Algerian Democratic and Popular Republic
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

University 20 august 1955-Skikda
Faculty of Sciences
Department of Mathematics
Ref:………………

جامعة 20أوت – 1955سكيكدة
كلية العلوم
قسم الرياضيات
المرجع:………..

Thesis

A view to obtaining the diploma of

Doctorate of 3° cycle (LMD) in Mathematics

Option: ***Mathematics***

Theoretical and numerical study of stochastic Keller-Segel problem

Presented by:

Slimani Ali

Publicly discussed: .. /../….

In front of the Jury:

1.	**Khemis Rabah**	**MCA**	**University 20 August 1955, Skikda**	**President**
2.	**Ouaoua Amar**	**MCA**	**University 20 August 1955, Skikda**	**Examiner**
3.	**Ellagoune Fateh**	**Professor**	**University 8 May 1945, Guelma**	**Examiner**
4.	**Zennir Khaled**	**Professor**	**Qassim University Saudi Arabia**	**Examiner**
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University year: 2022/2023

République Algérienne Démocratique et Populaire
وزارة التعليم العالي والبحث العلمي
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université20 Août 1955 -Skikda
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جامعة 20أوت – 1955سكيكدة
كلية العلوم
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Thèse

En vue de l'obtention du diplôme de

Doctorat de 3° cycle (LMD) en Mathématiques
Option : ***Mathématique***

Etude théorique et numérique le problème de keller-Segel stochastique

Présentée par :

Slimani Ali

Soutenue publiquement : .. /../….

Devant les Jury :

1. Khemis Rabah	**MCA, Université 20 Août 1955, Skikda**	**Président**
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Année Universitaire: 2022/2023

Thesis *for the fulfilment of the requirements of Doctorate of 3rd cycle degree in Mathematics*
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Supervisor: Professor. G. AMAR

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UNIVERSITY OF 20 AUGUST 1955-SKIKDA, 2023.

بسم الله الرحمن الرحيم

Dedications

With the help of God Almighty and the success of the Almighty, the completion of this research. And I present the dedication first to my parents who are my role models in life, the owners of the first and last credit after the grace of God in achieving this achievement through their support and encouragement for my education. To those who were the best support for me (My brothers, sisters and family) My honorable professors, gentlemen, supervisors, and members of the faculty who are stingy for you one day. In giving information and any valuable idea My colleagues and comrades on the path and those who never stingy to help me. To everyone who contributed and helped me in my academic life.....

To all of the above: I dedicate this my doctoral dissertation, which I ask God Almighty to accept as useful knowledge.

♡

Acknowledgements

In the first of all, praise and thanks be to Allah, the Almighty, for his showers of blessings throughout my research work to carry out the research. I am extremely grateful to my parents for their love, prayers, care, and sacrifices to educate me and prepare me for my future. A special thank you to my parents for their love, understanding, prayers and continued support in carrying out this research work. This work was carried out at the Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), Department of Mathematics, University of August 20, 1955 of Skikda under the direction of Professor A Guesmia and Doctor L. Bouzettouta. At the end of this work, I would like to express my deep gratitude to our dear professor and my supervisor Professor A. Guesmia for his follow-up and for his enormous support, which he never ceased to lavish on us throughout the project period. I would also like to thank my co-supervisor Doctor L. Bouzettouta for the time he devoted and for the valuable information he provided me with interest and understanding. I also address my sincere thanks to the members of the juries Docter R. Khemis, Doctor A. Ouaoua, Professor F. Ellagoune and Professor Kh. Zennir" for kindly reviewing and judging this work. My thanks go to all the staff I contacted during my internship at the University of Cadiz Professor Rafa Rodriguez Galvan and Professor Francisco Ortegon Gallego, with whom I found the warm welcome, help and assistance I need. I will not let this opportunity pass, without thanking all the teachers and staff of the University August 20, 1955, skikda, and particularly those of the mathematics section and Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), and in general Faculty of Sciences, University August 20, 1955, skikda for their help and valuable advice and for their interest in my training. Finally, my thanks to all those who have contributed directly or indirectly to the success of this project.

♡

ABSTRACT

In this thesis, we use a system of nonlinear PDEs, or the conventional d-dimensional parabolic-parabolic equation, to explain the Keller-Segel chemotaxis model. These PDEs include a convection-diffusion equation for the cell density and a reaction-diffusion equation for the chemoattractant concentration. The Keller-Segel chemotaxis model explains how the density of a cell population and the concentration of an attractant change over time. This thesis uses a variety of approaches and strategies to investigate the parabolic Keller-Segel equations. In the first, we talk about the biological and mathematical modeling of the phenomenon of chemical entrapment, and we create a non-linear fractional stochastic Keller-Segel model, where we demonstrate the existence and uniqueness and regularity properties of the mild solution to the investigated time- and space-fractional problem and the required results under specific presumptions. We also studied a stochastic chemotaxis Keller-Segel model perturbed with a Gaussian process, where we proved the local and global existence of solutions in time for a nonlinear stochastic Keller-Segel model with zero Dirichlet boundary conditions, and we also studied the phenomenon of the Keller-Segel model coupled with Boussinesq equations. The primary goals of this work are to investigate the global existence and uniqueness of a weak solution of the problem using the Galerkin method. Finally, we studied the numerical solution of one-dimensional Keller-Segel equations via the new homotopy perturbation method.

Keywords: Keller-Segel PDEs, chemotaxis models, stochastic Keller-Segel model, fractional derivative, mild solution, regularity properties, homotopy perturbation method (HPM), systems of partial differential equations, Numerical solution.

RÉSUMÉ

Dans cette thèse, nous utilisons un système d'EDPs non linéaires, ou l'équation parabolique-parabolique d-dimensionnelle conventionnelle pour expliquer le modèle de chimiotaxie de Keller-Segel. Ces PDEs comprennent une l'équation de convection-diffusion pour la densité cellulaire et une équation de réaction-diffusion pour la concentration de chimioattractant. Le modèle de chimiotaxie de Keller-Segel explique comment la densité d'une population cellulaire et la concentration d'un attractif changent avec le temps. Cette thèse utilise une variété d'approches et de stratégies pour étudier les équations paraboliques-paraboliques de Keller-Segel. Dans la première, nous parlons de la modélisation biologique et mathématique du phénomène de piégeage chimique, et nous créons un modèle Keller-Segel stochastique fractionnaire non linéaire, où nous démontrons l'existence et l'unicité de la solution douce aux problèmes de temps et d'espace étudiés, les résultats requis doivent, sous des présomptions spécifiques. Nous avons également étudié un modèle de chimiotaxie stochastique de Keller-Segel perturbé par un processus gaussien, oú nous avons prouvé l'existence locale et globale de solution en temps pour un modèle non linéaire de Keller-Segel stochastique avec des conditions aux limites de Dirichlet nulles, et nous avons également étudié le phénomène du modèle de Keller-Segel couplé aux équations de Boussinesq. Les principaux objectifs de ce travail sont d'étudier l'existence et l'unicité de la solution faible de problème, en utilisant la méthode Galerkin. Enfin, nous étudions la résolution numérique d'équations unidimensionnelles de Keller-Segel via la nouvelle méthode de perturbation d'homotopie.

Mots-clé: EDPs de Keller-Segel, modèles de chimiotaxie, modèle stochastique de Keller-Segel, dérivée fractionnaire, solution douce, propriétés de régularité, méthode de perturbation homotopie, systèmes d'équations aux dérivées partielles, Solution numérique.

ملخص

في هذه الأطروحة، نستخدم نظام PDEs غير الخطية، أو المعادلات التقليدية المكافئة-المكافئة ذات الأبعاد d لشرح نموذج كيلر-سيجل الكيميائي. تشتمل أجهزة PDEs على معادلة انتشار الحمل لكثافة الخلية ومعادلة التفاعل والانتشار لتركيز الجاذب الكيميائي. يوضح نموذج كيلر-سيجل الكيميائي كيف تتغير كثافة مجموعة الخلايا وتركيز الجاذب بمرور الوقت. تستخدم هذه الأطروحة مجموعة متنوعة من الأساليب والاستراتيجيات للتحقيق في معادلات كيلر-سيجل المكافئة.

في البداية، نتحدث عن النمذجة البيولوجية والرياضية لظاهرة الانحباس الكيميائي، بعدها نقوم بإنشاء نموذج عشوائي غير خطي كيلر-سيجل الكسري، حيث نظهر وجود وخصائص وتفرد وانتظام الحل المعتدل الذي تم فحصه للمشكلة في الزمان الكسري والمكان الكسري، نقوم بدراسة النتائج المطلوبة في ظل افتراضات محددة. كما درسنا أيضًا نموذج المحور الكيميائي العشوائي كيلر-سيجل المضطرب بعملية غاوس، حيث أثبتنا الوجود المحلي والعالمي للحل في الوقت لنموذج كيلر-سيجل العشوائي غير الخطي وبشروط حدود ديريكلي الصفرية، ودرسنا أيضًا ظاهرة نموذج كيلر سيجل مقرونًا بمعادلات بوسينيسك، حيث تتمثل الأهداف الأساسية لهذا العمل في التحقق من وجود الحل الضعيف وتفرده للمشكلة باستخدام طريقة غالوركينك. أخيرًا، درسنا الحل العددي لمعادلة كيلر-سيجل أحادية البعد بطريقة اضطراب التماثل الجديدة.

الكلمات المفتاحية: معادلة كيلر سجل، نموذج المحور الكيميائي، نموذج كيلر سجل العشوائي، المشتق الجزئي، الحل المعتدل، خواص الانتظام، طريقة اضطراب التماثل، نظام المعادلات التفاضلية الجزئية، الحل العددي.

Notations

- $\rightarrow$ designates the strong convergence.
- $\rightharpoonup$ indicates the weak convergence.
- $\hookrightarrow$ indicates the continuous embedding.
- ∇ stands for the gradient operator.
- div is the divergence operator.
- $\dfrac{\partial}{\partial x}$ partial derivative.
- $\dfrac{\partial}{\partial n}$ outward normal derivative.
- Δ_p is the p-Laplace operator.
- Δ_p^{-1} is the p-Laplace inverse operator.
- Δ^s is the fractional Laplace operator of order s.
- $\Gamma(z)$ is the Gamma function
- $\beta(z, \omega)$ is the Beta function
- E_α is the Metage-Leffler function(one parameter)
- $E_{\alpha,\beta}$ is the Metage-Leffler function(two parameters)
- $\mathcal{L}$ is the Laplace transform
- $\mathbb{F}$ is the Fourier transform
- $\mathcal{H}^\alpha(\mathbb{R}^\mathbb{N})$ is the homogeneous Sobolev space of order α

- $\mathcal{S}'$ is the Schwartz space
- M_α is the Mainardi function
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a space of probability
- $f_X(x)$ is called probability density of variable X
- $\mathbb{E}(X)$ is expectation
- $(\mathcal{F}_t)_{t\geq 0}$ is a filtration
- F^X is the natural filtration
- $X = (X_t)_{t\in T}$ is a stochastic process
- $(W_t)_{t\in T}$ is the Standard Brownian movement
- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}^n$ is the real space of dimension n.
- $\Omega \subset \mathbb{R}^n$ open set in $\mathbb{R}^n$.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.
- Ω^c the complement of Ω.
- $< ., . >$ denotes the scalar product.
- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.
- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

- $L^p_{loc}(\Omega)$ the space of local p-integrable functions on Ω.

- $W^{m,p}(\Omega)$ Sobolev space with norm $\|\cdot\|_{m,p}$.

- $W^{m,p}_{loc}(\Omega)$ the local Sobolev space.

- $W^{m,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

- $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $W^{s,p}(\Omega)$ fractional Sobolev space with norm $\|\cdot\|_{s,p}$.

- $W^{s,p}_0(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|.\|_{W^{s,p}_0(\Omega)}$.

- $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, $W^{s,2}_0(\mathbb{R}^n) = H^s_0(\mathbb{R}^n)$.

CONTENTS

GENERAL INTRODUCTION

An overview of mathematics and its applications

In the past decades mathematical thinking has been intensively applied on natural life sciences, especially in the field of ecology, physical processes in nature and many biological phenomena in general. The common goal is to map observable features of the real physical and biological processes to an abstract mathematical model and a corresponding discrete numerical formulation in order to gain new insights in the underlying real world objectives by means of reasonable simulations. Moreover, in several cases the mathematical description of the real world system is the only possibility to provide reliable predictive analysis for the underlying process of nature, which results in templates for, e.g., industrial or medical purposes. Many of those mathematical models are described by a (system of) partial differential equation(s) (PDEs). Well established laws of nature and their mathematical counterparts have led to most of the development of suitable PDEs, e.g., heat conduction, fluid dynamics or deformation of solids. However, sometimes the development and understanding of a particular mathematical model can only be tackled by a 'trial-and-error' approach, which is a two-step procedure. In a first step, the simulated results are compared to experimental data and in a second step, these comparisons are used to modify the mathematical model, i.e., the underlying PDE. Hence, simulations of PDEs are of tremendous importance when trying to understand real world processes.

With the recent advances in experimental biology, e.g., in live imaging, scientist are now in a promising position to examine biological processes in unprecedented detail. Although the quality of measurements in experimental biology is unfortunately still not as well developed compared to the precision of measuring tools in the field of (mechanical) engineering, the experimental investigation of biological processes lately experiences a huge wealth of breaking assets. In the course of these biological accomplishments, the advent and development of mathematical biology as a novel interdisciplinary research branch emerged and provides a new perspective on biological phenomena. The mathematical formulation of biological processes and their precise analysis allows biological experimentalists to verify results retrospectively and develop prospective conjectures. However, a pure theoretical analysis of mathematical models is crucially limited. Particularly for recent models that describe multi-dimensional signaling pathways incorporating several entities, their complexity cannot be fully captured and analyzed with tools provided by theoretical analysts. The urgent need of a detailed study of complex models drove numerical analysts to consider biologically motivated systems, the foundation of computational biology as a particular discipline of mathematical biology. The tremendous potential of these interdisciplinary research branches in today's science is nicely formulated in Cohen's essay mathematics is biology's next microscope, only better biology is mathematics' next physics, only better [19].

The mathematical theory of nonlinear partial differential equations (PDEs) describing various physical phenomena forms the basis of the study for many models in applied mathematics and mathematical physics. Theoretical study of PDEs starts with the wellposedness of the equations. That includes the existence of a solution to the equation, the uniqueness of the solution, and the continuous dependence on the data. These play a vital role in mathematics and has crucial effects in the real world.

History study of the phenomena of chemotaxis

Although migration of cells was detected from the early days of the development of microscopy (Leeuwenhoek), erudite description of chemotaxis was first made by T.W. Engelmann (1881) and W.F. Pfeffer (1884) in bacteria and H.S. Jennings (1906) in ciliates. The Nobel Prize Laureate E. Metchnikoff also contributed to the study of the field with investigations of the process as an initial step of phagocytosis. The significance of chemotaxis in biology and clinical pathology was widely accepted in the 1930s. The most fundamental definitions belonging to the phenomenon were also drafted by this time. The most important aspects in quality control of chemotaxis assays were described by H. Harris in the 1950s. In the 1960s and 1970s, the revolution of modern cell biology and biochemistry provided a series of novel techniques which became available to investigate the migratory responder cells and subcellular fractions responsible for chemotactic activity. The pioneering works of J. Adler represented a significant turning point in understanding the whole process of intracellular signal transduction of bacteria. Chemotaxis is one of the most basic cell physiological responses. Development of receptor systems for the detection of harmful and favorable substances in the environment was most essential to unicellular organisms from the very early stages of phylogeny. Comprehensive analysis of chemotactic activity of the eukaryotic protozoan tetrahymena pyriformis and consensus sequences of appearance of amino acids in the primordial soup suggest that there was a good correlation between the chemotactic character of these relative simple organic molecules and their development on the earth. In this way the earliest molecules are suggested to be highly chemoattractant (e.g. Gly, Glu, Pro), while latter ones are thought to be strongly chemorepellent (e.g. Tyr, Trp, Phe) amino acids.

Our biological motivations and biochemical concept of phenomena of chemotaxis

Let us begin with the literal translation and the description of the underlying biochemical process. The word chemotaxis is originally deduced from the Greek for in order to give meaning to the notion of chemo-taxis, we will start from the suffix taxis (pl. taxes), an ancient Greek word for arrangement. Taxis represents oriented movement of a motile organism in response to a stimulus (e.g. light, temperature, food). The movement can be directed towards or away from the stimulus.

Thus chemotaxis describes the phenomenon of directing the migration according to a gradient of some chemical substance. The literature differentiates positive and negative chemotaxis, which simply refers to the direction upward or downward the chemical gradient. In these cases, the chemical itself is called chemoattractant or chemorepellent, respectively. The character of the reaction on the chemical gradient is called chemosensitivity, namely a large or small chemosensitivity allows a rapid or slow reaction in terms of migration. The ability of organisms to sense and direct their motion towards (or away from) a chemical gradient is an essential property. In the first case, we have positive taxis and in the later negative taxis. It is important to emphasize that only the motile organisms are capable of performing such movements.

Motile essentially means able to move by itself. For example, bacteria cells use structures called flagella to enable these movements. Taxes should not be confused with tropism and kinesis. These are another classes of movements in response to a stimulus. The first one represents the movements that include growth towards or away from the stimulus. The difference is that in taxes the organism must have motility and the exhibited movement is not growth, but rather a guided change of position. On the other hand, in kinesis, the presence of stimulus influences the changes of velocity of the organism, but not its direction in movement.

Taxes are also classified by the type of stimulus governing them, which is indicated by a prefix. Photo-taxis is governed by light, thermo-taxis by temperature. If the presence of oxygen triggers the movements, we have aero-taxis. Finally, a chemical stimulus is responsible for chemo-taxis. Since the end of 17th century and Leeuwenhoek's advances in the field of microbiology, scientists have been studying the movements of organisms. However, bacterial chemotaxis was discovered two centuries after by Engelmann [32] and Pfeffer [74, 73]. By Pfeffer's original definition, chemotaxis is defined as anything that causes the oriented movement of an organism or a cell relative to a chemical gradient. In his work, Pfeffer also gave the basis for assays on how to detect chemotaxis, i.e. the capillary method [74, 73]. Chemical that prompts positive chemotaxis was called the chemo-attractant, while chemical that causes the organism to flee away from the source was called chemo-repellent. Chemo-attractants usually represent favourable environment for the organism, e.g. food, while the chemo-repellents are noxious substances, such as poisons. One interesting consequence of positive chemotaxis is cell aggregation. The chemo-attractants produced by the fellow species increase self-attraction among the population and further stimulate cell aggregation [13].

Exemplary let us provide three paradigms in which chemotaxis plays a vitally important role. First of all, in the stage of early development of higher organisms, e.g., mammals, chemotaxis allows the mobilization and organization of stem cells that eventually leads to differentiation into, e.g., highly specialized bone, neuronal or blood cells [21, 33, 61]. A second common example is the detection and localization of food sources or prey recognition. For instance, nutrients or prey serve as a source of chemical signals (either directly or indirectly via production/secretion of chemoattractants) for simplex life forms like, e.g., bacteria, slime molds or nematodes [15, 46, **?**]. The last example of how chemotaxis provides a necessary ability for organisms to react in their environment is the immune system [53, 65]. Let us focus on the human immune system. Once an inflammation arises, our immune system counters this invasion of toxic substances or harming bacteria by releasing leukocytes. The path that leukocytes take to localize the site of infection is determined by traces of chemokines that have been released by resident cells at the affected tissue. These chemokines act as attractive chemicals, hence guiding the leukocytes to the origin of the inflammation.

Biologically, chemotaxis is a process which involves a complex network of intracellular chemical signaling pathways that is activated by chemical-receptor bindings. We recommend the corresponding chapter in the book of Alberts et al. [7] [Chapter 15] for a detailed reference for chemical signaling pathways. Here, we will only briefly recapitulate the main concept of chemotaxis.

The sensory chemical (ligand) receptors that activate the complex signaling cascade mainly happen to be located at the cell's membrane (due to hydrophilic chemical molecules). If these receptors are active and bind a corresponding (extracellular) ligand, they allow an intracellular signaling cascade to be activated, see Figure (1). The detailed mechanism of which regulative entities are exactly involved and how they interact with each other are highly depending on the underlying organism. Even for bacterial chemotaxis this has only marginally been explored up to the present. In [85] we read:

> "Of the estimated many millions of bacterial species which are assumed to exist in nature, less than 100 have been studied in detail".

The common resulting effect of these different signaling pathways is the simple (re-)mobilization of a motor for the motility, e.g., adjustment of the flagella rotation in bacteria. It was observed that a clockwise (CW) rotation of the flagella motor causes the flagella to fly apart, whereas a counterclockwise (CCW) rotation results in a bundling of the flagella. Afterwards, the receptors adapt to the new extracellular concentration of ligands (e.g., by temporal methylation of the receptor) in order to allow further gradient detections. (1) depicts this process very roughly.

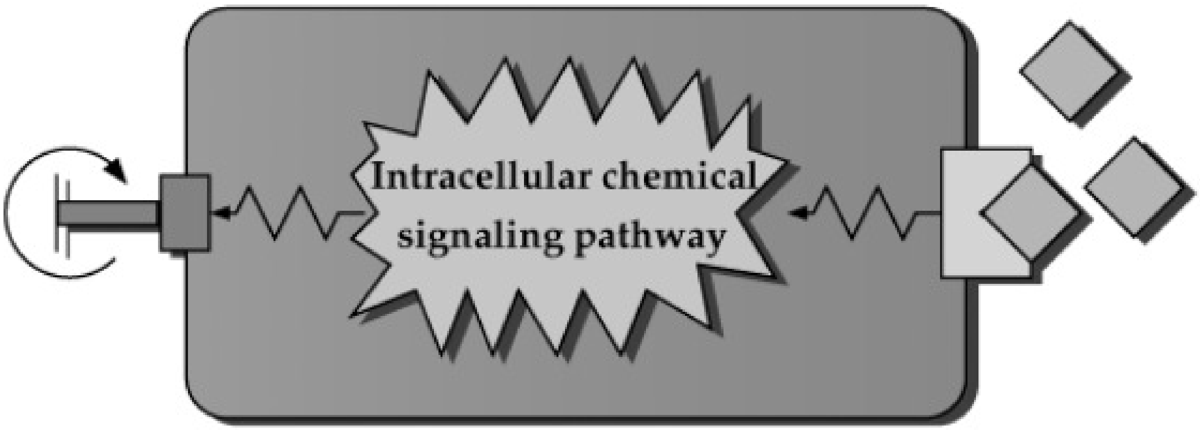


Figure 1: Intracellular chemical signaling pathway.

The figure (1): A rough sketch of a chemotaxis-induced signaling pathway of bacteria. The chemical ligands on the right activate the receptor, which initiates an intracellular signaling cascade. The result of this signaling pathway is the control of the flagella motor switching (from CW to CCW), which is depicted on the left. switching in the flagella motor rotation characterizes the bacterial motion. We can consider the bacterial chemotaxis-biased random walk to be split into two states. In one state the bacteria tumbles due to CW rotation of its flagella. In this state the bacteria re-orientates by selecting a random new walking direction. In a second state a counter-clockwise rotation of the flagella drives the bacteria to run in the previously selected direction. The result is a so-called run-andtumble walk, where the time between two turns (two tumbling phases) depends on the detected gradient of the chemical. A schematic sketch of this mechanism is depicted in figure (1). A short remark regarding the detection of chemical gradients seems indicated. There is a notable differentiation how chemical gradients can be identified during the processing of the signaling pathways. Because of the simple fact of their extremely small size, bacteria sense chemical gradients in a different manner than larger organisms, e.g., the experimentally well investigated slime mold Dictyostelium discoideum. While slime molds (some mm in size) can measure the gradient directly by sensing the non-uniformly distributed active ligand-receptor bindings along the membrane, bacteria (of only a few $\mu\, m$ in size) calculate the gradient by comparing the concentration along a walking path, since a non-uniform chemical concentration on the bacterial scale is already perturbed by the noise of the ubiquitous Brownian motion and hence cannot be detected directly.

The study of the phenomena of chemotaxis may be divided into two periods: before 1960's and after. As mentioned in [3], the work before 1960's was carried out in complex media and was of a quite subjective nature. The review of this period is given in [16, 89, 95]. In the second period, the first priority was to develop conditions for obtaining motility and chemotaxis in defined media [4, 5, 6]. Then it was important to find quantitative methods that objectively detect chemotaxis [87, 6]. This work, mostly by

Adler, altered the attention from phenomenological to quantitative research and initiated studies to reveal the molecular mechanism of bacterial chemotaxis. Afterwards, the number of groups studying bacterial chemotaxis has been continuously rising. Bacterial motility and chemotaxis have been studied most intensively in Escherichia coli and its close relative Salmonella enterica serovar Typhimurium. We refer to [31] for a very complete and thorough further reading, which deals not only with bacterial chemotaxis, but also with chemotaxis as a mean of cell-cell communication, chemotaxis in amoeba, blood cells, sperm cells and nervous system.
Many literature and authors worked about the Keller-Segel system in many subject mathematics and many case of the model keeler segel, we mention of which without giving details (see for example [?, 13, 25]...).

Plan of the thesis

This manuscript includes a general introduction, five chapters and an appendix with preliminary ideas.

Chapter :1

In this chapter, we present and talk about the biological and mathematical modeling of the phenomenon chemotaxis keller segel model, where we study the phenomenon from the biological point of view and give realistic examples of it, we also clarify the mathematical modeling of the phenomenon and modeling the phenomenon mathematically.

Chapter 2:

In this chapter, we investigate a nonlinear stochastic chemotaxis model with fractional derivatives and multiplicative noise-affected Dirichlet boundary conditions. We combine analysis techniques, fractional calculus, and semigroup theory to demonstrate the existence and uniqueness of the mild solution to the time and space-fractional problem. We also examine the regularity characteristics of the mild solution for this model.

Chapter 3:

In this present chapter, we consider stochastic chemotaxis Keller-Segel model impact by gaussian process. We study of local-global existence solution in time of nonlinear stochastic Keller-Segel model with zeros Dirichlet boundary conditions, for this we use analysis techniques lemmas and semigroup theory.

Chapter 4:

In this chapter, we study the phenomenon of Keller Segel model coupled with a Boussinesq equations. The main objectives of this work is the study of existence and uniqueness of weak solution for the problem, for this we use the technical of Galerkin method.

Chapter 5:

in this chapter, we study at numerical solution of Keller Segel model, by NMHPM for solving one dimensional Keller-Segel model for different types. Some properties show biologically acceptable dependency on parameter values, and numerical solutions are provided.
The end part, we present preliminary notions. In the first, we assume some preliminaries that play an important role in this thesis and we recall some necessary materials needed in the proof of our results,

such as basic results concerning the fontamotal spaces and some theorems on these last, as well as existence and uniqueness theorems. Fanily we have a general conclusion for all the works its study in this thesis. Also, we mention some future works.

CHAPTER 1

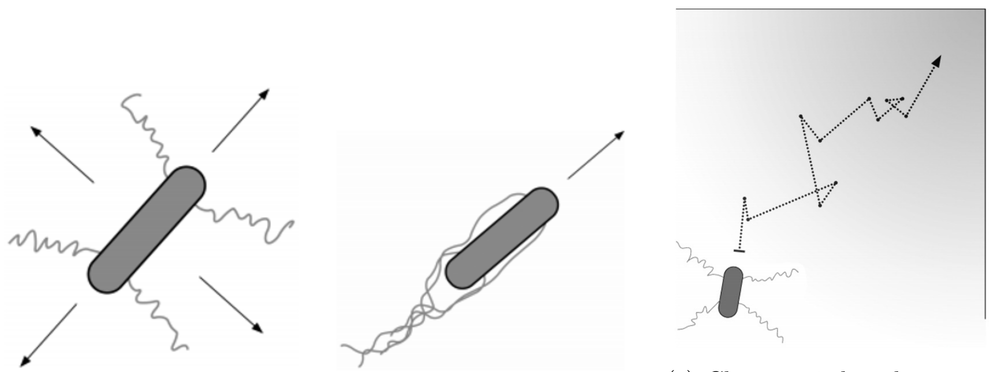
MODELING OF KELLER-SEGEL CHEMOTAXIS MODEL

In this chapter, we present and talk about the biological and mathematical modeling of the phenomenon of chemical entrapment, where we study the phenomenon from a biological point of view and give realistic examples of it. We also explain the mathematical modeling of the phenomenon.

1.1 The motivation for treating chemotaxis models

Now that we have classified the biochemical process on which the PDEs under consideration are based on, let us consider the motivation behind the numerical investigation of such models. The first PDE system that described a chemotaxis-driven population development goes back in time to the early 1970's. It was introduced by Keller and Segel [46] and was motivated by experiments.

Figure (1.1): Schematic illustration of (bacterial) chemotaxis, (a) state of tumbling (CW flagella rotation), (b) state of running (CCW flagella rotation), in (c) we sketched an exemplary chemotaxis path of a bacteria, where the upper right corner is the location of an attracting chemical. Note that the runs upwards the gradient are longer than the downwards runs. of Bonner [11] with the slime mold Dictyostelium discoideum, or in short 'dicty', as it is often tenderly called by researchers. Moreover, encouraged by the research of Adler [?], Keller and Segel extended their model to chemotaxis in bacteria.



(a) Tumbling bacteria. (b) Running bacteria. (c) Chemotaxis-based random walk.

Figure 1.1: Schematic illustration of (bacterial) chemotaxis.

Both models consist of one PDE for the cell density, commonly denoted by u, complemented by a second PDE for the chemoattractant, usually referred to as v or c. Since the publishing date of the chemotaxis model is not as long ago as the publishing of the Navier-Stokes (1827/1845) or even Euler (1755) equations, it provides us a numerical field where so much can be discovered, investigated and postulated. Indeed, since the theoretical aspects of many chemotaxis models are not fully understood yet, an extensive numerical treatment of those models is highly demanded in order to gain new insights in both theory, e.g., in terms of uniqueness or boundedness of solutions, and practical applications, e.g., prediction of cell/chemical distribution for experimental assays or even clinical studies. Already for the biologically well studied mechanism of bacterial chemotaxis (cf. the brief introduction given before), Alberts et al. remarked the huge potential of further numerical investigations of a corresponding model for the chemical signaling pathways involved. In their book, we read [7][Chapter 15, p. 944]:

Even in this relatively simple signaling network, however, computer-based simulations are required to comprehend how the system works as an integrated network. Cell signaling will provide an especially rich area of investigation for a new generation of computational biologists, as the network properties of these pathways are not understandable without powerful computational tools.

We will not postulate possible applications for numerical frameworks of chemotaxis PDE models in more detail since they highly depend on the governing model. Instead, let us mention the reason why chemotaxis models cannot be treated via some standard numerical scheme in the sense of a 'black-box' solver, rendering the present investigation redundant. The character of chemotaxis-driven PDEs is the agglomeration of cell concentrations in limited space with possibly sharp interfaces. Moreover, the speed of agglomeration can vary in different time-scales. Standard numerical schemes are not able to cover these characteristics within suitable CPU and memory bounds. In this context, we can confer to the treatment of Navier-Stokes equations at large Reynolds numbers. After discretization, it is well known that the temporal resolution highly depends on the spatial mesh-size (rf. CFL condition). When choosing a bad resolution, e.g., to save memory and/or CPU expenses, the numerical simulation provides very poor results, if at all. A similar behavior, although arising from a different subject, can be observed for chemotaxisdominated PDEs. A large Reynolds number corresponds to a large chemosensitivity. Moreover, because of the composition of the chemosensitivity as a function which usually depends on the chemical substance, a positive feedback, in terms of agglomeration of

$$u \rightarrow \text{increase of } v \rightarrow \text{even stronger agglomeration of } u,$$

enhances the agglomeration and possibly even leads to a locally unbounded increase of cell concentration. These properties already necessitate highly specialized numerical solvers that allow an efficient computation at a highly accurate resolution.

After such an extensional research in the field, natural question that poses itself is what the significance of chemotaxis is. It has been established that chemotaxis plays a role in some of the most important biological processes, not only for humans, but for almost all species.

Naturally, we start from the role of chemotaxis in reproduction, as it is the essential process for existance of life. It is firstly discovered in marine species [63] that chemotaxis is responsible of guided movement of spermatozoa to the egg during fertilization. The research spread to all species, from non-mammals to mammals. It has been established that for humans and some other mammals, chemotaxis besides the previous role in guiding, has a selective role as well. Namely, not all of the spermatozoa have the ability to fertilize the egg. The ones that do have it are chemotactically responsive. Chemotaxis is in charge for selecting them and then guiding towards the egg. For a full review on sperm chemotaxis we refer to Chapter 7 in [31].

Not only does the chemotaxis have a reproductive role, but it also appears in the embryonic phase once the fertilization is successfully completed. During the development of the embryo, cell migration has a crucial role in morphogenetic processes and formation of nervous system [37]. Many of these migration are caused by chemotaxis. The development and especially wiring of nervous system depends on the precise guidance of axonal growth cones to their targets. Mechanism underlying it is again chemotaxis [25].

Furthermore, we find its role in functioning of the immune system. Certainly, movement and quick response are essential when it come to the immune system. In order to threat an infection, the white blood cells need to migrate towards it. They are attracted by the change of chemical gradient that the infection produces [66].

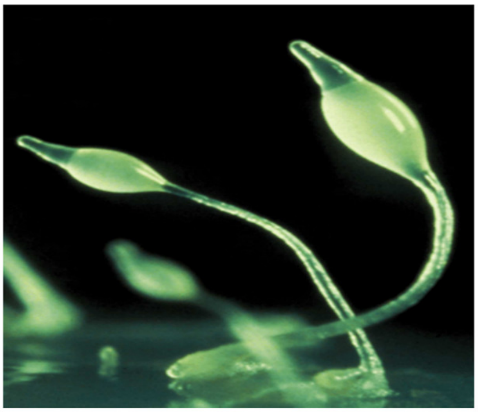
So far, we have only seen the positive aspects of chemotactic movements. However, a negative aspect is the participation of chemotaxis in cancer metastasis and progression. Once the tumor had affected a certain tissue, cancer cells use chemotaxis to migrate towards the surrounding tissue and invade blood vessels [77].

An interesting role of chemotaxis can be found in agronomy and the use of bio-fertilizers. Namely, certain groups of bacteria in the rhizosphere region of soil positively influences plant growth. Bacteria successfully colonizes the rhizosphere thanks to chemotactic attraction from the root exudates of the plants [68]. We conclude this part with one fascinating way to use chemotaxis in medical purposes. Particularly, in construction of nanorobots for human drug delivery. The idea is to design autonomously moving artificial cells which would carry drugs and be capable of chemotactic movements. These movements would rely on artificial chemotaxis. This concept is described and analyzed in [50].

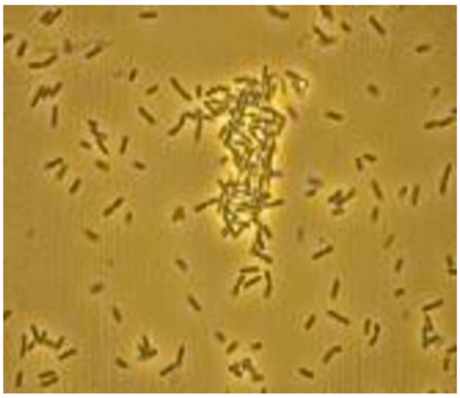
1.2 Some examples of Chemotaxis model

Chemotaxis is a fascinating biological phenomenon which occurs when the movement of a cell or organism is affected by a chemical in the environment [**?**]. Positive chemotaxis occurs when a chemical called the chemoattractant attracts the organism, and negative chemotaxis occurs when a so-called chemorepellent repells the organism. From fertilization of an egg in the earliest states of development to immune system function to cancer growth and metastasis, chemotaxis arises in many different biological processes [**?**]. In this subsection we show some examples and figures of Chemotaxis model(Keller-Segel model).

An example is the laboratory model organism "Dictyostelium discoideum" which is found on leaf litter in forests and feeds on bacteria and yeast. In the case of nutritional deficiency, this amoeba secretes a chemoattractant to form a pseudo-plasmode (Fig. (1.2a)) resembling a small slug and consisting of thousands of agglomerated amoebae. This pseudo-plasmode may persist for several days in order to seek more favorable nutritional conditions. Another example is given by the bacteria "Bacillus subtilis" which ensure their nutrients by moving towards environments rich in oxygen (Fig. (1.2b)), which makes it necessary to have a deep understanding of directed biological movements according to certain chemical species present in the environment as well as to develop appropriate tools for numerical simulations.

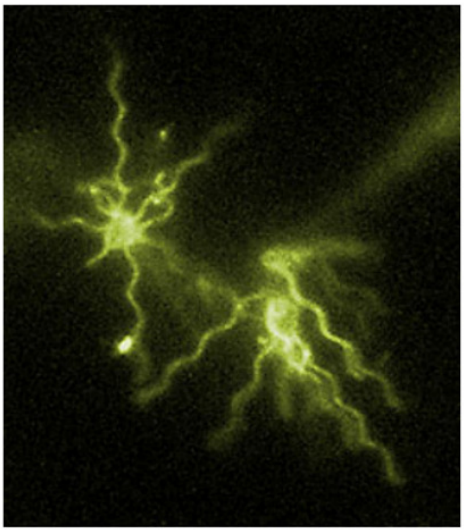


(a) Cells "Dictyostelium Discoideum" agglomerates.

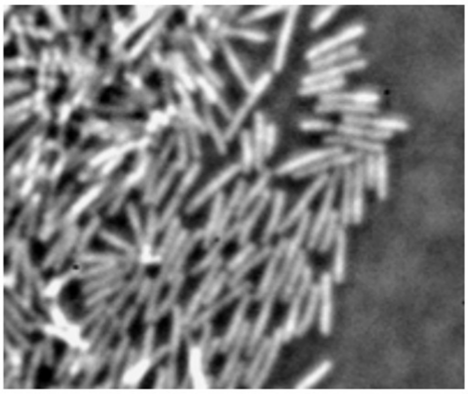


(b) Bacteria swimming towards a medium rich in oxygen.

Many bacteria, such as Escherichia coli, Rhodobacter sphaeroides and Bacillus subtilus (see [30] for a complete list), are able to respond to changes in the surrounding environment by a biased random walk. This allows cells to interact with each other by secreting a chemical substance to attract cells around them. The directed movement of cells and organisms in response to chemical gradients is called chemotaxis. This occurs for instance during the starvation stage of the slime mold Dyctiostelium discoideum. More generally, chemotaxis is widely observed in various biological fields (morphogenesis, bacterial self-organisation and inflammatory processes among others). The bacterium Escherischia coli is traditionally chosen for studying bacterial chemotaxis as its biochemistry as well as the dynamics of its movement are well understood.



(a) fluorescently labeled E. coli, source Howard Berg's website.



(b) An E. coli swarm, source Howard Berg's website.

