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**HOMOGENIZATION OF THE HYPERBOLIC-PARABOLIC
EQUATIONS IN DOMAINS WITH SMALL HOLES**

Option : AFA

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Abstract

In this memory we study a class of hyperbolic-parabolic problems in perforated domains with a homogeneous Neumann condition on the boundary of the holes. We focus on the homogenization of these equations by the periodic unfolding method in perforated domains.

Key words : hyperbolic-parabolic equation; perforated domains; homogenization; periodic unfolding method

Résumé

Nous tudions dans cette mémoire une class de Problème hyperbolique-parabolique dans un domaine perforé avec des petits trous avec des conditions de Neumann hmogène. Nous nous concentrons sur l'homogenisation de ces équation pour la method d'eclatement periodique dans les domaines perforés.

Mots clés : problème hypèrbolique-parabolique, domaines perforés, homogénéisation, méthode déclatement périodique.

ملخص

في هذا العمل قمنا بدراسة مشكلة القطع المكافئ الزائدي في وسط مسامي مع شروط نيومان متجانسة. ركزنا هنا على دراسة تجانس هذا النوع من المعادلات باستعمال طريقة الانفجار الدوري في مجالات مسامية .

كلمات مفتاحية: مسائل القطع المكافئ الزائدي، وسط مسامي، التجانس ، طريقة الانفجار الدوري .

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INTRODUCTION

In this memory, we adapt the periodic unfolding method to time-dependent problems in periodically perforated domains with small holes of the same size as the period. Then we apply the method to prove some homogenization and corrector results for the hyperbolic-parabolic equations in these domains.

The periodic unfolding method was first introduced in Cioranescu, Damlamian and Griso [4] for the case of fixed domains (see [7] for more details) and then extended to perforated ones in Cioranescu, Donato and Zaki [5]. Later, this method was adapted to two-component domains which are separated by a periodic interface in Donato, Nguyen and Tardieu [8]. Recently, Cioranescu, Damlamian, Donato, Grisco and Zaki [9] gave a comprehensive presentation of the periodic unfolding method for perforated domains, both when the unit hole is a compact subset of the open unit cell and when this is impossible to achieve.

Concerning the time-dependent periodic unfolding method for the fixed domains, we refer to Gaveau [11], where some elementary results were listed without proofs. Here, in order to treat the time-dependent problems in perforated domains, we adapt the results in [9] to time-dependent functions. For simplicity, we only treat the perforated domains when the unit hole is a compact subset of the open unit cell.

When working in perforated domains, the main difficulties are related to the fact that the equations and their solutions are defined on domains, which strongly depend on ε . In order

to speak about convergence when $\varepsilon \rightarrow \mathbf{0}$ that is to "homogenize"), one needs to introduce extension operators to fixed domain and to construct test functions, specific for each situation. To do that, several restrictions on the geometry of the holes and of the domain are necessary.

These difficulties are solved by the periodic unfolding method ([4] and [9]), as can be seen for example, in Cioranescu et al . [9]. The advantage of the unfolding method, it is the fact that it separates the scales. So one can treat in particular, holes at different scales in the same period. Such kind of problems, because of their complexity, cannot be solved by the classical methods in homogenization.

More precisely, we introduce the time-dependent unfolding operator $\mathcal{T}_\varepsilon^*$, which map functions defined on the oscillating domain into functions defined on the fixed domain. As shown in [9] and [5], the unfolding operator $\mathcal{T}_\varepsilon^*$ can, in some sense, replace the extension operator. We also state some related results needed in this work and give their proofs for the sake of completeness. And we give the example the application of this method for the wave equation with homogenous Dirichlet-Neumann boundary condition.

The aim of this work is to study the hyperbolic-parabolic equation with homogenous Dirichlet-Neumann boundary in the perforated domain.

This problem models many kinds of phenomena arising in electricity and magnetism, in the theory of elasticity, in vibrations theory and in hydrodynamics [14, 15].

The main purpose of this paper is to derive the homogenization result under these assumptions, which are weaker than those imposed in Migorski [12]. Our method is also quite different.

