\left|\int_{\Omega}(r(x)-1)\left|\rho_k\right|^{r(x)-2}\rho_k^{'}\rho_k^{''}dx\right| \leq (r_2-1)\left\|\rho_k\right\|_{2(r(x)-1)}^{r(x)-2}\left\|\rho_k^{'}\right\|_{2(r(x)-1)}\left\|\rho_k^{''}\right\|_2 \tag{5.25}$$

We have $\rho_k \in L^{\infty}\left([0,\ T],\ H_0^2(\Omega)\right)$, then

$$\int_{\Omega}\left|\rho_k\right|^{2r(x)-2}dx \leq \int_{\Omega}\left|\rho_k\right|^{2r_1-2}dx + \int_{\Omega}\left|\rho_k\right|^{2r_2-2}dx < +\infty,$$

since, $2(r_1-1) \leq 2(r(x)-1) \leq 2(r_2-1) \leq 2\frac{n}{n-2}$. The inequality (5.25), becomes

$$\left|\int_\Omega (r(x)-1)|\rho_k|^{r(x)-2}\rho_k'\rho_k''dx\right| \leq c_1 \left\|\rho_k'\right\|_{2(r(x)-1)}\left\|\rho_k''\right\|_2 \tag{5.26}$$

We have from Young's inequality and Poincare's inequality that

$$\left|\int_\Omega (r(x)-1)|\rho_k|^{r(x)-2}\rho_k'\rho_k''dx\right| \leq c_\delta \left\|\nabla\rho_k'\right\|_2^2 + \delta\left\|\rho_k''\right\|_2^2 \tag{5.27}$$

Substituting (5.27) into (5.24) and integrating over $(0,\ t)$ for all $t \in [0,\ T]$, we obtain

$$\int_\Omega \left|\rho_k''\right|^2 dx + \int_\Omega \left|\Delta\rho_k'\right|^2 dx + \int_\Omega \left|\nabla\rho_k'\right|^2 dx$$
$$\leq \left\|\rho_k''(0)\right\|_2^2 + \left\|\Delta\rho_k'(0)\right\|_2^2 + \left\|\nabla\rho_k'(0)\right\|_2^2 + c_2\int_0^t \left(\left\|\nabla\rho_k'\right\|_2^2 + \left\|\rho_k''\right\|_2^2\right) ds \tag{5.28}$$

It follows from (5.14) and the fact $\left\|\nabla\rho_k'(0)\right\|_2^2 \leq c_3\left\|\Delta\rho_k'(0)\right\|_2^2$ that

$$\left\|\nabla\rho_k'(0)\right\|_2^2 + \left\|\Delta\rho_k'(0)\right\|_2^2 \leq c_4 \tag{5.29}$$

where c_4 is a positive constant independent of k. Multiplying both sides of (5.12) by $\rho_{lk}''(t)$, and then summing over over l from 1 to k and putting $t=0$, we get

$$\left\|\rho_k''(0)\right\|_2^2 + \left(\Delta^2\rho_k(0),\ \rho_k''(0)\right) - \left(\Delta\rho_k(0),\ \rho_k''(0)\right)$$
$$+\left(\left|\rho_k'(0)\right|^{m(x)-2}\rho_k'(0),\ \rho_k''(0)\right) = \left(|\rho_k(0)|^{r(x)-2}\rho_k(0),\ \rho_k''(0)\right)$$

We have from Young's inequality, (5.13) and (5.14) that

$$\left\|\rho_k''(0)\right\|_2 \leq c_5 \tag{5.30}$$

where c_5 is a positive constant independent of k.

By (5.29) and (5.30), (5.28) becomes

$$\int_{\Omega} \left|\rho_k^{''}\right|^2 dx + \int_{\Omega} \left|\Delta\rho_k^{'}\right|^2 dx + \int_{\Omega} \left|\nabla\rho_k^{'}\right|^2 dx$$

$$\leq c_6 + c_7 \int_0^t \left(\left\|\rho_k^{''}\right\|_2^2 + \left\|\Delta\rho_k^{'}\right\|_2^2 + \left\|\nabla\rho_k^{'}\right\|_2^2 \right) ds. \tag{5.31}$$

We gain from (5.31) and Gronwall's lemma that

$$\left\|\rho_k^{''}\right\|_2^2 + \left\|\Delta\rho_k^{'}\right\|_2^2 + \left\|\nabla\rho_k^{'}\right\|_2^2 \leq c_8, \tag{5.32}$$

for all $t \in [0,\ T]$, and c_8 is a positive constant independent of k.