Lab experiments with observations and hypothesis in Budrene and Berg's experiment the bacteria were exposed to elements of the tricarboxylic acid (TCA) cycle which resulted in unique and exciting patterns that were not seen in previous lab experiments . The TCA substance is referred to as the stimulant and the bacteria were not chemotactic towards it. Instead, the bacteria itself produced a potent chemoattractant called aspertate (Budrene and Berg (1991)). The biological experiment was further classified into two types: liquid and semi-solid experiments.

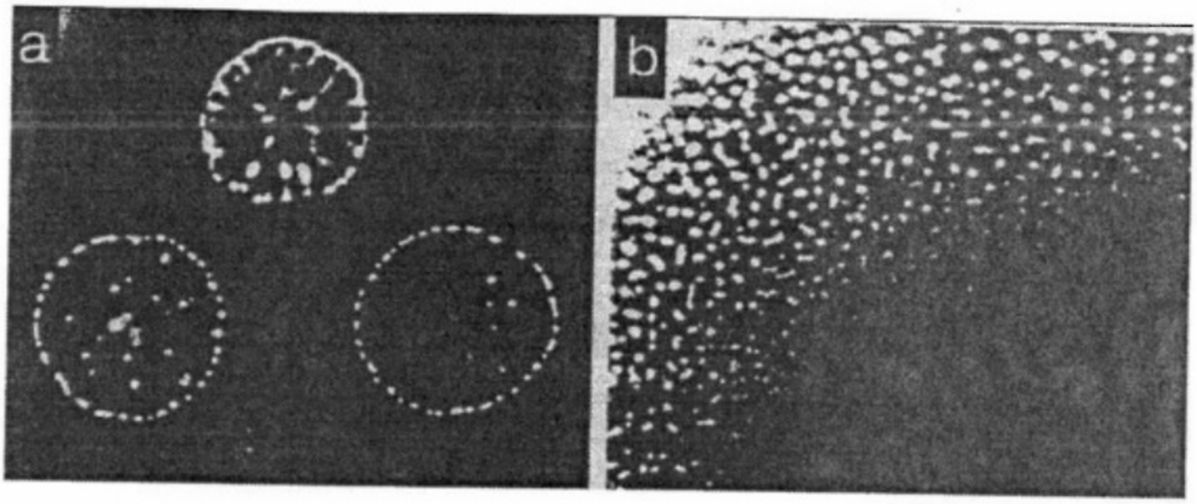


Figure 1.4: Patterns obtained in liquid medium. courtesy of E.O Budrene and H.C.Berg.

A cell–cell repulsion model on ahyperbolic Keller–Segel equation, ypical examples of such co-culture experiment include the s tudy of the i nteraction between cancer cellsand normal cells, which plays a crucial role in tumor development, and comparative studies of the r esistance of diff erent types of cancer cells to a chemotherapeutic drugs.

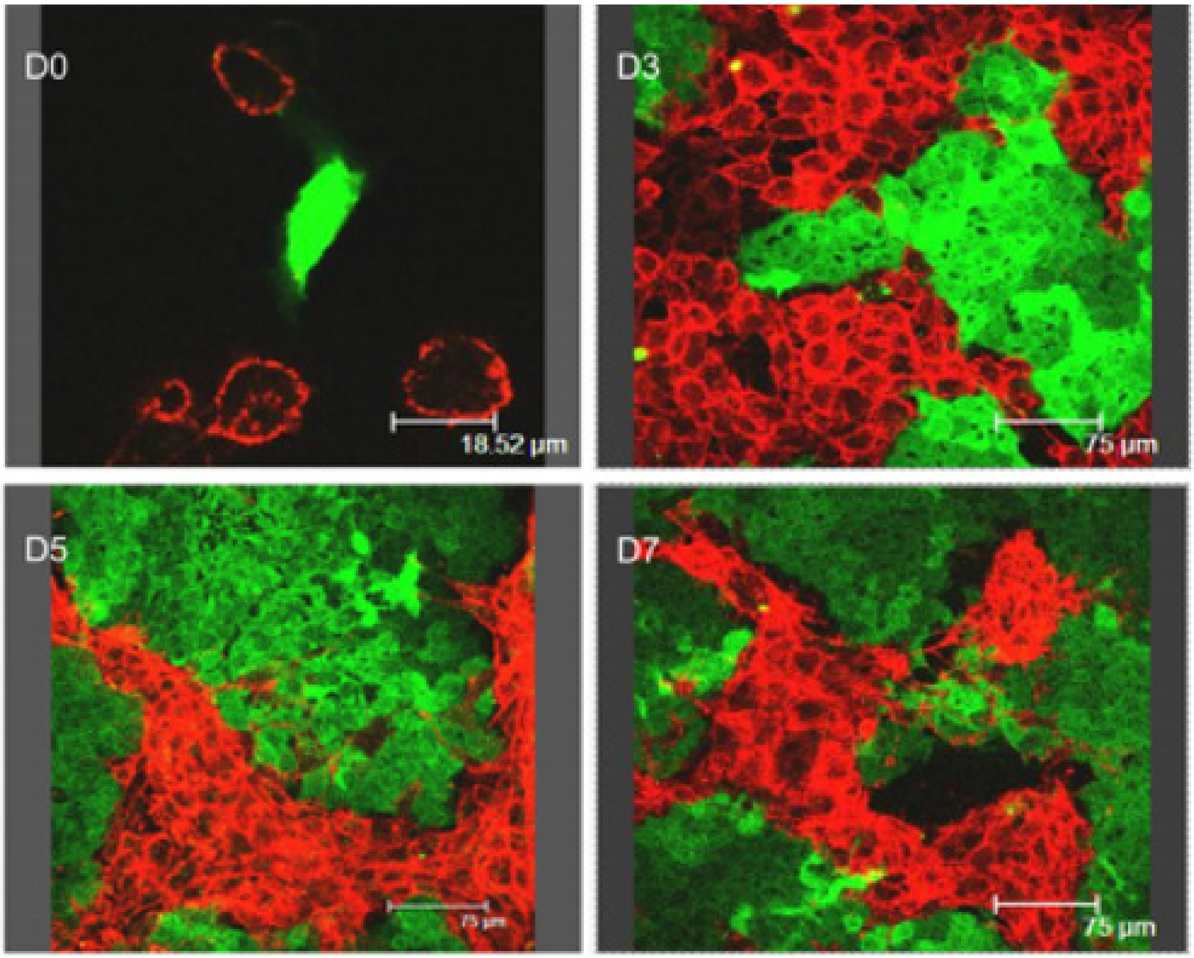


Figure 1.5: Direct immunodetection of P-gp transfers in co-cultures of sensitive (MCF-7) and resistant (MCF-7/Doxo) variants of the human breast cancer cell line.

In nature the dictyostelium discoideum spread on the soil and then come together by chemo-taxis to form a motile pseudoplasmodium. This slug creeps to a few centimetres below the soil surface where it forms a fruiting body with spores and a stalk. The spores are then blown away by the wind to colonise a new place, see (1.6).

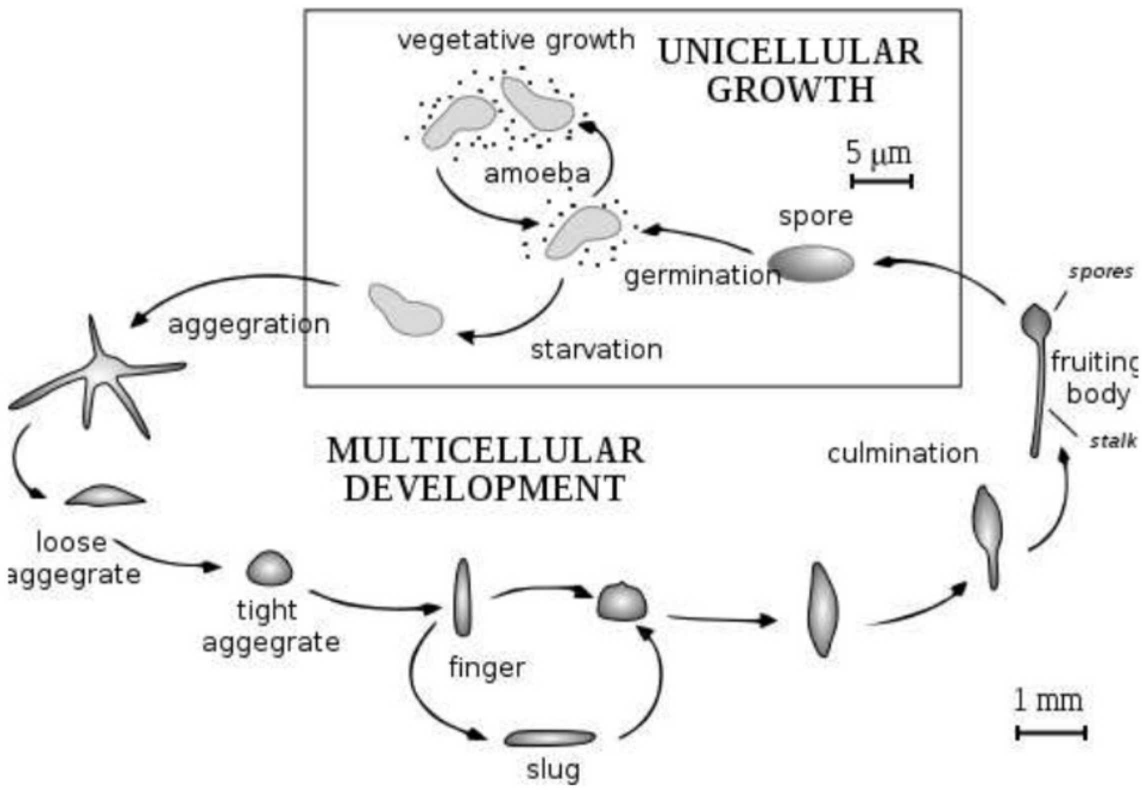


Figure 1.6: Dictyostelium Discoideum cycle.

There are also several studies and examples of the chemotaxis phenomena, which is one of the most prominent concerns of researchers, whether in the field of mathematics or the field biology and medcine.

1.3 Modelling of chemotaxis and the Keller–Segel approach

As the biological research of the phenomenon grew and altered its interest towards experiments, the need for mathematical models for chemotaxis emerged. Mathematical models help in better understanding of experimental results and allow biologists to study different characteristics of bacterial systems without the need to intensively repeat the experiments. When one desires to mathematically model chemotaxis, first the goal and nature of the results should be clearly defined. That is to say, are we interested in the particular behaviour of one individual (cell, bacteria) of the population or of the whole population at once. This leads us to two main approaches when modelling chemotactic movements, the microscopic and macroscopic approaches, respectively.

As the microscopic models focus on the individual cell, it is important to understand the biological processes that are happening within it when the cell becomes chemotactically active. We will try to illustrate it on the example of E. Coli, as its chemotaxis is understood best. When there is no stimuli in its environment, E. Coli swims in a random walk. The random walk takes on a biased character, towards the attractant or away from the repellent, as soon as the presence of stimuli is sensed. The movement itself is a series of "runs" and "tumbles". Runs are movements following a (fairly) straight line, which are suddenly interrupted by a change in the direction, a tumble. When E. Coli exhibits positive chemotaxis, the number of tumbles decrease. The opposite happens with the negative chemotaxis. If there is a change of gradient in the extra-cellular environment, the bacterium is unable to detect it along its own length, because its size is too small. Instead, the cell is equipped with membrane receptors, which are able to distinguish very low attractant concentrations. Once the attractant is detected, the receptor passes the signal inside the cell. Thanks to the intra-cellular proteins, called Che proteins (from Chemotaxis proteins), a signaling cascade occurs and finally arrives to flagellar motors. Then, the flagella are rotated clockwise or counterclockwise, depending on the type of the stimulus. Clockwise rotation leads to tumbling and counterclockwise to runs. An important part of the process is also the adaptation, which includes resetting of receptors, as if they have not been stimulated at all. Furthermore, since the bacteria are able to sense a tiny change in gradients, they need to be able to amplify the signal (gain process).

The mathematical models for one cell try to represent above mentioned processes, individually or together. So far, none of the models was able to reproduce well all of them together. One of the reasons is that they all occur in different time scales. The models which do a good job in representing ligand binding and adaptation, can not represent well also the chemoreceptor sensitivity and gain and vice versa. For a review on these and many other processes and how they have been modeled in the literature, we refer to the thorough and comprehensive review by Tindall [86].

Now, we will see how a population exhibiting chemotactic activity behaves on the example of slime molds. Slime molds are populations of amoebae that grow by cell division. The cells wander around their environment exhausting food supplies which they are able to find using chemotaxis. Once the nourishment is consumed, cells disperse uniformly around the area at their disposal. A while later, some of the cells begin emitting a signal that attracts other cells who start moving towards it and are triggered to emit the same attracting signal. The cells aggregate, forming a slug that may move, respond to chemical stimuli and detect food sources. Eventually, the slug produces fruiting bodies and releases spores in order to recommence the life cycle. The pioneer of biological research of slime molds was Bonner (see e.g. [10]). What is fascinating about slime molds even today, is that individually, they are very simple organisms that exhibit "intelligent" behaviour once they aggregate. In the study [84], the authors were even able to reproduce a map of Tokyo rail system once the different stimuli were put in the right places.

Motivated by describing the onset of slime mold aggregation using a macroscopic approach, Evelyn F. Keller and Lee A. Segel propose in [44] a model of four coupled parabolic equations.

Namely, the authors start from the individual properties of the cells in order to derive a model for the aggregation stage. Let $\rho(x,t)$ denote the density of the amoebae at point x in time t, $c(x,t)$ denotes the concentration of the chemo-attractant (acrasin), $W(x,t)$ denotes the concentration of the enzyme that degrades the chemo-attractant (acrasinase) and, finally, $v(x,t)$ denotes the concentration of a biochemical complex V formed by acrasin and arcasinase. The individual properties taken into account are the following:

1. The amoeba moves according to a random motion analogous to a diffusion that is biased towards the direction of the positive gradient of the attractant.

2. The acrasin is produced by the amoebae with rate $f(c)$.

3. The acrasinase is produced by the amoebae with rate $g(c, W)$.

4. The complex V dissociates into arcasinase and a degraded product $(d.p)$:

$$c + W \leftrightarrows^{k_1}_{k_{-1}} V \to W + d.p.$$

5. Acrasin, arcasinase and the complex V diffuse according to Fick's law.

In order to derive the equation for ρ, the authors use the mass balance equation and the fact that the flux of amobea mass is proportional to $\nabla\rho$ (by Fick's law) and ∇c (by Fourier's law). Birth and death are not taken into account. Thus,

$$\frac{\partial}{\partial t}\rho(t,x) = \nabla.(D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c).$$

Here, D_1 represents the strength of the random movement and χ the impact of the chemo-attractant gradient to the flow of the population. The chemo-attractant diffuses according to Fick's law and its dynamics involves its production and consumption rates as described above,

$$\frac{\partial}{\partial t}c(t,x) = D_c \triangle c + f(c)\rho - k_1 cW + k_{-1}\nu.$$

The equations for W and ν are derived in the same way. The authors arrive to the following system:

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) = \nabla.(D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}c(t,x) = D_c \triangle c + f(c)\rho - k_1 cW + k_{-1}\nu, \quad t>0,\ x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}W(t,x) = D_W \triangle W + \rho g(c,W) - k_1 cW + (k_{-1}+k_2)\nu, \quad t>0,\ x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}\nu(t,x) = D_\nu \triangle \nu + k_1 cW - (k_{-1}+k_2)\nu, \quad t>0,\ x\in\mathbb{R}^d, \\ \rho(0,x)=\rho_0(x),\ c(0,x)=c_0,\ W(0,x)=W_0,\ \nu(0,x)=\nu_0,\ x\in\mathbb{R}^d. \end{cases} \tag{1.1}$$

Here k_{-1}, k_1 and k_2 are positive constants.

Then, the authors argue that the aggregation occurs as, in some point of maturation, the individual properties of the cells change. Thus, a uniform distribution is no longer favorable and it becomes unstable. The objective is to see how such change in individual cells impacts the whole population, rather to explain why and how such change happens. In order to do so, the authors propose a simplified version of the latter system "as it is useful for the sake of clarity to employ the simplest reasonable model" [[?], p. 403]. They assume that the bio-chemical complex V is in a steady state w.r.t. the chemical reaction: $k_1 cW - (k_{-1}+k_2)\nu = 0$ and that the total concentration of the free and bound degradant is constant: $W + \nu = 0$. Thus, (1.1) transforms into the following system of non-linear parabolic equations:

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) = \nabla.(D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}c(t,x) = D_c \triangle c + f(c)\rho - k_1 cW + k_{-1}\nu, \quad t>0,\ x\in\mathbb{R}^d, \\ \rho(0,x)=\rho_0(x),\ c(0,x)=c_0,\ x\in\mathbb{R}^d. \end{cases} \tag{1.2}$$

Then, the authors study how a small time dependent perturbation of the uniform configuration influences a linearized version of (1.2) for $d = 2$. They find conditions under which the uniform state is temporarily or definitely perturbed.

1.4 PDE analysis of the Keller-Segel system

As the Keller-Segel system is designed to model the onset of cell aggregation when triggered by chemical stimulus, it is no surprise that the solutions may blow-up in finite time. The definition of the blow-up in finite time for a solution (ρ, c) is the following : there exists a time $T_0 < \infty$ such that ρ_t converges to a measure not belonging to $L^1(\mathbb{R}^d)$ as $t \to T_0$. In general, the question of well-posedness of (1.2) is a subject of an extensive amount of PDE literature over the past almost 40 years. A very complete review of the results obtained until early 2000's can be found in Horstmann [42, 41]. Then, we suggest to the interested reader the review of Perthame [71] which after a theoretical review of the Keller-Segel system shows its connection with kinetic models for chemotaxis and the work of Hillen and Painter [40] reviewing results on different variations of (1.2).

The principal conclusion when investigating the literature about the Keller-Segel system is that whether we have global well-posedness or a blow up in finite time is highly correlated with the space dimension of the problem. In addition, various results obtained depend also on the prescribed initial and possible boundary conditions, type of the domain and value of parameter α.

Here we will summarize some of the results in the literature and will classify them in three groups: $d = 1$, $d = 2$, $d \geq 3$. Chemotaxis has attracted significant interest due to its critical role in a wide range of biological phenomena. Recently, there has benn a growing number of researchers beginning to utilise Keller-Segel models and fractional KellerSegel models in simulations of important real-world applications.

CHAPTER 2

STOCHASTIC CHEMOTAXIS MODEL WITH FRACTIONAL DERIVATIVE DRIVEN BY MULTIPLICATIVE NOISE

In this chapter, we introduce stochastic model of chemotaxis by fractional Derivative generalizing the deterministic Keller Segel model. In this chapter, we study of nonlinear stochastic chemotaxis model with Dirichlet boundary conditions, fractional Derivative and disturbed by multiplicative noise. The required results is prove the existence and uniqueness of mild solution to time and space-fractional , for this we use analysis techniques and fractional calculus and semigroup theory, also studying the regularity properties of mild solution for this model.

2.1 Introduction

In this study, we consider on the following generalized SKSM with time-space fractional derivative on a bounded domain $D \subset \mathbb{R}^d (1 \leq d \leq 3)$:

$$\begin{cases} {}^cD_t^\beta u + (-\Delta)^{\frac{\alpha}{2}} u - \nabla(u\nabla c) = g(u)\dot{W}(t),\ (t,x) \in [0,T] \times D, \\ {}^cD_t^\beta c + (-\Delta)^{\frac{\alpha}{2}} c - c\nabla c = f(c)\dot{W}(t),\ (t,x) \in [0,T] \times D, \end{cases} \tag{2.1}$$

with subject to the initial conditions:

$$\begin{cases} u(0,x) = u_0(x),\ x \in D, \\ c(0,x) = c_0(x),\ x \in D, \end{cases} \tag{2.2}$$

and the Dirichlet boundary conditions:

$$\begin{cases} u(t,x) \mid_{\partial D} = 0,\ t \in [0,T], \\ c(t,x) \mid_{\partial D} = 0,\ t \in [0,T], \end{cases} \tag{2.3}$$

where $u = u(t,x)$ denots the population density of biological individuals, $c = c(t,x)$ denots the concentration of chemical substance, and $\nabla(u\nabla c)$ is called a chemotactic term that is used to model the fact that cells are attracted by chemical stimulus. In which the terms $g(u)\dot{W}(t) = g(u)\frac{dW(t)}{dt}$, and $f(c)\dot{W}(t) = f(c)\frac{dW(t)}{dt}$ They describe the case-dependent random noise, where $W(t)_{t\in[0T]}$ is $\mathcal{F}_t-$ adapted Wiener process defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the expectation $\mathbb{E}$ and associate with the normal filtration $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$, The operator $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (1,2)$ stands for the fractional power of the Laplacian (see [51]). We denote by ${}^cD_t^\beta$ the Caputo derivative of order β, which is defined by (see [83])

$$
\begin{aligned}
{}^cD_t^\beta u(t,x) &= \begin{cases} \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{\partial u(s,x)}{\partial s} \frac{ds}{(t-s)^\beta}, \ 0<\beta<1, \\ \frac{\partial u(s,x)}{\partial s}, \ \beta = 1, \end{cases} \\
{}^cD_t^\beta c(t,x) &= \begin{cases} \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{\partial c(s,x)}{\partial s} \frac{ds}{(t-s)^\beta}, \ 0<\beta<1, \\ \frac{\partial c(s,x)}{\partial s}, \ \beta = 1, \end{cases}
\end{aligned}
\tag{2.4}
$$

where $\Gamma(.)$ stands for the gemma function

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt.$$

The rest of the chapter is organized as follows. In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, the existence and uniqueness of mild solution to the problem of time-space fractional (2.1)-(2.3) and in Section 4, the spatial and temporal regularity properties of mild solution to this time-space fractional (2.9) are proved. In Section 5, the existence and uniqueness of mild solution to the problem of time-space fractional (2.1)-(2.3). Finally, the spatial and temporal regularity properties of mild solution to this time-space fractional (2.10) are proved. We use stochastic analysis techniques, fractional calculus and semigroup theory.
Next, we mention some notations and preliminaries the task at chapter.

2.2 Notations and preliminaries

Denote the basic functional space $L^p(D), 1 \leq p < \infty$ and $H^s(D)$ by the usual Lebesgue and Sobolev spaces, respectively. We assume that A is the negative Laplacian $-\Delta$ in a bounded domain D with zero Dirichlet boundary conditions in a Hilbert space $H = L^2(D)$, which are given by

$$A = -\Delta, \quad D(A) = H_0^1(D) \cap H^2(D).$$

Since the operator A is self-adjoint on H with discrete spectral, i.e., there exists the eigenvectors e_n with corresponding eigenvalues λ_n such that

$$Ae_n = \lambda_n e_n, \ e_n = \sqrt{2}\sin(n\pi), \ \lambda_n = \pi^2 n^2, \ n \in \mathbb{N}^+.$$

For any $s > 0$, let $\dot{H}^s$ be the domain of the fractional power $A^{\frac{s}{2}} = (-\Delta)^{\frac{s}{2}}$, which can be defined by

$$A^{\frac{s}{2}} e_n = \lambda_n^{\frac{s}{2}}, \ n = 1, 2, ...,$$

and

$$\dot{H}^s = D(A^{\frac{s}{2}}) = \{v \in L^2(D), s.t. \parallel v \parallel_{\dot{H}^s}^2 = \sum_{n=1}^\infty \lambda_n^{\frac{s}{2}} v_n^2 < \infty\},$$

where $v_n := \langle v, e_n \rangle$ with the inner product $\langle ., . \rangle$ in $L^2(D)$. We denote that
$\parallel v \parallel_{\dot{H}^s} = \parallel A^{\frac{s}{2}} v \parallel$, and the corresponding dual space $\dot{H}^{-s}$ with the inverse operator $A^{-\frac{s}{2}}$. We also denote A_s for $A^{\frac{s}{2}}$ and the bilinear operators $B(u,c) = \nabla(u\nabla c)$, and $D(B) = H_0^1$ and $L(c,v) = c\nabla v$, and

$D(L) = H_0^1$ with a slight abuse of notation $L(c,c) = L(c)$. Then the eqs (2.1) and (2.3) can be rewritten as the following abstract formulation:

$$\begin{cases} {}^cD^\beta u(t) = -A_\alpha u(t) + B(u(t), c(t)) + g(u(t))\frac{dW(t)}{dt}, \ t > 0, \\ u(0) = u_0 \end{cases} \tag{2.5}$$

and

$$\begin{cases} {}^cD^\beta c(t) = -A_\alpha c(t) + L(c(t)) + f(c(t))\frac{dW(t)}{dt}, \ t > 0, \\ c(0) = c_0, \end{cases} \tag{2.6}$$

where $\{W(t)\}_{t\geq 0}$ is a $\mathbb{Q}-$ Wiener process with linear bounded covariance operator $\mathbb{Q}$ such that $Tr(\mathbb{Q}) < \infty$. Further, there exists the eigenvalues λ_n and corresponding eigenfunctions e_n satisfy $\mathbb{Q}_n = \lambda_n e_n$, $n = 1, 2, ...$, then the Wiener process is given by

$$W(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \beta_n(t) e_n,$$

in which $\{\beta_n\}_{n\geq 1}$ is a sequence of real-valued standard Brownian motions. Let $L_0^2 = L^2(\mathbb{Q}^{\frac{1}{2}}(H), H)$ denote the space of Hilbert-Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}(H)$ to H with the norm

$$\| \Phi \|_{L_0^2} = \| \Phi\mathbb{Q}^{\frac{1}{2}} \|_{H^s} = (\sum_{n=1}^{\infty} \Phi\mathbb{Q}^{\frac{1}{2}} e_n)^{\frac{1}{2}},$$

i.e., $L_0^2 = \{\Phi \in L(H) : \sum\limits_{n=1}^{\infty} \| \Phi\mathbb{Q}^{\frac{1}{2}} \|^2 < \infty\}$, where $L(H)$ is the space of bounded linear operators from H to H. For an arbitrary Banach space B, we denote $\| \, . \, \|_{L^p(\Omega;B)}$ by the norm in $L^p(\Omega, \mathcal{F}, \mathbb{P}; B)$, which defined as

$$\| v \|_{L^p(\Omega;B)} = (\mathbb{E}[\| v \|_B^p])^{\frac{1}{p}}, \ \forall v \in L^p(\Omega, \mathcal{F}, \mathbb{P}; B),$$

for any $p \geq 2$. We shall also need the following result with respect to the fractional operator A_α (see Ref [?]).

Lemma 2.1 *For any $\alpha > 0$, an analytic semigroup $\mathcal{S}_\alpha(t) = e^{-tA_\alpha}$, $t \geq 0$ is generated by the operator $-A_\alpha$ on L^p, and for any $\nu \geq 0$, there exists a constant $C_{\alpha\nu}$ dependent on α and ν such that*

$$\| A_\nu \mathcal{S}_\alpha(t) \|_{\mathcal{L}(L^p)} \leq C_{\alpha\nu} t^{-\frac{\nu}{\alpha}}, \ t > 0, \tag{2.7}$$

in which $\mathcal{L}(B)$ denotes the Banach space of all linear bounded operators from B to itself.

Next, we will introduce the following lemma to estimate the stochastic integrals, which contains theBurkholder-Davis-Gundy inequality.

Lemma 2.2 *([48]) For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$, and for any predictable stochastic process $v : [0,T] \times \Omega \to L_0^2$, which satisfies*

$$\mathbb{E}[(\int_0^T \| v(s) \|_{L_0^2}^2 \, ds)^{\frac{p}{2}}] < \infty,$$

then we have

$$\mathbb{E}[\| \int_{t_1}^{t_2} v(s) dW(s) \|^p] \leq C(p)\mathbb{E}[(\int_{t_1}^{t_2} \| v(s) \|_{L_0^2}^2 \, ds)^{\frac{p}{2}}], \tag{2.8}$$

where $C(p) = [\frac{p(p-1)}{2}]^{\frac{p}{2}} (\frac{p}{p-1})^{p(\frac{p}{2}-1)}$ is a constant.

Now, we give the following definition of mild solution for our time-space fractional stochastic Keller-Segel model.

Definition 2.1 A $\mathcal{F}_t$ adapted process $(u(t), c(t))_{t\in[0,T]}$ is called a mild solution (2.1), if $(u(t), c(t))_{t\in[0,T]} \in \mathbb{C}\left([0,T]; \dot{H}^\nu\right)$ P - a. e, and it holds,

$$\begin{aligned} u(t) &= E_\beta(t)u_0 + \textstyle\int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) B\left(u(s), c(s)\right) ds \\ &+ \textstyle\int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) g(u(s)) dW(s), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} c(t) &= E_\beta(t)c_0 + \textstyle\int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) L(c(s)) ds \\ &+ \textstyle\int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) f(c(s)) dW(s), \end{aligned} \tag{2.10}$$

respectefily for a. s. $\omega \in \Omega$, where the generalized Mittag-Leffler operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are defined as

$$E_\beta(t) = \int_0^\infty M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta\theta M_\beta(\theta) S_\alpha(t^\beta \theta) d\theta,$$

which contain the Mainardi's Wright-type function with $\beta \in (0,1)$ given by

$$M_\beta(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1 - \beta(1+n))},$$

in which the Mainardi function $M_\beta(\theta)$ act as a bridge between the classical integral-order and fractional derivatives of differential equations, for more details see [83, 57]. Here, the derivation of mild solution (2.9) and (2.10) can be found in Appendix A (2.6) and Appendix B (2.6) (respectively).

Lemma 2.3 *[?] For any $\beta \in (0,1)$ and $-1 < \varepsilon < \infty$, it is not difficult to verity that*

$$M_\beta(\theta) \geq 0, \ \ \textit{and} \ \int_0^\infty \theta^\varepsilon M_\beta(\theta) d\theta = \frac{\Gamma(1+\varepsilon)}{\Gamma(1+\beta\varepsilon)}, \tag{2.11}$$

for all $\theta \geq 0$.

Theorem 2.1 *For any $t > 0$, $E_\beta(t)$ and $E_{\beta\beta}(t)$ are linear and bounded operators. Moreover, for $0 \leq \nu < \alpha < 2$, there exist constants $C_\alpha = C(\alpha, \beta, \nu) > 0$ and $C_\beta = C(\alpha, \beta, \nu) > 0$ such that*

$$\| E_\beta(t) v \|_{\dot{H}^\nu} \leq C_\alpha t^{\frac{-\beta\nu}{\alpha}} \| v \|, \ \ \| E_{\beta\beta}(t) v \|_{\dot{H}^\nu} \leq C_\beta t^{\frac{-\beta\nu}{\alpha}} \| v \| . \tag{2.12}$$

Proof. For $t > 0$ and $0 \leq \nu < \alpha < 2$, by means of the Lemma (2.1) and Lemma (2.3), we have

$$\begin{aligned} \| E_\beta(t) v \|_{\dot{H}^\nu} &\leq \textstyle\int_0^\infty M_\beta(\theta) \| A_\nu \mathcal{S}_\alpha(t^\beta \theta) v \| d\theta \\ &\leq \textstyle\int_0^\infty C_{\alpha\nu} t^{\frac{-\beta\nu}{\alpha}} \theta^{\frac{-\nu}{\alpha}} M_\beta(\theta) \| v \| d\theta \\ &= \tfrac{C_{\alpha\nu}\Gamma(1-\frac{\nu}{\alpha})}{\Gamma(1-\frac{\beta\nu}{\alpha})} t^{\frac{-\beta\nu}{\alpha}} \| v \|, \ v \in L^2(D) \end{aligned}$$

and

$$\begin{aligned}
\| E_{\beta\beta}(t)v \|_{\dot{H}^\nu} &\leq \int_0^\infty \beta\theta M_\beta(\theta) \| A_\nu \mathcal{S}_\alpha(t^\beta\theta)v \| d\theta \\
&\leq \int_0^\infty C_{\alpha\nu}\beta t^{\frac{-\beta\nu}{\alpha}} \theta^{1-\frac{\nu}{\alpha}} M_\beta(\theta) \| v \| d\theta \\
&= \frac{C_{\alpha\nu}\beta\Gamma(2-\frac{\nu}{\alpha})}{\Gamma(1+\beta(1-\frac{\nu}{\alpha}))} t^{\frac{-\beta\nu}{\alpha}} \| v \|, \ v \in L^2(D),
\end{aligned}$$

which imply that the estimates (2.12) hold, so it is easy to know that $E_\beta(t)$ and $E_{\beta\beta}(t)$ are linear and bounded operators. ∎

Theorem 2.2 *For any $t > 0$, the operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are strongly continuous. Moreover, for any $0 \leq t_1 < t_2 \leq T$ and for $0 < \nu < \alpha < 2$, there exist constants $C_{\alpha\nu} = C(\alpha, \beta, \nu) > 0$ and $C_{\beta\nu} = C(\alpha, \beta, \nu) > 0$ such that*

$$\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \leq C_{\alpha\nu}(t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \| \tag{2.13}$$

and

$$\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \leq C_{\beta\nu}(t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \| . \tag{2.14}$$

Proof. for any $0 \leq t_1 < t_2 \leq T$, it is easy to deduce that

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{d\mathcal{S}_\alpha(t^\beta\theta)}{dt} dt &= \mathcal{S}_\alpha(t_2^\beta\theta) - \mathcal{S}_\alpha(t_1^\beta\theta) \\
&= -\int_{t_1}^{t_2} \beta t^{\beta-1}\theta A_\alpha \mathcal{S}_\alpha(t^\beta\theta) dt,
\end{aligned} \tag{2.15}$$

for $0 < \nu < \alpha < 2$, making use of the above expression, the Lemma (2.1) and Lemma (2.3), we can arrive at

$$\begin{aligned}
\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} &= \| A_\nu(E_\beta(t_2) - E_\beta(t_1))v \| \\
&= \| \int_0^\infty M_\beta(\theta) A_\nu((\mathcal{S}_\alpha(t_2^\beta\theta) - \mathcal{S}_\alpha(t_1^\beta\theta))v)v d\theta \| \\
&\leq \int_0^\infty \beta\theta M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_{\alpha+\nu}\mathcal{S}_\alpha(t^\beta\theta)v \|_{L^2} dt d\theta \\
&\leq \int_0^\infty C_{\alpha\nu}\beta\theta^{-\frac{\nu}{\alpha}} M_\beta(\theta)(\int_{t_1}^{t_2} t^{\frac{-\beta\nu}{\alpha}-1} dt) \| v \| d\theta \\
&= \frac{\alpha C_{\alpha\nu}\Gamma(1-\frac{\nu}{\alpha})}{\nu\Gamma(1-\frac{\beta\nu}{\alpha})}(t_2^{\frac{-\beta\nu}{\alpha}} - t_1^{\frac{-\beta\nu}{\alpha}}) \| v \| \\
&\leq \frac{\alpha C_{\alpha\nu}\Gamma(1-\frac{\nu}{\alpha})}{\nu T^{\frac{2\beta}{\alpha}}\Gamma(1-\frac{\beta\nu}{\alpha})}(t_2 - t_1)^{\frac{\beta\nu}{\alpha}} \| v \|, \ v \in L^2(D)
\end{aligned}$$

and

$$\begin{aligned}
\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} &= \| A_\nu (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \| \\
&= \| \int_0^\infty \beta\theta M_\beta(\theta) A_\nu (\mathcal{S}_\alpha(t_2^\beta\theta) - \mathcal{S}_\alpha(t_1^\beta\theta))v d\theta \| \\
&\le \int_0^\infty \beta^2\theta^2 M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_{\alpha+\nu}\mathcal{S}_\alpha(t^\beta\theta)v \|_{L^2} dtd\theta \\
&\le \int_0^\infty C_{\alpha\nu}\beta^2\theta^{1-\frac{\nu}{\alpha}} M_\beta(\theta) (\int_{t_1}^{t_2} t^{\frac{-\beta\nu}{\alpha}-1} dt) \| v \| d\theta \\
&= \frac{\alpha C_{\alpha\nu}\Gamma(2-\frac{\nu}{\alpha})}{\nu\Gamma(1+\beta(1-\frac{\nu}{\alpha}))} (t_2^{\frac{-\beta\nu}{\alpha}} - t_1^{\frac{-\beta\nu}{\alpha}}) \| v \| \\
&\le \frac{\alpha C_{\alpha\nu}\Gamma(1-\frac{\nu}{\alpha})}{\nu T_0^{\frac{2\beta\nu}{\alpha}}\Gamma(1+\beta(1-\frac{\nu}{\alpha}))} (t_2-t_1)^{\frac{\beta\nu}{\alpha}} \| v \|, \ v \in L^2(D).
\end{aligned}$$

It is obviously to see that the term $\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \to 0$ and $\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \to 0$ as $t_1 \to t_2$. Which mean that the operators $E_\beta(t)$ and $E_{\beta\beta}(t)$ are strongly continuous. ■

Remark 2.1 Assume $\nu = 0$ in theorem (2.2), then there exist constants $C_\alpha = C(\alpha,\beta) > 0$ and $C_\beta = C(\alpha,\beta) > 0$ such that

$$\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} \le C_\alpha (t_2 - t_1) \| v \| \tag{2.16}$$

and

$$\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} \le C_\beta (t_2 - t_1) \| v \| . \tag{2.17}$$

Proof. for any $0 < T_0 \le t_1 < t_2 \le T$, the same as the proof of Theorem (2.2), we get

$$\begin{aligned}
\| (E_\beta(t_2) - E_\beta(t_1))v \|_{\dot{H}^\nu} &= \| \int_0^\infty M_\beta(\theta)((\mathcal{S}_\alpha(t_2^\beta\theta) - \mathcal{S}_\alpha(t_1^\beta\theta))vd\theta)vd\theta \|_{L^2} \\
&\le \int_0^\infty \beta\theta M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_\alpha \mathcal{S}_\alpha(t^\beta\theta)v \|_{L^2} dtd\theta \\
&\le \int_0^\infty C_{\alpha\alpha}\beta\theta M_\beta(\theta) (\int_{t_1}^{t_2} t^{-1} dt) \| v \|_{L^2} d\theta \\
&\le C_{\alpha\alpha}\beta(\ln t_2 - \ln t_1) \| v \| \\
&= \frac{C_{\alpha\alpha}}{T_0}(t_2 - t_1) \| v \|, \ v \in L^2(D)
\end{aligned}$$

and

$$\begin{aligned}
\| (E_{\beta\beta}(t_2) - E_{\beta\beta}(t_1))v \|_{\dot{H}^\nu} &= \| \int_0^\infty \beta\theta M_{\beta\beta}(\theta)((\mathcal{S}_\alpha(t_2^\beta\theta) - \mathcal{S}_\alpha(t_1^\beta\theta))vd\theta)vd\theta \|_{L^2} \\
&\leq \int_0^\infty \beta^2\theta^2 M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \| A_\alpha \mathcal{S}_\alpha(t^\beta\theta)v \|_{L^2} dtd\theta \\
&\leq \int_0^\infty C_{\alpha\alpha}\beta^2\theta M_\beta(\theta)(\int_{t_1}^{t_2} t^{-1}dt) \| v \| d\theta \\
&\leq \tfrac{C_{\alpha\alpha}\beta\Gamma(2)}{\Gamma(1+\beta)}(\ln t_2 - \ln t_1) \| v \| \\
&\leq \tfrac{C_{\alpha\alpha}\beta\Gamma(2)}{T_0\Gamma(1+\beta)}(t_2 - t_1) \| v \|, \ v \in L^2(D).
\end{aligned}$$

This completes the proof. ■

2.3 Existence and uniqueness of mild solution

Our main purpose of this section is to prove the existence and uniqueness of mild solution to the problem (2.5). To do this, the following assumptions are imposed.