The work of Migorski [12] was done by the Tatar's oscillating test function method, while our study mainly relies on the time-dependent periodic unfolding method in perforated domains [10]. This method was originally introduced in Cioranescu, Damlamian and Griso [4] (see [9] for more details) and extended to perforated domains in Cioranescu, Donato and Zaki [5, 6] (see Cioranescu, Damlamian, Donato, et al. [9] for more general situations and see [10] for the time-dependent case).

This memory is organized as follows: In the first chapter, we define first the periodic unfolding for one and two parameters in perforated domains and some properties.

In the second chapter, we define the time periodic unfolding in perforated domains and some

properties and we treat an the hyperbolic model problem with a Robin boundary condition. Finally, in third chapter we are concerned with the asymptotic behavior of hyperbolic-parabolic problem in perforated domains with small holes.

CHAPTER

1

THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains \mathcal{T}_ε for the classical homogenization.

1.1 The periodic unfolding operator \mathcal{T}_ε

In this section, we introduce the case of perforated domains introduced by Cioranescu et al. In the following we denote:

- Ω an open set in \mathbb{R}^N .
- $Y = \prod_{i=1}^N [0; l_i[$ the reference cell $l_i > 0$ for all $1 \leq i \leq N$, or more generally a set having property to a basis (b_1, \dots, b_N) defining the periods .
- T an open set included in Y such that ∂T does not contain the summits of Y .

We can be, sometimes, transported to this situation by a simple change of period, we define:

$$\mathbf{T}_\varepsilon = \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + \mathbf{T}) \quad \text{and} \quad \Omega_\varepsilon = \Omega \setminus \mathbf{T}_\varepsilon.$$

We assume in the following that Ω_ε is a connected set and we take the regularity hypothesis

$$|\partial\Omega| = 0,$$

and we define :

$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N, \varepsilon(\xi + \mathbf{Y}) \subset \Omega\}.$$

To do so, let us first define the following domain:

$$\widetilde{\Omega}_\varepsilon = \text{int} \left(\bigcup_{\xi \in \Lambda_\varepsilon} (\xi + \mathbf{Y}) \right),$$

where

$$\Lambda_\varepsilon = \{\xi \in \mathbb{Z}^N; \varepsilon(\xi + \overline{\mathbf{Y}}) \cap \Omega \neq \emptyset\}.$$

The set $\widetilde{\Omega}_\varepsilon$ is the smallest finite union of $\varepsilon\mathbf{Y}$ cells containing Ω .

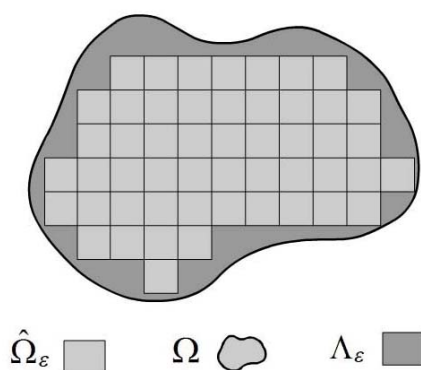


Fig. 1. The sets $\widehat{\Omega}_\varepsilon$, Ω and Λ_ε

In the sequel, we will use the following notation:

- $\tilde{\varphi}$ for the extension by 0 outside Ω_ε for any function $\varphi \in L^p(\Omega_\varepsilon)$.

- χ_ε for the characteristic function of Ω_ε .

By analogy to the $\mathbf{1D}$ notation, for $z \in \mathbb{R}^N$, $[z]_{\mathbf{Y}}$ denotes the unique integer combination $\sum_{j=1}^N k_j \mathbf{b}_j$, such that $[z]_{\mathbf{Y}}$ belongs to \mathbf{Y} . Set $z_{\mathbf{Y}} = z - [z]_{\mathbf{Y}}$. Then, for almost every $\mathbf{x} \in \mathbf{R}^N$, there exists a unique element in \mathbb{R}^N , denoted by $\left[\frac{\mathbf{x}}{\mathbf{y}} \right]_{\mathbf{Y}}$, such that:

$$\mathbf{x} - \varepsilon \left[\frac{\mathbf{x}}{\mathbf{y}} \right]_{\mathbf{Y}} = \varepsilon \left\{ \frac{\mathbf{x}}{\mathbf{y}} \right\}_{\mathbf{Y}}, \quad (1.1)$$

where

$$\left\{ \frac{\mathbf{x}}{\mathbf{y}} \right\}_{\mathbf{Y}} \in \mathbf{Y}.$$

This decomposition (1.1) is essential in the definition of the unfolding operator in the next.

definition 1.1 *Let $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1, +\infty]$. We define the function*

$$\mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) = \tilde{\varphi} \left(\varepsilon \left[\frac{\mathbf{x}}{\mathbf{y}} \right]_{\mathbf{Y}} + \varepsilon \mathbf{y} \right), \quad (1.2)$$

for every $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbf{Y}$.