We conclude from (5.32) that

$$\begin{cases} \left\{\rho_k^{'}\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ \left\{\rho_k^{''}\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \end{cases} \tag{5.33}$$

Similarly, we have

$$\begin{cases} \rho_k^{'} \rightharpoonup \rho^{'} \text{ weakly star in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ \rho_k^{''} \rightharpoonup \rho^{''} \text{ weakly star in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \end{cases} \tag{5.34}$$

Setting up $k \longrightarrow \infty$ and passing to the limit in (5.12), we obtain

$$\left(\rho^{''}(t),\ v_l\right) + \left(\Delta^2\rho(t),\ v_l\right) - \left(\Delta\rho(t),\ v_l\right) + \left(\left|\rho^{'}(t)\right|^{m(x)-2}\rho^{'}(t),\ v_l\right)$$

$$= \left(\left|\rho(t)\right|^{r(x)-2}\rho(t),\ v_l\right), \quad l = 1,\ 2, ..., k, \tag{5.35}$$

Since $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$, we deduce that ρ satisfies the equation of (5.1). From (5.20), (5.34) and Lemma 3.1.7 in [64], with $B = H_0^2(\Omega)$ and $L^2(\Omega)$, respectively, we infer that

$$\begin{cases} \rho_k(0) \rightharpoonup \rho(0) \text{ weakly in } H_0^2(\Omega), \\ \rho_k^{'}(0) \rightharpoonup \rho^{'}(0) \text{ weakly in } L^2(\Omega). \end{cases} \tag{5.36}$$

We get from (5.13), (5.14) and (5.36) that $\rho(0) = \rho_0$, $\rho^{'}(0) = \rho_1$.

Thus, the proof of existence is complete.

Uniqueness:

Now it remains to prove uniqueness.

Let ρ^1, ρ^2 be two solutions in the class described in the statement of this theorem, and $w = \rho^1 - \rho^2$. satisfies

$$w_{tt} + \Delta^2 w - \Delta w + \left|\rho_t^1\right|^{m(x)-2} \rho_t^1 - \left|\rho_t^2\right|^{m(x)-2} \rho_t^2 = \left|\rho^1\right|^{r(x)-2} \rho^1 - \left|\rho^2\right|^{r(x)-2} \rho^2 \tag{5.37}$$

and

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x)$$

Multiplying (5.37) by w_t, then integrating with respect to x, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega |w_t|^2 dx + \frac{1}{2}\int_\Omega |\Delta w|^2 dx + \frac{1}{2}\int_\Omega |\nabla w|^2 dx \\ &+ \int_0^t \int_\Omega \left(\left|\rho_t^1\right|^{m(x)-2} \rho_t^1 - \left|\rho_t^2\right|^{m(x)-2} \rho_t^2\right) w_t dx ds \\ &= \int_0^t \int_\Omega \left(\left|\rho^1\right|^{r(x)-2} \rho^1 - \left|\rho^2\right|^{r(x)-2} \rho^2\right) w_t dx ds \end{aligned}$$

By using the inequality

$$\left(|a|^{m(x)-2} a - |b|^{m(x)-2} b\right)(a-b) \geq 0, \forall a, b \in \mathbb{R} \text{ and } a.e.x \in \Omega.$$

This implies

$$\begin{aligned} &\|w_t\|_2^2 dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \\ &\leq C \int_0^t \int_\Omega \left(\left|\rho^1\right|^{r(x)-2} \rho^1 - \left|\rho^2\right|^{r(x)-2} \rho^2\right) w_t dx ds \end{aligned} \tag{5.38}$$

By repeating the estimate as in [39], we arive at

$$\int_\Omega |w_t|^2\,dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \leq C\int_0^t \left(\int_\Omega |w_t|^2\,dx + \|\nabla w\|_2^2\right) ds \quad (5.39)$$

Then

$$\int_\Omega |w_t|^2\,dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \leq C\int_0^t \left(\int_\Omega |w_t|^2\,dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2\right) ds \quad (5.40)$$

Gronwall's inequality yields

$$\|w_t\|_2^2 + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 = 0.$$

Thus, $w = 0$. The shows the uniqueness.