2.3.1 Assumption A

The measurable function $g : \Omega \times H \to L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$\| g(v) \|_{L_0^2} \leq C \| v \|, \quad \| g(u) - g(v) \|_{L_0^2} \leq C \| u - v \|, \tag{2.18}$$

for all $u, v \in H$.

2.3.2 Assumption B

Let C, C_1 are a positive real numbers, then the bounded bilinear operator $B : L_0^2(D) \to H^{-1}(D)$ satisfies the following properties:

$$\begin{aligned}
\| B(u,c) \|_{\dot{H}^{-1}} &\leq C \| u \| \| c \| \\
&\leq C_1 \| u \|^2
\end{aligned}$$

and

$$\| B(u,c) - B(v,c) \|_{\dot{H}^{-1}} \leq CC_1(\| u \| + \| v \|) \| u - v \|, \tag{2.19}$$

where C_1 depend a norm the c in $L_0^2(D)$, and for all $u, v, c \in L_0^2(D)$.

2.3.3 Assumption C

Assume that the initial value $u_0 : \Omega \to \dot{H}^\nu$ is a $\mathcal{F}_0$-measurable random variable, it holds that

$$\| u_0 \|_{L^p(\Omega, \dot{H}^\nu)} < \infty, \tag{2.20}$$

for any $0 \leq \nu < \alpha < 2$.

Theorem 2.3 *Let Assumptions (2.3.1) to (2.3.3) be satisfied for some $p \geq 2$, then there exists a unique mild solution $(u(t))_{t\in[0,T]}$ in the space $L^p(\Omega, \dot{H}^\nu)$ with $0 \leq \nu < \alpha < 2$.*

Proof. We fix an $\omega \in \Omega$ and use the standard Picard's iteration argument to prove the existence of mild solution. In the beginning, we construct the sequence of stochastic process $\{u(t)\}_{n\geq 0}$ as

$$\begin{cases} u_{n+1}(t) = E_\beta(t)u_0 + N_1(u_n(t)) + N_2(u_n(t)), \\ u_0(t) = u_0, \end{cases} \tag{2.21}$$

where

$$\begin{cases} N_1(u_n(t)) = \int\limits_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) B(u_n(s), c(s)) ds, \\ N_2(u_n(t)) = \int\limits_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) g(u_n(s)) dW(s). \end{cases} \tag{2.22}$$

The proof will be split into three steps.

Step1 For each $n \geq 0$, we show that

$$\sup \mathbb{E}[\| u_n(t) \|^p_{\dot{H}^\nu}] < \infty,$$

Note that

$$\begin{aligned} \mathbb{E}[\| u_{n+1}(t) \|^p_{\dot{H}^\nu}] &\leq 3^{p-1}\mathbb{E}[\| E_\beta(t)u_0 \|^p_{\dot{H}^\nu}] + 3^{p-1}\mathbb{E}[\| N_1(u_n(t)) \|^p_{\dot{H}^\nu}] \\ &+ 3^{p-1}\mathbb{E}[\| N_2(u_m(t)) \|^p_{\dot{H}^\nu}]. \end{aligned} \tag{2.23}$$

The application of the Lemma (2.1) gives

$$\begin{aligned} \mathbb{E}[\| E_\beta(t)u_0 \|^p_{\dot{H}^\nu}] &\leq \mathbb{E}[\int\limits_0^\infty M_\beta(\theta)(\| A_\nu \mathcal{S}_\alpha(t^\beta\theta)u_0 \|^2)^{\frac{1}{2}} d\theta] \\ &= \mathbb{E}[\int\limits_0^\infty M_\beta(\theta)(\textstyle\sum_{n=1}^\infty \langle A_\alpha e^{-t^\beta\theta A_\alpha} u_0, e_n\rangle^2)^{\frac{1}{2}} d\theta] \\ &= \mathbb{E}[\int\limits_0^\infty M_\beta(\theta)(\textstyle\sum_{n=1}^\infty \langle A_\alpha u_0, e^{-t^\beta\theta\lambda^{\frac{\alpha}{2}}} e_n\rangle^2)^{\frac{1}{2}} d\theta] \\ &\leq \mathbb{E}[\int\limits_0^\infty M_\beta(\theta) \| u_0 \| d\theta] = \mathbb{E}[\| u_0 \|_{\dot{H}^\nu}]. \end{aligned} \tag{2.24}$$

Applying the following Hölder inequality to the second term of the right-hand side of (2.23)

$$\begin{aligned} \mathbb{E}[\| N_1(u_n(t)) \|^p_{\dot{H}^\nu}] &\leq \mathbb{E}[(\int\limits_0^t \| (t-s)^{\beta-1} A_1 E_{\beta\beta}(t-s) A_{\nu-1} B(u_n(s), c(s)) \| ds)^p] \\ &\leq C^p_\beta(\int\limits_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta}{\alpha})}{p-1}} ds)^{p-1} \int\limits_0^t \mathbb{E}[\| A_{\nu-1} B(u_n(s), c(s)) \|^p] ds \\ &\leq K_1 \int\limits_0^t \mathbb{E}[\| u_n(s) \|^p_{\dot{H}^\nu}] ds, \end{aligned} \tag{2.25}$$

where $K_1 = C^p_\beta C^p C^p_1 [\frac{p-1}{p\beta(1-\frac{1}{alpha})-1}]^{p-1} T^{p\beta(1\frac{1}{\alpha})-1} (\max\limits_{t\in[0,T]} \mathbb{E}[\| u_n(t) \|^p_{\dot{H}^\nu}$.

Making use of the Hölder inequality and Lemma (2.2) to the third term of the right-hand side of (2.23), we get

$$\begin{aligned} \mathbb{E}[\| N_2(u_n(t)) \|^p_{\dot{H}^\nu}] &\leq C(p)\mathbb{E}[(\int\limits_0^t \| (t-s)^{\beta-1} E_{\beta\beta}(t-s) \|^2 \| A_\nu g(u_n(s)) \|^2_{L^2_0} ds)^{\frac{p}{2}}] \\ &\leq K_2 \int\limits_0^t \mathbb{E}[\| u_n(s) \|^2_{\dot{H}^\nu}] ds, \end{aligned} \tag{2.26}$$

where $K_2 = C(p)C_\beta^p C^p C_1^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}}$.

Using the above estimates (2.23) and (2.26), we have

$$\mathbb{E}[\| u_{n+1}(t) \|_{\dot{H}^\nu}^p] \le 3^{p-1}\mathbb{E}[\| u_0 \|_{\dot{H}^\nu}^p] + 3^{p-1}(K_1 + K_2)\int_0^t \mathbb{E}[\| u_n(s) \|_{\dot{H}^\nu}^p]ds.$$

By means of the extension of Gronwall's lemma, it holds that

$$\sup_{t\in[0,T]} \mathbb{E}[\| u_{n+1}(t) \|_{\dot{H}^\nu}^p] < \infty, \text{ for each } n \ge 0.$$

Step2 Show that the sequence $\{u_n(t)\}_{n\ge0}$ is a Cauchy sequence in the space $L^p(\Omega;\dot{H}^\nu)$. For any $n \ge m \ge 1$, applying the similar arguments employed to obtain (2.25) and (2.26), we get

$$\begin{aligned}
\mathbb{E}[\| u_n(t) - u_m(t) \|_{\dot{H}^\nu}^p] &\le 2^{p-1}\mathbb{E}[\| N_1(u_{n-1}(t)) - N_1(u_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\
&+ 2^{p-1}\mathbb{E}[\| N_2(u_{n-1}(t)) - N_2(u_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\
&\le K\int_0^t \mathbb{E}[\| u_{n-1}(s) - u_{m-1}(s) \|_{\dot{H}^\nu}^p]ds,
\end{aligned} \tag{2.27}$$

in which

$$\begin{aligned}
K &= 2^{p-1}\{C_\beta^p C^p C_1^p\}[\frac{p-1}{p(\beta-\frac{\beta}{\nu})-1}]^{p-1}T^{p(\beta-\frac{\beta}{\alpha})}(\max_{t\in[0,T]} \mathbb{E}[\| u_{n-1}(t) \|_{\dot{H}^\nu}^p]) \\
&+ \max_{t\in[0,T]} \mathbb{E}[\| u_{m-1}(t) \|_{\dot{H}^\nu}^p]) + C(p)C_\beta^p C^p C_1^p[\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}}T^{\frac{p(2\beta-1)-2}{2}}\}.
\end{aligned}$$

A direct application of Gronwall's lemma yields

$$\sup_{t\in[0,T]} \mathbb{E}[\| u_n(t) - u_m(t) \|_{\dot{H}^\nu}^p] = 0,$$

for all $T > 0$. Taking limits to the stochastic sequence $\{u_n(t)\}_{n\ge0}$ in (2.21) as $n \to \infty$, we finish the proof of the existence of mild solution to (2.5).

Step3 We show the uniqueness of mild solution. Assume u and v are two mild solutions of the problem (2.5), using the similar calculations as in **Step 2**, we can obtain

$$\sup_{t\in[0,T]} \mathbb{E}[\| u(t) - v(t) \|_{\dot{H}^\nu}^p] = 0, \tag{2.28}$$

for all $T > 0$, which implies that $u = v$, it follows that the uniqueness of mild solution. Obviously, when $\nu = 0$, the above three steps still work. Thus the proof of Theorem (2.3) is completed. ■

2.4 Regularity of mild solution

In this section, we will prove the spatial and temporal regularity properties of mild solution to time-space fractional SKSM based on the analytic semigroup.

Theorem 2.4 *Let Assumptions (2.3.1) to (2.3.3) hold with $1 \le \nu < \alpha < 2$ and $p \ge 2$ let $u(t)$ be a unique mild solution of the problem (2.5) with $\mathbb{P}(u(t) \in \dot{H}^\nu) = 1$ for any $t \in [0,T]$, then there exists a constant C such that*

$$\sup_{t\in[0,T]} \| u(t) \|_{L^p(\Omega;\dot{H}^\nu)} \le C(\| u_0 \|_{L^p(\Omega;H)} + \sup_{t\in[0,T]} \| u(t) \|_{L^p(\Omega;\dot{H}^1)}). \tag{2.29}$$

Proof. For any $0 \leq t \leq T$ and $1 \leq \nu < \alpha < 2$, we have

$$\begin{aligned}
\| u(t) \|_{L^p(\Omega;\dot{H}^\nu)} &= (\mathbb{E}[\| u(t) \|_{\dot{H}^\nu}^p])^{\frac{1}{p}} = \| A_\nu u(t) \|_{L^p(\Omega;H)} \\
&\leq \| A_\nu E_\beta(t) u_0 \|_{L^p(\Omega;H)} \\
&+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) B(u(s),c(s)) ds \|_{L^p(\Omega;H)} \\
&+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) g(u(s)) dW(s) \|_{L^p(\Omega;H)} \\
&= I + II + III.
\end{aligned} \tag{2.30}$$

Using Theorem (2.1), the first term can be estimated by

$$I = \| A_\nu E_\beta(t) u_0 \|_{L^p(\Omega;H)} \leq C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| u_0 \|_{L^p(\Omega;H)} < \infty. \tag{2.31}$$

It is easy to know that

$$\int_0^T C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| u_0 \|_{L^p(\Omega;H)} dt = \frac{\alpha C_\alpha}{\alpha - \beta\nu} T^{\frac{\alpha-\beta\nu}{\alpha}} \| u_0 \|_{L^p(\Omega;H)} . \tag{2.32}$$

The application of Theorem (2.1) and Assumptions (2.3.2), we get

$$\begin{aligned}
(II)^p &\leq \mathbb{E}[(\| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) B(u(s),c(s)) \| ds)^p] \\
&\leq C_\beta^p (\int_0^t (t-s)^{\frac{p[\beta-1-\frac{\beta(\nu+1)}{\alpha}]}{p-1}} ds)^{p-1} \int_0^t \mathbb{E}[\| A_{-1} B(u(s),c(s)) \|_{\dot{H}^1}^p] ds \\
&\leq C_2 \sup_{t\in[0,T]} \mathbb{E}[\| u(s) \|_{\dot{H}^1}^p],
\end{aligned} \tag{2.33}$$

where $C_2 = C_\beta^p C^p C_1^p \{\frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}]-1}\}^{p-1} T^{p[\beta-\frac{\beta(\nu+1)}{\alpha}]-1} + (\max_{t\in[0,T]} \mathbb{E}[\| u(t) \|_{\dot{H}^1}])$.
By means of Theorem (2.1), Assumptions (2.3.1) and Lemma (2.2), we can deduce

$$\begin{aligned}
(III)^p &\leq C(p)\mathbb{E}[(\| \int_0^t \| (t-s)^{\beta-1} A_{\nu-1} E_{\beta\beta}(t-s) \|^2 \| A_1 g(u(s)) \|_{L_2^0}^2 ds)^{\frac{p}{2}}] \\
&\leq C(p) C_\beta^p (\int_0^t (t-s)^{\frac{2p[\beta-1-\frac{\beta(\nu-1)}{\alpha}]}{p-2}} ds)^{\frac{p-2}{2}} \int_0^t \mathbb{E} \| A_1 g(u(s)) \|_{L_2^0}^p ds \\
&\leq C_3 \sup_{t\in[0,T]} \mathbb{E}[\| u(s) \|_{\dot{H}^1}^p],
\end{aligned} \tag{2.34}$$

where $C_3 = C(p) C_\beta^p C^p C_1^p [\frac{p-2}{p[2\beta-1-\frac{\beta(\nu-1)}{\alpha}]-2}]^{\frac{p-2}{2}} T^{\frac{p[2\beta-1-\frac{\beta(\nu-1)}{\alpha}]-2}{2}}$.

Thus, we conclude the proof of Theorem (2.4) by combining with the estimates (2.30)- (2.34). ■
Next, we will devote to the temporal regularity of the mild solution.

Theorem 2.5 *Let Assumptions* (2.3.1) *to* (2.3.3) *hold with* $0 < \nu < \alpha < 2$ *and* $p \geq 2$ *for any* $0 \leq t_1 < t_2 \leq T$*, the unique mild solution* $u(t)$*to the problem* (2.5) *is Hölder continuous with respect to the norm* $\| . \|_{L^p(\Omega;\dot{H}^\nu)}$ *and satisfies*

$$\| u(t_2) - u(t_1) \|_{L^p(\Omega;\dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma. \tag{2.35}$$

Proof. Foor any $0 \leq t_1 < t_2 \leq T$, for the mild solution (2.9), we have

$$\begin{aligned} u(t_2) - u(t_1) &= E_\beta(t_2)u_0 - E_\beta(t_1)u_0 \\ &+ \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \\ &- \int_0^{t1} (t_1 - s)^{\beta-1} E_{\beta\beta}(t_1 - s) B(u(s), c(s)) ds \\ &+ \int_0^{t_2} (t_2 - s)^{\beta-1\beta-1} E_{\beta\beta}(t_2 - s) g(u(s)) dW(s) \\ &- \int_0^{t_1} (t_1 - s)^{\beta-1\beta-1} E_{\beta\beta}(t_1 - s) g(u(s)) dW(s) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{2.36}$$

where

$$I_1 = E_\beta(t_2)u_0 - E_\beta(t_1)u_0,$$

and

$$\begin{aligned} I_2 &= \int_0^{t_2} (t_2 - s)^{\beta-1} E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \\ &- \int_0^{t1} (t_1 - s)^{\beta-1} E_{\beta\beta}(t_1 - s) B(u(s), c(s)) ds \\ &= \int_0^{t1} (t_1 - s)^{\beta-1} [E_{\beta\beta}(t_2 - s) - E_{\beta\beta}(t_1 - s)] B(u(s), c(s)) ds \\ &+ \int_0^{t1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \\ &+ \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \\ &= I_{21} + I_{22} + I_{23} \end{aligned} \tag{2.37}$$

and

$$
\begin{aligned}
I_3 &= \int_0^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)g(u(s))dW(s)\\
&- \int_0^{t_1}(t_1-s)^{\beta-1}E_{\beta\beta}(t_1-s)g(u(s))dW(s)\\
&= \int_0^{t_1}(t_1-s)^{\beta-1}[E_{\beta\beta}(t_2-s)-E_{\beta\beta}(t_1-s)]g(u(s))dW(s)\\
&+ \int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]E_{\beta\beta}(t_2-s)g(u(s))dW(s)\\
&+ \int_{t_1}^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)g(u(s))dW(s)\\
&= I_{31}+I_{32}+I_{33}.
\end{aligned}
\tag{2.38}
$$

For any $0<\nu<\alpha<2$ and $p\geq 2$, by by virtue of Theorem (2.2), it follows that

$$
\begin{aligned}
\mathbb{E}[\| I_1 \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| A_\nu[E_\beta(t_2)-E_\beta(t_1)]u_0 \|^p]\\
&\leq C^p_{\alpha,\nu}(t_2-t_1)^{\frac{p\beta\nu}{\alpha}}\mathbb{E}[\| u_0 \|^p].
\end{aligned}
\tag{2.39}
$$

For the first term I_{21} in (2.37), applying the Assumption (2.3.2) and Theorem (2.2) and Hölder's inequality, we have

$$
\begin{aligned}
\mathbb{E}[\| I_{21} \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| \int_0^{t1}(t_1-s)^{\beta-1}A_\nu[E_{\beta\beta}(t_2-s)-E_{\beta\beta}(t_1-s)]B(u(s),c(s))ds \|^p]\\
&\leq C^p_{\beta\nu}(t_2-t_1)^{\frac{p\beta(\nu+1)}{\alpha}}(\int_0^{t1}(t_1-s)^{\frac{p(\beta-1)}{p-1}}ds)^{p-1}\int_0^{t1}\mathbb{E}[\| A_{-1}B(u(s),c(s)) \|^p_{\dot{H}^1}]ds\\
&\leq C^pC_1^pC^p_{\beta\nu}T^p(\tfrac{p-1}{p\beta-1})^{p-1}(\sup_{t\in[0,T]}\mathbb{E}[\| u(s) \|^{2p}_{\dot{H}^1}])(t_2-t_1)^{\frac{p\beta(\nu+1)}{\alpha}}.
\end{aligned}
\tag{2.40}
$$

Using the Assumptions (2.3.2), Theorem (2.1) and Hölder's inequality, we get

$$
\begin{aligned}
\mathbb{E}[\| I_{22} \|^p_{\dot{H}^\nu} &= \mathbb{E}[\| \int_0^{t1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]A_\nu E_{\beta\beta}(t_2-s)B(u(s),c(s))ds \|^p]\\
&\leq C^p_\beta(\int_0^{t1}\{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]\times(t_2-s)^{-\frac{\beta(\nu+1)}{\alpha}}\}^{\frac{p}{p-1}}ds)^{p-1}\\
&\times \int_0^{t1}\mathbb{E}[\| A_1B(u(s),c(s)) \|^p_{\dot{H}^1}]ds\\
&\leq C^pC_1^pC^p_\beta T\{\frac{p-1}{p[\beta-\frac{\beta(\nu+1)}{\alpha}]-1}\}^{p-1}(\sup_{t\in[0,T]}\mathbb{E}[\| u(s) \|^{2p}_{\dot{H}^1}])(t_2-t_1)^{\frac{p\beta(\alpha-v-1)-\alpha}{\alpha}},
\end{aligned}
\tag{2.41}
$$

and

$$\begin{aligned}
\mathbb{E}[\| I_{23} \|^p_{\dot{H}^\nu}] &= E[\| \int\limits_{t_1}^{t_2} (t_2 s)^{\beta-1} A_\nu E_{\beta\beta}(t_2 - s) B(u(s), c(s)) ds \|^p] \\
&\leq C^p_\beta (\int\limits_{t_1}^{t_2} [(t_2 - s)^{\beta - 1 - \frac{\beta(\nu+1)}{\alpha}}]^{\frac{p}{p-1}} ds)^{p-1} \\
&\times \int\limits_{t_1}^{t_2} \mathbb{E}[\| A_1 B(u(s), c(s)) \|^p_{\dot{H}^1}] ds \\
&\leq C^p C^p_1 C^p_\beta \{\frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}] - 1}\}^{p-1} (\sup_{t\in[0,T]} \mathbb{E}[\| u(s) \|^{2p}_{\dot{H}^1}]) (t_2 - t_1)^{\frac{p\beta(\alpha - v - 1)}{\alpha}}.
\end{aligned} \tag{2.42}$$

Next, by following the similar arguments as in the proof of (2.40)- (2.42) and using the Lemma (2.2), there holds

$$\begin{aligned}
\mathbb{E}[\| I_{31} \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| \int\limits_0^{t_1} (t_1 - s)^{\beta-1} A_\nu [E_{\beta\beta}(t_2 - s) - E_{\beta\beta}(t_1 - s)] g(u(s)) dW(s) \|^p] \\
&\leq C(p) \mathbb{E}[(\int\limits_0^{t_1} \| (t_1 - s)^{\beta-1} A_\nu [E_{\beta\beta}(t_2 - s) - E_{\beta\beta}(t_1 - s)] \|^2 \| g(u(s)) \|^2_{L^2_0} ds)^{\frac{p}{2}}] \\
&\leq C(p) C^p_{\beta\nu} (t_2 - t_1)^{\frac{p\beta\nu}{\alpha}} (\int\limits_0^{t_1} (t_1 - s)^{\frac{2p(\beta-1)}{p-2}} ds)^{\frac{p-2}{2}} \int\limits_0^{t_1} \mathbb{E} \| g(u(s)) \|^p_{L^2_0} ds \\
&\leq C(p) C^p_{\beta\nu} T^{\frac{2p\beta - p - 1}{2}} (\frac{p-1}{2p\beta - p - 2})^{p-1} (\sup_{t\in[0,T]} E[\| u(t) \|^p]) (t_1 - t_2)^{\frac{p\beta\nu}{\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\| I_{32} \|] &= \mathbb{E}[\| \int\limits_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] A_\nu E_{\beta\beta}(t_2 - s) g(u(s)) dW(s) \|^p] \\
&\leq C(p) \mathbb{E}[(\int\limits_0^{t_1} \| ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}) A_\nu E_{\beta\beta}(t_2 - s) \|^2 \| g(u(s)) \|^2_{L^2_0})^{\frac{p}{2}}] ds \\
&\leq C(p) C^p_\beta (\int\limits_0^{t_1} \{[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \times (t_2 - s)^{-\frac{\beta\nu}{2}}\}^{\frac{2p}{p-2}} ds)^{\frac{p-2}{2}} \\
&\times \int\limits_0^{t_1} \mathbb{E} \| g(u(s)) \|^p_{L^2_0} ds \\
&\leq C(p) C^p_\beta C^p T [\frac{\alpha(p-2)}{2p\beta(\alpha - \nu) - (p+2)\alpha}]^{\frac{p-2}{2}} (\sup_{t\in[0,T]} \mathbb{E}[\| u(t) \|^p]) (t_2 - t_1)^{\frac{2p\beta(\alpha-\nu) - (p+2)\alpha}{2\alpha}},
\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}
\mathbb{E}[\parallel I_{33} \parallel] &= \mathbb{E}[\parallel \int_{t_1}^{t_2} (t_2-s)^{\beta-1} A_\nu E_{\beta\beta}(t_2-s) g(u(s)) dW(s) \parallel^p] \\
&\leq C(p)\mathbb{E}[\int_{t_1}^{t_2} (t_2-s)^{\beta-1} A_\nu E_{\beta\beta}(t_2-s) \parallel^2 \parallel g(u(s)) \parallel_{L_0^2}^2 ds)^{\frac{p}{2}}] \\
&\leq C(p)C_\beta^p (\int_{t_1}^{t_2} [(t_2-s)^{\beta-1-\frac{\beta\nu}{\alpha}}]^{\frac{2p}{p-2}} ds)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E} \parallel g(u) \parallel_{L_0^2}^p ds \\
&\leq C(p)C_\beta^p C^\rho [\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}} (\sup_{t\in[0,T]} \mathbb{E}[\parallel u(t) \parallel^p])(t_2-t_1)^{\frac{2p\beta(\alpha-\nu)-p\alpha}{2\alpha}}.
\end{aligned} \tag{2.44}$$

■

Taking expectation on the both side of (2.36), and in view of the estimates (2.39)- (2.44), we conclude that

$$\parallel u(t_2) - u(t_1) \parallel_{L^p(\Omega;\dot{H}^v)} \leq C(t_2-t_1)^\gamma, \tag{2.45}$$

in which we tak $\gamma = \min\{\frac{\beta\nu}{\alpha}, \frac{p\beta(\alpha-\nu-1)-\alpha}{p\alpha}, \frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2p\alpha}\}$, whene $0 < t_2 - t_1 < 1$.

Otherwise, if $t_2 - t_1 \geq 1$ then we set $\gamma = \max\{\frac{\beta(\nu+1)}{\alpha}, \frac{\beta(\alpha-\nu-1)}{\alpha}, \frac{2p\beta(\alpha-\nu)-p\alpha}{2p\alpha}\}$. This completes the proof of Theorem (2.5).

2.5 Existence and uniqueness of mild solution

Our main purpose of this section is to prove the existence and uniqueness of mild solution to the problem (2.10). To do this, the following assumptions are imposed.

2.5.1 Assumption 1

The measurable function $f : \Omega \times H \to L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$\parallel f(v) \parallel_{L_0^2} \leq C \parallel v \parallel, \quad \parallel f(u) - f(v) \parallel_{L_0^2} \leq C \parallel u - v \parallel, \tag{2.46}$$

for all $u, v \in H$.

2.5.2 Assumption 2

Let C, is a positive real number, then the bounded bilinear operator
$L : L_0^2(D) \to H^{-1}(D)$ satisfies the following properties:

$$\parallel L(c) \parallel_{\dot{H}^{-1}} \leq C \parallel c \parallel^2, \tag{2.47}$$

and

$$\parallel L(c) - L(v) \parallel_{\dot{H}^{-1}} \leq C(\parallel c \parallel + \parallel v \parallel) \parallel c - v \parallel, \tag{2.48}$$

and for all $v, c \in L_0^2(D)$.

2.5.3 Assumption 3

Assume that the initial value $c_0 : \Omega \to \dot{H}^\nu$ is a $\mathcal{F}_0$-measurable random variable, it holds that

$$\parallel c_0 \parallel_{L^p(\Omega, \dot{H}^\nu)} < \infty, \tag{2.49}$$

for any $0 \leq \nu < \alpha < 2$.

Theorem 2.6 *Let Assumptions (2.5.1) to (2.5.3) be satisfied for some $p \geq 2$, then there exists a unique mild solution $(c(t))_{t \in [0,T]}$ in the space $L^p(\Omega, \dot{H}^\nu)$ with $0 \leq \nu < \alpha < 2$.*

Proof. We fix an $\omega \in \Omega$ and use the standard Picard's iteration argument to prove the existence of mild solution. In the beginning, we construct the sequence of stochastic process $\{c_n(t)\}_{n \geq 0}$ as

$$\begin{cases} c_{n+1}(t) & = E_\beta(t)c_0 + N_1(c_n(t)) + N_2(c_n(t)), \\ c_0(t) & = c_0, \end{cases} \tag{2.50}$$

where

$$\begin{cases} N_1(c_n(t)) = \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) L(c_n(s)) ds, \\ N_2(c_n(t)) = \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) f(c_n(s)) dW(s). \end{cases} \tag{2.51}$$

The proof will be split into three steps.
Step1 For each $n \geq 0$, we show that

$$\sup \mathbb{E}[\parallel c_n(t) \parallel^p_{\dot{H}^\nu}] < \infty,$$

note that

$$\begin{aligned} \mathbb{E}[\parallel c_{n+1}(t) \parallel^p_{\dot{H}^\nu}] \;&\leq\; 3^{p-1}\mathbb{E}[\parallel E_\beta(t)c_0 \parallel^p_{\dot{H}^\nu}] + 3^{p-1}\mathbb{E}[\parallel N_1(c_n(t)) \parallel^p_{\dot{H}^\nu}] \\ &+\; 3^{p-1}\mathbb{E}[\parallel N_2(c_n(t)) \parallel^p_{\dot{H}^\nu}]. \end{aligned} \tag{2.52}$$

The application of the Lemma (2.1) gives

$$\begin{aligned} \mathbb{E}[\parallel E_\beta(t)c_0 \parallel^p_{\dot{H}^\nu}] \;&\leq\; \mathbb{E}[\int_0^\infty M_\beta(\theta)(\parallel A_\nu S_\alpha(t^\beta\theta)c_0 \parallel^2)^{\frac{1}{2}} d\theta] \\ &=\; \mathbb{E}[\int_0^\infty M_\beta(\theta)(\textstyle\sum_{n=1}^\infty \langle A_\alpha e^{-t^\beta\theta A_\alpha} c_0, e_n\rangle^2)^{\frac{1}{2}} d\theta] \\ &=\; \mathbb{E}[\int_0^\infty M_\beta(\theta)(\textstyle\sum_{n=1}^\infty \langle A_\alpha c_0, e^{-t^\beta\theta\lambda^{\frac{\alpha}{2}}} e_n\rangle^2)^{\frac{1}{2}} d\theta] \\ &\leq\; \mathbb{E}[\int_0^\infty M_\beta(\theta) \parallel c_0 \parallel_{\dot{H}^\nu} d\theta] = \mathbb{E}[\parallel c_0 \parallel_{\dot{H}^\nu}]. \end{aligned} \tag{2.53}$$

Applying the following Hölder inequality to the second term of the right-hand side of (2.52)

$$\begin{aligned} \mathbb{E}[\parallel N_1(c_n(t)) \parallel^p_{\dot{H}^\nu}] \;&\leq\; \mathbb{E}[(\int_0^t \parallel (t-s)^{\beta-1} A_1 E_{\beta\beta}(t-s) A_{\nu-1} L(c_n(s)) \parallel ds)^p] \\ &\leq\; C^p_\beta(\int_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta}{\alpha})}{p-1}} ds)^{p-1} \int_0^t \mathbb{E}[\parallel A_{\nu-1} L(c_n(s)) \parallel^p] ds \\ &\leq\; K_1 \int_0^t \mathbb{E}[\parallel c_n(s) \parallel^p_{\dot{H}^\nu}]] ds, \end{aligned} \tag{2.54}$$

where $K_1 = C_\beta^p C^p [\frac{p-1}{p\beta(1-\frac{1}{\alpha})-1}]^{p-1} T^{p\beta(1-\frac{1}{\alpha})-1} (\max_{t\in[0,T]} \mathbb{E}[\| c_n(t) \|_{\dot{H}^\nu}^p)$.
Making use of the Hölder inequality and Lemma (2.2) to the third term of the right-hand side of (2.52), we get

$$\begin{aligned} \mathbb{E}[\| N_2(c_n(t)) \|_{\dot{H}^\nu}^p] &\leq C(p)\mathbb{E}[(\int_0^t \| (t-s)^{\beta-1} E_{\beta\beta}(t-s) \|^2 A_\nu f(c_n(s)) \|_{L_0^2}^2 ds)^{\frac{p}{2}}] \\ &\leq K_2 \int_0^t \mathbb{E}[\| c_n(s) \|_{\dot{H}^\nu}^2] ds, \end{aligned} \tag{2.55}$$

where $K_2 = C(p) C_\beta^p C^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}}$.

Using the above estimates (2.52)- (2.55), we have

$$\mathbb{E}[\| c_{n+1}(t) \|_{\dot{H}^\nu}^p] \leq 3^{p-1}\mathbb{E}[\| c_0 \|_{\dot{H}^\nu}^p] + 3^{p-1}(K_1+K_2)\int_0^t \mathbb{E}[\| c_n(s) \|_{\dot{H}^\nu}^p] ds.$$

By means of the extension of Gronwall's lemma, it holds that

$$\sup_{t\in[0,T]} \mathbb{E}[\| c_{n+1}(t) \|_{\dot{H}^\nu}^p] < \infty, \text{ for each } n \geq 0.$$

Step1: Show that the sequence $\{c_n(t)\}_{n\geq 0}$ is a Cauchy sequence in the space $L^p(\Omega; \dot{H}^\nu)$. For any $n \geq m \geq 1$, applying the similar arguments employed to obtain (2.54) and (2.55), we get

$$\begin{aligned} \mathbb{E}[\| c_n(t) - c_m(t) \|_{\dot{H}^\nu}^p] &\leq 2^{p-1}\mathbb{E}[\| N_1(c_{n-1}(t)) - N_1(c_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\ &+ 2^{p-1}\mathbb{E}[\| N_2(c_{n-1}(t)) - N_2(c_{m-1}(t)) \|_{\dot{H}^\nu}^p] \\ &\leq K \int_0^t \mathbb{E}[\| c_{n-1}(s) - c_{m-1}(s) \|_{\dot{H}^\nu}^p] ds, \end{aligned} \tag{2.56}$$

in which

$$\begin{aligned} K &= 2^{p-1}\{C_\beta^p C^p\}[\frac{p-1}{p(\beta-\frac{\beta}{\nu})-1}]^{p-1} T^{p(\beta-\frac{\beta}{\alpha})} (\max_{t\in[0,T]} \mathbb{E}[\| c_{n-1}(t) \|_{\dot{H}^\nu}^p]) \\ &+ \max_{t\in[0,T]} \mathbb{E}[\| c_{m-1}(t) \|_{\dot{H}^\nu}^p]) + C(p) C_\beta^p C^p [\frac{p-2}{p(2\beta-1)-2}]^{\frac{p-2}{2}} T^{\frac{p(2\beta-1)-2}{2}}\}. \end{aligned} \tag{2.57}$$

A direct application of Gronwall's lemma yields

$$\sup_{t\in[0,T]} \mathbb{E}[\| c_n(t) - c_m(t) \|_{\dot{H}^\nu}^p] = 0, \ for\ all\ T > 0.$$

Taking limits to the stochastic sequence $\{c_n(t)\}_{n\geq 0}$ in (2.50) as $n \to \infty$, we finish the proof of the existence of mild solution to (2.10).
Step 3: We show the uniqueness of mild solution. Assume c and v are two mild solutions of the problem (2.10), using the similar calculations as in **Step 2**, we can obtain

$$\sup_{t\in[0,T]} \mathbb{E}[\| c(t) - v(t) \|_{\dot{H}^\nu}^p] = 0, \tag{2.58}$$

for all $T > 0$, which implies that $c = v$, it follows that the uniqueness of mild solution. Obviously, when $\nu = 0$, the above three steps still work. Thus the proof of Theorem (2.6) is completed. ■

2.6 Regularity of mild solution

In this section, we will prove the spatial and temporal regularity properties of mild solution to time-space fractional SKSM based on the analytic semigroup.

Theorem 2.7 *Let Assumptions (2.5.1) to (2.5.3) hold with $1 \leq \nu < \alpha < 2$ and $p \geq 2$, let $c(t)$ be a unique mild solution of the problem (2.10) with $\mathbb{P}(c(t) \in \dot{H}^{\nu}) = 1$ for any $t \in [0,T]$, then there exists a constant C such that*

$$\sup_{t\in[0,T]} \| c(t) \|_{L^p(\Omega;\dot{H}^{\nu})} \leq C(\| c_0 \|_{L^p(\Omega;H)} + \sup_{t\in[0,T]} \| c(t) \|_{L^p(\Omega;\dot{H}^1)}). \tag{2.59}$$

Proof. For any $0 \leq t \leq T$ and $1 \leq \nu < \alpha < 2$, we have

$$\begin{aligned} \| c(t) \|_{L^p(\Omega;\dot{H}^{\nu})} &= (\mathbb{E}[\| c(t) \|^p_{\dot{H}^{\nu}}])^{\frac{1}{p}} = \| A_\nu c(t) \|_{L^p(\Omega;H)} \\ &\leq \| A_\nu E_\beta(t) c_0 \|_{L^p(\Omega;H)} \\ &+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) L(c(s)) ds \|_{L^p(\Omega;H)} \\ &+ \| A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta\beta}(t-s) f(c(s)) dW(s) \|_{L^p(\Omega;H)} \\ &= I + II + III. \end{aligned} \tag{2.60}$$

Using Theorem (2.1), the first term can be estimated by

$$I = \| A_\nu E_\beta(t) c_0 \|_{L^p(\Omega;H)} \leq C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega;H)} < \infty. \tag{2.61}$$

It is easy to know that

$$\int_0^T C_\alpha t^{-\frac{\beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega;H)} dt = \frac{\alpha C_\alpha}{\alpha - \beta\nu} T^{\frac{\alpha-\beta\nu}{\alpha}} \| c_0 \|_{L^p(\Omega;H)} . \tag{2.62}$$

The application of Theorem (2.1) and Assumptions (2.5.2), we get

$$\begin{aligned} (II)^p &\leq \mathbb{E}[(\| \int_0^t (t-s)^{\beta-1} A_\nu E_{\beta\beta}(t-s) L(c(s)) \| ds)^p] \\ &\leq C_\beta^p (\int_0^t (t-s)^{\frac{p[\beta-1-\frac{\beta(\nu+1)}{\alpha}]}{p-1}} ds)^{p-1} \int_0^t \mathbb{E}[\| A_{-1} L(c(s)) \|^p_{\dot{H}^1}] ds \\ &\leq C_2 \sup_{t\in[0,T]} \mathbb{E}[\| c(s) \|^p_{\dot{H}^1}], \end{aligned} \tag{2.63}$$

where $C_2 = C_\beta^p C^p \{\frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}]-1}\}^{p-1} T^{p[\beta-\frac{\beta(\nu+1)}{\alpha}]-1} + (\max_{t\in[0,T]} \mathbb{E}[\| c(t) \|_{\dot{H}^1}])$.
By means of Theorem (2.1), Assumptions (2.5.1) and Lemma (2.2), we can deduce

$$\begin{aligned}
(III)^p &\leq C(p)\mathbb{E}[(\| A_\nu \int_0^t \| (t-s)^{\beta-1} A_{\nu-1} E_{\beta\beta}(t-s) \|^2 \| A_1 f(c(s)) \|^2_{L_2^0} ds)^{\frac{p}{2}}] \\
&\leq C(p) C_\beta^p (\int_0^t (t-s)^{\frac{2p[\beta-1-\frac{\beta(\nu-1)}{\alpha}]}{p-2}} ds)^{\frac{p-2}{2}} \int_0^t \mathbb{E} \| A_1 f(c(s)) \|^p_{L_2^0} ds \\
&\leq C_3 \sup_{t\in[0,T]} \mathbb{E}[\| c(s) \|^p_{\dot{H}^1}],
\end{aligned} \tag{2.64}$$

where $C_3 = C(p) C_\beta^p C^p [\frac{p-2}{p[2\beta-1-\frac{\beta(\nu-1)}{\alpha}]-2}]^{\frac{p-2}{2}} T^{\frac{p[2\beta-1-\frac{\beta(\nu-1)}{\alpha}]-2}{2}}$.