Remark 1.1 *Notice that the oscillations due to perforations are shifted into the second variable \mathbf{y} which belongs to the fixed domain \mathbf{Y} , while the first variable \mathbf{x} belongs to \mathbb{R}^N .*

One see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on Ω_ε one needs first, while using standard methods, to extend it to a fixed domain. With \mathcal{T}_ε , such extensions are more necessary.

The main properties given in [3] for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are listed in the proposition below. To do so, let us first define the following domain:

$$\tilde{\Omega}_\varepsilon = \text{int} \left(\bigcup_{\xi \in \Lambda_\varepsilon} (\xi + \mathbf{Y}) \right),$$

where

$$\Lambda_\varepsilon = \{ \xi \in \mathbb{Z}^N; \varepsilon(\xi + \bar{\mathbf{Y}}) \cap \Omega \neq \emptyset \}.$$

The set $\tilde{\Omega}$ is the smallest finite union of $\varepsilon \mathbf{Y}$ cells containing Ω .

Proposition 1.1 [3] *The unfolding operator \mathcal{T}_ε has the following properties:*

1. \mathcal{T}_ε is a linear operator.
2. $\mathcal{T}_\varepsilon\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)_Y = \varphi(\mathbf{x}); \forall \varphi \in L^p(\Omega_\varepsilon)$ and $\mathbf{x} \in \mathbb{R}^N$.
3. $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi), \forall \varphi, \psi \in L^p(\Omega_\varepsilon)$.
4. Let φ in $L^p(\mathbf{Y})$ or $L^p(\mathbf{Y})$ be a Y -periodic function. Set $\varphi_\varepsilon(\mathbf{x}) = \varphi\left(\frac{\mathbf{x}}{\varepsilon}\right)$. Then, $\mathcal{T}_\varepsilon(\varphi_\varepsilon)(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{y})$, a.e. in $\tilde{\Omega}_\varepsilon$.
5. One the integration formula:

$$\int_{\Omega_\varepsilon} \varphi d\mathbf{x} = \frac{1}{|\mathbf{Y}|} \int_{\tilde{\Omega}_\varepsilon * \mathbf{Y}} \mathcal{T}_\varepsilon(\varphi) d\mathbf{x}d\mathbf{y}, \forall \varphi \in L^1(\Omega_\varepsilon).$$

6. For every $\varphi \in L^2(\Omega_\varepsilon)$, belongs to $L^2(\mathbb{R}^N * \mathbf{Y})$. It also belongs to $L^2(\tilde{\Omega}_\varepsilon * \mathbf{Y})$.
7. For every $\varphi \in L^2(\Omega_\varepsilon)$, one has :

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N * \mathbf{Y})} = \sqrt{|\mathbf{Y}|} \|\varphi\|_{L^2(\Omega_\varepsilon)}.$$

8. $\nabla \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(\mathbf{x}, \mathbf{y})$ for every $(\mathbf{x}, \mathbf{y}) \in \Omega \times \mathbf{Y}$.
9. If $\varphi \in H^1(\tilde{\omega}_\varepsilon)$, then $\mathcal{T}_\varepsilon(\varphi)$ is in $L^2(\mathbb{R}^N, H^1(\mathbf{Y}))$.
10. One has the estimate:

$$\|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega \times \mathbf{Y}))^N} = \varepsilon \sqrt{|\mathbf{Y}|} \|\nabla_x \varphi\|_{(L^2(\Omega_\varepsilon))^N}.$$

Proof.