5.3 Global existence

In this section, we state and prove a global existence result.

Theorem 5.2 *Under the assumptions of lemma 5.4, the local solution of* (5.1) *is global.*

Proof. We have

$$\begin{aligned} E(\rho(t)) &= \frac{1}{2}\left(\|\rho_t(t)\|_2^2 + \|\nabla\rho(t)\|_2^2 + \|\Delta\rho(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|\rho(t)|^{r(x)}\,dx. \\ &\geq \frac{1}{2}\|\rho_t(t)\|_2^2 + \frac{r_1-2}{2r_1}\|\nabla\rho(t)\|_2^2 + \frac{r_1-2}{2r_1}\|\Delta\rho(t)\|_2^2. \end{aligned}$$

So that

$$\|\rho_t(t)\|_2^2 + \|\nabla\rho(t)\|_2^2 \leq C\ E(t). \quad (5.41)$$

By Lemma 5.3, we obtain

$$\|\rho_t(t)\|_2^2 + \|\nabla\rho(t)\|_2^2 \leq C\ E(0). \quad (5.42)$$

This implies that the local solution is global in time.

5.4 Stability result

In this section our main result is based a Komornik's inequality [14]

We start our main results by introducing the following lemma:

Lemma 5.5 *Suppose that the assumptions of Lemma 5.4 hold, then, there exists a positive constant c such that*

$$\int_{\Omega} |\rho(t)|^{m(x)}\, dx \leq cE(t).$$

Proof.

$$\begin{aligned}\int_{\Omega} |\rho(t)|^{m(x)}\, dx \leq\ & Max\left\{\|\rho(t)\|_{m(.)}^{m_1},\ \|\rho(t)\|_{m(.)}^{m_2}\right\} \\ &= \alpha\ Max\left\{\|\rho(t)\|_{r(.)}^{m_1},\ \|\rho(t)\|_{r(.)}^{m_2}\right\} \\ &+ (1-\alpha)\, Max\left\{\|\rho(t)\|_{m(.)}^{m_1},\ \|\rho(t)\|_{m(.)}^{m_2}\right\}.\end{aligned}$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^{m(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{m(.)}(\Omega)$, we obtain

$$\begin{aligned}&\int_{\Omega} |\rho(t)|^{m(x)}\, dx \leq \alpha\ Max\left\{\lambda_{1,*}^{m_1}\|\nabla\rho(t)\|_2^{m_1},\ \lambda_{1,*}^{m_2}\|\nabla\rho(t)\|_2^{m_2}\right\} \\ &+ (1-\alpha)\, Max\left\{\lambda_{2,*}^{m_1}\|\Delta\rho(t)\|_2^{m_1},\ \lambda_{2,*}^{m_2}\|\Delta\rho(t)\|_2^{m_2}\right\} \\ &\leq \alpha\ Max\left\{\lambda_{1,*}^{m_1}\|\nabla\rho(t)\|_2^{m_1-2},\ \lambda_{1,*}^{m_2}\|\nabla\rho(t)\|_2^{m_2-2}\right\} \times \|\nabla\rho(t)\|_2^2 \\ &+ (1-\alpha)\, Max\left\{\lambda_{2,*}^{m_1}\|\Delta\rho(t)\|_2^{m_1-2},\ \lambda_{2,*}^{m_2}\|\Delta\rho(t)\|_2^{m_2-2}\right\} \times \|\Delta\rho(t)\|_2^2 \\ &\leq \alpha\ Max\left\{\lambda_{1,*}^{m_1}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_1-2}{2}},\ \lambda_{1,*}^{m_2}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_2-2}{2}}\right\} \times \|\nabla\rho(t)\|_2^2\end{aligned}$$