Thus, we conclude the proof of Theorem (2.7) by combining with the estimates (2.60)- (2.64). ■ Next, we will devote to the temporal regularity of the mild solution.

Theorem 2.8 *Let Assumptions* (2.5.1) *to* (2.5.3) *hold with* $0 < \nu < \alpha < 2$ *and* $p \geq 2$*, for any* $0 \leq t_1 < t_2 \leq T$*, the unique mild solution* $c(t)$*to the problem* (2.10) *is Hölder continuous with respect to the norm* $\| . \|_{L^p(\Omega;\dot{H}^\nu)}$ *and satisfies*

$$\| c(t_2) - c(t_1) \|_{L^p(\Omega;\dot{H}^\nu)} \leq C(t_2 - t_1)^\gamma. \tag{2.65}$$

Proof. Foor any $0 \leq t_1 < t_2 \leq T$, for the mild solution (2.10), we have

$$\begin{aligned}
c(t_2) - c(t_1) &= E_\beta(t_2)c_0 - E_\beta(t_1)c_0 \\
&+ \int_0^{t_2} (t_2-s)^{\beta-1} E_{\beta\beta}(t_2-s) L(c(s)) ds \\
&- \int_0^{t1} (t_1-s)^{\beta-1} E_{\beta\beta}(t_1-s) L(c(s)) ds \\
&+ \int_0^{t_2} (t_2-s)^{\beta-1\beta-1} E_{\beta\beta}(t_2-s) f(c(s)) dW(s) \\
&- \int_0^{t_1} (t_2-s)^{\beta-1\beta-1} E_{\beta\beta}(t_1-s) G(u(s)) dW(s) \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{2.66}$$

where

$$I_1 = E_\beta(t_2)c_0 - E_\beta(t_1)c_0,$$

and

$$
\begin{aligned}
I_2 &= \int_0^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)L(c(s))ds \\
&- \int_0^{t1}(t_1-s)^{\beta-1}E_{\beta\beta}(t_1-s)L(c(s))ds \\
&= \int_0^{t1}(t_1-s)^{\beta-1}[E_{\beta\beta}(t_2-s)-E_{\beta\beta}(t_1-s)]L(c(s))ds \\
&+ \int_0^{t1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]E_{\beta\beta}(t_2-s)L(c(s))ds \\
&+ \int_{t_1}^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)L(c(s))ds \\
&= I_{21}+I_{22}+I_{23},
\end{aligned}
\tag{2.67}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)f(c(s))dW(s) \\
&- \int_0^{t_1}(t_1-s)^{\beta-1}E_{\beta\beta}(t_1-s)f(c(s))dW(s) \\
&= \int_0^{t_1}(t_1-s)^{\beta-1}[E_{\beta\beta}(t_2-s)-E_{\beta\beta}(t_1-s)]f(c(s))dW(s) \\
&+ \int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]E_{\beta\beta}(t_2-s)g(u(s))dW(s) \\
&+ \int_{t_1}^{t_2}(t_2-s)^{\beta-1}E_{\beta\beta}(t_2-s)f(c(s))dW(s) \\
&= I_{31}+I_{32}+I_{33}.
\end{aligned}
\tag{2.68}
$$

For any $0<\nu<\alpha<2$ and $p\geq 2$, by virtue of Theorem (2.2), it follows that

$$
\begin{aligned}
\mathbb{E}[\| I_1 \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| A_\nu[E_\beta(t_2)-E_\beta(t_1)]c_0 \|^p] \\
&\leq C^p_{\alpha\nu}(t_2-t_1)^{\frac{p\beta\nu}{\alpha}}\mathbb{E}[\| c_0 \|^p].
\end{aligned}
\tag{2.69}
$$

For the first term I_{21} in (2.67), applying the Assumption (2.5.2) and Theorem (2.8) and Hölder's inequality, we have

$$
\begin{aligned}
\mathbb{E}[\| I_{21} \|^p_{\dot{H}^\nu}] &= E[\| \int_0^{t1}(t_1-s)^{\beta-1}A_\nu[E_{\beta\beta}(t_2-s)-E_{\beta\beta}(t_1-s)]L(c(s))ds \|^p] \\
&\leq C^p_{\beta\nu}(t_2-t_1)^{\frac{p\beta(\nu+1)}{\alpha}}(\int_0^{t1}(t_1-s)^{\frac{p(\beta-1)}{p-1}}ds)^{p-1}\int_0^{t1}\mathbb{E}[\| A_{-1}L(c(s)) \|^p_{\dot{H}^1}]ds \\
&\leq C^pC^p_{\beta\nu}T^p(\tfrac{p-1}{p\beta-1})^{p-1}(\sup_{t\in[0,T]}\mathbb{E}[\| c(s) \|^{2p}_{\dot{H}^1}])(t_2-t_1)^{\frac{p\beta(\nu+1)}{\alpha}}.
\end{aligned}
\tag{2.70}
$$

Using the Assumptions (2.5.2), Theorem (2.8) and Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}[\| I_{22} \|^p_{\dot{H}^\nu} &= \mathbb{E}[\| \int\limits_0^{t1} [(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}] A_\nu E_{\beta\beta}(t_2-s) L(c(s)) ds \|^p] \\
&\leq C^p_\beta (\int\limits_0^{t1} \{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}] \\
&\times (t_2-s)^{-\frac{\beta(\nu+1)}{\alpha}}\}^{\frac{p}{p-1}} ds)^{p-1} \times \int\limits_0^{t1} \mathbb{E}[\| A_1 L(c(s)) \|^p_{\dot{H}^1}] ds \\
&\leq C^p C^p_\beta T \{\frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}]-1}\}^{p-1} (\sup_{t\in[0,T]} \mathbb{E}[\| c(s) \|^{2p}_{\dot{H}^1}])(t_2-t_1)^{\frac{p\beta(\alpha-v-1)-\alpha}{\alpha}},
\end{aligned} \tag{2.71}$$

and

$$\begin{aligned}
\mathbb{E}[\| I_{23} \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| \int\limits_{t_1}^{t_2} (t_2-s)^{\beta-1} A_\nu E_{\beta\beta}(t_2-s) L(c(s)) ds \|^p] \\
&\leq C^p_\beta (\int\limits_{t_1}^{t_2} [(t_2-s)^{\beta-1-\frac{\beta(\nu+1)}{\alpha}}]^{\frac{p}{p-1}} ds)^{p-1} \\
&\times \int\limits_{t_1}^{t_2} \mathbb{E}[\| A_1 L(c(s)) \|^p_{\dot{H}^1}] ds \\
&\leq C^p C^p_\beta \{\frac{p-1}{p[\beta - \frac{\beta(\nu+1)}{\alpha}]-1}\}^{p-1} (\sup_{t\in[0,T]} \mathbb{E}[\| c(s) \|^{2p}_{\dot{H}^1}])(t_2-t_1)^{\frac{p\beta(\alpha-v-1)}{\alpha}}.
\end{aligned} \tag{2.72}$$

Next, by following the similar arguments as in the proof of (2.70)- (2.72) and using the Lemma (2.2), there holds

$$\begin{aligned}
\mathbb{E}[\| I_{31} \|^p_{\dot{H}^\nu}] &= \mathbb{E}[\| \int\limits_0^{t_1} (t_1-s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2-s) - E_{\beta,\beta}(t_1-s)] f(u(s)) dW(s) \|^p] \\
&\leq C(p)\mathbb{E}[(\int\limits_0^{t_1} \| (t_1-s)^{\beta-1} A_\nu [E_{\beta,\beta}(t_2-s) - E_{\beta,\beta}(t_1-s)] \|^2 \| f(c(s)) \|^2_{L^2_0} ds)^{\frac{p}{2}}] \\
&\leq C(p) C^p_{\beta\nu} (t_2-t_1)^{\frac{p\beta\nu}{\alpha}} (\int\limits_0^{t_1} (t_1-s)^{\frac{2p(\beta-1)}{p-2}} ds)^{\frac{p-2}{2}} \int\limits_0^{t_1} \mathbb{E} \| f(c(s)) \|^p_{L^2_0} ds \\
&\leq C(p) C^p_{\beta\nu} T^{\frac{2p\beta-p-1}{2}} (\frac{p-1}{2p\beta-p-2})^{p-1} (\sup_{t\in[0,T]} \mathbb{E}[\| c(t) \|^p])(t_1-t_2)^{\frac{p\beta\nu}{\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\| I_{32} \|] &= \mathbb{E}[\| \int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]A_\nu E_{\beta\beta}(t_2-s)f(c(s))dW(s) \|^p] \\
&\le C(p)\mathbb{E}[(\int_0^{t_1} \| ((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1})A_\nu E_{\beta\beta}(t_2-s) \|^2 \| f(c(s)) \|^2_{L_0^2})^{\frac{p}{2}}]ds \\
&\le C(p)C_\beta^p(\int_0^{t_1}\{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]\times(t_2-s)^{-\frac{\beta\nu}{2}}\}^{\frac{2p}{p-2}}ds)^{\frac{p-2}{2}} \\
&\times \int_0^{t_1}\mathbb{E} \| f(c(s)) \|^p_{L_0^2} ds \\
&\le C(p)C_\beta^p C^p T[\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}}(\sup_{t\in[0,T]}\mathbb{E}[\| c(t) \|^p])(t_2-t_1)^{\frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2\alpha}},
\end{aligned} \tag{2.73}$$

and

$$\begin{aligned}
\mathbb{E}[\| I_{33} \|] &= \mathbb{E}[\| \int_{t_1}^{t_2}(t_2-s)^{\beta-1}A_\nu E_{\beta\beta}(t_2-s)f(c(s))dW(s) \|^p] \\
&\le C(p)\mathbb{E}[\int_{t_1}^{t_2}(t_2-s)^{\beta-1}A_\nu E_{\beta\beta}(t_2-s) \|^2 \| f(c(s)) \|^2_{L_0^2} ds)^{\frac{p}{2}}] \\
&\le C(p)C_\beta^p(\int_{t_1}^{t_2}[(t_2-s)^{\beta-1-\frac{\beta\nu}{\alpha}}]^{\frac{2p}{p-2}}ds)^{\frac{p-2}{2}}\int_{t_1}^{t_2}\mathbb{E} \| f(u) \|^p_{L_0^2} ds \\
&\le C(p)C_\beta^p C^\rho[\frac{\alpha(p-2)}{2p\beta(\alpha-\nu)-(p+2)\alpha}]^{\frac{p-2}{2}}(\sup_{t\in[0,T]}\mathbb{E}[\| c(t) \|^p])(t_2-t_1)^{\frac{2p\beta(\alpha-\nu)-p\alpha}{2\alpha}}
\end{aligned} \tag{2.74}$$

Taking expectation on the both side of (2.66), and in view of the estimates (2.69)- (2.74), we conclude that

$$\| c(t_2)-c(t_1) \|_{L^p(\Omega;\dot{H}^v)} \le C(t_2-t_1)^\gamma, \tag{2.75}$$

in which we tak $\gamma=\min\{\frac{\beta\nu}{\alpha},\frac{p\beta(\alpha-\nu-1)-\alpha}{p\alpha},\frac{2p\beta(\alpha-\nu)-(p+2)\alpha}{2p\alpha}\}$, whene $0<t_2-t_1<1$.

Otherwise, if $t_2-t_1\ge 1$ then we set $\gamma=\max\{\frac{\beta(\nu+1)}{\alpha},\frac{\beta(\alpha-\nu-1)}{\alpha},\frac{2p\beta(\alpha-\nu)-p\alpha}{2p\alpha}\}$. ■
This completes the proof of Theorem (2.8).

Appendix A

Considering the following abstract formulation of time-space fractional stochastic of equation (2.5)

$$\begin{cases} {}^cD_t^\beta u(t) &= -A_\alpha u(t)+B(u(t),c(t))+g(u(t))\frac{dW(t)}{dt}, \quad t>0, \\ u(0) &= u_0. \end{cases} \tag{2.76}$$

We derive the mild solution to (2.76) by means of Laplace transform, which denoted by $\hat{}$. Let $\lambda>0$, and we define that

$$\hat{u}(\lambda)=\int_0^\infty e^{-\lambda s}u(s)ds,\ \hat{B}(\lambda)=\int_0^\infty e^{-\lambda s}B(u(s),c(s))ds,$$

and

$$\hat{G}(\lambda) = \int\limits_0^\infty e^{-\lambda s}[g(u(s))\frac{dW(s)}{ds}]ds = \int\limits_0^\infty e^{-\lambda s} g(u(s))dW(s).$$

Upon Laplace transform, using the formula ${}^c\hat{D}_t^\beta u(\lambda) = \lambda^\beta \hat{u} - \lambda^{\beta-1}u_0$. Then applying the Laplace transform to (2.76), we obtain

$$\begin{aligned}
\hat{u}(\lambda) &= \tfrac{1}{\lambda}u_0 + \tfrac{1}{\lambda^\beta}(-A_\alpha)\hat{u}(\lambda) + \tfrac{1}{\lambda^\beta}[\hat{B}(\lambda) + \hat{G}(\lambda)] \\
&= \lambda^{\beta-1}(\lambda^\beta I + A_\alpha)^{-1}u_0 + (\lambda^\beta I + A_\alpha)^{-1}[\hat{B}(\lambda) + \hat{G}(\lambda)] \\
&= \lambda^{\beta-1}\int\limits_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)u_0 ds + \int\limits_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)[\hat{B}(\lambda) + \hat{G}(\lambda)]ds,
\end{aligned} \tag{2.77}$$

in which I is the identity operator, and $\mathcal{S}_\alpha(t) = e^{-tA_\alpha}$ is an analytic semigroup generated by the operator $-A_\alpha$. We introduce the following one-sided stable probability density function:

$$W_\beta = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{\beta n-1}\frac{\Gamma(\beta_n+1)}{n!}\sin(n\pi\beta), \quad \theta \in (0,\infty), \tag{2.78}$$

whose Laplace transform is given by

$$\int\limits_0^\infty e^{-\lambda\theta}W_\beta(\theta)d\theta = e^{-\lambda^\beta},\ 0 < \beta < 1. \tag{2.79}$$

Making use of above expression (2.79), then the terms on the right-hand side of (2.77) can be written as

$$\begin{aligned}
\lambda^{\beta-1}\int\limits_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)u_0 ds &= \int\limits_0^\infty \lambda^{\beta-1}e^{-\lambda^\beta t^\beta}\mathcal{S}_\alpha(t^\beta)u_0 dt \\
&= \int\limits_0^\infty \beta(\lambda t)^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)u_0 dt \\
&= \int\limits_0^\infty \tfrac{1}{\lambda}\tfrac{d}{dt}[e^{-(\lambda t)^\beta}]\mathcal{S}_\alpha(t^\beta)u_0 dt \\
&= \int\limits_0^\infty\int\limits_0^\infty \theta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)u_0 d\theta dt \\
&= \int\limits_0^\infty e^{-\lambda t}[\int\limits_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})u_0 d\theta]dt,
\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)\hat{B}(\lambda)ds \\
&= \int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)\hat{B}(\lambda)dt \\
&= \int_0^\infty\int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}B(u(s),c(s))dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}B(u(s),c(s))d\theta dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda(t+s)}\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})\tfrac{t^{\beta-1}}{\theta^\beta}B(u(s),c(s))d\theta dsdt \\
&= \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}B(u(s),c(s))d\theta ds]dt,
\end{aligned} \tag{2.81}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)\hat{G}(\lambda)ds \\
&= \int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)\hat{G}(\lambda)dt \\
&= \int_0^\infty\int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}g(u(s))dW(s)dt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}g(u(s))d\theta dW(s)dt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda(t+s)}\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})\tfrac{t^{\beta-1}}{\theta^\beta}g(u(s))d\theta dW(s)dt \\
&= \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}g(u(s))d\theta dW(s)]dt.
\end{aligned} \tag{2.82}$$

Together with (2.77) and (2.80)- (2.82) helps us to get

$$\begin{aligned}
\hat{u}(\lambda) &= \int_0^\infty e^{-\lambda t}[\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})u_0 d\theta]dt \\
&+ \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}B(u(s),c(s))d\theta ds]dt \\
&+ \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}g(u(s))d\theta dW(s)]dt,
\end{aligned} \tag{2.83}$$

Now, by means of inverse Laplace transform to (2.83), we have achieved that

$$\begin{aligned}
u(t) &= \int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})u_0 d\theta \\
&+ \beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}B(u(s),c(s))d\theta ds \\
&+ \beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}g(u(s))d\theta dW(s) \\
&= \int_0^\infty \tfrac{1}{\beta}\theta^{-\frac{1}{\beta}-1}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha(t^\beta\theta)u_0 d\theta \\
&+ \int_0^t\int_0^\infty \theta^{-\frac{1}{\beta}}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1}B(u(s),c(s))d\theta ds \\
&\quad \int_0^t\int_0^\infty \theta^{-\frac{1}{\beta}}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1}g(u(s))d\theta dW(s).
\end{aligned} \tag{2.84}$$

Here, we also introduce the Mainardi's Wright-type function

$$\begin{aligned}
M_\beta(\theta) &= \sum_{n=0}^\infty \frac{(-1)^n\theta^n}{n\theta!\Gamma(1-\beta(1+n))} \\
&= \tfrac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}\theta^{n-1}}{(n-1)!}\Gamma(n\beta)\sin(n\pi\beta),
\end{aligned}$$

where $0 < \beta < 1$ and $\theta \in (0,\infty)$. Further, the relationships between the probability density function $W_\beta(\theta)$ and Mainardi's Wright-type function $M_\beta(\theta)$ are shown that

$$M_\beta(\theta) = \frac{1}{\beta}\theta^{-\frac{1}{\beta}-1}W_\beta(\theta^{-\frac{1}{\beta}}).$$

We denote the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\beta\beta}(t)$ as

$$E_\alpha(t) = \int_0^\infty M_\beta(\theta)\mathcal{S}_\alpha(t^\beta\theta)d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta\theta M_\beta(\theta)\mathcal{S}_\alpha(t^\beta\theta)d\theta.$$

Therefore, the equation (2.84) can be written as

$$\begin{aligned}
u(t) &= E_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1}E_{\beta\beta}(t-s)B\left(u(s),c(s)\right)ds \\
&+ \int_0^t (t-s)^{\beta-1}E_{\beta\beta}(t-s)g(u(s))dW(s),
\end{aligned} \tag{2.85}$$

Up to now, we have deduced the mild solution (2.85) to the time-space fractional stochastic equation (2.5).

Appendix B

Considering the following abstract formulation of time-space fractional stochastic of equation (2.10)

$$\begin{cases} {}^cD_t^\beta c(t) &= -A_\alpha c(t) + L(c(t)) + f(c(t))\frac{dW(t)}{dt} \quad, t>0, \\ c(0) &= c_0, \end{cases} \tag{2.86}$$

We derive the mild solution to (2.86) by means of Laplace transform, which denoted by $\hat{}$. $\lambda > 0$, and we define that

$$\hat{c}(\lambda) = \int_0^\infty e^{-\lambda s} c(s)ds, \ \hat{L}(\lambda) = \int_0^\infty e^{-\lambda s} L(c(s))ds,$$

and

$$\hat{H}(\lambda) = \int_0^\infty e^{-\lambda s}[f(c(s))\frac{dW(s)}{ds}]ds = \int_0^\infty e^{-\lambda s} f(c(s))dW(s).$$

Upon Laplace transform, using the formula ${}^c\hat{D}_t^\beta c(\lambda) = \lambda^\beta \hat{c} - \lambda^{\beta-1}c_0$. Then applying the Laplace transform to (2.86), we obtain

$$\begin{aligned} \hat{c}(\lambda) &= \tfrac{1}{\lambda}c_0 + \tfrac{1}{\lambda^\beta}(-A_\alpha)\hat{c}(\lambda) + \tfrac{1}{\lambda^\beta}[\hat{L}(\lambda) + \hat{H}(\lambda)] \\ &= \lambda^{\beta-1}(\lambda^\beta I + A_\alpha)^{-1}c_0 + (\lambda^\beta I + A_\alpha)^{-1}[\hat{L}(\lambda) + \hat{H}(\lambda)] \\ &= \lambda^{\beta-1}\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)c_0 ds + \int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)[\hat{L}(\lambda) + \hat{H}(\lambda)]ds \end{aligned} \tag{2.87}$$

in which I is the identity operator, and $\mathcal{S}_\alpha(t) = e^{-tA_\alpha}$ is an analytic semigroup generated by the operator $-A_\alpha$. We introduce the following one-sided stable probability density function:

$$W_\beta = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{\beta n-1}\frac{\Gamma(\beta_n + 1)}{n!}\sin(n\pi\beta), \ \theta \in (0,\infty), \tag{2.88}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}W_\beta(\theta)d\theta = e^{-\lambda^\beta}, \ 0<\beta<1. \tag{2.89}$$

Making use of above expression (2.89), then the terms on the right-hand side of (2.87) can be written as

$$\begin{aligned} \lambda^{\beta-1}\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)c_0 ds &= \int_0^\infty \lambda^{\beta-1}e^{-\lambda^\beta t^\beta}\mathcal{S}_\alpha(t^\beta)c_0 dt \\ &= \int_0^\infty \beta(\lambda t)^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)c_0 dt \\ &= \int_0^\infty \tfrac{1}{\lambda}\tfrac{d}{dt}[e^{-(\lambda t)^\beta}]\mathcal{S}_\alpha(t^\beta)c_0 dt \\ &= \int_0^\infty\int_0^\infty \theta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)c_0 d\theta dt \\ &= \int_0^\infty e^{-\lambda t}[\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})c_0 d\theta]dt, \end{aligned} \tag{2.90}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)\hat{L}(\lambda)ds \\
=& \int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)\hat{L}(\lambda)dt \\
=& \int_0^\infty\int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}L(c(s))dsdt \\
=& \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}L(c(s))d\theta dsdt \\
=& \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda(t+s)}\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})\tfrac{t^{\beta-1}}{\theta^\beta}L(c(s))d\theta dsdt \\
=& \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}L(c(s))d\theta ds]dt,
\end{aligned} \tag{2.91}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\beta s}\mathcal{S}_\alpha(s)\hat{H}(\lambda)ds \\
=& \int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)\hat{H}(\lambda)dt \\
=& \int_0^\infty\int_0^\infty \beta t^{\beta-1}e^{-(\lambda t)^\beta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}f(c(s))dW(s)dt \\
=& \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda t\theta}\mathcal{S}_\alpha(t^\beta)e^{-\lambda s}t^{\beta-1}f(c(s))d\theta dW(s)dt \\
=& \int_0^\infty\int_0^\infty\int_0^\infty \beta W_\beta(\theta)e^{-\lambda(t+s)}\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})\tfrac{t^{\beta-1}}{\theta^\beta}f(c(s))d\theta dW(s)dt \\
=& \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}f(c(s))d\theta dW(s)]dt.
\end{aligned} \tag{2.92}$$

Together with (2.87) and (2.90)- (2.92) helps us to get

$$\begin{aligned}
\hat{c}(\lambda) =& \int_0^\infty e^{-\lambda t}[\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})c_0 d\theta]dt \\
&+ \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}L(c(s))d\theta ds]dt \\
&+ \int_0^\infty e^{-\lambda t}[\beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}f(c(s))d\theta dW(s)]dt.
\end{aligned} \tag{2.93}$$

Now, by means of inverse Laplace transform to (2.93), we have achieved that

$$\begin{aligned}
c(t) &= \int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{t^\beta}{\theta^\beta})c_0 d\theta \\
&+ \beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}L(c(s))d\theta ds \\
&+ \beta\int_0^t\int_0^\infty W_\beta(\theta)\mathcal{S}_\alpha(\tfrac{(t-s)^\beta}{\theta^\beta})\tfrac{(t-s)^{\beta-1}}{\theta^\beta}f(c(s))d\theta dW(s) \\
&= \int_0^\infty \tfrac{1}{\beta}\theta^{-\frac{1}{\beta}-1}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha(t^\beta\theta)c_0 d\theta \\
&+ \int_0^t\int_0^\infty \theta^{-\frac{1}{\beta}}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1}L(c(s))d\theta ds \\
&\quad \int_0^t\int_0^\infty \theta^{-\frac{1}{\beta}}W_\beta(\theta^{-\frac{1}{\beta}})\mathcal{S}_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1}f(c(s))d\theta dW(s).
\end{aligned} \tag{2.94}$$

Here, we also introduce the Mainardi's Wright-type function

$$\begin{aligned}
M_\beta(\theta) &= \sum_{n=0}^\infty \frac{(-1)^n\theta^n}{n\theta!\Gamma(1-\beta(1+n))} \\
&= \tfrac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}\theta^{n-1}}{(n-1)!}\Gamma(n\beta)\sin(n\pi\beta),
\end{aligned}$$

where $0<\beta<1$ and $\theta\in(0,\infty)$. Further, the relationships between the probability density function $W_\beta(\theta)$ and Mainardi's Wright-type function $M_\beta(\theta)$ are shown that

$$M_\beta(\theta) = \frac{1}{\beta}\theta^{-\frac{1}{\beta}-1}W_\beta(\theta^{-\frac{1}{\beta}}).$$

We denote the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\beta\beta}(t)$ as

$$E_\alpha(t) = \int_0^\infty M_\beta(\theta)\mathcal{S}_\alpha(t^\beta\theta)d\theta,$$

and

$$E_{\beta\beta}(t) = \int_0^\infty \beta\theta M_\beta(\theta)\mathcal{S}_\alpha(t^\beta\theta)d\theta.$$

Therefore, the equation (2.84) can be written as

$$\begin{aligned}
c(t) &= E_\beta(t)c_0 + \int_0^t (t-s)^{\beta-1}E_{\beta\beta}(t-s)L(c(s))ds \\
&+ \int_0^t (t-s)^{\beta-1}E_{\beta\beta}(t-s)f(c(s))dW(s).
\end{aligned} \tag{2.95}$$

Up to now, we have deduced the mild solution (2.95) to the time-space fractional stochastic equation (2.10).

CHAPTER 3

EXISTENCE AND UNIQUENESS OF GLOBAL SOLUTION OF STOCHASTIC KELLER-SEGEL MODEL

In this present chapter, we consider stochastic chemotaxis Keller-Segel model impact by gaussian process. The first we study of local existence in time of nonlinear stochastic Keller-Segel model with zeros Dirichlet boundary conditions and we add condition for prove local solution is global solution, for this we use analysis techniques lemmas and semigroup theory.

Now we consider the Keller-Segel model with a Random which a space time white noise as follows

$$\begin{cases} u_t = \kappa \frac{\partial^2 u}{\partial x^2} - b\frac{\partial}{\partial x}\left(u^2 \frac{\partial}{\partial x} f(c)\right) + \frac{\partial^2 \tilde{W}}{\partial t \partial x}, \quad x \in [0,1],\ t > 0, \\ c_t = a\frac{\partial^2 c}{\partial x^2} - \frac{\partial}{\partial x} f(c) + \frac{\partial^2 \tilde{W}}{\partial t \partial x}, \quad x \in [0,1],\ t > 0. \end{cases} \tag{3.1}$$

Where the population density of biological individuals and the concentration of a chemical substance are denoted by $u = u(x,t)$, $c = c(x,t)$ (respectively), κ, a are diffusion rate of u and positive chemotactic coefficient respectively, b is positive coefficient, $f : \mathbb{R} \to \mathbb{R}$ is nonlinear sensitivity function, $\tilde{W}(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ is the zero mean Gaussiam process with covariance function:

$$E[\tilde{W}(t_1,x_1), \tilde{W}(t_2,x_2)] = (t_1 \wedge t_2)(x_1 \wedge x_2), \quad t_1,\ t_2 \geq 0, \quad x_1,\ x_2 \in \mathbb{R}$$

and cylindrical Wiener process W can be considered by

$$W(t) = \frac{\partial \tilde{W}}{\partial x} = \sum_{n=1}^{\infty} \beta_n e_n, \tag{3.2}$$

where $\{\beta_n\}$, $\{e_n\}$ are real Brownian motions and is an orthonormal basis of $L^2(0,1)$ respectively, (Ω, F, P) is probability space adapted to a filtration $\{F_t\}$ $t \geq 0$. The series (3.2) does not converge in $L^2(0,1)$ but it is convergent in any Hilbert space V such that the embedding $L^2(0,1) \subset V$ is Hilbert-Schmidt [23]. We will write (3.1) in the following format:

$$du(t,x) = \left(\kappa \frac{\partial^2 u}{\partial x^2} - b\frac{\partial}{\partial x}\left(u^2 \frac{\partial}{\partial x} f(c)\right)\right) dt + dW, \quad x \in [0,1], t > 0, \tag{3.3}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{3.4}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{3.5}$$

and

$$dc(t,x) = \left(a\frac{\partial^2}{\partial x^2}c - \frac{\partial}{\partial x}f\left(c\right) \right) dt + dW, \quad x \in [0,1], t > 0. \tag{3.6}$$

with Dirichlet boundary conditions

$$c(0,t) = c(1,t) = 0, \tag{3.7}$$

and the initial condition

$$c(x,0) = c_0(x), \quad x \in [0,1]. \tag{3.8}$$

Where (3.2) is used to describe W. where we make the assumption that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfy condition

$$\mid \frac{\partial}{\partial x} f(c) \mid_{L^{\frac{p}{2}}(0,1)} \leq Lip_1. \tag{3.9}$$

To simplify our work, we write problem (3.3) with boundary and initial conditions (3.4) and (3.5) (respectively) on the form problem as

$$\begin{cases} du(t,x) & = \left(\kappa\frac{\partial^2 u}{\partial x^2} - b\frac{\partial}{\partial x}\left(u^2\frac{\partial}{\partial x}f(c)\right)\right) dt + dW, \quad x \in [0,1], \ t > 0, \\ u(0,t) & = u(1,t) = 0, \ t > 0 \\ u(x,0) & = u_0(x), \quad x \in [0,1], \end{cases} \tag{3.10}$$

and we write problem (3.6) with boundary and initial conditions (3.7) and (3.8) (respectively) on the form problem as

$$\begin{cases} dc(t,x) & = \left(a\frac{\partial^2}{\partial x^2}c - \frac{\partial}{\partial x}f\left(c\right) \right) dt + dW, \quad x \in [0,1], t > 0, \\ c(0,t) & = c(1,t) = 0, \ t > 0, \\ c(x,0) & = c_0(x), \quad x \in [0,1]. \end{cases} \tag{3.11}$$

Our aim in this chapter is to prove the problems (3.10) and (3.11) have a unique global solutions, for this we use semigroup theory.
In the next section, we present the notations preliminaries and we prove local existence of problem (3.10).

3.1 Local Existence in Time of problem

Define the unbounded self-adjoint operator A on $L^2(0,1)$ by

$$Au = \kappa\frac{\partial^2}{\partial x^2}u,$$

for u on the domain

$$H(A) = \{u \in H^2(0,1) : u(0) = u(1) = 0\}.$$

We assume that e^{tA}, $t \geq 0$, the semigroup on $L^2(0,1)$ generated by A, that we put e^{tA}, $t \geq 0$, as a contraction semigroup on $L^2(0,1)$, for any $p \geq 1$. Finally, we denote by $\{e_h\}$ the complete orthonormal system on $L^2(0,1)$ which diagonalizes A and $\{\lambda_h\}$ the eigenvalues. We obtain

$$e_h(x) = \sqrt{\frac{2}{\pi}}\sin(h\pi x), \ h = 1, 2, ...$$

and

$$\lambda_h = -\pi^2 h^2, \ h = 1, 2, ...$$

Now, we rewrite (3.10) as follows

$$\begin{cases} du = \left(Au - b\frac{\partial}{\partial x}\left(u^2\frac{\partial}{\partial x}f(c)\right)\right)dt + dW, \\ u(0) = u_0. \end{cases} \tag{3.12}$$

Remember, the solution of linear problem is

$$\begin{cases} du(t,x) = Audt + dW, \\ u(0) = u_0, \end{cases} \tag{3.13}$$

is unique and given bu the so-called stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s). \tag{3.14}$$

It can be prove that W_A is a Gaussian process and it is mean square continuous with values in $L^2(0,1)$. We put

$$U(t) = u(t) - W_A(t),\ t > 0.$$

Then u check (3.12) if and only if U is a solution of this problem

$$\begin{cases} \frac{dU}{dt} = AU - b\frac{\partial}{\partial x}\left((U+W_A)^2\frac{\partial}{\partial x}f(c)\right), \\ U(0) = u_0. \end{cases} \tag{3.15}$$

Now, we will study of (3.15) where in W_A is an α-Hölder continuous function with respect to (t,x) for any $\alpha \in [0, 1/4[$ then the equation (3.15) can be write as follows

$$U(t) = e^{tA}u_0 - b\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left((U+W_A)^2\frac{\partial}{\partial x}f(c)\right)ds. \tag{3.16}$$

Then if U verifie (3.16) we call that it is a mild solution of (3.15). We will solve equation (3.16) in the space $C([0,T^*]; L^p(0,1))$ for all $p > 1$ and $T^* > 0$. We put

$$\sum_p(M,T^*) = \{U \in C([0,T^*]; L^p(0,1)) : \ \mid U(t)\mid_{L^p[0,1]} \le M,\ \forall t \in [0,T^*]\},$$

and consider an initial datum u_0 $\mathcal{F}_0$-measurable and belonging to $L^p(0,1)$, for $t > 0$, $\omega \in \Omega$. We will see, in the proof of the Lemma (3.1) below that if $w(t)$ is a bounded function from $[0,T]$ into $L^p(0,1)$, for $t > 0$, is as follows. As a result, the integral in (3.16) is converges in $L^p(0,1)$ a.s. Thus, (3.16) has the same value as an equality in $L^p(0,1)$.

Lemma 3.1 *For any $p \ge 2$ and $M > \mid u_0\mid_{L^p(0,1)}$, there exists a stoping time $T^* > 0$ such that (3.16) has a unique solution in $\sum_p(M,T^*)$.*

Proof. for any U in $\sum_p(M,T^*)$ as well as describe $w = LU$ by

$$w(t) = e^{tA}u_0 - b\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left((U+W_A)^2\frac{\partial}{\partial x}f(c)\right)ds,$$

where $L : C([0,T^*]; L^p(0,1)) \to C([0,T^*]; L^p(0,1))$, is non-linear operator. Then

$$\mid w(t)\mid_{L^p(0,1)} \le \mid e^{tA}u_0\mid_{L^p(0,1)} + b\int_0^t \mid e^{(t-s)A}\frac{\partial}{\partial x}\left((U+W_A)^2\frac{\partial}{\partial x}f(c)\right)\mid_{L^p(0,1)} ds,$$

for any $s_1 < s_2$ in $\mathbb{R}$ and $r \geq 1$, e^{tA} maps $W^{s_1,r}(0,1)$ into $W^{s_2,r}(0,1)$, for all $t > 0$. Moreover, the following estimate holds

$$| e^{tA}w |_{W^{s_2,r}(0,1)} \leq C_1 \left(t^{\frac{s_1-s_2}{2}} + 1 \right) | w |_{W^{s_1,r}(0,1)}, \quad \text{for all } w \in W^{s_1,r}(0,1). \tag{3.17}$$

The constant C_1 depends only on s_1, s_2 and r, see [76] Using the Sobolev embedding theorem we have

$$| e^{(t-s)A}\frac{\partial}{\partial x}((U+W_A)^2\frac{\partial}{\partial x}f(c)) |_{L^p(0,1)} \leq c_2 | e^{(t-s)A}\frac{\partial}{\partial x}((U+W_A)^2\frac{\partial}{\partial x}f(c)) |_{W^{\frac{1}{p},\frac{p}{2}}(0,1)},$$

for (3.17) with $s_1 = -1$, $s_2 = \frac{1}{p}, r = \frac{p}{2}$, therefore, we have

$$\begin{aligned} | e^{(t-s)A}\tfrac{\partial}{\partial x}((U+W_A)^2\tfrac{\partial}{\partial x}f(c)) |_{L^p(0,1)} &\leq c_1c_2((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1) | \tfrac{\partial}{\partial x}((U+W_A)^2\tfrac{\partial}{\partial x}f(c)) |_{W^{-1,\frac{p}{2}}(0,1)} \\ &\leq c_1c_2((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1) | (U+W_A)^2 |_{L^{\frac{p}{2}}(0,1)} | \tfrac{\partial}{\partial x}f(c) |_{L^{\frac{p}{2}}(0,1)}, \end{aligned}$$

and with condition of f in (3.9), we have

$$| e^{(t-s)}\tfrac{\partial}{\partial x}((U+W_A)^2\tfrac{\partial}{\partial x}f(c)) |_{L^p(0,1)} \leq c_1c_2Lip_1((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1) | (U+W_A)^2 |_{L^{\frac{p}{2}}(0,1)} .$$

We obtain

$$\begin{aligned} | w(t) |_{L^p(0,1)} &\leq | u_0 |_{L^p(0,1)} + bc_1c_2Lip_1 \textstyle\int_0^t ((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1) | (U+W_A)^2 |_{L^{\frac{p}{2}}(0,1)} ds \\ &\leq | u_0 |_{L^p(0,1)} + bc_1c_2Lip_1 \textstyle\int_0^t ((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)(| U |_{L^{\frac{p}{2}}(0,1)} + | W_A |_{L^{\frac{p}{2}}(0,1)})ds \\ &\leq | u_0 |_{L^p(0,1)} + bc_1c_2Lip_1(M+\chi_p)^2 \textstyle\int_0^t ((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)ds \\ &\leq | u_0 |_{L^p(0,1)} + bc_1c_2Lip_1(M+\chi_p)^2(\tfrac{2p}{p-1}t^{\frac{1}{2}-\frac{1}{2p}}+t), \end{aligned}$$

where in Lip_1 is the Lipschitz constant of f which depend on $M+\chi_p$, and $\chi_p = \sup_{t\in[0,T]} | W_A(t) |_{L^{\frac{p}{2}}(0,1)}$.
Hemce $| w(t) |_{L^p(0,1)} \leq M \ for\ all\ t \in [0,T^*]$, provided

$$| u_0 |_{L^p(0,1)} + bc_1c_2Lip_1(M+\chi_p)^2(\frac{2p}{p-1}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*) \leq M. \tag{3.18}$$

It is clear that every $M >| u_0 |_{L^p(0,1)}$ there exists a T^* satisfying (3.18).
Now we consider $U_1,\ U_2 \in \sum_p(M,T^*)$ and we put $w_i = LU_i, i = 1,2$ and $w = w_1 - w_2$, Then

$$w(t) = b\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}[((U_1+W_A)^2-(U_2+W_A)^2)\frac{\partial}{\partial x}f(c)]ds,$$

As a result, we arrive at the following conclusions

$$| w(t) |_{L^p(0,1)} \leq bc_1c_2Lip_1 \textstyle\int_0^t ((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1) | (U_1+W_A)^2-(U_2+W_A)^2 |_{L^{\frac{p}{2}}(0,1)} ds$$

Application the Hölder's inequality, we obtain

$$\begin{aligned} | (U_1+W_A)^2-(U_2+W_A)^2 |_{L^{\frac{p}{2}}(0,1)} &=| (U_1+U_2+2W_A)(U_1-U_2) |_{L^{\frac{p}{2}}(0,1)} \\ &\leq | (U_1+U_2+2W_A) |_{L^p(0,1)} | U_1-U_2 |_{L^p(0,1)} \\ &\leq 2(M+\chi_p) | U_1-U_2 |_{L^p(0,1)}, \end{aligned}$$

hence

$$| w(t) |_{L^p(0,1)} \quad \leq 2bc_1c_2Lip_1(M+\chi_p)(\tfrac{2p}{p-1}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*),$$

where in Lip_1 is the Lipschitz constant of f which depend on $M+\chi_p$, and for all $t \in [0,T^*]$ given

$$| LU_1 - LU_2 |_{c([0,T^*];L^p(0,1))} \leq 2bc_1c_2Lip_1(M+\chi_p)(\frac{2p}{p-1}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*) | U_1 - U_2 |_{c([0,T^*];L^p(0,1))},$$

We considerT^* such that

$$2bc_1c_2Lip_1(M+\chi_p)(\frac{2p}{p-1}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*) < 1$$

and (3.18) holds so that L is a strict contraction on $\sum_p(M,T^*)$. ■
In the next section we will prove that $T^* = T$ a.s. for $\omega \in \Omega$.