1. Let $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned} \mathcal{T}_\varepsilon(\alpha\varphi + \beta\psi) &= \alpha\varphi + \beta\psi \left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon} \right]_y + \varepsilon\mathbf{y} \right) \\ &= \alpha\varphi \left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon} \right]_y + \varepsilon\mathbf{y} \right) + \beta\psi \left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon} \right]_y + \varepsilon\mathbf{y} \right) \\ &= \alpha\mathcal{T}_\varepsilon(\varphi) + \beta\mathcal{T}_\varepsilon(\psi). \end{aligned}$$

Then \mathcal{T}_ε is a linear operator.

3. Let $\varphi, \psi \in L^2(\Omega_\varepsilon)$ we have by definition 1.1

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi\psi) &= \varphi\psi\left(\varepsilon\left[\frac{\mathbf{x}}{\varepsilon}\right]_y + \varepsilon\mathbf{y}\right) \\ &= \varphi\left(\varepsilon\left[\frac{\mathbf{x}}{\varepsilon}\right]_y + \varepsilon\mathbf{y}\right)\psi\left(\varepsilon\left[\frac{\mathbf{x}}{\varepsilon}\right]_y + \varepsilon\mathbf{y}\right) \\ &= \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi).\end{aligned}$$

4. and by definition for all $\varphi_\varepsilon = \varphi\left(\frac{\mathbf{x}}{\varepsilon}\right)$, $\varphi \in L^2(\mathbf{Y})$, we have:

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi_\varepsilon)(\mathbf{x}, \mathbf{y}) &= \varphi\left(\varepsilon\left[\frac{\left\{\frac{\mathbf{x}}{\varepsilon}\right\}}{\varepsilon}\right] + \varepsilon\left\{\frac{\left\{\frac{\mathbf{x}}{\varepsilon}\right\}}{\varepsilon}\right\}\right) \\ &= \varphi\left\{\frac{\mathbf{x}}{\varepsilon}\right\} = \varphi(\mathbf{y}).\end{aligned}$$

5. according to the definition 1.1 we have :

$$\begin{aligned}\frac{1}{|\bar{\mathbf{Y}}|} \int_{\Omega \times \mathbf{Y}} \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y} &= \frac{1}{|\mathbf{Y}|} \int_{\tilde{\Omega}_\varepsilon \times \mathbf{Y}} \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y} \\ &= \frac{1}{|\mathbf{Y}|} \sum_{\xi \in \Xi_\varepsilon} \int_{(\varepsilon\xi + \varepsilon\mathbf{Y}) \times \mathbf{Y}} \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y},\end{aligned}$$

on each set $(\varepsilon\xi + \varepsilon\mathbf{Y}) \times \mathbf{Y}$ with $\xi \in \Xi_\varepsilon$, the function $\mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) = \varphi(\varepsilon\xi + \varepsilon\mathbf{y})$ is constant in \mathbf{x} , it is consequence of (2.1). So, for each integral in the sum of the right member, we have :

$$\begin{aligned}\int_{(\varepsilon\xi + \varepsilon\mathbf{Y}) \times \mathbf{Y}} \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) d\mathbf{x}d\mathbf{y} &= |\varepsilon\xi + \varepsilon\mathbf{Y}| \int_{\mathbf{Y}} \varphi(\varepsilon\xi + \varepsilon\mathbf{y}) d\mathbf{y} \\ &= \varepsilon^N |\mathbf{Y}| \int_{\mathbf{Y}} \varphi(\varepsilon\xi + \varepsilon\mathbf{y}) d\mathbf{y} \\ &= |\mathbf{Y}| \int_{(\varepsilon\xi + \varepsilon\mathbf{Y})} \varphi(\mathbf{x}) d\mathbf{x},\end{aligned}\tag{1.3}$$

we suppose $\mathbf{x} = \varepsilon\xi + \varepsilon\mathbf{y}$, this implies $d\mathbf{x} = \varepsilon^N d\mathbf{y} \implies d\mathbf{y} = \frac{1}{\varepsilon^N} d\mathbf{x}$

by suming in Ξ , the right member be $\int_{\tilde{\Omega}_\varepsilon} \varphi(\mathbf{x}) d\mathbf{x}$

so we get: $\sum_{\xi \in \Xi} \varepsilon^N |\mathbf{Y}| \int_{(\varepsilon\xi + \varepsilon\mathbf{Y})} \varphi(\mathbf{x}) \frac{1}{\varepsilon^N} d\mathbf{x} = \int_{\tilde{\Omega}} \varphi(\mathbf{x}) d\mathbf{x}$.