$$+(1-\alpha) Max\left\{\lambda_{2,*}^{m_1}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_1-2}{2}},\ \lambda_{2,*}^{m_2}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_2-2}{2}}\right\}$$

$$\times \|\Delta\rho(t)\|_2^2$$

$$= c_1\|\nabla\rho(t)\|_2^2 + c_2\|\Delta\rho(t)\|_2^2.$$

By using (5.9), we obtain

$$\int_\Omega |\rho(t)|^{m(x)}\, dx \le cE(t).$$

Theorem 5.3 *Let the assumptions of Lemma 5.4, then, there exists a positive constant* $C>0$, *such that*

$$E(t) \le \frac{C}{(1+t)^{\frac{2}{m_2-2}}},\quad \textit{for all } t\ge 0.$$

Now, we carry out the proof of theorem 5.3.

Proof. Multiplying first equation of (5.1) by $\rho(t)\ E^{\frac{m_2-2}{2}}(t)$ and integrating over $\Omega\times(S,\ T)$, we obtain

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\left[\rho_{tt}(t)+\Delta^2\rho(t)-\Delta\rho(t)+|\rho_t(t)|^{m(x)-2}\rho_t(t)\right]dxdt$$

$$=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\rho(t)|^{r(x)}\,dxdt.$$

So that

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega\left[(\rho(t)\rho_t(t))_t-|\rho_t(t)|^2+|\Delta\rho(t)|^2+|\nabla\rho(t)|^2\right.$$

$$\left.+\rho(t)|\rho_t(t)|^{m(x)-2}\rho_t\right]dxdt=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega|\rho(t)|^{r(x)}\,dxdt.$$

We add and substract the term

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\theta_1 |\nabla\rho(t)|^2+\theta_2|\Delta\rho(t)|^2+(2+\theta_1+\theta_2)|\rho_t(t)|^2\right]dxdt,$$

and use (5.10), to get

$$\begin{aligned}
&(1-\theta_1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[|\nabla\rho(t)|^2+|\rho_t(t)|^2\right]dxdt\\
&+(1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[|\Delta\rho(t)|^2+|\rho_t(t)|^2\right]dxdt\\
&+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[(\rho(t)\rho_t(t))_t-(3-\theta_1-\theta_2)|\rho_t(t)|^2\right]\\
&+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)|\rho_t(t)|^{m(x)-2}\rho_t(t)\,dxdt\\
&=-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\theta_1|\nabla\rho(t)|^2+\theta_2|\Delta\rho(t)|^2-|\rho(t)|^{r(x)}\right]dxdt\leq 0.
\end{aligned}\tag{5.43}$$

It is clear that

$$\begin{aligned}
&\gamma\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\nabla\rho(t)|^2+\frac{1}{2}|\Delta\rho(t)|^2+\frac{|\rho_t(t)|^2}{2}-\frac{|\rho(t)|^{r(x)}}{r(x)}\right]dxdt\\
&\leq(1-\theta_1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\nabla\rho(t)|^2+\frac{|\rho_t(t)|^2}{2}\right]dxdt\\
&+(1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\Delta\rho(t)|^2+\frac{|\rho_t(t)|^2}{2}\right]dxdt,
\end{aligned}\tag{5.44}$$

where $\gamma = Min\left((1-\theta_1),\ (1-\theta_2)\right)$. By (5.43), (5.44) and definition of $E(t)$, we get

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left(\rho(t)\rho_t(t)\right)_t dxdt$$
$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\left|\rho_t(t)\right|^{m(x)-2}\rho_t(t)\,dxdt$$
$$+(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left|\rho_t(t)\right|^2 dxdt. \tag{5.45}$$

Using the definition of $E(t)$ and the following expression

$$\frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\rho_t(t)\,dx\right) = E^{\frac{m_2-2}{2}}(t)\int_\Omega \left(\rho(t)\rho_t(t)\right)_t dx$$
$$+\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega \rho(t)\rho_t(t)\,dxdt. \tag{5.46}$$