3.2 Global Existence

In this section, we prove the global existence of the problem (3.10). We are still considering equation (3.16) as a deterministic one, working a.s. for $\omega \in \Omega$.

Theorem 3.1 *(Global existence.) Let u_0 be given which is F_0-measurable and such that for some $p \geq 2$, $u_0 \in L^p(0,1)$ a.s. If $\kappa \geq \frac{3}{2}bLip_1$, then there exists a unique mild solution of equation (3.12), which belongs a.s. to $c([0,T^*];L^p(0,1))$.*

We derive an a priori estimate that yields global existence in the following lemma.

Lemma 3.2 *If $U \in c([0,T^*];L^p(0,1))$ satisfies (3.16) and $\kappa \geq \frac{3}{2}bLip_1$, then*

$$| U |_{L^p(0,1)} \leq e^{p(p-1)bLip_1(\chi_\infty^2+\frac{1}{2}\chi_\infty^4)t} | u_0 |_{L^p(0,1)},$$

where $\chi_\infty = \sup_{t\in[0,T]} | W_A(t) |_{L^\infty(0,1)}$.

Proof. put $\{u_0^n\}$ be sequence in $c^\infty(0,1)$ such that $u_0^n \to u_0$ in $L^p(0,1)$ and we put $\{W^n\}$ be sequence for regular processes such that

$$W_A^n(t) = \int_0^t e^{(t-s)A}dW^n(s) \to W_A(t), \text{in } c([0,T]\times(0,1)),$$

for $\omega \in \Omega$. Let $\{U^n\}$ be the solution of

$$U^n(t) = e^{tA}u_0^n - b\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}((U^n+W_A^n)^2\frac{\partial}{\partial x}f(c))ds,$$

given by Lemma (3.1). It is clear to U^n does exist on an interval of time $[0,T_n]$ such that $T_n \to T^*$ a.s. and that U_n converges to U in $c([0,T^*];L^p(0,1))$ a.s. Moreover, U_n is a regular a.s. and hold

$$\frac{\partial U^n}{\partial t} - \kappa\frac{\partial^2 U^n}{\partial x^2} + b\frac{\partial}{\partial x}((U^n+W_A^n)^2\frac{\partial}{\partial}f(c)) = 0. \tag{3.19}$$

Multipling (3.19) by $| U^n |^{p-2} U^n$ and integrating over $[0,1]$, we have

$$\frac{1}{p}\frac{\partial}{\partial t} | U^n |^p_{L^p(0,1)} + \kappa(p-1)\int_0^1 | U^n |^{p-2} (\frac{\partial}{\partial x}U^n)^2dx + b\int_0^1 \frac{\partial}{\partial x}((U^n+W_A^n)^2\frac{\partial}{\partial x}f(c)) | U^n |^{p-2} U^n dx = 0. \tag{3.20}$$

We integrate by parts the last integral

$$b\int_0^1 \frac{\partial}{\partial x}((U^n+W_A^n)^2\frac{\partial}{\partial x}f(c)) \mid v^n \mid^{p-2} U^n dx = -b(p-1)\int_0^1((U^n+W_A^n)^2\frac{\partial}{\partial x}f(c)) \mid U^n \mid^{p-2} \frac{\partial}{\partial x}U^n dx,$$

then

$$\begin{aligned}
\mid b\int_0^1 \tfrac{\partial}{\partial x}((U^n+W_A^n)^2\tfrac{\partial}{\partial x}f(c)) \mid U^n \mid^{p-2} U^n dx \mid &= b(p-1) \mid \int_0^1((U^n+W_A^n)^2\tfrac{\partial}{\partial x}f(c)) \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx \mid \\
&\le b(p-1)\int_0^1 \mid ((U^n+W_A^n)^2\tfrac{\partial}{\partial x}f(c)) \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n \mid dx \\
&\le b(p-1)\int_0^1 Lip_1 \mid (U^n+W_A^n)^2 \mid\mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx \\
&\le b(p-1)Lip_1\int_0^1(\mid U^n+W_A^n \mid^2) \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx \\
&= b(p-1)Lip_1\int_0^1 \mid U^n \mid^p \tfrac{\partial}{\partial x}U^n dx \\
&+2b(p-1)Lip_1\int_0^1 W_A^nU^n \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx \\
&+b(p-1)Lip_1\int_0^1(W_A^n)^2 \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx
\end{aligned}$$

The first term is zero, indeed

$$\int_0^1 \mid U^n \mid^p \frac{\partial}{\partial x}U^n dx = \frac{1}{p+1}\int_0^1 \frac{\partial}{\partial x}(\mid U^n \mid^p U^n)dx = 0.$$

In the same way, we can prove that the second term is also zero

$$\begin{aligned}
2bLip_1(p-1)\int_0^1 \mid W_A^n \mid\mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx &\le 2bLip_1(p-1) \mid W_A^n \mid_{L^\infty(0,1)}\mid U^n \mid^{\frac{p-2}{2}}_{L^{p-2}(0,1)} \\
&\times(\int_0^1 \mid U^n \mid^{p-2} (\tfrac{\partial}{\partial x}U^n)^2dx)^{\frac{1}{2}} \\
&\le bLip_1(p-1)\chi^2_{n,\infty} \mid U^n \mid^{p-2}_{L^p(0,1)} \\
&+bLip_1(p-1)\int_0^1 \mid U^n \mid^{p-2} (\tfrac{\partial}{\partial x}U^n)^2dx,
\end{aligned}$$

where $\chi_{n,\infty} = \sup_{t\in[0,T]} \mid W_A^n(t) \mid_{L^\infty(0,1)}$ for as, $\omega \in \Omega$.
We will be writing for the third term.

$$\begin{aligned}
b(p-1)Lip_1\int_0^1(W_A^n)^2 \mid U^n \mid^{p-2} \tfrac{\partial}{\partial x}U^n dx &\le b(p-1)Lip_1\chi^2_{n,\infty} \mid U^n \mid^{\frac{p-2}{2}}_{L^p(0,1)} (\int_0^1 \mid U^n \mid^{p-2} (\tfrac{\partial}{\partial x}U^n)^2dx)^{\frac{1}{2}} \\
&\le bLip_1\tfrac{(p-1)}{2}\int_0^1 \mid^U n \mid^{p-2} (\tfrac{\partial}{\partial x}U^n)^2dx \\
&+bLip_1\tfrac{(p-1)}{2}\chi^4_{n,\infty} \mid U^n \mid^{p-2}_{L^p(0,1)},
\end{aligned}$$

where $\chi_{n,\infty} = \sup_{t\in[0,T]} \mid W_A^n(t) \mid_{L^\infty(0,1)}$ for $\omega \in \Omega$.

Getting back to (3.20) we have

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}\mid U^n\mid^p_{L^p(0,1)} + \kappa(p-1)\int_0^1 \mid U^n\mid^{p-2}(\frac{\partial}{\partial x}U^n)^2dx \quad &\leq bLip_1(p-1)\chi^2_{n,\infty}\mid U^n\mid^{p-2}_{L^p(0,1)}\\ &+bLip_1(p-1)\int_0^1\mid U^n\mid^{p-2}(\frac{\partial}{\partial x}U^n)^2dx\\ &+bLip_1\frac{(p-1)}{2}\int_0^1\mid U^n\mid^{p-2}(\frac{\partial}{\partial x}U^n)^2dx\\ &+bLip_1\frac{(p-1)}{2}\chi^4_{n,\infty}\mid U^n\mid^{p-2}_{L^p(0,1)}.\end{aligned}$$

It is as follows

$$\frac{1}{p}\frac{\partial}{\partial t}\mid U^n\mid^p_{L^p(0,1)} + (p-1)\left(\kappa-\frac{3}{2}bLip_1\right)\int_0^1\mid U^n\mid^{p-2}(\frac{\partial}{\partial x}U^n)^2dx \leq bLip_1(p-1)(\chi^2_{n,\infty}+\frac{1}{2}\chi^4_{n,\infty})\mid U^n\mid^{p-2}_{L^p(0,1)}.$$

Taking κ and b, Lip_1 such that

$$\kappa\geq\frac{3}{2}bLip_1,$$

we have

$$\frac{\partial}{\partial t}\mid U^n\mid^p_{L^p(0,1)}\leq p(p-1)bLip_1(\chi^2_{n,\infty}+\frac{1}{2}\chi^4_{n,\infty})\mid U^n\mid^{p-2}_{L^p(0,1)},$$

using to Gronwall's lemma

$$\mid U^n\mid^p_{L^p(0,1)}\leq e^{p(p-1)bLip_1(\chi^2_{n,\infty}+\frac{1}{2}\chi^4_{n,\infty})t}\mid u^n_0\mid^p_{L^p(0,1)},$$

using the limit as $n\to\infty$, we get

$$\mid U\mid^p_{L^p(0,1)}\leq e^{p(p-1)bLip_1(\chi^2_{\infty}+\frac{1}{2}\chi^4_{\infty})t}\mid u_0\mid^p_{L^p(0,1)}.$$

As a result

$$\mid U\mid_{L^p(0,1)}\leq e^{p(p-1)bLip_1(\chi^2_{\infty}+\frac{1}{2}\chi^4_{\infty})t}\mid u_0\mid_{L^p(0,1)}.$$

After that, the lemma is asserted. ■

Proof of Theorem (3.1). It is easy to deduce from Lemma (3.1) and Lemma (3.2). The next section, we prove local existence in time of (3.11).

3.3 Local Existence in Time of problem

Denote the unbounded self-adjoint operator B on $L^2(0,1)$ by

$$Bu = a\frac{\partial^2}{\partial x^2}c,$$

where c on the domain

$$H(B)=\{c\in H^2(0,1): c(0)=c(1)=0\}.$$

Denote e^{tB}, $t\geq 0$, the semigroup on $L^2(0,1)$ generated by B. we put e^{tB}, $t\geq 0$, as a contraction semigroup on $L^2(0,1)$, for any $p\geq 1$. Finally, we denote by $\{e_h\}$ the complete orthonormal system on $L^2(0,1)$ which diagonalizes B and $\{\lambda_h\}$ the eigenvalues. We obtain

$$e_h(x)=\sqrt{\frac{2}{\pi}}\sin(k\pi x),\ h=1,2,...$$

and

$$\lambda_h=-\pi^2h^2,\ h=1,2,...$$

The equation (3.11) is written as

$$\begin{cases} dc = (Bc - \partial_x f(c))\, dt + dW, \\ c(0) = c_0. \end{cases} \tag{3.21}$$

Remember the solution of linear problem

$$\begin{cases} dc(t,x) = Bcdt + dW, \\ c(0) = c_0, \end{cases} \tag{3.22}$$

is unique as well as by

$$W_B(t) = \int_0^t e^{(t-s)B} dW(s). \tag{3.23}$$

It can be shown that W_B is a Gaussian process and it is mean square continuous with values in $L^2(0,1)$. We put

$$v(t) = c(t) - W_B(t),\ t > 0.$$

Then c fulfills (3.21) if and only if v is a solution of

$$\begin{cases} \frac{dv}{dt} = Bv - \partial_x f(v + W_B)dt, \\ v(0) = c_0. \end{cases} \tag{3.24}$$

Now, we will study of (3.24) where in W_B is an α-Hölder continuous function with respect to (t,x) for any $\alpha \in [0, 1/4[$, then the equation (3.24) can be write as follows

$$v(t) = e^{tB} c_0 - \int_0^t e^{(t-s)B} \partial_x f(v + W_B) ds. \tag{3.25}$$

Then if v verify (3.25) It is a mild solution, according to us (3.24). We will to solve (3.25) in the space $C([0,T^*]; L^p(0,1))$ for $p > 1$ and for some $T^* > 0$.
We put

$$\sum_p (M, T^*) = \{v \in C([0,T^*]; L^p(0,1)) : \ \mid v(t) \mid \leq M,\ \forall t \in [0,T^*]\}.$$

If $z(t)$ is a bounded function from $[0,T]$ to $L^p(0,1)$, then for $t > 0$, the proof of the Lemma (3.3) is as follows. As a result, the integral in (3.25) converges in $L^p(0,1)$ a.s. As a result, (3.25) has the same value as an equality in $L^p(0,1)$.

Lemma 3.3 *For any $p \geq 2$ and $M > \mid c_0 \mid_{L^p(0,1)}$, there exists a stoping time $T^* > 0$ such that (3.25) has a unique solution in $\sum_p (M, T^*)$.*

Proof. for any v in $\sum_p (M, T^*)$ and take $z = Gv$ by

$$z(t) = e^{tB} c_0 - \int_0^t e^{(t-s)B} \partial_x f(v + W_B) ds,$$

where in $G : C([0,T^*]; L^p(0,1)) \to C([0,T^*]; L^p(0,1))$ is non-linear operator. Then

$$\mid z(t) \mid_{L^p(0,1)} \leq \mid e^{tB} c_0 \mid_{L^p(0,1)} + \int_0^t \mid e^{(t-s)B} \partial_x f(v + W_B) \mid_{L^p(0,1)} ds.$$

We put, e^{tB}, $t\geq 0$ is a contraction semigroup on $L^p(0,1)$ which has a regularizing, for any $s_1<s_2$ in $\mathbb{R}$ and $r\geq 1$, e^{tB} maps $W^{s_1,r}(0,1)$ into $W^{s_2,r}(0,1)$, for all $t>0$. Furthermore, the following calculation is correct:

$$\mid e^{tB}z\mid_{W^{s_2,r}(0,1)}\leq C_1\left(t^{\frac{s_1-s_2}{2}}+1\right)\mid z\mid_{W^{s_1,r}(0,1)},\quad \text{for all } z\in W^{s_1,r}(0,1). \tag{3.26}$$

The constant C_1 depends only on s_1, s_2 and r see [76] Using the Sobolev embedding theorem we obtain

$$\mid e^{(t-s)B}\partial_x f(v+W_B)\mid_{L^p(0,1)}\leq c_2\mid e^{(t-s)B}\partial_x f(v+W_B)\mid_{W^{\frac{1}{p},\frac{p}{2}}(0,1)},$$

and for (3.17) with $s_1=-1$, $s_2=\frac{1}{p}, r=\frac{p}{2}$
Consequently

$$\begin{aligned}\|\ e^{(t-s)B}\partial_x f(v+W_B)\mid_{L^p(0,1)}\mid_{L^p(0,1)} &\leq c_1c_2((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)\mid \partial_x f(v+W_B)\mid_{W^{-1,\frac{p}{2}}(0,1)}\\ &\leq c_1c_2((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)\mid f(v+W_B)\mid_{L^{\frac{p}{2}}(0,1)}.\end{aligned}$$

Consequently

$$\begin{aligned}\mid z(t)\mid_{L^p(0,1)} &\leq\mid c_0\mid_{L^p(0,1)}+c_1c_2\textstyle\int_0^t((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)\mid f(v+W_B)\mid_{L^{\frac{p}{2}}(0,1)}ds\\ &\leq\mid c_0\mid_{L^p(0,1)}+c_1c_2Lip_1\textstyle\int_0^t((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)(1+\mid v+W_B\mid_{L^{\frac{p}{2}}(0,1)}ds\\ &\leq\mid c_0\mid_{L^p(0,1)}+c_1c_2Lip_1\textstyle\int_0^t((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)(1+\mid v\mid_{L^{\frac{p}{2}}(0,1)}+\mid W_B\mid_{L^{\frac{p}{2}}(0,1)})ds\\ &\leq\mid c_0\mid_{L^p(0,1)}+c_1c_2Lip_1(1+M+\upsilon_p)(\tfrac{2p}{p-1}t^{\frac{1}{2}-\frac{1}{2p}}+t),\end{aligned}$$

where in Lip_1 is the Lipschitz constant of f which depend on $M+\upsilon_p$, and
$\upsilon_p=\sup\limits_{t\in[0,T]}\mid W_B(t)\mid_{L^{\frac{p}{2}}(0,1)}$.
Consequently $\mid z(t)\mid_{L^p(0,1)}\leq M\ forall\ t\in[0,T^*]$, we have

$$\mid c_0\mid_{L^p(0,1)}+c_1c_2Lip_1(M+\upsilon_p)(\frac{2p}{p-1}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*)\leq M. \tag{3.27}$$

It is obvious that every $M>\mid c_0\mid_{L^p(0,1)}$ there exists a T^* fulfilling (3.27).
Now we will take $v_1,\ v_2\in\sum_p(M,T^*)$ and set $z_i=Gv_i, i=1,2$ and $z=z_1-z_2$, Then

$$z(t)=\int_0^t e^{(t-s)B}\frac{\partial}{\partial x}[f(v_1+W_B)-f(v_2+W_B)]ds,$$

we obtain

$$\mid z(t)\mid_{L^p(0,1)}\quad\leq c_1c_2\textstyle\int_0^t((t-s)^{\frac{-1}{2}-\frac{1}{2p}}+1)\mid f(v_1+W_B)-f(v_2+W_B)\mid_{L^{\frac{p}{2}}(0,1)}ds.$$

According to the proposed hypothesis of f, we obtain

$$\begin{aligned}\mid f(v_1+W_B)-f(v_2+W_B)\mid_{L^{\frac{p}{2}}(0,1)} &\leq Lip_2\mid v_1-v_2\mid_{L^{\frac{p}{2}}(0,1)}\\ &\leq Lip_2\mid v_1-v_2\mid_{L^p(0,1)}\\ &=c_3\mid v_1-v_2\mid_{L^p(0,1)},\end{aligned}$$

where in Lip_2 is the Lipschitz constant of f which depend on $M + v_p$, consequently

$$\begin{aligned} | z(t) |_{L^p(0,1)} & \leq c_1 c_2 c_3 \int_0^t ((t-s)^{\frac{-1}{2} - \frac{1}{2p}} + 1) \mid v_1 - v_2 \mid_{L^p(0,1)} ds \\ & \leq C \max_{0 \leq s \leq t} \mid v_1(s) - v_2(s) \mid_{L^p(0,1)} \int_0^t ((t-s)^{\frac{-1}{2} - \frac{1}{2p}} + 1) ds \\ & \leq C(\tfrac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^*) \mid v_1 - v_2 \mid_{C([0,T^*];L^p(0,1))}, \end{aligned}$$

and for any $t \in [0, T^*]$ we have

$$| Gv_1 - Gv_2 |_{c([0,T^*];L^p(0,1))} \leq C(\frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^*) \mid v_1 - v_2 \mid_{c([0,T^*];L^p(0,1))} .$$

We consider T^* as a result

$$C(\frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^*) < 1.$$

and (3.27) holds so that L is a strict contraction on $\sum_p (M, T^*)$. ■
We will demonstrate in the next section that $T^* = T$ a.s. for ω in Ω,

3.4 Global Existence

In this section, we prove the global existence of the problem (3.11). We are also using (3.25) as a deterministic equation, operating a.s. for ω in Ω

Theorem 3.2 *(Global existence.) Let c_0 be given which is F_0-measurable and such that for some $p \geq 2$, $c_0 \in L^p(0, 2\pi)$ a.s. If $a \geq \frac{Lip_1}{2}$, then there exists a unique mild solution of equation (3.21), which belongs a.s. to $c([0, T^*]; L^p(0, 1))$.*

We derive an a priori estimate that yields global existence in the following lemma.

Lemma 3.4 *If $v \in c([0, T^*]; L^p(0, 1))$ salisfies (3.25) and $a \geq \frac{Lip_1}{2}$, then*

$$| v |_{L^p(0,1)} \leq e^{tLip_1 \frac{(p-1)}{2} v_\infty^2} \mid c_0 \mid_{L^p(0,1)},$$

where in $v_\infty = \sup\limits_{t \in [0,T]} \mid W_B(t) \mid_{L^\infty(0,1)}$.

Proof. We put $\{c_0^n\}$ be sequence in $c^\infty(0, 1)$ such that $c_0^n \to c_0$ in $L^p(0, 1)$ and put $\{W^n\}$ be sequence for regular processes with

$$W_B^n(t) = \int_0^t e^{(t-s)B} dW^n(s) \to W_B(t), \text{in } c([0, T] \times (0, 1)),$$

a.s. for $\omega \in \Omega$. We put $\{v^n\}$ is the solution of

$$v^n(t) = e^{tB} c_0^n - \int_0^t e^{(t-s)B} \frac{\partial}{\partial x} f(v^n + W_B^n) ds,$$

we obtain by Lemma (3.3). It is easy to see that v^n does exist on an interval of time $[0, T_n]$ such that $T_n \to T^*$ a.s. and that v_n converges to U in $c([0, T^*]; L^p(0, 1))$. Furthermore, v_n is regular a.s. and fulfills

$$\frac{\partial v^n}{\partial t} - a \frac{\partial^2 v^n}{\partial x^2} + \partial_x f(v^n + W_B^n) = 0. \tag{3.28}$$

Multipling (3.28) by $| v^n |^{p-2} v^n$ and integrating over $[0,1]$, we obtain

$$\frac{1}{p}\frac{\partial}{\partial t} | v^n |^p_{L^p(0,1)} + a(p-1)\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x}v^n)^2 dx + \int_0^1 \partial_x f(v^n + W_B^n) | v^n |^{p-2} v^n dx = 0. \qquad (3.29)$$

We integrate by parts the last integral

$$\int_0^1 \partial_x f(v^n + W_B^n) | v^n |^{p-2} v^n dx = -(p-1)\int_0^1 f(v^n + W_B^n) | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx,$$

then

$$\begin{aligned}
| \textstyle\int_0^1 \frac{\partial}{\partial x} f(v^n + W_B^n) | v^n |^{p-2} v^n dx |= &\ (p-1) | \textstyle\int_0^1 f(v^n + W_B^n) | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx | \\
\leq &\ (p-1)\textstyle\int_0^1 | f(v^n + W_B^n) | v^n |^{p-2} \frac{\partial}{\partial x} v^n | dx \\
\leq &\ (p-1)\textstyle\int_0^1 Lip_1(1 + | v^n + W_B^n |) | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx \\
\leq &\ (p-1)\textstyle\int_0^1 Lip_1 | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx \\
+ &\ (p-1)\textstyle\int_0^1 Lip_1 | v^n |^{p-1} \frac{\partial}{\partial x} v^n dx \\
+ &\ (p-1)\textstyle\int_0^1 Lip_1 | W_B^n || v^n |^{p-2} \frac{\partial}{\partial x} v^n dx.
\end{aligned}$$

The first term issezo, indeed,

$$\int_0^1 | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx = -\int_0^1 (p-2) | v^n |^{p-2} \frac{\partial}{\partial x} | v^n | dx.$$

Consequently

$$(p-1)\int_0^1 | v^n |^{p-2} \frac{\partial}{\partial x} v^n dx = 0.$$

Similarly, we can show that the second term is zero. We bound the third term as follows using the Hölder's and Couchy's inequalities.

$$\begin{aligned}
Lip_1(p-1)\textstyle\int_0^1 | W_B^n || v^n |^{p-2} \frac{\partial}{\partial x} v^n dx &\leq Lip_1(p-1) | W_B^n |_{L^\infty(0,1)} | v^n |^{\frac{p-2}{2}}_{L^{p-2}(0,1)} \\
&\times (\textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx)^{\frac{1}{2}} \\
&\leq Lip_1(p-1) v_{n,\infty} | v^n |^{\frac{p-2}{2}}_{L^p(0,1)} (\textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx)^{\frac{1}{2}} \\
&\leq Lip_1 \frac{(p-1)}{2} v_{n,\infty}^2 | v^n |^{p-2}_{L^p(0,1)} + Lip_1 \frac{(p-1)}{2} \textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx,
\end{aligned}$$

where in $v_{n,\infty} = \sup_{t \in [0,T]} | W_B^n |_{L^\infty(0,1)}$ for all, $\omega \in \Omega$.
Using (3.29), we get

$$\begin{aligned}
\tfrac{1}{p}\tfrac{\partial}{\partial t} | v^n |^p_{L^p(0,1)} + a(p-1)\textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx &\leq Lip_1 \frac{(p-1)}{2} v_{n,\infty}^2 | v^n |^{p-2}_{L^p(0,1)} \\
&+ Lip_1 \frac{(p-1)}{2} \textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx,
\end{aligned}$$

as follows

$$\tfrac{1}{p}\tfrac{\partial}{\partial t} | v^n |^p_{L^p(0,1)} + (p-1)\left(a - \tfrac{Lip_1}{2}\right)\textstyle\int_0^1 | v^n |^{p-2} (\frac{\partial}{\partial x} v^n)^2 dx \leq Lip_1 \frac{(p-1)}{2} v_{n,\infty}^2 | v^n |^{p-2}_{L^p(0,1)}.$$

If we set a and Lip_1 such that

$$a \geq \frac{Lip_1}{2},$$

we have

$$\frac{\partial}{\partial t} \mid v^n \mid_{L^p(0,1)}^p \leq Lip_1 \frac{p(p-1)}{2} v_{n,\infty}^2 \mid v^n \mid_{L^p(0,1)}^{p-2},$$

application Gronwall's lemma, we obtain

$$\mid v^n \mid_{L^p(0,1)}^p \leq e^{tLip_1 \frac{p(p-1)}{2} v_{n,\infty}^2} \mid c_0^n \mid_{L^p(0,1)}^p .$$

Taking the limit as $n \to \infty$, we obtain

$$\mid v \mid_{L^p(0,1)}^p \leq e^{tLip_1 \frac{p(p-1)}{2} \nu_{\infty}^2} \mid c_0 \mid_{L^p(0,1)}^p .$$

As a result

$$\mid v \mid_{L^p(0,1)} \leq e^{tLip_1 \frac{(p-1)}{2} v_{\infty}^2} \mid c_0 \mid_{L^p(0,1)},$$

following that, the lemma is asserted. ∎

Proof of Theorem (3.2). It is easily deduced from Lemma (3.3) and Lemma (3.4).

CHAPTER 4

EXISTENCE AND UNIQUENESS OF WEAK SOLUTION FOR KELLER-SEGEL MODEL COUPLED WITH BOUSSINESQ EQUATIONS

In this chapter, we study the phenomenon of Keller Segel model coupled with a Boussinesq equations. The main objectives of this chapter is to study the existence and uniqueness of weak solution for the problem (4.1), for this we use the technical of Galerkin method.
Now we consider Keller–Segel model coupled to Boussinesq equations define as

$$\left\{ \begin{array}{r} n_t - u\nabla n - \Delta n - \nabla(n\nabla c) - \nabla(n\nabla\theta) = 0,\ (x,t) \in \Omega \times \mathbb{R}^+, \\ c_t - u\nabla c - \triangle c - \tau c - \rho u - b\theta = 0,\ (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta_t - u\nabla\theta - k\triangle\theta - n\theta = 0,\ (x,t) \in \Omega \times \mathbb{R}^+, \\ u_t - u\nabla u - v\Delta u - \nabla p - (\theta + n)f = 0,\ (x,t) \in \Omega \times \mathbb{R}^+. \end{array} \right. \tag{4.1}$$

Where $n = n(x,t)$ denots the density of the cells in position $x \in \mathbb{R}^d$ at time t, $c = c(x,t)$ is the concentration of chemical attractant, $\theta = \theta(x,t)$ denots temperature of the fluid, $u = u(x,t)$ is field denoting the velocity, τ, ρ, b are positive constants, $k \geq 0$ is the thermal diffusivity, $v \geq 0$ is the kinematic veseosity, $f(x,t)$ is the extermal potential force where $\nabla f = 0$, $p(x,t)$ is the pressuve. The main objectives of this chapter is to study of the problem Keller-Segel coupled with Boussiqnice equation and we demonstrate the existence and uniqueness of weak solution for KSB problem with Dirichlet boundary conditions and initials conditions defined as:

$$
\left\{
\begin{array}{l}
P1 \left\{
\begin{array}{l}
n_t - u\nabla n - \Delta n - \nabla(n\nabla c) - \nabla(n\nabla\theta) = 0, \ (x,t) \in \Omega\times\mathbb{R}^+, \\
n = 0, \ x \in \Gamma, \\
n(0,x) = n_0, \ x \in \ \Omega,
\end{array}
\right. \\
\\
P2 \left\{
\begin{array}{l}
c_t - u\nabla c - \triangle c - \tau c - \rho u - b\theta = 0, \ (x,t) \in \Omega\times\mathbb{R}^+, \\
c = 0, \ x \in \Gamma, \\
c(0,x) = c_0, \ x \in \ \Omega,
\end{array}
\right. \\
\\
P3 \left\{
\begin{array}{l}
\theta_t - u\nabla\theta - k\triangle\theta - n\theta = 0, \ (x,t) \in \Omega\times\mathbb{R}^+, \\
\theta = 0, \ x \in \Gamma, \\
\theta(0,x) = \theta_0, \ x \in \Omega.
\end{array}
\right. \\
\\
P4 \left\{
\begin{array}{l}
u_t - u\nabla u - v\Delta u - \nabla p - (\theta + n)f = 0, \ (x,t) \in \Omega\times\mathbb{R}^+, \\
u = 0, \ x \in \Gamma, \\
u(0,x) = u_0, \ x \in \Omega,
\end{array}
\right.
\end{array}
\right. \tag{4.2}
$$

4.1 Existence and uniqueness of weak solution of the problem

To simplify to prove the weak solution of the problem (4.2) a decomposition into four subproblems $(P1)$ and $(P2)$ and $(P3)$ and $(P4)$ are adopted. We use the Galerkin method we can demonstrate the existence and uniqueness of a weak solution of subproblems $(P1)$ and $(P2)$ and $(P3)$ and $(P4)$ therefore we have the existence and uniqueness of a weak solution of the problem (4.2). The following initial-boundary conditions assumption is used to prove the solution of problem (4.2),

$$n_0 \in L^2(\Omega), \tag{4.3}$$

$$c_0 \in L^2(\Omega), \tag{4.4}$$

$$\theta_0 \in L^2(\Omega). \tag{4.5}$$

$$u_0 \in L^2(\Omega). \tag{4.6}$$

4.1.1 Existence and uniqueness of weak solution of the problem (P1)

In subsection, we prove the existence and uniqueness of weak solution result of the problem $(P1)$.

Definition 4.1 We say $n \in L^2(0,T;H_0^1(\Omega))$ with $n_t \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the problem $(P1)$ if and only if

$$\langle n_t, \Phi\rangle + B(n,\Phi,t) = 0, \tag{4.7}$$

where

$$B(n,\Phi,t) = \int_\Omega [(\nabla n\nabla\Phi) + n\nabla(u\Phi) + (n\nabla c + n\nabla\theta)\nabla\Phi]dx, \tag{4.8}$$

for all $\Phi \in H_0^1(\Omega)$,$0 \leq t \leq T$,and

$$n(0,x) = n_0 \in L^2(\Omega). \tag{4.9}$$

Remark 4.1 Note that $n \in C([0,T];L^2(\Omega))$ as $n \in L^2(0,T;H_0^1(\Omega))$ and $n_t \in L^2(0,T;H^{-1}(\Omega))$ Then equality (4.9) makes sense.

To demonstrate the existence of weak solution of problem $(P1)$ we use the method of Galerkin, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m, \text{ its linearly independent}, \\ \text{ the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{4.10}$$

We are looking for $n_m = n_m(t)$ solution $\prec\prec$approached$\succ\succ$ of the problem in the form

$$n_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{4.11}$$

and g_{im} to be determined by the conditions:

$$\begin{cases} \langle n_m^{'}, w_j \rangle + B(n_m, w_j, t) = 0, \\ \qquad 1 \leq j \leq m. \end{cases} \tag{4.12}$$

The nonlinear differential equation system is to be completed by the conditional:

$$n_m(0) = n_{0m}, \ n_{0m} = \textstyle\sum_{i=1}^{m} \alpha_{im} w_j \to u_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty.$$

We propose now to send m to infinity and show a subsequence of our solutions n_m of the approximation problems (4.12) and (4.1.1) converges to a weak solution of $(P1)$. For this we will need some uniform estimates.

4.1.1.1 Energy estimates

Theorem 4.1 *(Energy estimates.) There exists a constant C depending only on Ω, T, such that*

$$\max_{0 \leq t \leq T} \| n_m \|_{L^2(\Omega)} + \| n_m \|_{L^2(0,T;H_0^1(\Omega))} + \| n_m' \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \| n_0 \|_{L^2(\Omega)} . \tag{4.13}$$

Proof. In order to prove the estimation (4.13) we will estime each terms in the left side of (4.11) one by one as follows:

1. Multiplying equation (4.12) by $g_{jm}(t)$ and summing for $k = 1...m$, and then recalling (4.11) we find

$$\langle n_m^{'}, n_m \rangle + B(n_m, n_m, t) = 0, \tag{4.14}$$

and we have

$$\frac{1}{2} \frac{d}{dt} [\| n_m \|_{L^2(\Omega)}^2] + B(n_m, n_m, t) = 0, \tag{4.15}$$

and we put $\| n \| = \sqrt{B(n,n)}$ ($=$ is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2} \frac{d}{dt} (\| n_m \|_{L^2(\Omega)}^2) + \| n_m \|_{H_0^1(\Omega)}^2 = 0, \tag{4.16}$$

so we have

$$\max_{0 \leq t \leq T} \| n_m \|_{L^2(\Omega)} \leq \| n_0 \|_{L^2(\Omega)} . \tag{4.17}$$

2. Integrate inequality (4.16) from 0 to T and we employ the inequality (4.17) to find

$$\| n_m \|_{L^2(0,T;H_0^1(\Omega))}^2 = \int_0^T \| n_m \|_{H_0^1(\Omega)}^2 \, dt.$$

3. Fix any $v \in H_0^1(\Omega)$, with $\| v \|^2_{H_0^1(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$, $(k = 1, ..., m)$, we use (4.12) we deduce for all $0 \leq t \leq T$ that

$$(n_m^{'}, v^1) + B(n_m, v^1, t) = 0, \tag{4.18}$$

then (4.18) implies

$$\langle n_m^{'}, v \rangle = (n_m^{'}, v) = (n_m^{'}, v^1) = -B(n_m, v^1, t),$$

consequently

$$| \langle n_m^{'}, v \rangle | \leq C \| n_m \|^2_{H_0^1(\Omega)},$$

since
$\| v^1 \|^2_{H_0^1(\Omega)} \leq \| v \|^2_{H_0^1(\Omega)} \leq 1$, we have

$$\| n_m^{'} \|_{H^{-1}(\Omega)} \leq C \| n_m \|_{H_0^1(\Omega)},$$

therefore

$$\| n_m^{'} \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| n_m^{'} \|^2_{H^{-1}(\Omega)} \, dt \leq C \int_0^T \| n_m \|^2_{H_0^1(\Omega)} \, dt \leq C \| n_0 \|^2_{L^2(\Omega)} .$$

■

4.1.1.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P1)$.

Theorem 4.2 *(Existence of weak solution.) Under hypothesis (4.3), there exists a weak solution of problem $(P1)$.*

Proof. According to the energy estimates (4.13), we see that the sequence $\{n_m\}_{m=1}^{\infty}$is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{n_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. Consequently there exists a subsequence which is also noted by $\{n_m\}_{m=1}^{\infty}$ and a function $n \in L^2(0, T; H_0^1(\Omega))$, with $n^{'} \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} n_m &\rightarrow n \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ n_m^{'} &\rightarrow n^{'} \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{4.19}$$

2. Next fix an integer N and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} g^{(k)}(t) w_k, \tag{4.20}$$

where $\left\{g^{(k)}\right\}_{k=1}^{N}$ are given smooth functions, we choose $m \geq N$ and multipling equation (4.12) by $g^{(k)}(t)$ $\forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle n_m^{'}, v \rangle + B(n_m, v, t) dt = 0, \tag{4.21}$$

we recall (4.19) and to find upon passing to weak limits that

$$\int_0^t \langle n^{'}, v \rangle + B(n, v, t) dt = 0, \quad \forall v \in L^2(0, T; H_0^1(\Omega)), \tag{4.22}$$

as functions of the form (4.20) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle n', v\rangle + B(n,v,t) = 0, \quad \forall v \in H_0^1(\Omega) \text{ and } \forall t \in [0,T], \tag{4.23}$$

and from Remark (4.1) we have $n \in C(0,T;L^2(\Omega))$.

3. In order to prove $n(0) = n_0$, we first note from (4.9) that

$$\int_0^t -\langle n, v'\rangle + B(n,v,t) = (n(0), v(0)), \tag{4.24}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (4.21) we obtain

$$\int_0^t -\langle n_m, v'\rangle + B(n_m,v,t)dt = (n_0, v(0)), \tag{4.25}$$

we use again (4.24), we obtain

$$\int_0^t -\langle n, v'\rangle + B(n,v,t)dt = (n_0, v(0)), \tag{4.26}$$

since $n_m(0) \to n_0$ in $L^2(\Omega)$. Comparing (4.24) and (4.26), we conclude $n(0) = n_0$. ■

Theorem 4.3 *(Uniqueness of weak solutions.) A weak solution of problem $(P1)$ is unique.*

Proof. We suppose there exists two weak solution n_1 and $n_2$and we put $N = n_2 - n_1$ then N is also a solution of problem $(P1)$ with $N_0 = (n_2 - n_1)(0) \equiv 0$. Setting $v = N$ in identity (4.15) we are

$$\frac{d}{dt}(\frac{1}{2} \parallel N \parallel^2_{L^2(\Omega)}) + B(N,N,t) = 0,$$

so, we have

$$\frac{d}{dt}(\frac{1}{2} \parallel N \parallel^2_{L^2(\Omega)}) \leq 0,$$

then integrate with respect to t to find

$$\parallel N \parallel^2_{L^2(\Omega)} \leq \parallel N_0 \parallel^2_{L^2(\Omega)} = 0,$$

thus $N \equiv 0$. ■

4.1.2 Existence and uniqueness of weak solution of problem (P2)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem $(P2)$.