Now we remplace this expretion in (1.3), we get the result.

7. for all $\varphi \in L^2(\Omega_\varepsilon)$,

$$\begin{aligned} \|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^N \times Y)} &= \left(\int |\mathcal{T}_\varepsilon(\varphi)|^2 dx dy \right)^{\frac{1}{2}} \\ &= \left(\mathbf{1} \int_{\Omega \times y} |\varphi|^2 dx dy \right)^{\frac{1}{2}} \\ &= \left(\int \mathbf{1} dy \right)^{\frac{1}{2}} \left(\int |\varphi|^2 dx \right)^{\frac{1}{2}} \\ &= \sqrt{|Y|} \|\varphi\|_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

8. for all $\varphi \in L^2(\Omega_\varepsilon)$,

$$\begin{aligned} \nabla_y \mathcal{T}_\varepsilon(\varphi)(x, y) &= \nabla_y \varphi \left(\varepsilon \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_y + \varepsilon y \right) \\ &= \varepsilon \nabla_x \varphi \left(\varepsilon \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_y + \varepsilon y \right) \\ &= \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(x, y). \end{aligned}$$

10. we apply proposition 2.1, we obtain:

$$\begin{aligned} \|\nabla_y \mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega \times Y))^N} &= \|\varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)\|_{(L^2(\Omega \times Y))^N} \\ &= \varepsilon \sqrt{|Y|} \|\nabla_x \varphi\|_{(L^2(\Omega_\varepsilon))^N}. \end{aligned}$$

THE TIME-DEPENDENT UNFOLDING OPERATOR

2.1 The time-dependent unfolding operator in perforated domains

In this section, we adapt the unfolding operator in [9] to time-dependent functions. We present the unfolding operator $\mathcal{T}_\varepsilon^*$ which maps functions defined on the oscillating domain $\Omega_\varepsilon^* \times (0, T)$ into functions defined on the fixed domain $\Omega \times Y^* \times (0, T)$. This avoids the use any extension operator. We also give some properties by extending the previous ones (see [9] for more details and [7] for classical unfolding theory).

For any $z \in \mathbb{R}^n$, we use $[z]_Y$ to denote the unique integer combination $\sum_{j=1}^n k_j \mathbf{b}_j$ of the period such that $z - [z]_Y \in Y$. Set

$$\{z\}_Y = z - [z]_Y \in Y \text{ a.e for } z \in \mathbb{R}^n.$$

Then for each $\mathbf{x} \in \mathbb{R}^n$, we have:

$$\mathbf{x} = \varepsilon \left(\left[\frac{\mathbf{x}}{\varepsilon} \right]_Y + \left\{ \frac{\mathbf{x}}{\varepsilon} \right\} \right) \text{ a.e. for } \mathbf{x} \in \mathbb{R}^n.$$

Let us first recall unfolding operator \mathcal{T}_ε for the fixed domain $\Omega \times (0, T)$ introduced in [11], where the properties of \mathcal{T}_ε are shown without proofs.

definition 2.1 For $p \in [1, +\infty)$ and $q \in [1, \infty]$, let ϕ be in $L^q(0, T; L^p(\Omega))$. The unfolding operator $\mathcal{T}_\varepsilon : L^q(0, T; L^p(\Omega)) \mapsto L^q(0, T; L^p(\Omega \times Y))$ is defined as follows:

$$\mathcal{T}(\phi)(\mathbf{x}, \mathbf{y}, t) = \begin{cases} \phi \left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon} \right]_Y + \varepsilon \mathbf{y}, t \right) & \text{a.e. for } (\mathbf{x}, \mathbf{y}, t) \in \tilde{\Omega}_\varepsilon \times Y \times (0, T), \\ 0 & \text{a.e. for } (\mathbf{x}, \mathbf{y}, t) \in \Lambda_\varepsilon \times Y \times (0, T). \end{cases}$$