Then, inequality (5.45), becomes

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le -\int_S^T \frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\rho_t(t)\,dx\right)dt$$
$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\left|\rho_t(t)\right|^{m(x)-2}\rho_t(t)\,dxdt$$
$$+\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega \rho(t)\rho_t(t)\,dxdt$$
$$+(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left|\rho_t(t)\right|^2 dxdt. \tag{5.47}$$

We estimate the terms in the right-hand side of (5.47) as follow:

For the first term, we have

$$
\begin{aligned}
&-\int_S^T \frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\,\rho_t(t)\,dx\right)dxdt \\
&\le \left|E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(S)\,\rho_t(S)\,dx - E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(T)\,\rho_t(T)\,dx\right| \\
&\le E^{\frac{m_2-2}{2}}(t)\left|\int_\Omega \rho(x,S)\,\rho_t(x,S)\,dx\right| + E^{\frac{m_2-2}{2}}(t)\left|\int_\Omega \rho(x,T)\,\rho_t(x,T)\,dx\right| \\
&\le cE^{\frac{m_2}{2}}(S) + cE^{\frac{m_2}{2}}(T) \le cE^{\frac{m_2-2}{2}}(0)\,E(S) \\
&\le cE(S).
\end{aligned} \tag{5.48}
$$

For the second term, we use the following Young inequality:

$$
XY \le \frac{\varepsilon}{\lambda_1}X^{\lambda_1} + \frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ \ X,\ Y\ge 0,\ \ \varepsilon>0 \text{ and } \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1.
$$

with $\lambda_1(x) = m(x)$, $\lambda_2(x) = \frac{m(x)}{m(x)-1}$. By Lemma 5.3 and Lemma 5.5, we obtain

$$
\begin{aligned}
&-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \rho(t)\,|\rho_t(t)|^{m(x)-2}\,\rho_t(t)\,dxdt \\
&\le \int_S^T E^{\frac{m_2-2}{2}}(t)\left(\varepsilon c\int_\Omega |\rho(t)|^{m(x)}\,dx + c_\varepsilon\int_\Omega |\rho_t(t)|^{m(x)}\,dx\right)dt \\
&\le \varepsilon c\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\rho(t)|^{m(x)}\,dxdt + c_\varepsilon\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)dt \\
&\le \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S).
\end{aligned} \tag{5.49}
$$

By Young's, Poincare's inequalities and (5.41), we obtain

$$\begin{aligned}
&\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega \rho(t)\rho_t(t)\,dxdt\\
&\le \frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\left(-E'(t)\right)\int_\Omega\left(\frac{1}{2}|\rho(t)|^2+\frac{1}{2}|\rho_t(t)|^2\right)dxdt\\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)dt\\
&\le cE^{\frac{m_2}{2}}(S)-E^{\frac{m_2}{2}}(T)\\
&\le cE^{\frac{m_2}{2}}(0)E(S)\le cE(S)
\end{aligned}\tag{5.50}$$

For the last term of (5.47), we have

$$\begin{aligned}
&(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\rho_t(t)|^2\,dxdt\\
&\le (3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\int_{\Omega^-}|\rho_t(t)|^2\,dx+\int_{\Omega^+}|\rho_t(t)|^2\,dx\right]dt\\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\left(\int_{\Omega^-}|\rho_t(t)|^{m_2}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega^+}|\rho_t(t)|^{m_1}\,dx\right)^{\frac{2}{m_1}}\right]dt\\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\left(\int_{\Omega}|\rho_t(t)|^{m(x)}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega}|\rho_t(t)|^{m(x)}\,dx\right)^{\frac{2}{m_1}}\right]dt.
\end{aligned}$$

This implies

$$\begin{aligned}
&(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\rho_t(t)|^2\,dxdt\\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt+c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt.
\end{aligned}\tag{5.51}$$

First, we use Young's inequality with $\lambda_1 = m_2/(m_2-2)$ and $\lambda_2 = m_2/2$, we have

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}} dt \leq \varepsilon c \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon \int_S^T \left(-E'(t)\right) dt.$$

This implies

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}} dt \leq \varepsilon c \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon E(S). \tag{5.52}$$