Definition 4.2 We say $c \in L^2(0,T;H_0^1(\Omega))$ with $c_t \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the problem $(P2)$ if and only if

$$\langle c_t, q\rangle + L(c,q,t) = 0, \tag{4.27}$$

where

$$L(c,q,t) = \int_\Omega [(\nabla c \nabla q) + c\nabla(uq) + \tau cq + \rho nq + b\theta q]dx, \tag{4.28}$$

for all $q \in H_0^1(\Omega)$,$0 \leq t \leq T$, and

$$c(0,x) = c_0 \in L^2(\Omega). \tag{4.29}$$

Remark 4.2 Note that $c \in C([0,T];L^2(\Omega))$ as $c \in L^2(0,T;H_0^1(\Omega))$ and $c_t \in L^2(0,T;H^{-1}(\Omega))$, then equality (4.29) makes sense.

To demonstrate existence of weak solution of problem (P2) we use the Galerkin method, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent }, \\ \text{ the finite linear combination of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases}$$

We are looking for $c_m = c_m(t)$ solution $\prec\prec$approached$\succ\succ$ of the problem in the form

$$c_m(t) = \sum_{i=1}^{m} d_{im}(t) w_i, \tag{4.30}$$

the d_{im} to be determined by the conditions:

$$\begin{cases} \langle c'_m, w_j \rangle + L(c_m, w_j, t) = 0, \\ 1 \leq j \leq m. \end{cases} \tag{4.31}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$c_m(0) = c_{0m}, \ c_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to c_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty. \tag{4.32}$$

We now propose to send m to infinity and to show a subsequence of our solutions c_m approximation problems (4.31) and (4.32) converges towards a weak solution of the problem $(P2)$. For this we need uniform estimates.

4.1.2.1 Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions c_m of the approximation problems (4.31) and (4.32) converges to a weak solution of problem $(P2)$. For this we will need some uniform estimates.

Theorem 4.4 *(Energy estimates.) They exists a constant C depending only on Ω, T such that*

$$\max_{0 \leq t \leq T} \| c_m \|_{L^2(\Omega)} + \| c_m \|_{L^2(0,T;H_0^1(\Omega))} + \| c'_m \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \| c_0 \|_{L^2(\Omega)} . \tag{4.33}$$

Proof. In order to prove the estimation (4.33) we will estime each terms in the left side of (4.31) one by one as follows:

1. Multiplying equation (4.31) by $d_{jm}(t)$ and summing for j, we find

$$\langle c'_m, w_j \rangle + L(c_m, c_m, t) = 0, \tag{4.34}$$

and we have

$$\frac{1}{2} \frac{d}{dt} [\| c_m \|^2_{L^2(\Omega)}] + L(c_m(t), c_m(t)) = 0,$$

we put $\| v \| = \sqrt{L(v,v)}$ ($=$ is norm in $H_0^1(\Omega)$), so we have

$$\frac{1}{2} \frac{d}{dt} (\| c_m \|^2_{L^2(\Omega)}) + \| c_m \|^2_{H_0^1(\Omega)} = 0, \tag{4.35}$$

we obtain

$$\frac{d}{dt} (\| c_m \|^2_{L^2(\Omega)}) \leq 0,$$

and we have

$$\| c_m \|^2_{L^2(\Omega)} \leq \| c_m(0) \|^2_{L^2(\Omega)} \leq \| c_0 \|^2_{L^2(\Omega)},$$

so, we obtain

$$\max_{0\leq t\leq T} \parallel c_m \parallel_{L^2(\Omega)} \leq \parallel c_0 \parallel_{L^2(\Omega)} . \tag{4.36}$$

2. Integrate inequality (4.35) from 0 to T and we use (4.36) to find

$$\parallel c_m \parallel^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \parallel c_m \parallel^2_{H^1_0(\Omega)} dt.$$

3. Fix any $v \in H^1_0(\Omega)$, with $\parallel v \parallel^2_{H^1_0(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, ..., m)$. we use (4.31) from all $0 \leq t \leq T$ that

$$(c^{'}_m, v^1) + L(c_m, v^1, t) = 0. \tag{4.37}$$

Then (4.37) implies

$$(c^{'}_m, v) = \langle c^{'}_m, v\rangle = \langle c^{'}_m, v^1\rangle = -L(c_m, v^1, t),$$

consequently

$$| \langle c^{'}_m, v\rangle | \leq C \parallel c_m \parallel^2_{H^1_0(\Omega)},$$

then
$\parallel v^1 \parallel^2_{H^1_0(\Omega)} \leq \parallel v \parallel^2_{H^1_0(\Omega)} \leq 1$, thus

$$\parallel c^{'}_m \parallel_{H^{-1}(\Omega)} \leq C \parallel c_m \parallel_{H^1_0(\Omega)},$$

and therefore

$$\parallel c^{'}_m \parallel^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \parallel c^{'}_m \parallel^2_{H^{-1}(\Omega)} dt \leq C \int_0^T \parallel c_m \parallel^2_{H^1_0(\Omega)} dt \leq C \parallel c_0 \parallel^2_{L^2(\Omega)} .$$

■

4.1.2.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P2)$.

Theorem 4.5 *(Existence of weak solution.) Under hypothesis* (4.4)*, there exists a weak solution of* $(P2)$.

Proof. According to the energy estimates (4.33), we see that the sequence $\{c_m\}_{m=1}^{\infty}$is bounded in $L^2(0, T; H^1_0(\Omega))$ and $\{c^{'}_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $c \in L^2(0, T; H^1_0(\Omega))$ with $c^{'} \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} c_m &\rightarrow c \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \\ c^{'}_m &\rightarrow c^{'} \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{4.38}$$

2. Next fix an integer N and choose a function $v \in C^1(0, T; H^1_0(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{4.39}$$

where $\left\{d^{(k)}\right\}_{k=1}^{N}$ are given smooth functions. We choose $m \geq N$, multiply equation (4.31) by $d^{(k)}(t)$. $\forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle c^{'}_m, v\rangle + L(c_m, v, t)dt = 0, \tag{4.40}$$

we recall (4.38) to find upon passing to weak limits that

$$\int_0^t \langle c^{'}, v\rangle + L(c,v,t)dt = 0,\ \forall v \in L^2(0,T;H_0^1(\Omega)). \tag{4.41}$$

As functions of the form (4.30) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle c', v\rangle + L(c,v,t) = 0,\ \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0,T] \tag{4.42}$$

and from Remark (4.2) we have $c \in C(0,T;L^2(\Omega))$.
3. In order to prove for prouver $c(0) = c_0$, we first note from (4.29) that

$$\int_0^t -\langle c, v'\rangle + L(c,v,t)dt = (c(0), v(0)), \tag{4.43}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (4.43) we obtain

$$\int_0^t -\langle c_m, v'\rangle + L(c_m,v,t)dt = (c_0, v(0)), \tag{4.44}$$

we use again (4.43), we obtain

$$\int_0^t -\langle c, v'\rangle + L(c,v,t)dt = (c_0, v(0)), \tag{4.45}$$

since $c_m(0) \to c_0$ in $L^2(\Omega)$. Comparing (4.43) and (4.45), we conclude $c(0) = c_0$. ■

Theorem 4.6 (*Uniqueness of weak solutions.*) *A weak solution of problem* $(P2)$ *is unique.*

Proof. We suppose there exists two weak solution c_1 et c_2 and we put that $C = c_2 - c_1$ then C is also a solution of $(P2)$ with $C_0 = (c_2 - c_1)(0) \equiv 0$. Setting $v = C$ in identity (4.42) we have

$$\frac{d}{dt}(\frac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) + L(C,C,t) = 0,$$

and as $\parallel C \parallel = \sqrt{L(C,C)}$ (= norm in $H_0^1(\Omega)$), there $L(C,C,t) = \parallel C \parallel^2_{H_0^1(\Omega)} \geq 0$, then we have

$\frac{d}{dt}(\frac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) \leq 0,$
then integrate with respect to t to find

$$\parallel C \parallel^2_{L^2(\Omega)} \leq \parallel C_0 \parallel^2_{L^2(\Omega)} = 0,$$

then $C \equiv 0$. ■

4.1.3 Existence and uniqueness of weak solution of the problem (P3)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem $(P3)$

Definition 4.3 We say $\theta \in L^2(0,T;H_0^1(\Omega))$ with $\theta_t \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the problem $(P3)$ if and only if

$$\langle c_t, w\rangle + A(c,w,t) = 0, \tag{4.46}$$

where

$$A(c,w,t) = \int_\Omega [k(\nabla\theta\nabla w) + \theta\nabla(uw) + n\theta w]dx, \tag{4.47}$$

for all $w \in H_0^1(\Omega)$,$0 \leq t \leq T$, and

$$\theta(0,x) = \theta_0 \in L^2(\Omega). \tag{4.48}$$

Remark 4.3 Note that $\theta \in C([0,T];L^2(\Omega))$ as $\theta \in L^2(0,T;H_0^1(\Omega))$ and $\theta_t \in L^2(0,T;H^{-1}(\Omega))$ then equality (4.48) makes sense.

To demonstrate existence of weak solution of problem $(P3)$ we use the Galerkin method, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent }, \\ \text{ the finite linear combination of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{4.49}$$

We are looking for $\theta_m = \theta_m(t)$ solution $\prec\prec$ approached$\succ\succ$ of the problem in the form

$$\theta_m(t) = \sum_{i=1}^{m} d_{im}(t) w_i, \tag{4.50}$$

the d_{im} to be determined by the conditions:

$$\begin{cases} \langle \theta'_m, w_j \rangle + A(\theta_m, w_j, t) = 0, \\ 1 \le j \le m. \end{cases} \tag{4.51}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$\theta_m(0) = \theta_{0m}, \ \theta_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to \theta_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty. \tag{4.52}$$

We now propose to send m to infinity and to show a subsequence of our solutions θ_m approximation problems (4.51) and (4.52) converges towards a weak solution of the problem $(P3)$. For this we need some uniform estimates.

4.1.3.1 Energy estimates

Theorem 4.7 *(Energy estimates.) They exists a constant C depending only on Ω, T such that*

$$\max_{0 \le t \le T} \| \theta_m \|_{L^2(\Omega)} + \| \theta_m \|_{L^2(0,T;H_0^1(\Omega))} + \| \theta'_m \|_{L^2(0,T;H^{-1}(\Omega))} \le C \| \theta_0 \|_{L^2(\Omega)} . \tag{4.53}$$

Proof. In order to prove the estimation (4.53) we will estime each terms in the left side of (4.51) one by one as follows:

1. Multiplying equation (4.51) by $d_{jm}(t)$ and summing for j we find

$$\langle \theta'_m, w_j \rangle + A(\theta_m, \theta_m, t) = 0, \tag{4.54}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\| \theta_m \|^2_{L^2(\Omega)}] + A(\theta_m(t), \theta_m(t)) = 0,$$

and we put $\| v \| = \sqrt{A(v,v)}$ (= is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{d}{dt}(\| \theta_m \|^2_{L^2(\Omega)}) + \| \theta_m \|^2_{H_0^1(\Omega)} = 0, \tag{4.55}$$

we have

$$\frac{d}{dt}(\| \theta_m \|^2_{L^2(\Omega)}) \le 0,$$

and we have

$$\| \theta_m \|^2_{L^2(\Omega)} \le \| \theta_m(0) \|^2_{L^2(\Omega)} \le \| \theta_0 \|^2_{L^2(\Omega)},$$

so we are

$$\max_{0\le t\le T} \parallel \theta_m \parallel_{L^2(\Omega)} \le \parallel \theta_0 \parallel_{L^2(\Omega)} . \tag{4.56}$$

2. Integrate inequality (4.55) from 0 to T and we use (4.56) to find

$$\parallel \theta_m \parallel^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \parallel \theta_m \parallel^2_{H^1_0(\Omega)} dt.$$

3. Fix any $v \in H^1_0(\Omega)$, with $\parallel v \parallel^2_{H^1_0(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, ..., m)$. we use (4.51) from all $0 \le t \le T$ that

$$(\theta'_m, v^1) + A(\theta_m, v^1, t) = 0. \tag{4.57}$$

Then (4.57) implies

$$(\theta'_m, v) = \langle \theta'_m, v\rangle = \langle \theta'_m, v^1\rangle = -A(\theta_m, v^1, t),$$

consequently

$$\mid \langle \theta'_m, v\rangle \mid \le C \parallel \theta_m \parallel^2_{H^1_0(\Omega)},$$

and as

$\parallel v^1 \parallel^2_{H^1_0(\Omega)} \le \parallel v \parallel^2_{H^1_0(\Omega)} \le 1,$

thus

$$\parallel \theta'_m \parallel_{H^{-1}(\Omega)} \le C \parallel \theta_m \parallel_{H^1_0(\Omega)},$$

and therefore

$$\parallel \theta'_m \parallel^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \parallel \theta'_m \parallel^2_{H^{-1}(\Omega)} dt \le C \int_0^T \parallel \theta_m \parallel^2_{H^1_0(\Omega)} dt \le C \parallel \theta_0 \parallel^2_{L^2(\Omega)} .$$

■

4.1.3.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P3)$.

Theorem 4.8 *(Existence of weak solution.) Under hypothesis (4.57), there exists a weak solution of $(P3)$.*

Proof. According to the energy estimates (4.53), we see that the sequence $\{\theta_m\}_{m=1}^{\infty}$is bounded in $L^2(0, T; H^1_0(\Omega))$ and $\{\theta'_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $\theta \in L^2(0, T; H^1_0(\Omega))$ with $\theta' \in L^2(0, T; H^{-1}(\Omega))$, such that

$$\begin{aligned} \theta_m &\to \theta \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \\ \theta'_m &\to \theta' \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \tag{4.58}$$

2. Next fix an integer N and choose a function $v \in C^1(0, T; H^1_0(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{4.59}$$

where $\left\{d^{(k)}\right\}_{k=1}^{N}$ are given smooth functions. We choose $m \ge N$, multiply equation (4.51) by $d^{(k)}(t)$ $\forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle \theta'_m, v\rangle + A(\theta_m, v, t)dt = 0, \tag{4.60}$$

we recall (4.58) to find upon passing to weak limits that

$$\int_0^t \langle \theta^{'}, v\rangle + A(\theta, v, t)dt = 0,\ \forall v \in L^2(0, T; H_0^1(\Omega)). \tag{4.61}$$

As functions of the form (4.59) are dense in $L^2(0, T; H_0^1(\Omega))$. Hence in particular

$$\langle \theta', v\rangle + A(\theta, v, t) = 0,\ \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0, T] \tag{4.62}$$

and from Remark (4.3) we have $\theta \in C(0, T; L^2(\Omega))$.
3. In order to prove for to prouve $\theta(0) = \theta_0$, we first note from (4.48) that

$$\int_0^t -\langle \theta, v'\rangle + A(\theta, v, t)dt = (\theta(0), v(0)), \tag{4.63}$$

for each $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (4.60) we obtain

$$\int_0^t -\langle \theta_m, v'\rangle + A(\theta_m, v, t)dt = (\theta_0, v(0)), \tag{4.64}$$

we use again (4.63), we obtain

$$\int_0^t -\langle \theta, v'\rangle + A(\theta, v, t)dt = (\theta_0, v(0)), \tag{4.65}$$

since $\theta_m(0) \to \theta_0$ in $L^2(\Omega)$. Comparing (4.63) and (4.65), we conclude $(\theta_0) = \theta_0$. ■

Theorem 4.9 (*Uniqueness of weak solutions.*) *A weak solution of problem* $(P3)$ *is unique.*

Proof. We suppose there exists two weak solution θ_1 et θ_2 and we put that $\theta = \theta_2 - \theta_1$ then θ is also a solution of $(P3)$ with $\theta_0 = (\theta_2 - \theta_1)(0) \equiv 0$. Setting $v = \theta$ in identity (4.62) we have

$$\frac{d}{dt}(\frac{1}{2} \parallel \theta \parallel^2_{L^2(\Omega)}) + A(\theta, \theta, t) = 0,$$

and as $\parallel \theta \parallel = \sqrt{A(\theta, \theta)}$ (norm in $H_0^1(\Omega)$), there $A(\theta, \theta, t) = \parallel \theta \parallel^2_{H_0^1(\Omega)} \geq 0$, then we have

$\frac{d}{dt}(\frac{1}{2} \parallel \theta \parallel^2_{L^2(\Omega)}) \leq 0,$
then integrate with respect to t to find

$$\parallel \theta \parallel^2_{L^2(\Omega)} \leq \parallel \theta_0 \parallel^2_{L^2(\Omega)} = 0,$$

then $\theta \equiv 0$. ■

4.1.4 Existence and uniqueness of weak solution of problem (P4)

We state and prove the existence and uniqueness of weak solution result of the problem $(P4)$

Definition 4.4 We say $u \in L^2(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of the problem $(P4)$ if and only if

$$\langle u_t, z\rangle + K(u, z, t) = 0,$$

where

$$K(c, z, t) = \int_\Omega [(\nu \nabla u \nabla q) + u\nabla(uz) + p\nabla z + (\theta + n)fz]dx, \tag{4.66}$$

for all $z \in H_0^1(\Omega)$,$0 \leq t \leq T$,and

$$u(0, x) = u_0 \in L^2(\Omega). \tag{4.67}$$

Remark 4.4 Note that $u \in C([0,T]; L^2(\Omega))$ as $u \in L^2(0,T; H_0^1(\Omega))$ and $u_t \in L^2(0,T; H^{-1}(\Omega))$, then equality (4.67) makes sense.

To demonstrate existence of weak solution of problem (P1) we use the Galerkin method, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent }, \\ \text{ the finite linear combination of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{4.68}$$

We are looking for $u_m = u_m(t)$ solution $\prec\prec$approached$\succ\succ$ of the problem in the form

$$u_m(t) = \sum_{i=1}^{m} d_{im}(t) w_i, \tag{4.69}$$

the d_{im} to be determined by the conditions:

$$\begin{cases} \langle c'_m, w_j\rangle + K(c_m, w_j, t) = 0, \\ 1 \leq j \leq m. \end{cases} \tag{4.70}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$u_m(0) = u_{0m},\ u_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to u_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty. \tag{4.71}$$

We now propose to send m to infinity and to show a subsequence of our solutions u_m approximation problems (4.70) and (4.71) converges towards a weak solution of the problem $(P4)$, for this we need uniform estimates.

4.1.4.1 Energy estimates

Theorem 4.10 *(Energy estimates.) They exists a constant C depending only on Ω, T such that*

$$\max_{0\leq t\leq T} \| u_m \|_{L^2(\Omega)} + \| u_m \|_{L^2(0,T;H_0^1(\Omega))} + \| u'_m \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \| u_0 \|_{L^2(\Omega)} . \tag{4.72}$$

Proof. In order to prove the estimation (4.72) we will estime each terms in the left side of (4.70) one by one as follows:

1. Multiplying equation (4.70) by $d_{jm}(t)$ and summing for j we find

$$\langle u'_m, w_j\rangle + K(u_m, u_m, t) = 0, \tag{4.73}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\| u_m \|^2_{L^2(\Omega)}] + K(u_m(t), u_m(t)) = 0, \tag{4.74}$$

and we put $\| u \| = \sqrt{K(u,u)}$ ($=$ is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{d}{dt}(\| u_m \|^2_{L^2(\Omega)}) + \| u_m \|^2_{H_0^1(\Omega)} = 0, \tag{4.75}$$

we have

$$\frac{d}{dt}(\| u_m \|^2_{L^2(\Omega)}) \leq 0,$$

and we have

$$\| u_m \|^2_{L^2(\Omega)} \leq \| u_m(0) \|^2_{L^2(\Omega)} \leq \| u_0 \|^2_{L^2(\Omega)}, \tag{4.76}$$

so, we obtain

$$\max_{0\le t\le T} \| u_m \|_{L^2(\Omega)} \le \| u_0 \|_{L^2(\Omega)} . \tag{4.77}$$

2. Integrate inequality (4.75) from 0 to T and we use (4.77) to find

$$\| u_m \|^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \| u_m \|^2_{H^1_0(\Omega)} dt. \tag{4.78}$$

3. Fix any $v \in H^1_0(\Omega)$, with $\| v \|^2_{H^1_0(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, ..., m)$. we use (4.70) from all $0 \le t \le T$ that

$$(u'_m, v^1) + K(u_m, v^1, t) = 0. \tag{4.79}$$

Then (4.79) implies

$$(u'_m, v) = \langle u'_m, v\rangle = \langle u'_m, v^1\rangle = -K(u_m, v^1, t),$$

consequently

$$| \langle u'_m, v\rangle | \le C \| u_m \|^2_{H^1_0(\Omega)},$$

we have also

$$\| v^1 \|^2_{H^1_0(\Omega)} \le \| v \|^2_{H^1_0(\Omega)} \le 1,$$

thus

$$\| u'_m \|_{H^{-1}(\Omega)} \le C \| u_m \|_{H^1_0(\Omega)},$$

and therefore

$$\| u'_m \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| u'_m \|^2_{H^{-1}(\Omega)} dt \le C \int_0^T \| u_m \|^2_{H^1_0(\Omega)} dt \le C \| u_0 \|^2_{L^2(\Omega)} .$$

■

4.1.4.2 Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P4)$.

Theorem 4.11 *(Existence of weak solution.) Under hypothesis (4.71), there exists a weak solution of $(P4)$.*

Proof. According to the energy estimates (4.72), we see that the sequence $\{u_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T;H^1_0(\Omega))$ and $\{u'_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ consequently there exists a subsequence which is also noted by $\{u_m\}_{m=1}^{\infty}$ and a function $u \in L^2(0,T;H^1_0(\Omega))$ with $u' \in L^2(0,T;H^{-1}(\Omega))$, such that

$$\begin{aligned} u_m &\to u \text{ weakly in } L^2(0,T;H^1_0(\Omega)), \\ u'_m &\to u' \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{4.80}$$

2. Next fix an integer N and choose a function $v \in C^1(0,T;H^1_0(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{4.81}$$

where $\left\{d^{(k)}\right\}_{k=1}^{N}$ are given smooth functions. We choose $m \ge N$, multiply equation (4.70) by $d^{(k)}(t)$ $\forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle u'_m, v\rangle + K(u_m, v, t) dt = 0, \tag{4.82}$$

we recall (4.80) to find upon passing to weak limits that

$$\int_0^t \langle u^{'}, v\rangle + K(u,v,t)dt = 0,\ \forall v \in L^2(0,T;H_0^1(\Omega)). \tag{4.83}$$

As functions of the form (4.81) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle u', v\rangle + K(u,v,t) = 0,\ \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0,T] \tag{4.84}$$

and from Remark (4.4) we have $u \in C(0,T;L^2(\Omega))$.
3. In order to prove for prouver $u(0) = u_0$, we first note from (4.67) that

$$\int_0^t -\langle u, v'\rangle + K(u,v,t)dt = (u(0), v(0)), \tag{4.85}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (4.82) we obtain

$$\int_0^t -\langle u_m, v'\rangle + K(u_m,v,t)dt = (u_0, v(0)), \tag{4.86}$$

we use again (4.85), we obtain

$$\int_0^t -\langle u, v'\rangle + K(u,v,t)dt = (u_0, v(0)), \tag{4.87}$$

since $u_m(0) \to u_0$ in $L^2(\Omega)$. Comparing (4.85) and (4.87), we conclude $u(0) = u_0$. ∎

Theorem 4.12 (*Uniqueness of weak solutions.*) *A weak solution of problem* $(P4)$ *is unique.*

Proof. We suppose there exists two weak solution u_1 et u_2 and we put that $U = u_2 - u_1$ then U is also a solution of $(P4)$ with $U_0 = (u_2 - u_1)(0) \equiv 0$. Setting $v = U$ in identity (4.75) we have

$$\frac{d}{dt}(\frac{1}{2} \parallel U \parallel^2_{L^2(\Omega)}) + L(U,U,t) = 0,$$

and as $\parallel U \parallel = \sqrt{K(U,U)}$ ($=$ norm in $H_0^1(\Omega)$), there $K(U,U,t) = \parallel U \parallel^2_{H_0^1(\Omega)} \geq 0$, then we have

$$\frac{d}{dt}(\frac{1}{2} \parallel U \parallel^2_{L^2(\Omega)}) \leq 0,$$

then integrate with respect to t to find

$$\parallel U \parallel^2_{L^2(\Omega)} \leq \parallel U_0 \parallel^2_{L^2(\Omega)} = 0,$$

then $U \equiv 0$. ∎

CHAPTER 5

NUMERICAL SOLUTION OF ONE-DIMENSIONAL KELLER-SEGEL EQUATIONS VIA NEW HOMOTOPY PERTURBATION METHOD

In this chapter we stusy a numerical solution of one-dimensional Keller-Segel equations via new homotopy perturbation method, where for solving a system of nonlinear partial differential equations (PDE) emerging in an attractor one-dimensional Chemotaxis model, we used a relatively new analytical method called the new modified homotopy perturbation method (NMHPM). We use NMHPM for solving one dimensional Keller-Segel model for different types. Some properties show biologically acceptable dependency on parameter values, and numerical solutions are provided. NMHPM is tability and reduced computing time provide it a broader range of applications. The algorithm provides analytical approximations for different types of the Keller-Segel equations. Some numerical illustrations are given to show the efficiency of the algorithm.

5.1 Alternative frame work

The algorithm of homotopy perturbation method introduced by He [62, 72] and a modification algorithm of the HPM introduced by Momani [64]. The basic concepts of the homotopy perturbation method for the following nonlinear differential equation as follows

$$L(u) + N(u) = f(r),\ r \in \Omega, \tag{5.1}$$

with the boundary condition:

$$B(u, \frac{\partial u}{\partial n}) = 0,\ r \in \Gamma, \tag{5.2}$$

where L is linear differential operator and N is nonlinear differential operator, $f(r)$, is a known analytic function, B is aboundary operator and n is the unit outward normal and Γ is the boundary of the domain Ω. In this HPM defined the homotopy as

$$v(r,p) : \Omega \times [0,1] \to \mathbb{R}, \tag{5.3}$$

which corresponds to

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{5.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5.5}$$

where $r \in \Omega$ and $p \in [0,1]$ are the attached parameters, u_0 is initial appoarch value which fulfil the initial condition. From equation (5.4) and (5.5), it is obtained

$$H(v,0) = L(v) - L(u_0) = 0, \tag{5.6}$$

and

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{5.7}$$

He [38] assumes that the solutions of (5.4) and (5.5), can be expressed as the power series of p:

$$v = \sum_{i=0} p^i v_i = v_0 + pv_1 + p^2v^2 + ... \tag{5.8}$$

the appoach solution of (5.1) is

$$u = \lim_{p\to 1} \sum_{i=0} v_i = v_0 + v_1 + v_2 + ... \tag{5.9}$$

Ferthermore, HPM method is modified by Odibat and Momani [20] into modified homotopy perturbation method (MHPM) by including u^m into both sides of the homotopy equation (5.10). In this chapter we study for new modification HPM (NMHPM) for solving one-dimennsional Keller-Segel model of different types. Now we consider system of nonlinear coupled equations is given below:

$$\begin{cases} u_t + L_1(u) + N_1(u,c) = f_u(r), \ r \in \Omega, \\ c_t + L_2(c) + N_2(v,c) = f_c(r), \ r \in \Omega, \end{cases} \tag{5.10}$$

where L_1, L_2 are linear operators, N_1, N_2 are non-bilinear operators, $f_u(r)$, $f_c(r)$ are analyticl functions. The initial conditions are

$$\begin{cases} u(x,0) = u_0(x), \\ c(x,0) = c_0(x), \end{cases} \tag{5.11}$$

from equation (5.10) we obtained the homotopy equation (5.12)

$$\begin{cases} H(v,p) = (1-p)[L_1(v) - L_1(u_0)] + p[v_t - L_1(v) - N_1(v,u) - f_u(r)] = 0, \\ H(\bar{v},p) = (1-p)[L_2(\bar{v}) - L_2(c_0)] + p[\bar{v}_t - L_2(\bar{v}) - N_2(\bar{v},c) - f_c(r)] = 0. \end{cases} \tag{5.12}$$

Hence the solution of (5.4) and (5.5) in from of p- power series is

$$\begin{cases} v = \sum\limits_{i=0} p^i v_i = v_0 + pv_1 + p^2 v_2 + ... \\ \bar{v} = \sum\limits_{i=0} p^i \bar{v}_i = \bar{v}_0 + p\bar{v}_1 + p^2 \bar{v}_2 + ... \end{cases} \tag{5.13}$$

by substituting (5.12) to (5.13) and take $p = 1$, then solution function of (5.9) will beas follows

$$u(t) = \sum_{n=0} v_n(t), \tag{5.14}$$

and

$$c(t) = \sum_{n=0} \bar{v}_n(t), \tag{5.15}$$

or

$$S(t) = (u(t), c(t)) = \sum_{n=0} (v_n(t), \bar{v}_n(t)).$$

The convergence of the series (5.14), (5.15) has been proved in [8]. Hence the convergence of series $u(t)$, $v(t)$ are proved and as $S(t) = (u(t), v(t))$, so $S(t)$ is converge.

5.2 Application

We consider Keller-Segel model as follows

$$\begin{cases} u_t - u_{xx} + (uc_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ c_t - c_{xx} - u + c = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{5.16}$$

with subject to the initial conditions of

$$\begin{cases} u(x,0) = u_0(x), \quad (x) \in \mathbb{R} \\ c(x,0) = c_0(x), \quad (x) \in \mathbb{R} \end{cases} \tag{5.17}$$

where $u = u(x,t)$ denotes the population density of biological individuals, $c = c(t,x)$ denotes the concentration of chemical substance. Now we solving KSM by NMHPM , we take into account the homotopy define as follows

$$\begin{cases} H(v,p) = (1-p)[v_t - \frac{\partial u_0}{\partial t}] + p(v_t - v_{xx} + (v\bar{v}_x)_x) = 0, \\ \\ H(\bar{v},p) = (1-p)[\bar{v}_t - \frac{\partial c_0}{\partial t}] + p(\bar{v}_t - \bar{v}_{xx} - v\bar{v}) = 0, \end{cases} \tag{5.18}$$

or

$$H((v,\bar{v}),p) = (1-p)\begin{pmatrix} v_t - \frac{\partial u_0}{\partial t} \\ \\ \bar{v}_t - \frac{\partial c_0}{\partial t} \end{pmatrix} + p\begin{pmatrix} v_t - v_{xx} + (v\bar{v}_x)_x \\ \\ \bar{v}_t - \bar{v}_{xx} - v\bar{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \tag{5.19}$$

substituting v and $\bar{v}$ from (5.13) into (5.18) or (5.30) terms, we can obtain:

$$p^0 \quad : \quad \begin{cases} \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t}, \\ \\ \frac{\partial \bar{v}_0}{\partial t} = \frac{\partial c_0}{\partial t}, \end{cases} \tag{5.20}$$

$$p^1 \quad : \quad \begin{cases} \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial v_0}{\partial x}\frac{\partial \bar{v}_0}{\partial x} + v_0\frac{\partial^2 \bar{v}_0}{\partial x^2} = 0, \\ \\ \frac{\partial \bar{v}_1}{\partial t} - \frac{\partial^2 \bar{v}_0}{\partial x^2} - v_0 + \bar{v}_0 = 0, \end{cases} \tag{5.21}$$

$$p^2 \quad : \quad \begin{cases} \frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial v_0}{\partial x}\frac{\partial \bar{v}_1}{\partial x} + \frac{\partial v_1}{\partial x}\frac{\partial \bar{v}_0}{\partial x} + v_1\frac{\partial^2 \bar{v}_0}{\partial x^2} + v_0\frac{\partial^2 \bar{v}_1}{\partial x^2} = 0, \\ \\ \frac{\partial \bar{v}_2}{\partial t} - \frac{\partial^2 \bar{v}_1}{\partial x^2} - v_0 + \bar{v}_0 - v_1 + \bar{v}_1 = 0, \end{cases} \tag{5.22}$$

$$\vdots$$

We can written (5.22) as follows

$$\begin{cases} \frac{\partial v_i}{\partial t} = \frac{\partial^2 v_{i-1}}{\partial x^2} - \sum\limits_{\substack{j=0 \\ i \neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x}\frac{\partial \bar{v}_{i-j-1}}{\partial x} + v_j\frac{\partial^2 \bar{v}_{i-j-1}}{\partial x^2} \right), \ i = 1:n, \\ \\ \frac{\partial \bar{v}_i}{\partial t} = \frac{\partial^2 \bar{v}_{i-1}}{\partial x^2} + \sum\limits_{\substack{j=0 \\ i \neq j}}^{i-1} v_j - \bar{v}_j \ , \ i = 1:n, \end{cases} \tag{5.23}$$

for solving (5.22) or (5.23) we integrate with t, we obtain

$$v_i = \int \left[\frac{\partial^2 v_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i\neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x} \frac{\partial \bar{v}_{i-j-1}}{\partial x} + v_j \frac{\partial^2 \bar{v}_{i-j-1}}{\partial x^2} \right) \right] dt, \; i = 1:n,$$

$$\bar{v}_i = \int \left[\frac{\partial^2 \bar{v}_{i-1}}{\partial x^2} + \sum_{\substack{j=0 \\ i\neq j}}^{i-1} v_j - \bar{v}_j \right] dt, \; i = 1:n,$$

with initial conditions $u_0(x)$ and $c_0(x)$ is a known.

5.3 Numerical solutions test for Keller-Segel model

Now we solving this problem with initial condition defined as

$$\begin{cases} v_0(x) = u_0(x) = \cos kx, & (x) \in \mathbb{R} \\ \bar{v}_0(x) = c_0(x) = \cos kx, & (x) \in \mathbb{R} \end{cases} \tag{5.24}$$

we have for $i = 1, 2...$

$$\begin{cases} v_1 = -t(k^2 sin(kx)^2 - k^2 cos(kx)^2 + k^2 cos(kx)), \\ \bar{v}_1 = -k^2 t cos(kx), \end{cases}$$

and

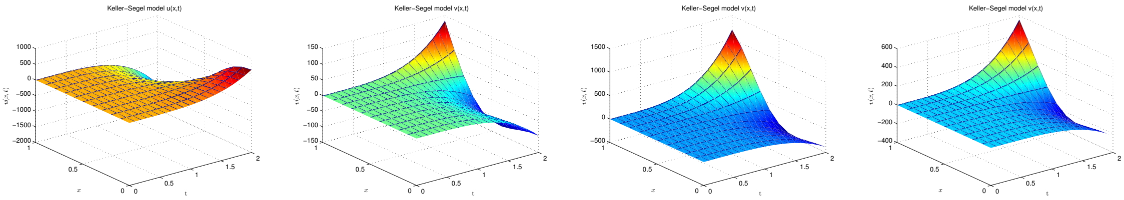
$$\begin{cases} v_2 = \frac{(k^4t^2cos(kx)^3)}{2} - \frac{(5k^4t^2cos(kx)^2)}{2} + \frac{(5k^4t^2sin(kx)^2)}{2} + \frac{(k^4t^2cos(kx))}{2} - \frac{(5k^4t^2cos(kx)sin(kx)^2)}{2} \\ \bar{v}_2 = \frac{(k^2t^2)}{2} - \frac{k^2t^2cos(kx)^2+(k^4t^2cos(kx))}{2} \end{cases}$$

.
.
.

The remaining terms can be obtained using the iterative formula. However, we only consider a few terms of the series of solutions and the asymptotic solution is given as:

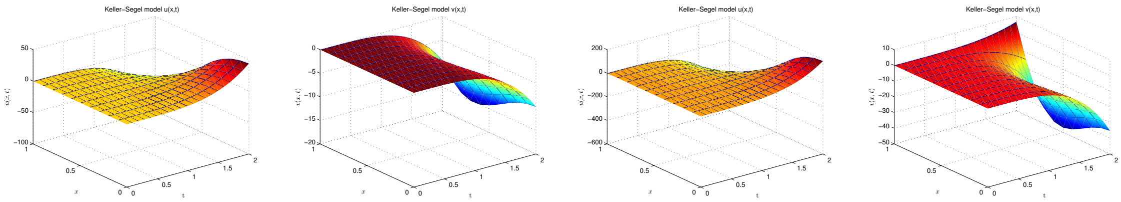
$$\begin{cases} u(x,t) = v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t) + ... \\ c(x,t) = \bar{v}_1(x,t) + \bar{v}_2(x,t) + \bar{v}_3(x,t) + \bar{v}_4(x,t) + ... \end{cases}$$

In the below, we show some results of the approximation solution of one dimensional Keller-Segel with different parameters types. Where following figures illustrate the biological behavior of the coupled solution for the following set of constants $k = 1, 1.2, 1.4, 1.6, 1.75, 1.8$.



(a) u(x,t) for k=1.4. (b) c(x,t) for k=1.4. (c) u(x,t) for k=1.6. (d) c(x,t) for k=1.6.

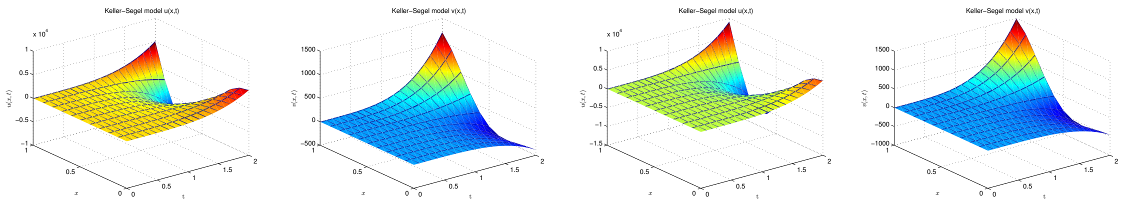
Figure 5.2: Coupled solution for k=1.4 and 1.6.



(a) u(x,t) for k=1. (b) c(x,t) for k=1 (c) u(x,t) for k=1.2. (d) c(x,t) for k=1.2.

Figure 5.1: Coupled solution for k=1 and 1.2.

[H]



(a) u(x,t) for k=1.7. (b) c(x,t) for k=1.7. (c) u(x,t) for k=1.8. (d) c(x,t) for k=1.8.

Figure 5.3: Coupled solution for k=1.7 and 1.8.