On the other hand, we use the Young's inequality $\lambda_1 = \frac{m_1}{m_1-2}$ and $\lambda_2 = \frac{m_1}{2}$, to obtain

$$\begin{aligned}\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}} dt &\leq \varepsilon c \int_S^T E^{\frac{m_1(m_2-2)}{2(m_1-2)}}(t)\, dt + c_\varepsilon \int_S^T \left(-E'(t)\right) dt \\ &\leq \varepsilon c \int_S^T E^{\frac{m_1(m_2-2)}{2(m_1-2)}}(t)\, dt + c_\varepsilon E(S).\end{aligned}$$

We notice that $\frac{m_1(m_2-2)}{2(m_1-2)} = \frac{m_2}{2} + \frac{m_2-m_1}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}} dt &\leq \varepsilon c \left(E(S)\right)^{\frac{m_2-m_1}{m_1-2}} \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon E(S) \\ &\leq \varepsilon c \left(E(0)\right)^{\frac{m_2-m_1}{m_1-2}} \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon E(S) \\ &\leq \varepsilon c \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon E(S).\end{aligned} \tag{5.53}$$

We substituting (5.52) and (5.53) in (5.51), we obtain

$$(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\rho_t(t)|^2\, dx dt \leq \varepsilon c \int_S^T E^{\frac{m_2}{2}}(t)\, dt + c_\varepsilon E(S). \tag{5.54}$$

By insert (5.48), (5.49), (5.50) and (5.53) in (5.47), we arrive at

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \leq \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m_2}{2}}(t)\,dt \leq cE(S).$$

By taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m_2}{2}}(t)\,dt \leq cE(S).$$

By Komornik's integral inequality 5.2 yields the result.

5.5 Numerical example

In this section, we present an application to illustrate numerically the stability result of Theorem 3.3. For this purpose, we numerically solve problem (5.1), for $n = 2$ where the domain is taken to be $\Omega = [-1,\ 1]^2$. We chosen $\rho_0(x_1, x_2) = (x_1 + 1)(x_1 - 1)(x_2 + 1)(x_2 - 1)$ and $\rho_1(x_1, x_2) = 0$, where will be chosen such that $E(0) > 0$, we take the exponent function $r(x_1, x_2) = 4$, and $m(x_1, x_2) = {x_1}^2 + {x_2}^2 + 2.5$ which satisfy condition (1.3), where $m_2 = 4.5$. We numerically verify that

$$E(t) \leq C(1+t)^{-0.8}.$$

5.5.1 Numerical method

We first introduce a suitable numerical scheme to discretize (5.1) using finite differences for the time variable $t \in [0, T]$ and the space variable

$x = (x_1, x_2) \in \Omega$. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $\quad t_n = n\ \delta t, \quad n = 1,\ 2, ..., N+1$, where δt is the time step.

Let $\rho^n(x_1, x_2) = \rho(x_1, x_2,\ t_n)$, and use the finite-difference formulas: the first-order backward difference for

$$\partial_t \rho^n(x_1, x_2) = \frac{\rho^n(x_1, x_2) - \rho^{n-1}(x_1, x_2)}{\delta t}.$$

and the second-order center difference for

$$\partial_{tt} \rho^n(x_1, x_2) = \frac{\rho^{n+1}(x_1, x_2) - 2\rho^n(x_1, x_2) + \rho^{n-1}(x_1, x_2)}{(\delta t)^2}.$$

Then the time discrete problem of (5.1) reads: Given u_0 and ρ_1, find $\{\rho^2,\ \rho^3, ..., \rho^{n+1}\}$ such that

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - |\partial_t \rho^n|^{m(x_1,x_2)-2}\,\partial_t \rho^n & \\ -\Delta^2 \rho^n + |\rho^n|^{r(x_1,x_2)-2}\,\rho^n, & \text{in } \Omega \\ \rho^{n+1} = 0, & \text{on } \partial\Omega \\ \rho^0 = \rho_0(x_1, x_2), \quad \rho^1 = \rho^0 + (\delta t)\,\rho_1(x_1, x_2), & \text{in } \Omega \end{cases} \tag{5.55}$$

Note that the above problem is linear in ρ^{n+1}, which is achieved by using the history data ρ^n and ρ^{n-1} in the second side of the equation. Problem(5.55) is solved iteratively as for given regular ρ^n, the solution ρ^{n+1} satisfies the boundary-value problem:

$$\begin{cases} \frac{\rho^{n+1}}{(\delta t)^2} - \Delta\rho^{n+1} = F(\rho^n,\ \rho^{n-1}), & \text{in } \Omega_h \\ \rho^{n+1} = 0, & \text{on } \partial\Omega_h \end{cases} \tag{5.56}$$

where $F(\rho^n,\ \rho^{n-1}) = \frac{2\rho^n - \rho^{n-1}}{(\delta t)^2} - \Delta^2\rho^n - |\partial_t \rho^n|^{m(x_1,x_2)-2}\,\partial_t\rho^n + |\rho^n|^{r(x_1,x_2)-2}\,\rho^n$.

5.5.2 Numerical results

In this subsection, we present and discuss the stability results of the numerical scheme(5.55). The numerical results are obtained using the Matlab codes.

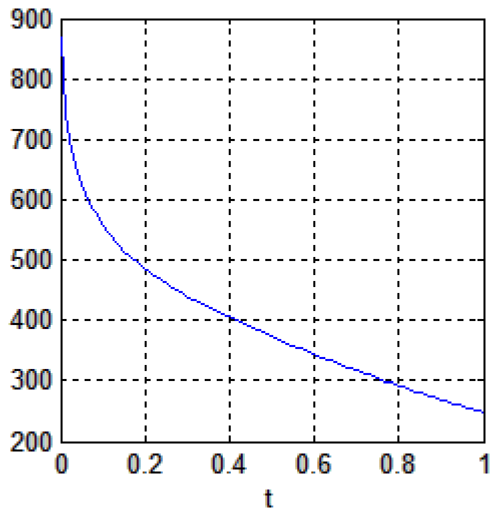

Figure 5.1: Energy: E(t)

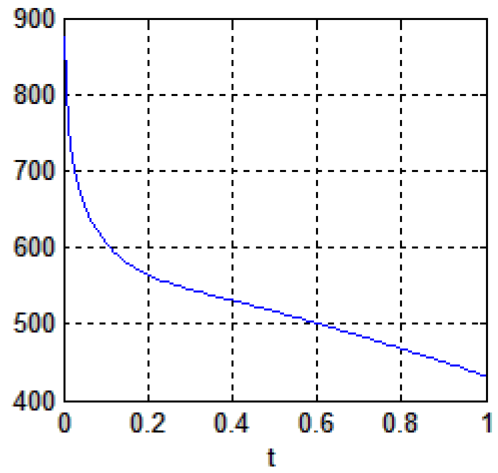

Figure 5.2: Polynomial decay:z=E(t)(1+t)$^{0.8}$

The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 100.
- Time step is: $\delta t = 0.01$.

Figures. 1 and 2 presents the energy $E(t)$ and $E(t)(1+t)^{0.8}$ respectively for the times $t_n \in \{1,\ 2, ..., 100\}$. The numerical solutions of problem (5.1) make the energy function $E(t)$ satisfy

$$E(t)(1+t)^{0.8} \leq 9 \times 10^2.$$

In conclusion, the above numerical application verifies and agrees with the stability results of Theorem 3.3.

CONCLUSION AND PERSPECTIVE

In this thesis, we have solved many problems related to differential equations of different orders. We tackled both lower-order and higher-order equations using various methods. Specifically, we utilized the concavity method to solve problems involving higher-order wave equations, considering cases where the source terms, once with variable exponent and once with logarithmic variable exponent, along with the total energy technique related the equation. We observed that the solution blow up in finite time in both cases. As for the remaining problems, we introduced the Nihari spaces as well as the total energy technique associated to the equation, which led to stabilizing the solution either based on the stability of the solution in terms of the decay exponential of the total energy or based on the stability of the solution in terms of the decay polynomial of the energy total of the problem. All that comes to mind for future work is attempting to study the same previous problems using alternative methods, for instance, the method of semi-groups, which is considered an additional method due to some potentially advantageous features for dealing with such cases.

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