5.4 Test method for classical Keller-Segal model

Now, we study numerical solution for it classical version by NHPM with initial conditions, we present the results of approximate coupled solution of different types parameters and source terms functions. We define new version Keller-Segel model as follows

$$\begin{cases} u_t(x,t) - \delta_u u_{xx} + (\chi u c_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ \\ c_t(x,t) - \delta_c c_{xx} + \rho u - \tau c = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{5.25}$$

where $u = u(x,t)$ denotes the density of the cells in position $x \in [0,]$ at time $t \in \mathbb{R}$, $c = c(x,t)$ is the concentration of chemical attractant in position $x \in \mathbb{R}$ at time $t \in [0,1]$, τ, ρ and χ, are positive constants, where diffusion coefficients δ_u, δ_c respectively are assumed as constants. With zero Dirichlet boubary conditions, and initial conditions define us

$$\begin{cases} u(x,0) = u_0(x), \ x \in \mathbb{R}, \\ c(x,0) = c_0(x), \ x \in \mathbb{R}. \end{cases}$$

Now, we study numerical coupled solutions of the problem (5.25) by NHPM with initial conditions as follows

$$\begin{cases} u_0(x) = (x^2+1)e^x, \\ \\ c_0(x) = (x^2+1)e^{-x} \end{cases} \tag{5.26}$$

and parameters $\tau = \rho = \chi = d_1 = d_2 = \delta_u = \delta_c = 1$.

Now, we compute the approximate solution of the problem associated with the initial condition and the parameters define in (5.26) and (5.4) (respectively) by NHPM, we find

$$\begin{cases} v_0(x) = (x^2+1)e^x, \\ \\ \bar{v}_0(x) = (x^2+1)e^{-x}, \end{cases} \tag{5.27}$$

we have for $i = 1, 2...$

$$\begin{cases} v_1 &= t(2e^x + e^x(x+1) - (e^{-x} - e^{-x}(x+1))(e^x + e^x(x+1)) + e^x(2e^{-x} - e^x(x+1))(x+1)), \\ \\ \bar{v}_1 &= -t(2e^{-x} + e^x(x+1) - 2e^{-x}(x+1)). \end{cases}$$

and

$$\begin{cases} v_2 &= \frac{(5t^2e^x)}{2} + \frac{(t^2e^{-x}))}{2} + t^2x + \frac{(3t^2)}{2} - t^2x^2e^{-x} + \frac{(t^2xe^x)}{2} + \frac{(3t^2x*e^{-x})}{2}(3t^2xe^{-x}), \\ \\ \bar{v}_2 &= 2t^2xe^{-x} - 2t^2e^{-x} - t^2x - \frac{t^2}{2} - \frac{(3t^2xe^x)}{2} - \frac{(7t^2e^x)}{2} \end{cases}$$

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Now, we show the approximate solution in below figures with different parameters

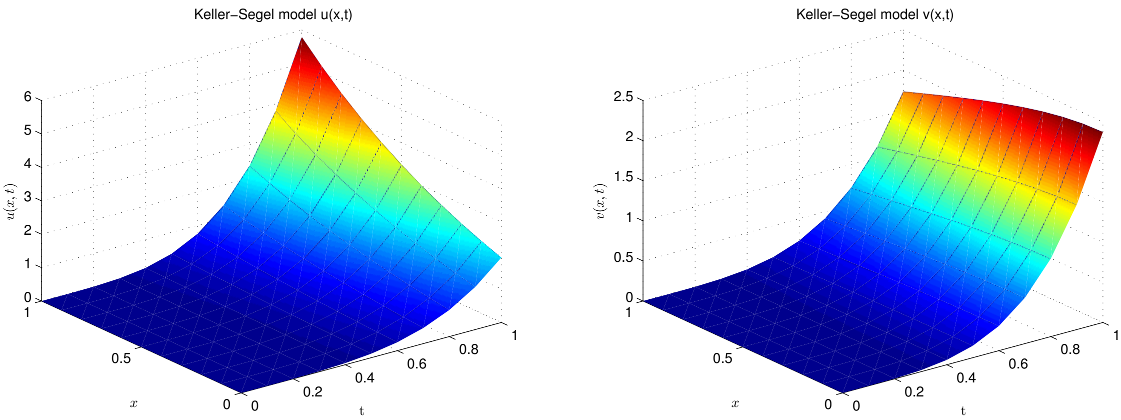

Figure 5.4: $\tau = 2.75,\ \rho = 10^{-3},\ \chi = 0.05,\ \delta_u = 1.5,\ \delta_c = 10^{-3}$.

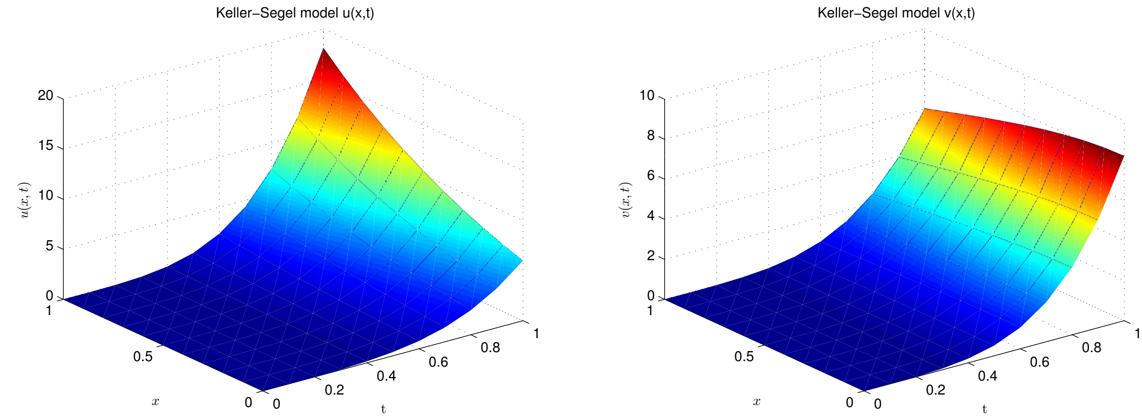

Figure 5.5: $\tau = 3.75,\ \rho = 5 \times 10^{-5},\ \chi = 5 \times 10^{-4},\ \delta_u = 2,\ \delta_c = 10^{-5}$.

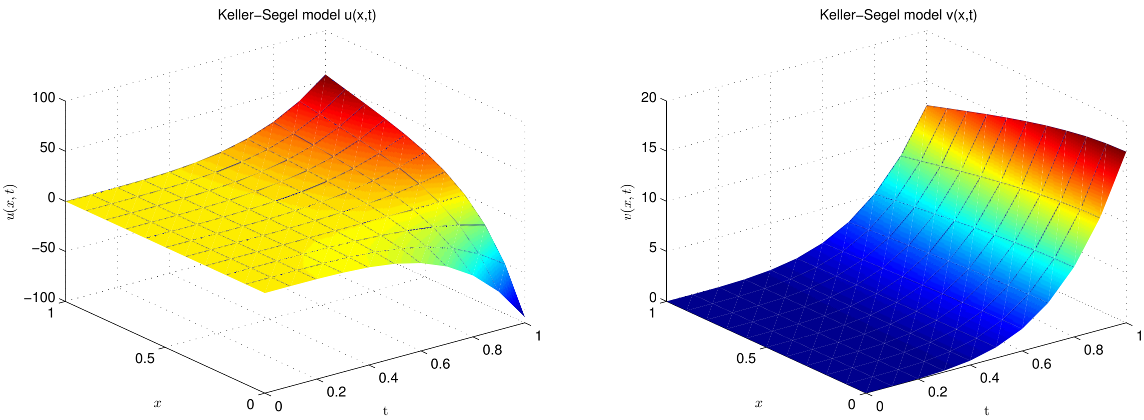

Figure 5.6: $\tau = 4.5,\ \rho = 10^{-9},\ \chi = 2.5,\ \delta_u = 2.5,\ \delta_c = 10^{-7}$.

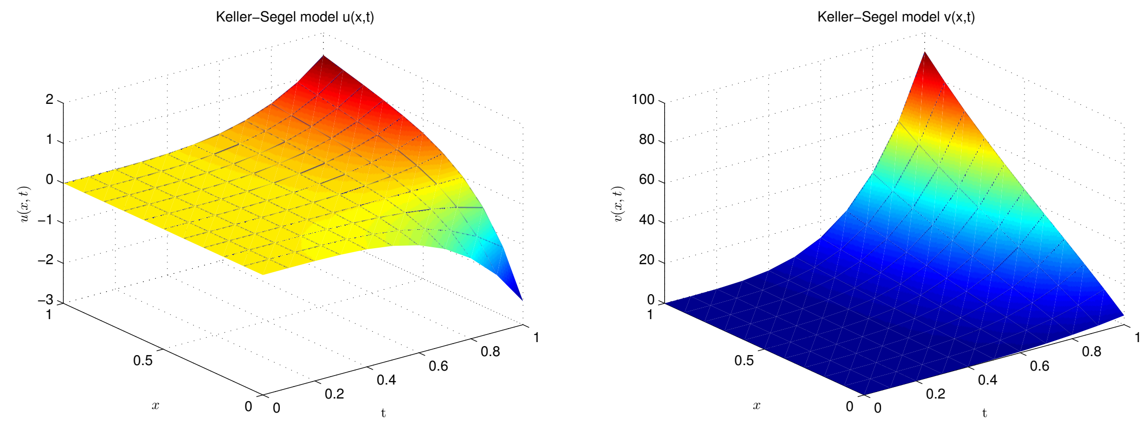

Figure 5.7: $\tau = 1,\ \rho = 1,\ \chi = 1,\ \delta_u = 1,\ \delta_v = 1$.

In this case we present a solution of the problem (5.8) with source terms functions $f_u = \cos(x)e^x + 1$,

$f_v = \cos(x)e^{-x} - 1$ and $\tau = 1,\ \rho = 1,\ \chi = 1,\ \delta_u = 1,\ \delta_c = 1$ we have

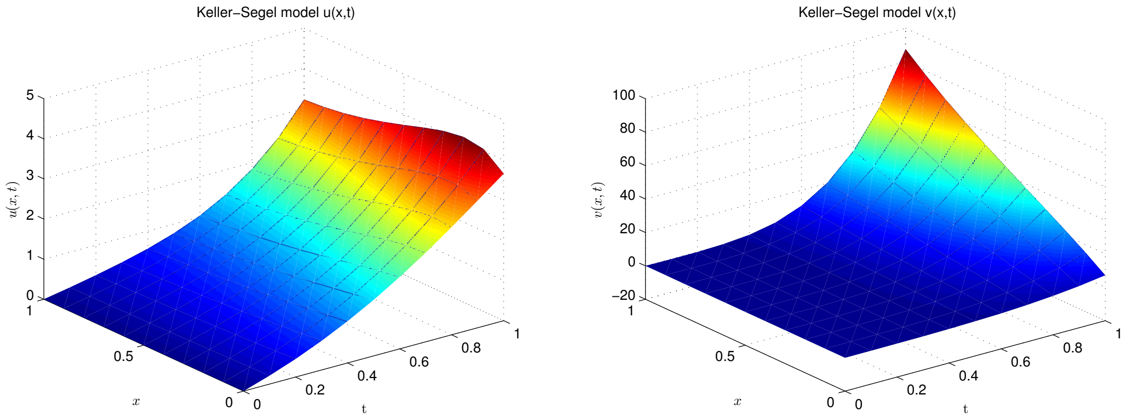

Figure 5.8: Coupled solution for source terms functions f_u and f_c

The behavior of the solution of the system of equations (5.8) with initial conditions in Equations (5.26) and various types of paremeters (5.4) is shown in the above figures (5.4), (5.5) and (5.6), (5.7). These solutions describe the biological cell density and chemical substance concentrations in places $x \in \mathbb{R}$ and $t \in [0,1]$ for a particular set of theoretical parameters chosen from the literature. While figure (5.8) show the behavior of a coupled solution for as source terms functions f_u and f_c of space, we can deduce that the cell density biological increases in space as the concentration of the chemical substance decreases, when source terms functions f_u and f_c are used, the cell density biological and chemical substance concentrations increase in space. The approximate solutions obtained, as seen in the graphicals depiction, mimic the behavior of the real-world situation.

5.5 Test method for new version Keller-Segal model

We consider new version Keller-Segel model define as

$$\begin{cases} u_t(x,t) - u_{xx} + (uc_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ \\ c_t(x,t) - c_{xx} + uc - u_x c_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{5.28}$$

with initial condition as follows

$$\begin{cases} u(x,0) = u_0(x) = me^{x}, \text{ in } (x) \in \mathbb{R}, \\ \\ c(x,0) = c_0(x) = ke^{-x}, \text{ in } (x) \in \mathbb{R}, \end{cases} \tag{5.29}$$

Now, we use homotopy (5.12) for to solve new version Keller-Segel model by NMHPM, so we have

$$H((v,\bar{v}),p) = (1-p)\begin{pmatrix} v_t - \frac{\partial u_0}{\partial t} \\ \\ \bar{v}_t - \frac{\partial c_0}{\partial t} \end{pmatrix} + p\begin{pmatrix} v_t - v_{xx} + (v\bar{v}_x)_x \\ \\ \bar{v}_t - \bar{v}_{xx} - v\bar{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \tag{5.30}$$

the NMMPM method giving the solution of new version Keller-Segel model as follows

$$\begin{cases} \frac{\partial v_i}{\partial t} = \frac{\partial^2 v_{i-1}}{\partial x^2} - \sum\limits_{\substack{j=0 \\ i\neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x} \frac{\partial \bar{v}_{i-j-1}}{\partial x} + v_j \frac{\partial^2 \bar{v}_{i-j-1}}{\partial x^2} \right), \; i = 1:n, \\ \\ \frac{\partial \bar{v}_i}{\partial t} = \frac{\partial^2 \bar{v}_{i-1}}{\partial x^2} + \sum\limits_{\substack{j=0 \\ i\neq j}}^{i-1} (v_{i-j-1}\bar{v}_j + \frac{\partial v_{i-j-1}}{\partial x} \frac{\partial \bar{v}_j}{\partial x}), \; i = 1:n, \end{cases} \tag{5.31}$$

$$\begin{aligned} v_i &= \int \left[\frac{\partial^2 v_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i\neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x} \frac{\partial \bar{v}_{i-j-1}}{\partial x} + v_j \frac{\partial^2 \bar{v}_{i-j-1}}{\partial x^2} \right) \right] dt, \; i = 1:n, \\ \bar{v}_i &= \int \left[\frac{\partial^2 \bar{v}_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i\neq j}}^{i-1} v_{i-j-1}\bar{v}_j + \frac{\partial v_{i-j-1}}{\partial x} \frac{\partial \bar{v}_j}{\partial x} \right] dt, \; i = 1:n, \end{aligned}$$

so, The solution reads

$$\begin{cases} v_0(x) = me^x, \\ \\ \bar{v}_0 = ke^{-x}, \end{cases} \tag{5.32}$$

$$\begin{cases} v_1(x,t) = me^x t, \\ \\ \bar{v}_1(x,t) = ke^{-x} t, \end{cases} \tag{5.33}$$

$$\begin{cases} v_2(x,t) = \frac{1}{2} me^x t^2, \\ \\ \bar{v}_2(x,t) = \frac{1}{2} ke^{-x} t^2, \end{cases} \tag{5.34}$$

$$\begin{cases} v_3(x,t) = \frac{1}{6} me^x t^3, \\ \\ \bar{v}_3(x,t) = \frac{1}{6} ke^{-x} t^3, \end{cases} \tag{5.35}$$

$$\begin{cases} v_4(x,t) = \frac{1}{24} me^x t^4, \\ \\ \bar{v}_4(x,t) = \frac{1}{24} ke^{-x} t^4, \end{cases} \tag{5.36}$$

$$\begin{matrix} . \\ . \\ . \end{matrix} \tag{5.37}$$

The iteration formula can be used to acquire the remaining terms. Only a few terms of the solutions are considered, and the asymptotic solution is as follows with $n = 0, 1, ...$

$$\begin{cases} v_n(x,t) = \frac{1}{\prod\limits_{h=2}^{n} h} me^x t^n \\ \\ \bar{v}_n(x,t) = \frac{1}{\prod\limits_{h=2}^{n} h} ke^{-x} t^n, \end{cases} \tag{5.38}$$

Thus, we obtain the approximation solution of the problem as follows:

$$\begin{cases} u_N(x,t) = \sum_{n=0}^{N} v_N = \sum_{n=0}^{N} \frac{1}{\prod_{n=2}^{N} n} m e^{x} t^{N} \\ \\ c_N = \sum_{n=0}^{N} \bar{v}_N(x,t) = \sum_{n=0}^{N} \frac{1}{\prod_{n=2}^{N} n} k e^{-x} t^{N}, \end{cases} \tag{5.39}$$

The biological behavior of the coupled solution for the following and fixed time t=0.5 and t=1 set is shown in the pictures below:

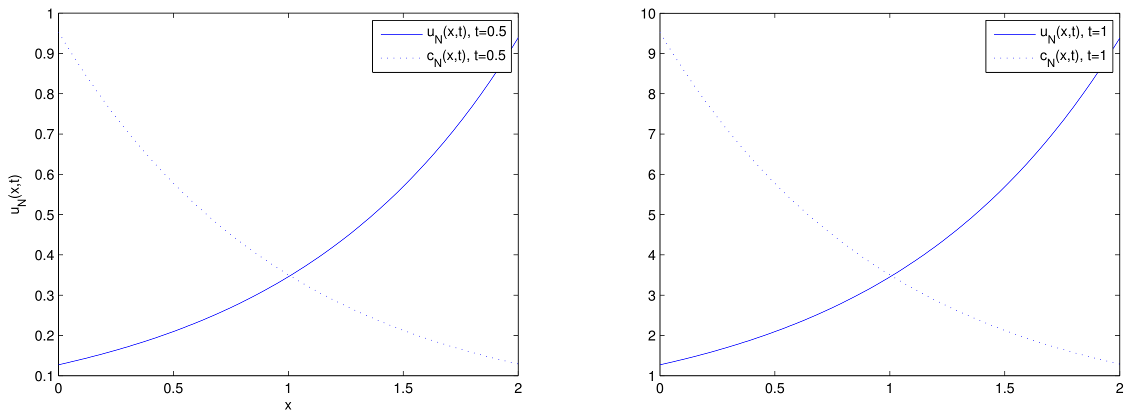

Figure 5.9: Coupled solutions as function of x for a fixed time

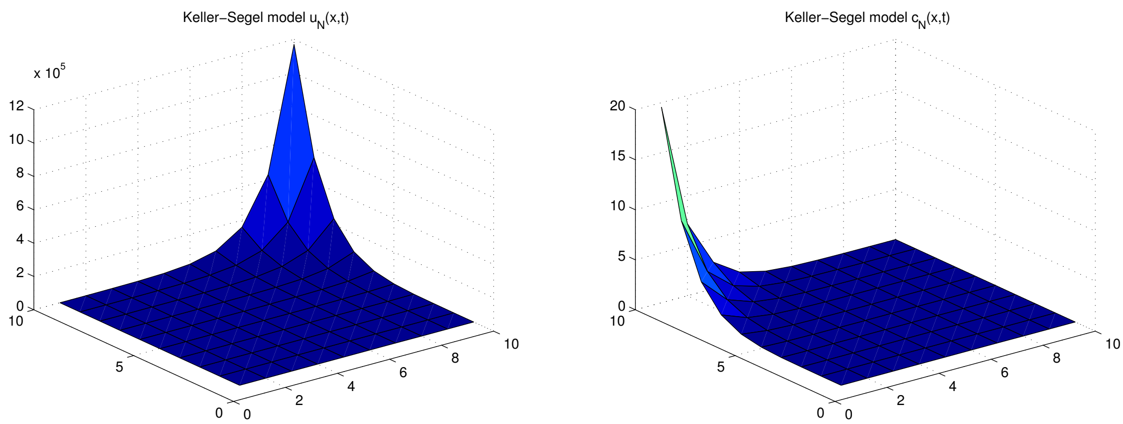

Figure 5.10: Coupled solutions of $u_N(x,t)$ and $c_N(x,t)$

conclution

In this work, we demonstrate how to solve nonlinear coupled partial differential equations emerging in an attractor one-dimensional Keller-Segel dynamics system using a relatively new analytical technique, the new homotopy perturpation method NHPM. The model mimics the regular biological diffusion behavior seen in the field, according to the analysis and conclusions of the nonlinear system of attractor one-dimensional Keller-Segel equation.

CHAPTER 6

APPENDIX PRELIMINARIES

In this chapter, we recall some necessary materials needed in the proof of our results, such as basic results which concerning the fontamotal spaces some theorems on these last and existence and uniqueness theorem.

6.1 Fractional calculus theory

This chapter is devoted to the presentation of fractional differential equations which are the subject of developments in the other chapters. In the first part, we give a historical reminder of the theory of fractional calculus. Then the emphasis is placed on two different approaches to fractional derivation in the Riemann-Liouville sense and in the Caputo sense after the introduction of some fractional calculus tools: the Gamma function, the Mittag-Leffler function, the Laplace transformation, ... Second, we introduce the definition of the fractional Laplacian and its properties. Then, the study of some fractional differential equations is carried out.

6.1.1 Historical

Fractional calculus is an extension of the classical notions of antiderivative and derivation of nonzero integer order to any real order. Although the fractional derivation has been defined by several approaches to the names of Grünwald-Letnikov, Riemann-Liouville, Caputo, this notion was introduced in the 15th century when Gottfried Leibniz defined the symbol of the derivation of positive integer order, Guillaume the Hospital asked him about the possibility of having a derivative of order $\frac{1}{2}$.
This question attracted the attention of mathematicians including Euler or Lagrange in the 15th century followed by Liouville in 1837, Riemann in 1847 as well as Grünwald 1867 and Letnikov in 1868. For more historical details, one can consult[67, 75]

6.1.2 Mathematical tools and specific functions

In this section, we present definitions and properties of some specific functions.

6.1.2.1 Gamma function

His function plays an important role in the theory of fractional calculus, it extends the factorial function to the set of complex numbers [45, 72, 56]. For the fractional calculation we use the definition of the

Gamma function as the Euler integral of the second space.

Definition 6.1 The Gamma function is set to $D(\Gamma) := z \in \mathbb{C}\backslash\{-1, -2, ...\}$ by

$$\Gamma(z) := \int_0^{+\infty} s^{z-1} e^{-s} ds, \ Re(z) > 0,$$

and

$$z\Gamma(z) = \Gamma(z+1). \tag{6.1}$$

Moreover, we have for $n \in \mathbb{N}$

$$\Gamma(n+1) = n!. \tag{6.2}$$

The assertion (6.1) can be proved by integration by parts, and (6.2) is a direct deduction.

Theorem 6.1 *Let $z \in \mathbb{C}$ such as $Re(z) > 0$, then we have*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{H_a} \mu^{-z} e^{\mu} d\mu, \tag{6.3}$$

with H_a is the Hankel contour, see [56].

6.1.2.2 Fonction Beta

We define another function which is related to the Gamma function, called the Beta function or the Euler integral of the first space.

Definition 6.2 The noted Beta function $\beta(z_1, z_2)$ is define to $D(\beta) := \{(z_1, z_2) \in \mathbb{C}, Re(z_1) > 0, Re(z_2) > 0\}$ by

$$\beta(z_1, z_2) := \int_0^1 s^{z_1-1}(1-s)^{z_2-1} ds.$$

Moreover, we have

$$\Gamma(z_1)\Gamma(z_2) := \Gamma(z_1 + z_2)\beta(z_1, z_2). \tag{6.4}$$

The demonstration of (6.4) is done by writing $\Gamma(z_1)\Gamma(z_2)$ as a double integral then we introduce the polar coordinates.

6.1.2.3 The Mittag-Leffler function

The exponential function e^z plays a very important role in the theory of differential equations of integer order, the generalization of this function is the Mittag-Leffler function denoted $E_\alpha(z)$, $z > 0$ is named in honor of the mathematician Magnus Gustaf Mittag-Leffler who introduced it at the beginning of the 19th century. Subsequently Agrawal generalized these functions by functions with two parameters ($\alpha > 0$ et $\beta \in \mathbb{C}$) and it called them two-parameter Mittag-Leffler functions [72, 35, 62].

Definition 6.3 Be $\alpha > 0$ and $\beta \in \mathbb{C}$, we have

$$E_\alpha(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \ z \in \mathbb{C},$$

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \ z \in \mathbb{C}.$$

In the case where $\alpha = 1$, we have

$$E_1(z) = e^z, z \in \mathbb{C}.$$

Theorem 6.2 *([78, 72]) For $\alpha \in (0,1)$, $\beta > 0$ et $W \geq 0$, we have*

$$E_{\alpha,\alpha}(W) \geq 0, \tag{6.5}$$

for $m \in \mathbb{N}$, we have

$$(\frac{d}{dt})^m (E_\alpha(Wt^\alpha)) = t^{\alpha-m} E_{\alpha,\alpha-m+1}(Wt^\alpha), \tag{6.6}$$

$$(\frac{d}{dt})^m (t^{\beta-1} E_{\alpha,\beta}(Wt^\alpha)) = t^{\beta-m-1} E_{\alpha,\beta-m}(Wt^\alpha). \tag{6.7}$$

We are now interested in the asymptotic behavior of Mittag-Leffler functions which will allow us to study the behavior of the solution of certain fractional differential equations, see [72, 47]. The following lemma [72] generally describes the behavior of the Mittag-Leffler function.

Lemma 6.1 *For $0 < \alpha < 2$, $\beta > 0$ and γ is such that $\frac{\alpha\pi}{2} < \gamma < \min(\pi, \alpha\pi)$ there is a constant $C > 0$ such as*

$$\mid E_{\alpha,\beta}(z) \mid \leq \frac{C}{1+ \mid Z \mid}, \; z \in \mathbb{C}, \; \gamma \leq \arg \mid z \mid \leq \pi.$$

In particular, we have for $\lambda \geq 0$

$$\mid t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \mid \leq C t^{\alpha-1}, \; t > 0,$$

$$\mid E_\alpha(-\lambda t^\alpha) \mid \leq \frac{C}{1+\lambda t^\alpha}, \; t \geq 0.$$

6.1.3 Laplace transform

The Laplace transform is used to solve differential equations [26, 72], the advantage of the Laplace transform is that most common operations on the function $f(t)$,such as the derivation (classical or fractional) or a shift on the variable t, have a simpler translation by the transform $\mathcal{L}(f)(s)$. Moreover, the Laplace transformation takes into account the initial conditions.

Definition 6.4 We say that $f : [0, +\infty[\to \mathbb{R}$ is order exponential σ, if it exists t_0, $M > 0$ and $\sigma > 0$ such as

$$\mid f(t) \mid < M e^{\sigma t}, \text{ pour tout } t \geq t_0.$$

In other terms, the function $f(t)$ must not grow faster than some exponential function as t tends to infinity.

Definition 6.5 For $f : [0, +\infty[\to \mathbb{R}$ locally integrable, we define $\mathcal{L}$ the Laplace transform of a function f of exponential order σ by

$$\mathcal{L}f(s) = \int_0^{+\infty} f(t) e^{-st} dt, \text{ for s} \in \mathbb{C}, \; Re(s) > \sigma.$$

Theorem 6.3 *Let f, g be two functions defined on $[0, +\infty[$ locally integrable and such that $\mathcal{L}f$ (resp. $\mathcal{L}g$) exists for $Re(s) > \sigma_1$ (resp. $Re(s) > \sigma_2$). So*

1. *Let* $c_1, c_2 \in \mathbb{R}$ *and* $h = c_1 f + c_2 g$, *we have*

$$\mathcal{L}h(s) = c_1\mathcal{L}f(s) + c_2\mathcal{L}g(s), \text{ for } Re(s) > \max(\sigma_1, \sigma_2).$$

2. *If* $h(t) = \int\limits_0^t f(t-\tau)g(\tau)d\tau$, *so*

$$\mathcal{L}h(s) = \mathcal{L}f(s)\mathcal{L}g(s) \text{ for } Re(s) > \max(\sigma_1, \sigma_2).$$

3. *If* $h(t) = \int\limits_0^t f(\tau)d\tau$, *so for* $Re(s) > \max(\sigma_1, 0)$ *we have*

$$\mathcal{L}h(s) = \frac{1}{s}\mathcal{L}f(s).$$

4. *That is* $m \in \mathbb{N}^*$ *and* $h = D^m f$, *then we have*

$$\mathcal{L}h(s) = s^m\mathcal{L}f(s) - \sum_{k=1}^{m} s^{m-1}f^{(k-1)}(0).$$

We can see that the Laplace transformation is a bijective and continuous operator (see [72, 90]), so we define the inverse Laplace transform ($\mathcal{L}^{-1}$) with Bromwich integral.

Theorem 6.4 *Let* f *be a locally integrable function on* $[0, +\infty[$ *and of exponential order* σ, *then*

$$f(t) = \mathcal{L}^{-1}(\mathcal{L}f)(t) = \frac{1}{2i\pi}\int\limits_{c-i\infty}^{c+i\infty} \mathcal{L}f(s)e^{st}ds, \ c = Re(s) > \sigma.$$

For $\alpha, \beta > 0$ *the Mittag-Leffler functions are related to the Laplace integral by the relation (see [72], chap1)*

$$\frac{1}{1-z} = \begin{cases} \int\limits_0^{+\infty} e^{-t}E_\alpha(zt^\alpha)dt & , \mid z \mid < 1, \\ \int\limits_0^{+\infty} e^{-t}t^{\beta-1}E_{\alpha,\beta}(zt^\alpha)dt & , \mid z \mid < 0. \end{cases}$$

Using a change of variable we deduce the following results which are useful when solving fractional differential equations

$$\mathcal{L}(E_\alpha(\lambda t^\alpha))(s) = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, \quad \mid s \mid > \mid \lambda \mid^{\frac{1}{\alpha}}, \ Re(s) > 0, \tag{6.8}$$

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha))(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \quad \mid s \mid > \mid \lambda \mid^{\frac{1}{\alpha}}, \ Re(s) > 0. \tag{6.9}$$

6.1.4 Fourier transform

Definition 6.6 For f an integrable function on $\mathbb{R}^n$, we define its Fourier transform denoted $\mathcal{F}(f) = \hat{f}$ by

$$\mathcal{F}(f)(\xi) = \int\limits_{\mathbb{R}^n} e^{-i\xi x}f(x)dx.$$

If $\hat{f}$ is also integrable, then the inverse Fourier transform denoted $\mathcal{F}^{-1}$ is given by

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{i\xi x}\hat{f}(\xi)d\xi.$$

6.1.5 Integration and fractional derivation

Many are the definitions of the operator of the fractional derivative, we present in this part those which are the most used. According to the Riemann-Liouville approach to fractional calculus, the notion of the fractional integral is a natural consequence of the well-known Cauchy formula which reduces the calculus of the primitive nth of a function $f(t)$ to a single integral

$$\begin{aligned} J^n f(t) &= \int_0^t d\tau_{n-1} \int_0^{\tau_{n-1}} d\tau_{n-2} \int_0^{\tau_{n-2}} d\tau_{n-3} ... \int_0^{\tau_1} f(\tau_0) d\tau_0 \\ &= \tfrac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \ t > 0, \ n \in \mathbb{N}. \end{aligned} \tag{6.10}$$

Using the Gamma function ($\Gamma(n) = (n-1)!$), Cauchy's formula can naturally extend to the real α as follows α as following

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t > 0, \ \alpha \in \mathbb{R}^+. \tag{6.11}$$

6.1.6 Fractional integral in the sense of Riemann Liouville

In order to obtain a formal definition, some terminology is introduced. We consider the following notation for $\beta \geq 0$

$$g_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{6.12}$$

and $g_0(t) = 0$. This function satisfies the semigroup property

$$g_\alpha * g_\beta = g_\alpha + g_\beta. \tag{6.13}$$

Definition 6.7 For $\alpha \geq 0$ and $f \in L^1([0,T])$, $T > 0$. The Riemann Liouville fractional integral of order α rated by $J_t^\alpha f$ is defined by

$$J_t^\alpha f = \begin{cases} (g_\alpha * f)(t), & \alpha > 0 \\ f(t), & \alpha = 0 \end{cases}, \text{ almost for all } [0,T]. \tag{6.14}$$

Using (6.13) and the associativity of the convolution product we conclude.

Theorem 6.5 *Let α, $\beta \geq 0$ and $f \in L^1([0,T])$, $T > 0$ so*

$$J_t^\alpha J_t^\beta f = J_t^{\alpha+\beta} f \text{ almost every where } [0,T]. \tag{6.15}$$

Moreover $\left\{J_t^\alpha : L^1([0,T]) \to L^1([0,T]), \ \alpha \geq 0\right\}$ forms a semi-group neutral element J_t^0.

6.1.6.1 Fractional derivative in the sense of Riemann-Liouville

As in the classical case the derivation operator D^n of order $n \in \mathbb{N}^*$ is the left inverse of J^n, so after having introduced the notion of fractional integration, that of fractional derivation becomes a requirement.

Knowing that the operator D^n ([26], Lemma 1.2) check for n, $m \in \mathbb{N}^*$ such as $m \geq n$ and f a function of $C^n([0,T])$

$$D^n f = D^m J^{m-n} f, \tag{6.16}$$

Riemann and Liouville gave the following generalization under certain conditions on f.

Definition 6.8 For $\alpha > 0$, $m \in \mathbb{N}^*$; $m-1 < \alpha < m$ and $f \in L^1([0,T])$ such as $g_{m-\alpha} * f \in W^{m,1}([0,T])$. The fractional derivative in the sense of Riemann Liouville of order α noted $D_t^\alpha f$ is defined by

$$D_t^\alpha f(t) = D^m J_t^{m-\alpha} f(t) = D^m (g_{m-\alpha} * f)(t), \tag{6.17}$$

almost every where on $[0,T]$ with $D^m = \frac{d^m}{dt^m}$.

Remark 6.1 In particular if $\alpha \in (0,1)$ and $f \in Ac([0,T])$ where $Ac([0,T])$ is the space of absolutely continuous functions on $[0,T]$ so $D_t^\alpha f$ exists almost every where $[0,T]$. Moreover $D_t^\alpha f \in L^p([0,T])$ for $1 \le p < \frac{1}{\alpha}$, see[[79], lemma 2.2].

Example 6.1 For $\alpha \in (0,1)$ and $\beta > -1$, we have

$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}.$$

The Riemann-Liouville fractional derivative of the constant function is given by

$$D_t^\alpha c = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C.$$

We see that the derivative of a constant is non-zero.

Theorem 6.6 *Under the assumptions of definition (6.8) we have*

$$D_t^\alpha J_t^\alpha f(t) = f(t) \tag{6.18}$$

and

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t) \tag{6.19}$$

, almost everywhere on $[0,T]$. In particular if $g_{m-\alpha} * f \in W_0^{m,1}([0,T])$ we will have

$$J_t^\alpha D_t^\alpha f(t) = f(t). \tag{6.20}$$

Proposition 6.1 *Si $f \in W^{m,1}([0,T])$ so $D_t^\alpha f$ exists almost everywhere $[0,T]$, moreover it is given by*

$$D_t^\alpha f(t) = \sum_{k=0}^{m-1} f^{(K)}(0) g_{k-\alpha+1}(t) + J_t^{m-\alpha} D^m f(t). \tag{6.21}$$

This is an immediate consequence of the representation of the elements of $W^{m,1}([0,T])$ and the definition of D_t^α(6.17).

The definition of fractional derivative in the sense of Riemann-Liouville played an important role in the development of the theory of fractional calculus but when modeling physical problems it leads to initial conditions containing the limiting values of fractional derivatives at meaning of Riemann-Liouville in $t = 0$. Although these problems admit mathematical solutions [72], in practice they have no physical interpretation. Another disadvantage of the Riemann-Liouville approach is that the derivative of a constant is not zero. However, some revision is required to resolve this conflict between mathematical theory and practical needs.

6.1.6.2 Fractional derivative in the sense of Caputo

In this section, we are interested in the approach proposed by Caputo [18] and after by Caputo and Mainardi [17] thanks to its usefulness for the formulation of the initial conditions with the values of the derivatives of integer order in $t = 0$ (speed, acceleration...) as in the case of differential equations of integer order.

Definition 6.9 For $\alpha > 0$, $m \in \mathbb{N}^*$, $m-1 < \alpha < m$ and $f \in C^{m-1}([0,T])$ sush us $g_{m-\alpha} * f \in W^{m,1}([0,T])$. The fractional derivative in the sense of Caputo of order α noted ${}^cD_t^\alpha f$ is defined by

$$ {}^cD_t^\alpha f(t) = D_t^\alpha\big(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t)\big), \tag{6.22}$$

for all on $t \in [0,T]$.

Theorem 6.7 *Let $\alpha > 0$, $m \in \mathbb{N}^*$, $m-1 < \alpha < m$ and $f \in C^m([0,T])$. So ${}^cD_t^\alpha f$ is continuous on $[0,T]$, moreover we have*

$$ {}^cD_t^\alpha f(t) = J_t^{m-\alpha} D^m f(t) \text{ for all } \quad t \in [0,T], \tag{6.23}$$

with $D^m = \frac{d^m}{dt^m}$.
In particular if $\alpha \in (0,1)$ we have ${}^cD_t^\alpha f(t) = J_t^{1-\alpha} Df(t)$.

The proof is done using ([45], Theorem 2.2) to show that ${}^cD_t^\alpha f \in C^m([0,T])$, and combining the proposition (6.1) and the fact that $D_t^\alpha g_{k+1} = g_{k+1-\alpha}$ to show (6.23).
The operator of the fractional derivation of Caputo${}^cD_t^\alpha$ is also the inverse left of J_t^α but is not inverse on the right.

Proposition 6.2 *For $f \in L^1([0,T])$ we have ${}^cD_t^\alpha J_t^\alpha f = f$. If $f \in C^{m-1}([0,T])$ is such that $g_{m-\alpha} * f \in W^{m,1}([0,T])$, then we have*

$$ J_t^\alpha \, {}^cD_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t). \tag{6.24}$$

In the case where $\alpha \in (0,1)$ and $f \in C([0,T])$ such as $g_{1-\alpha} * f \in W^{1,1}([0,T])$, then we have

$$ J_t^\alpha \, {}^cD_t^\alpha f(t) = f(t) - f(0). \tag{6.25}$$

6.1.7 Fractional Laplacian

In this part, we introduce the definition of the fractional Laplacian and some properties used in the study of anomalous diffusion systems. The fractional Laplacian is the most important example of the Lévy operator L which is a pseudo differential operator defined by

$$ Lu(x) = \mathcal{F}^{-1}(a(\zeta))\mathcal{F}(u)(\zeta)(x), \; x \in \mathbb{R}^{\mathbb{N}},$$

where $a(\zeta)$ is the symbol of the semi-group convolution of measures defined by the formula of Lévy-Khintchine formula ([14], Chap1, Theorem 1). The fractional Laplacian is a non-local operator defined for $N \geq 1$ and $0 < \alpha < 2$ by

$$ (-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{F}^{-1}(\mid \zeta \mid^\alpha \mathcal{F}u(\zeta))(x), \; x \in \mathbb{R}^{\mathbb{N}}, \tag{6.26}$$

for all $u \in D((-\Delta)^{\frac{\alpha}{2}}) = H^\alpha(\mathbb{R}^{\mathbb{N}})$ where $H^\alpha(\mathbb{R}^{\mathbb{N}})$ is the homogeneous Sobolev space of order α given by

$$ H^\alpha(\mathbb{R}^{\mathbb{N}}) = \left\{ u \in S'; \; (-\Delta)^{\frac{\alpha}{2}} u \in L^2(\mathbb{R}^{\mathbb{N}}) \right\}, \; si \; \alpha \notin \mathbb{N},$$

$$H^{\alpha}(\mathbb{R}^{\mathbb{N}}) = \left\{u \in L^2(\mathbb{R}^{\mathbb{N}});\ (-\Delta)^{\frac{\alpha}{2}} u \in L^2(\mathbb{R}^{\mathbb{N}})\right\},\ si\ \alpha \in \mathbb{N},$$

where S' is the Schwartz space. For more details see [43, 9].
By a calculation based on the properties of the Fourier transformation, we can show an equivalence between the definition of the fractional Laplacian given by (6.26) and the following definition (see [28]) :

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -C(\alpha) \lim_{\varepsilon \to 0} \int\limits_{|W| \geq \varepsilon} \frac{u(x-W) - u(x)}{\mid W \mid^{N+\alpha}} dW. \tag{6.27}$$

where $C(\alpha)$ is a positive constant that depends on α.
In practice, the fractional Laplacian describes the phenomenon of anomalous diffusion in evolution problems. Let's move on to the presentation of some tools concerning the semigroup generated by $(-\Delta)^{\frac{\alpha}{2}}$. We consider the linear equation of the anomalous diffusion (see [43, 2, 29])

$$u_t + (-\Delta)^{\frac{\alpha}{2}} u = 0,\ x \in \mathbb{R}^N,\ 0 < \alpha \leq 2, \tag{6.28}$$

whose fundamental solution S_α is given by the formula

$$S_\alpha(t)(x) = S_\alpha(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{i\zeta x - t|\zeta|^\alpha} d\zeta. \tag{6.29}$$

Using Young's inequality for convolution and the self-similarity formula

$$S_\alpha(x,t) = t^{-\frac{N}{\alpha}} S_\alpha(x t^{\frac{1}{\alpha}}, 1)$$

which is proved in [43], we obtain the following estimate:

Lemma 6.2 *For all $u \in L^r(\mathbb{R}^N)$ and $1 \leq r \leq q$, we have $S_\alpha * u \in L^q(\mathbb{R}^N)$ et*

$$\| S_\alpha(.,t) * u(x) \|_q \leq C t^{-\frac{N}{\alpha}(\frac{1}{r} - \frac{1}{q})} \| u \|_r, \qquad t > 0, \tag{6.30}$$

where C is a positive constant. In the case of a bounded domain $\Omega \subset \mathbb{R}^N$, we present the definition of the fractional Laplacian on Ω with homogeneous boundary conditions of Neumann type denoted $(-\Delta_N)^{\frac{\alpha}{2}}$. For $\lambda_k (k = 1, ..., +\infty)$,the eigenvalue of Laplacian in $L^2(\Omega)$ with homogeneous boundary conditions of Neumann type and u_k the eigenfunction associated with λ_k, we have

$$\begin{cases} (-\Delta_N)^{\frac{\alpha}{2}} u_k = \lambda_k^{\frac{\alpha}{2}} u_k \text{ on } \Omega, \\ \frac{\partial u_k}{\partial W} = 0 \text{ on } \partial\Omega. \end{cases}$$

Definition 6.10 For $u \in D((-\Delta_N)^{\frac{\alpha}{2}})$, $0 < \alpha \leq 2$, we have

$$(-\Delta_N)^{\frac{\alpha}{2}} u = \sum_{k=1}^{+\infty} \lambda_k^{\frac{\alpha}{2}} \langle u, u_k \rangle u_k, \tag{6.31}$$

where $D((-\Delta_N)^{\frac{\alpha}{2}}) = \left\{ u \in L^2(\Omega) / \frac{\partial u}{\partial W} = 0 \text{ et } \sum\limits_{k=1}^{+\infty} \mid \lambda_k^{\frac{\alpha}{2}} \langle u, u_k \rangle \mid^2 < \infty \right\}$.

Using the definition of the fractional Laplacian given by (6.31), we can show the following integration by parts formula:

Lemma 6.3 *For $u, v \in D((-\Delta_N)^{\frac{\alpha}{2}})$, $0 < \alpha \leq 2$, we have*

$$\int\limits_{\Omega} u(x)(-\Delta_N)^{\frac{\alpha}{2}} v(x)dx = \int\limits_{\Omega} (-\Delta_N)^{\frac{\alpha}{2}} u(x)v(x)dx. \tag{6.32}$$

Now, we present the Stroock-Varopoulos inequality for the fractional Laplacian operator whose proof is given in [88].

Lemma 6.4 *[[52], Theorem 1] For all $u \in L^p(\Omega)$ such us $(-\Delta_N)^{\frac{\alpha}{2}} u \in L^p(\Omega)$, $0 < \alpha < 2$ we have*

$$\int\limits_{\Omega} (| u |^{p-2} u)(-\Delta_N)^{\frac{\alpha}{2}} u dx \geq \frac{4(p-1)}{p^2} \int\limits_{\Omega} (| (-\Delta_N)^{\frac{\alpha}{4}} | u |^{\frac{P}{2}} |^2 dx. \tag{6.33}$$

For $p \in (0, 1)$, we note by A_p The realisation of $(-\Delta_N)^{\frac{\alpha}{2}}$ in $L^p(\Omega)$. It is well known that A_p is a sector operator that generates the semi-group $\left\{e^{-tA_p}\right\}_{t\geq 0}$. Semigroup theory has been developed in several works, see for example [39, 43, 91].

Lemma 6.5 *For all $u \in L^r(\Omega)$ and $1 \leq r \leq q \leq \infty$ we have*

$$\| e^{-tA_p} u \|_{L^q(\Omega)} \leq C t^{-\frac{N}{\alpha}(\frac{1}{r}-\frac{1}{q})} \| u \|_{L^r(\Omega)}, \qquad t \geq 0, \tag{6.34}$$

and

$$\| e^{-tA_p} u \|_{L^q(\Omega)} \leq C e^{-\lambda_1^{\frac{\alpha}{2}} t} \| u \|_{L^q(\Omega)}, \qquad t \geq 0, \tag{6.35}$$

where λ_1 is the smallest eigenvalue of the Laplacian.

6.1.8 Mittag-Leffler operators

In this section, we are interested in the Mittag-Leffler operator $E_\alpha(-t^\alpha A)$ defined for a positive sector operator A by

$$E_\alpha(-t^\alpha A) = \sum_{k=0}^{+\infty} \frac{(-t^\alpha A)^k}{\Gamma(\alpha k + 1)}.$$

This operator is used to construct the solution of the following problem:

$$\begin{cases} {}^c D_t^\alpha u(t) &= -Au(t), \ t > 0 \\ u(0) &= u_0 \in X \end{cases} \tag{6.36}$$

where $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is a positive sector operator and X is a Banach space.
We consider the space $C^\alpha([0, T], X)$ defined by

$$C^\alpha([0, T], X) = \{u \in C([0, T], X), \ {}^c D_t^\alpha u \in C([0, T], X)\}$$

with the norm $\| u \| = \| u \|_{C([0,T],\ X)} = \sup_{t\in[0,T]} \| u \|_X$.

Definition 6.11 We say that a continuous function $u : [0, +\infty) \to X$ is a global solution of the problem (6.36), si $u \in C^\alpha([0, T], X)$ for all $T > 0$ and verifies (6.36).

To show that the problem (6.36) has a unique global solution, we start by establish the equivalence between (6.36) and an integral equation. Then, we apply the Banach fixed point theorem.

Lemma 6.6 *Let $u \in C([0, T], X)$ for all $T > 0$. So u is a global solution of problem (6.36) if and only if it satisfies the following equation*

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha - 1} Au(s)ds, \ t \geq 0. \tag{6.37}$$

Proof. On the one hand, for $T>0$ and u a global solution of (6.36) we have $u\in C([0,T],\,X)$, ${}^cD_t^\alpha u\in C([0,T],\,X)$ and

$$^cD_t^\alpha u(t)=-Au(t),\ t\in[0,T]. \tag{6.38}$$

By applying the operator J_t^α then using (6.25) we obtain

$$\begin{aligned} u(t) &= u(0)-J_t^\alpha Au(t)\\ &= u_0-\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds,\ t\in[0,T]. \end{aligned}\tag{6.39}$$

Since u verifies the equation (6.39) for $t\in[0,T]$ with T arbitrary then, it verifies the equation (6.37) for all $t\geq 0$. On the other hand, we choose $T>0$ arbitrary such that $u\in C([0,T],\,X)$ and verifies the equation

$$u(t)=u_0-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds,\ t\in[0,T]. \tag{6.40}$$

Using the fact that $t\to Au(t)$ is continuous, we can apply ${}^cD_t^\alpha$ to equation (6.40) for obtain

$$^cD_t^\alpha u(t)=-Au(t),\ t\in[0,T], \tag{6.41}$$

which implies that ${}^cD_t^\alpha u\in C([0,T],\,X)$. Moreover from the equation (6.39) we see that $u(0)=u_0$. Since T is arbitrary, u is a global solution of the problem (6.36) ■

Theorem 6.8 *For $\alpha\in(0,1)$ and $u_0\in X$ then the problem (6.36) admits a unique global solution, given by*

$$u(t)=\sum_{k=0}^{+\infty}\frac{(-t^\alpha A)^k}{\Gamma(\alpha k+1)}u_0=E_\alpha(-t^\alpha A)u_0.$$

Proof. For $T>0$ we consider the space $E_T=\{u\in C([0,T],X),\ u(0)=u_0\}$ and the operator $B: E_T\to E_T$ defined by

$$B(u(t))=u_0-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds.$$

Let $u,\,v\in E_T$ we have

$$\begin{aligned} \parallel B(u(t))-B(v(t))\parallel_X &\leq \tfrac{\|A\|_{\mathcal{L}(X)}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\parallel u(s)-v(s)\parallel_X ds\\ &\leq \tfrac{t^\alpha}{\Gamma(\alpha+1)}\parallel A\parallel_{\mathcal{L}(X)}\sup_{s\in[0,T]}\parallel u(s)-v(s)\parallel_X . \end{aligned}$$

By iteration, we obtain for all $k\in\mathbb{N}$

$$\begin{aligned} \parallel B^k(u(t))-B^k(v(t))\parallel_X &\leq \tfrac{t^{\alpha k}}{\Gamma(\alpha k+1)}\parallel A\parallel_{\mathcal{L}}^k\sup_{s\in[0,T]}\parallel u(s)-v(s)\parallel_X\\ &\leq \tfrac{T^{\alpha k}}{\Gamma(\alpha k+1)}\parallel A\parallel_{\mathcal{L}}^k\sup_{s\in[0,T]}\parallel u(s)-v(s)\parallel_X\\ &\leq \omega_k\parallel u-v\parallel . \end{aligned}$$

we have $\lim\limits_{k\to+\infty}(\frac{1}{\Gamma(\alpha k+1)})^{\frac{1}{k}}=0$ since using Stirling's formula we obtain when k tends to infinity

$$\Gamma(\alpha k+1)=(\frac{\alpha k}{e})^{\alpha k}\sqrt{2\pi\alpha k}(1+O(1)).$$

So the series $\sum\limits_{k=0}^{+\infty}\omega_k$ is convergent.
By applying the theorem 1.10 in [45], B has a unique fixed point $u\in E_T$ for all $T>0$. By the lemma

(6.6) the problem (6.36) has a unique global solution. For the construction of the solution, we consider the sequence$(U_n)_{n\geq 0}$ defined by

$$\begin{cases} U_0 &= u_0 \\ U_n &= B^n u_0, \ n \geq 1. \end{cases}$$

For $n = 1$ we have

$$Bu_0 = u_0 - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A u_0 ds = u_0 - \frac{t^\alpha A}{\Gamma(\alpha+1)} u_0.$$

By iteration, we can obtain

$$U_n = \sum_{k=0}^{n} \frac{(-t^\alpha A)^k}{\Gamma(\alpha k+1)} u_0,$$

so $(U_n)_{n\geq 0}$ converges to the fixed point u, and

$$u(t) = \sum_{k=0}^{+\infty} \frac{(-t^\alpha A)^k}{\Gamma(\alpha k+1)} u_0 = E_\alpha(-t^\alpha A) u_0.$$

To study a nonlinear fractional differential equation, another operator intervenes, it is $\{E_{\alpha,\alpha}(-t^\alpha A)\}_{t\geq 0}$. In the following, we introduce the integral formulas of the Mittag-Leffler operators using the Mainardi function M_α [58, 56]. ■

We present some properties of this function in the following lemma.

Lemma 6.7 *For $\alpha \in (0,1)$, $-1 < r < \infty$ and $s \in \mathbb{C}$. The Mainardi function M_α verifies the following assertions*

(A) $M_\alpha(\theta) \geq 0, \ \theta > 0.$

(B) $\int_0^\infty M_\alpha(\theta) d\theta = 1.$

(C) $\int_0^\infty \theta^r M_\alpha(\theta) d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}.$

(D) $\mathcal{L}(M_\alpha(\theta))(s) = E_\alpha(-s).$

(E) $\mathcal{L}(\alpha\theta M_\alpha(\theta))(s) = E_{\alpha,\alpha}(-s).$

For the proof of this lemma see [56].

Theorem 6.9 *[[82], Lemma 9] For $\alpha \in (0,1)$ and A a positive sector operator, we have*

$$E_\beta(-t^\beta A) = \int_0^\infty M_\beta(\theta) e^{-\theta t^\beta A} d\theta, \ t \geq 0 \tag{6.42}$$

and

$$E_{\beta,\beta}(-t^\beta A) = \int_0^\infty \beta\theta M_\beta(\theta) e^{-\theta t^\beta A} d\theta, \ t \geq 0, \tag{6.43}$$

where $\left\{e^{-tA}\right\}_{t\geq 0}$ is the semigroup generated by $-A$.

Now, we present estimates of the Mittag-Leffler operators which are used in the chapter 3

Proposition 6.3 *For $1 \leq r \leq p \leq \infty$ such as $\frac{N}{2}(\frac{1}{r}-\frac{1}{p}) < 1$ we have*

$$\| E_\alpha(-t^\alpha A)u \|_p \leq Ct^{-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})} \| u \|_r, \ t > 0, \tag{6.44}$$

$$\| E_{\alpha,\alpha}(-t^\alpha A)u \|_p \leq Ct^{-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})} \| u \|_r, \ t > 0. \tag{6.45}$$

Moreover,

$$\| E_\alpha(-t^\alpha A)u \|_\infty \leq \| u \|_\infty \ E_\alpha(-t^\alpha \delta), \ t > 0 \tag{6.46}$$

and

$$\| E_{\alpha,\alpha}(-t^\alpha A)u \|_\infty \leq \| u \|_\infty \ E_{\alpha,\alpha}(-t^\alpha \delta), \ t > 0, \tag{6.47}$$

where $\delta < Re(\lambda)$ is such that $\lambda \in \sigma(A)$ (the spectrum of A) and C a positive constant.

Proof. Using the estimates of the semi-group $\{e^{-tA}\}_{t\geq 0}$ and lemma (6.7), we obtain

$$\begin{aligned}
\| E_{\alpha,\alpha}(-t^\alpha A)u \|_p &\leq \int_0^\infty \alpha\theta M_\beta(\theta) \| e^{-\theta t^\alpha A} u \|_p \, d\theta \\
&\leq Ct^{-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})} \int_0^\infty \alpha\theta^{-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})} M_\beta(\theta) d\theta \| u \|_r \\
&\leq C\alpha \frac{\Gamma(-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})+1)}{\Gamma(-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})+1)} t^{-\frac{N}{2}\alpha(\frac{1}{r}-\frac{1}{p})} \| u \|_r,
\end{aligned}$$

in the same way we show the first assertion.
Using the following estimate given in [39]

$$\| e^{-tA} \| \leq e^{-t\delta}, \ t > 0, \ \delta < Re(\lambda), \ \lambda \in \sigma(A), \tag{6.48}$$

we obtain

$$\begin{aligned}
\| E_\alpha(-t^\alpha A)u \|_\infty &\leq \int_0^\infty M_\beta(\theta) \| e^{-\theta t^\alpha A} u \|_\infty \, d\theta \\
&\leq \int_0^\infty M_\beta(\theta) e^{-\theta t^\alpha \delta} d\theta \| u \|_\infty \\
&\leq \| u \|_\infty \ \mathcal{L}(M_\beta(\theta))(t^\alpha \delta) = \| u \|_\infty \ E_\alpha(-t^\alpha \delta).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\| E_{\alpha,\alpha}(-t^\alpha A)u \|_\infty &\leq \int_0^\infty \alpha\theta M_\beta(\theta) \| e^{-\theta t^\alpha A} u \|_\infty \, d\theta \\
&\leq \int_0^\infty \alpha\theta M_\beta(\theta) e^{-\theta t^\alpha \delta} d\theta \| u \|_\infty \\
&\leq \| u \|_\infty \ \mathcal{L}(\alpha\theta M_\beta(\theta))(t^\alpha \delta) = \| u \|_\infty \ E_{\alpha,\alpha}(-t^\alpha \delta).
\end{aligned}$$

■

Remark 6.2 Although the study of problem (6.36) guarantees that the solution follows the same properties of the construction as that of the ordinary case (with a classical derivative), we can check that the Mittag-Leffler operator does not satisfy the property of semi-group [70, 69].

$$E_\alpha(-(t+s)^\alpha A) \neq E_\alpha(-t^\alpha A) E_\alpha(-s^\alpha A).$$

conclusion

In this chapter, we have introduced the theory of fractional calculus from a reminder of specific functions and two different definitions of fractional derivation. We also presented the fractional Laplacian as a nonlocal operator that describes anomalous diffusion in the fourth chapter. Finally, the Mittag-Leffler operators are introduced in the closing of this chapter. We exposed the definitions of these operators and some properties used below in the chapter two.

6.2 Stochastic Calculus

In this section, we introduce some basic elements of probability theory and stochastic process, functional analysis for which we refer to [54] for more details.

6.2.1 Fondamental notion of probability theory

This part is devoted to the presentation of fundamental notion of probability theory, which are the subject of developments in the other chapters. Where we talk and present some basic concepts for the title of this chapter, which play a role in carrying out our work in some of the chapters from this thesis.

6.2.1.1 Probability spaces and random variables

Definition 6.12 A tribe (or σalgebra) on Ω is an $\mathcal{F}$ subsets (called events) such as:

i $\varnothing \in \mathcal{F}$.

ii $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

iii $(A_n)_{n=1}^{\infty} \subset \mathcal{F} \Rightarrow \cup_{n=1}^{\infty} \in \mathcal{F}$.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of the sample space Ω, the σ-algebra (the set of events) $\mathcal{F}$, and the probability measure $\mathbb{P}$. Thus, a σ-algebra is a collection of subsets of A that contains the empty set. For $F \in \mathcal{F}$ is known as a measurable set, $(A, \mathcal{F})$ is a measurable space.

Definition 6.13 (measure) A measure μ on a measurable space $(A, \mathcal{F})$ is a mapping from $\mathcal{F}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that
(i) the empty set has measure zero, $\mu(\{\}) = 0$,
(ii) $\mu(\cup_{j\in\mathbb{N}} F_j) = \sum_{j\in\mathbb{N}} \mu(F_j)$ if $F_j \in \mathcal{F}$ are disjoint (i.e., $F_k \cap F_j = \{\}$ for $k \neq j$).

Together $(A, \mathcal{F}, \mu)$ is a measure space.

Definition 6.14 A probability on $(\Omega,\ \mathcal{F})$ is a $\mathbb{P}$ application of $\mathcal{F}$ in $[0, 1]$ as

a $\mathbb{P}(\Omega) = 1$.

b $\mathbb{P}(\cup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} \mathbb{P}(A_n)$ for A_n belonging to $\mathcal{F}$ two to two disjoints.

Definition 6.15 (probability) A measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a probability measure if it has unit total mass, $\mathbb{P}(\Omega) = 1$.

Definition 6.16 (Random variables, realisation) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\Psi, \mathcal{G})$ be a measurable space. Then, X is a Ψ-valued random variable if X is a measurable function from $(\Omega, \mathcal{F})$ to $(\Psi, \mathcal{G})$. To emphasise the σ-algebra on Ω, we may write that X is an $\mathcal{F}$-measurable random variable. The observed value of $X(\omega)$ for a given $\omega \in \Omega$ belongs to Ψ and is called a realisation of X.

6.2.1.2 Expectation

Let X be a Banach space-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If X is integrable, the expectation of X defined as

$$\mathbb{E}[X] := \int_{\Omega} X(\omega)\, dp(\omega).$$

6.2.1.3 Gaussian distribution

A random variable X is said to follow the Gaussian or normal distribution on $D = \mathbb{R}$ if, for some $\mu \in \mathbb{R}$ and $\sigma > 0$, its probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right).$$

We write $X \sim N\left(\mu, \sigma^2\right)$ (the random variable X follow the Gaussian distribution with mean μ and variance σ^2).

6.2.2 Notion of stochastic process

6.2.2.1 Stochastic process

For a set $\mathcal{I} \subset \mathbb{R}$, and a Hilbert space H,a H-valued stochastic process is a set of H-valued random variables $\{X(t) : t \in \mathcal{I}\}$. We some times drop the set $\mathcal{I}$ and simply write $X(t)$ to denote the process. This should not be interpreted as a simple function of t, to emphasise the dependence on ω and that $X : \mathcal{I} \times \Omega \to H$, we may write $X(t, \omega)$ or $X_t(\omega)$. For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a family of random processes $X_t(\omega)$ on this space with $t \in [0, T]$, where t usually is understood as time variable, and T is a fixed parameter, called also terminal time. For a fixed $\omega \in \Omega$, the time function $X_t(\omega)$, $t \in [0, T]$ is called a trajectory or realization corresponding to an elementary ω.

6.2.2.2 Brownian motion

The name Brownian refers to Robert Brown, who identified Brownian motion in the movement of pollen particles. Brownian motion is frequently called the Wiener process, after Norbert Wiener, who made a significant contribution to the mathematical theory. In this section, we focus on real-valued processes and use the name Brownian motion.

Definition 6.17 (Brownian motion) For $T \in \mathbb{R}^+$ given, a Brownian Motion is a stochastic process $(t, \omega) \in [0, T] \times \Omega \mapsto \beta_t(\omega)$, such that:

1. $\beta_0 = 0$ a.s.
2. The increments are stationary: for any $0 \leq s \leq t \leq T$, $\beta_t - \beta_s \sim \beta_{t-s}$
3. The increments are independent: for any $n \in \mathbb{N}^*$, for any $0 = t_0 \leq t_1 \leq ... \leq t_n \leq T$, $\left(\beta_{t_{i+1}} - \beta_{t_i}\right)_{0 \leq i \leq n-1}$ are independent.
4. The process is Gaussian: for any $n \in \mathbb{N}^*$, for any $0 \leq t_1 \leq ... \leq t_n \leq T$ and any $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$, $\lambda_1 \beta_{t_1} + ... + \lambda_n \beta_{t_n}$ is a real Gaussian random variable.
5. For any $t \geq 0$, $\beta_t \sim \mathcal{N}(0, t)$.
6. The trajectories are almost surely continuous: for $\mathbb{P}$-almost every $\omega \in \Omega$, $t \in [0, T] \mapsto \beta_t(\omega)$ is continuous.

Usually, the variable ω is not written.

6.2.2.3 Filtration

To handle the idea of past and future more generally, we use sub σ-algebras and introduce a sub σ-algebra $\mathcal{F}_t$ for each time t. Intuitively, the events in $\mathcal{F}_t$ are those observable by time t, because we have more observations as time passes, the σ-algebras $\mathcal{F}_t$ contain more events as t increases. We make this precise with the notion of filtration.

Definition 6.18 (filtration) A filtration $\{\mathcal{F}_t \ : t \geq 0\}$ is a family of sub σ-algebras of $\mathcal{F}$ that are increasing, that is, $\mathcal{F}_s$ is a sub σ-algebra of $\mathcal{F}_t$ for $s \leq t$.
A filtered probability space is a quadruple $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t \ : t \geq 0\}$ is a filtration of $\mathcal{F}$.

Stochastic processes that conform to the notion of time described by the filtration $\mathcal{F}_t$ are known as *adapted* processes.

Definition 6.19 (adapted) Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. A stochastic process $\{X(t) \ : t \in [0,T]\}$ is $\mathcal{F}_t$ -adapted if the random variable $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$.

6.2.2.4 The cylindrical Wiener process and Stochastic integration in Hilbert spaces.

In this part, we recall the definition of the cylindrical Wiener process and of stochastic integral on a separable Hilbert space H (its norm is denoted by $|.|_H$ or just $|.|$).

We first fix a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. A *cylindrical Wiener process* or *cylindrical Brownian motion* on H is defined with two elements:

- A complete orthonormal system of H, denoted by $(e_i)_{i\in I}$, where I is a subset of $\mathbb{N}$.
- A family $(\beta_i)_{i\in I}$of independent real Wiener processes with respect to the filtration $\left((\mathcal{F}_t)_{t\geq 0}\right)$.

then W defined by

$$W(t) = \sum_{i\in I} \beta_i(t)\, e_i. \tag{6.49}$$

When I is a finite set, we recover the usual definition of Wiener processes in the finite dimensional space $\mathbb{R}^{|I|}$.

However the subject here is the study of some Stochastic Partial Differential Equations, so that in the sequel the cunderlying Hilbert space H is infinite dimensional, for instance when $H = L^2(0,1)$, an example of complete orthonormal system is $(e_k) = \left(\sqrt{2}\sin(k\pi.)\right)_{k\geq 1}$. The formal $dW(t) = \sum_{i\in I} d\beta_i(t)\, e_i(x)$ is called *space-time white noise*. A fundamental remark is that the series in (6.49) does not converge in H, but is convergent in any larger Hilbert spaces K such that $H \subset K$ and the embedding is Hilbert-Schmidt. We recall now the definition of such operators.

Definition 6.20 (Hilbert-Schmidt operator) Let H, K two separable Hilbert spaces, and $U \in \mathcal{L}(H,K)$ a bounded linear operator. We say that U is a Hilbert-Schmidt operator if there exists a complete orthonormal system $(e_k)_{k\in I}$ of H, $I \subset \mathbb{N}^*$, such that $\sum_{k\in I} |Ue_k|_K^2 < \infty$.

In this case, the value $\sum_{k\in I} |Ue_k|_K^2 < \infty$ is finite for any complete orthonormal system, and does not depend on the choice of such a basis.

A typical example of Hilbert-Schmidt operator is the inclusion map $i : L^2(D) \to H^{-s}(\Omega)$ for $s > 1/2$.

6.3 Stochastic partial differential equations and their applications

Let (V, H, V') be a triple of Hilbert spaces and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a stochastic basis.

Suppose that $\mathcal{A}(t) : V \to V'$, $\mathcal{M}_k(t) : V \to H$ are linear bounded operators for all $k \in \mathbb{N}$, $t \in [0, T]$, and for some finite terminal time T. A general Linear Stochastic Partial Differential Equation is written as

$$du(t) = (\mathcal{A}u(t) + f(t))\, dt + (\mathcal{M}_k u(t) + g_k(t))\, d\beta_k(t), \qquad u(0) = u_0, \tag{6.50}$$

where $t \in [0, T]$, $u_0 \in L^2(\Omega; H)$ and u_0 is $\mathcal{F}_0$-measurable. We assume that β_k $(k \in \mathbb{N})$ are independent standard Wiener processes, f, g_k $(k \in \mathbb{N})$ are $\mathcal{F}_t$-adapted random processes such that $f \in L^2(\Omega \times (0, T); V')$ and $g_k \in L^2(\Omega \times (0, T); H)$, $k \in \mathbb{N}$. Depending on the noise term the equation (6.50) is classified as follows:

- Equations with *additive noise*, if $\mathcal{M}_k = 0$.
- Equations with *multiplicative noise*, if $\mathcal{M}_k \neq 0$.

Similar to classical deterministic PDEs, the solution of equation (6.50) can be specified in different ways. Since the solution is a stochastic process, besides of PDE part, where we have classical solution, strong/weak generalized solution and mild solution, also we can specify strong and weak probabilistic solution. In this work we consider only solution which are strong in probabilistic sense and weak/strong/mild in PDE sense.

We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a weak solution of equation (6.50), if for every $u \in V$ and all $t \in [0, T]$, the equality

$$\begin{aligned}\langle u(t), u\rangle_H &= \langle u_0, u\rangle_H + \int_0^t \langle \mathcal{A}u(s) + f(s), u\rangle_H\, ds \\ &\quad + \int_0^t \langle \mathcal{M}_k u(s) + g_k(s), u\rangle_H\, d\beta_k(s)\end{aligned} \tag{6.51}$$

holds with probability one.

We say that $\mathcal{F}_t$-adapted function $u \in L^2(\Omega \times (0, T); V)$ is a mild solution of equation (6.50) if

$$\begin{aligned}u(t) &= S(t)\, u_0 + \int_0^t S(t-s)\, f(s)\, ds \\ &\quad + \int_0^t S(t-s)\,(\mathcal{M}_k u(s) + g_k(s))\, d\beta_k(s), \quad t \in [0, T],\end{aligned}$$

holds true with probability one, where S is a strongly continuous semigroup with infinitesimal generator $\mathcal{A}$. As we observed, stochastic evolution equations in infinite dimensions are natural generalizations of classical PDEs and systems of stochastic ordinary differential equations. The theory related to all these equations has motivations coming from physics, chemistry, biology, medicine, finance etc. Although, the theory of SPDEs is already established and widely developed field in mathematics, and problems arising in this theory represent an interest for mathematics itself.

6.3.1 Solutions of linear SPDEs perturbed by space-time white noise: stochastic convolution

In an abstract form, we want to solve SPDEs written in the Hilbert space H

$$du(t) = Au(t)dt + dW(t), \tag{6.52}$$

with an initial condition $u(0) = u_0 \in H$, $(W(t))_{t\in[0,T]}$ a cylindrical Wiener process in H (it could be in another space U), and $A : H \to H$. We assume that A generates a strongly continuous semi-group $(S(t))_{t\in[0,T]}$.

Definition 6.21 u is a weak solution of the SPDE (6.52) if for any $\xi \in D(A^*)$ and any $t > 0$ we have

$$\langle u(t), \xi \rangle = \langle u_0, \xi \rangle + \int_0^t \langle u(s), A^* \xi \rangle \, ds + \langle W(t), \xi \rangle .$$

Theorem 6.10 *Assume that*

$$\int_0^t |S(s)|^2_{\mathcal{L}_2(H,H)} \, ds < \infty. \tag{6.53}$$

Then (6.52) admits a unique weak solution, which satisfies:

$$u(t) = S(t)\, u_0 + \int_0^t S(t-s)\, dW(s) .$$

A function u satisfying such a formula is a mild (or integral) solution. Moreover

$$\mathbb{E}\, |u(t)|^2_H = \int_0^t \|S(s)\|^2_{\mathcal{L}_2(H,H)} \, ds.$$

When $u_0 = 0$, the solution is denoted by W_A and is called the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) .$$

The condition (6.53) is precisely the one required to be able to define the stochastic integral in H. If it is removed, there exists no H-valued solution.

6.4 Some notion of functional analysis

Definition 6.22 (Sobolev spaces) Let D be a domain and Y be a Banach space. For $p \geq 1$, the Sobolev space $W^{r,p}(D, Y)$ is the set of functions whose weak derivatives up to order $r \in \mathbb{N}$ are in $L^p(D, Y)$. That is,

$$W^{r,p}(D, Y) := \{u : \mathcal{D}^\alpha u \in L^p(D, Y) \text{ if } |\alpha| \leq r\} .$$

If $p = 2$ and H is a Hilbert space, $H^r(D, H)$ is used to denote $W^{r,2}(D, H)$.
$W^{r,p}(D, Y)$ is a Banach space with norm

$$\|u\|_{W^{r,p}(D,Y)} := \left(\sum_{0 \leq |\alpha| \leq r} \|\mathcal{D}^\alpha u\|^P_{L^p(D,Y)} \right)^{1/p} .$$

Definition 6.23 ($L^p(\Omega, H)$ spaces) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let H be a Hilbert space with norm $\|.\|$. Then, $L^p(\Omega, H)$ with $1 \leq p < \infty$ is the space of H-valued $\mathcal{F}$-measurable random variables $X : \Omega \to H$ with $E[\|X\|^p] < \infty$ and is a Banach space with a norm

$$\|X\|_{L^p(\Omega,H)} := \left(\int_\Omega \|X(\omega)\|^p \, d\mathbb{P} \right)^{1/p} = E\left[\|X\|^p\right]^{1/p} .$$

We recall now some Gronwall's lemma.

Lemma 6.8 (Gronwall's inequality (differential form)) **i** *Let $f(.)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$f'(t) \leq g(t) f(t) + h(t),$$

where $g(t)$ and $h(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$f(t) \leq e^{\int_0^t g(s)ds} \left[f(0) + \int_0^t h(s)\, ds \right]$$

for all $0 \leq t \leq T$.

ii *In particular, if*

$$f' \leq gf \quad on \;\; [0,T] \quad and \;\; f(0) = 0,$$

then

$$f = 0 \quad on \;\; [0,T].$$

We also require the discrete Gronwall inequality

Lemma 6.9 (discrete Gronwall inequality) *Consider $z_n \geq 0$ such that $z_n \leq a + b\sum_{k=0}^{n-1} z_k$, for $n = 0, 1, ...$ and constants $a, b \geq 0$. If $b = 1$, then $z_n \leq z_0 + na$. If $b \neq 1$, then*

$$z_n \leq b^n z_0 + \frac{a}{1-b}(1 - b^n).$$

GENERAL CONCLUSION

Keller-Segel chemotaxis model is described by a system of nonlinear PDE : a convection diffusion equation for the cell density coupled with a reaction-diffusion equation for chemoattractant concentration as follows

$$\begin{cases} \frac{\partial u}{\partial t} = \overbrace{\nabla(m\nabla u)}^{\substack{\text{Movement of}\\ \text{radon cells}}} \overbrace{+\nabla(\zeta u \nabla c)}^{\substack{\text{directed cell}\\ \text{movement}}} & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \delta \frac{\partial c}{\partial t} = \underbrace{D_c \triangle c}_{\substack{\text{chemical}\\ \text{diffusion}}} \underbrace{+\tau c - \rho u}_{\substack{\text{chemical degradation}\\ \text{by cells}}} & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \end{cases} \tag{6.54}$$

this research work, we have studied the following Keller-Segel model

$$\begin{cases} {}^cD_t^\beta u + (-\Delta)^{\frac{\alpha}{2}} u - \nabla(u\nabla c) = g(u)\dot{\mathcal{W}}(t), \ (t,x) \in [0,\mathcal{T}] \times \mathcal{D}, \\ {}^cD_t^\beta c + (-\Delta)^{\frac{\alpha}{2}} c - c\nabla c = f(c)\dot{\mathcal{W}}(t), \ (t,x) \in [0,\mathcal{T}] \times \mathcal{D}, \end{cases} \tag{6.55}$$

and

$$\begin{cases} u_t = \kappa \frac{\partial^2 u}{\partial x^2} - b\frac{\partial}{\partial x}\left(u^2 \frac{\partial}{\partial x} f(c)\right) + \frac{\partial^2 \tilde{W}}{\partial t \partial x}, \\ \\ c_t = a\frac{\partial^2 c}{\partial x^2} - \frac{\partial}{\partial x} f(c) + \frac{\partial^2 \tilde{W}}{\partial t \partial x} \ . \end{cases} \tag{6.56}$$

and

$$\begin{cases} n_t + u\nabla n - \Delta n - \nabla(n\nabla c) - \nabla(n\nabla\theta) = 0, \ (x,t) \in \Omega \times \mathbb{R}^+, \\ c_t + u\nabla c - \triangle c - \tau c - \rho u - b\theta = 0, \ (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta_t + u\nabla\theta - k\triangle\theta - n\theta = 0, \ (x,t) \in \Omega \times \mathbb{R}^+, \\ u_t + u\nabla u - v\Delta u - \nabla p - (\theta + n)f = 0, \ (x,t) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{6.57}$$

For the Keller-Segel chemotaxis model, under some assumptions on initial and boundary data we have proved:

- In the first part, we discussed the biological and mathematical modeling of Keller-Segel chemotaxis model.
- The required results in (6.55) is prove the existence and uniqueness of mild solution to time and space-fractional , for this we use analysis techniques and fractional calculus and semigroup theory, also studying the regularity properties of mild solution for this model.

- In the problem (6.56), we consider stochastic chemotaxis Keller-Segel model impact by gaussian process. Where we study the local-global existence solution to time of nonlinear stochastic Keller-Segel-Model with zeros Dirichlet-Boundary-Conditions, we use analysis techniques, lemmas and semigroup theory.

- In the problem (6.57) we study the phenomenon of Keller Segel model coupled with a Boussinesq equations, where we prove of the global existence and uniqueness weak solution for the problem, for this we use the technical of Galerkin method.

- finally we demonstrate how to solve nonlinear coupled partial differential equations emerging in an attractor one-dimensional Keller-Segel dynamics system using a relatively new analytical technique, the new homotopy perturpation method NHPM.

CONTRIBUTIONS

The research works that has been presented has several contributions, among which the following are the most significant:

1. We covered the biology and mathematical modeling of the Keller-Segel chemotaxis model in the first section.
2. Existence and uniqueness of stochastic Keller-Segel chemotaxis model (P) solutions are determined through proof and analysis.Â
3. Proving the existence and uniqueness of solutions for the fractional stochastic chemotaxis model (FP) and the stochastic chemotaxis model perturbed with Gaussian pressure and Keller-Segel coupled with Boussinesq equations.
4. studying the approximate solutions of a one-dimensional Keller-Segel model by new homotopy perturbation methods.

FUTURE RESEARCHS

In light of the Keller-Segel chemotaxis model's significance in the sciences and its many uses. There are many unexplored areas of inquiry that could be the subject of future studies. Research in particular could be beneficial in the following areas:

- Investigating the theoretical study of solutions for the problem in different cases, such as fractional stochastic, shows that fractional problems can be solved using different methods with different boundary conditions.
- Investigating the numerical solution to various types of the problem, such as the stochastic case and fractional stochastic, fractional problems solve using different numerical methods with different boundary conditions, and nonlinear source terms.

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