In a similar way, we extend the unfolding operator for the perforated domain Ω_ε^* to the following operator $\mathcal{T}_\varepsilon^*$ for the perforated domain $\Omega_\varepsilon^* \times (0, T)$.

definition 2.2 For $p \in [1, +\infty[$ and $q \in [1, \infty[$, let ϕ be in $L^q(0, T; L^p(\Omega_\varepsilon^*))$. The unfolding operator $\mathcal{T}_\varepsilon^* : L^q(0, T; L^p(\Omega_\varepsilon^*)) \mapsto L^q(0, T; L^p(\Omega \times Y^*))$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(\phi)(\mathbf{x}, \mathbf{y}, t) = \begin{cases} \phi \left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon} \right]_Y + \varepsilon, \mathbf{y}, t \right) & \text{a.e for } (\mathbf{x}, \mathbf{y}, t) \in \tilde{\Omega}_\varepsilon \times Y^* \times (0, T), \\ 0 & \text{a.e for } (\mathbf{x}, \mathbf{y}, t) \in \Lambda_\varepsilon \times Y^* \times (0, T). \end{cases}$$

From this definition, the following properties are immediate:

$$\begin{aligned} (i) \quad & \mathcal{T}_\varepsilon^*(vw) = \mathcal{T}_\varepsilon^*(v)\mathcal{T}_\varepsilon^*(w), \forall v, w \in L^q(0, T; L^p(\Omega_\varepsilon^*)), \\ (ii) \quad & \mathcal{T}_\varepsilon^*(\psi\varphi) = \varphi\mathcal{T}_\varepsilon^*(\psi), \forall \psi \in L^p(\Omega_\varepsilon^*) \text{ and } \varphi \in L^q(0, T), \\ (iii) \quad & \nabla_{\mathbf{y}}(\mathcal{T}_\varepsilon^*(\phi)) = \varepsilon\mathcal{T}_\varepsilon^*(\nabla\phi), \forall \phi \in L^q(0, T; W^{1,p}(\Omega_\varepsilon^*)). \end{aligned} \tag{2.1}$$

Remark 2.1 Concerning \mathcal{T}_ε and $\mathcal{T}_\varepsilon^*$, we have the following:

$$\begin{aligned} \mathcal{T}_\varepsilon^*(\omega |_{(\Omega_\varepsilon^* \times (0, T))}) &= \mathcal{T}_\varepsilon(\omega) |_{\Omega \times Y^* \times (0, T)}, \\ \mathcal{T}_\varepsilon^*(\psi) &= \mathcal{T}_\varepsilon(\tilde{\psi}) |_{\Omega \times Y^* \times (0, T)}. \end{aligned}$$

Where ω and ψ are defined on $\Omega \times (0, T)$ and $\Omega_\varepsilon^* \times (0, T)$, respectively.

In definition 2.1 and definition 2.2, if ϕ is independent of \mathbf{t} , then \mathcal{T}_ε and $\mathcal{T}_\varepsilon^*$ are the classical unfolding operator defined in [4] and [9], respectively. For simplify, we always write $\mathcal{T}_\varepsilon^*(\phi)$ instead of $\mathcal{T}_\varepsilon^*(\phi |_{\Omega_\varepsilon^* \times (0, T)})$ for any function ϕ defined in $\Omega \times (0, T)$.

Next we list some properties of the unfolding operator $\mathcal{T}_\varepsilon^*$ used in this paper. The proofs are essentially the same as those in [9] and [7].

Proposition 2.1 For $p \in [1, +\infty)$ and $q \in [1, \infty]$ the operator $\mathcal{T}_\varepsilon^*$ is linear and continuous from $L^q(0, T; L^p(\Omega_\varepsilon^*))$ to $L^q(0, T; L^p(\Omega \times Y^*))$. Let $\phi \in L^q(0, T; L^1(\Omega_\varepsilon^*))$ and

$$\mathbf{w}, \mathbf{v} \in L^q(0, T; L^p(\Omega_\varepsilon^*)).$$

For a.e $\mathbf{t} \in (0, T)$, we have :

- (i) $\frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(\mathbf{x}, \mathbf{y}, t) d\mathbf{x} d\mathbf{y} = \int_{\widetilde{\Omega_\varepsilon^*}} \phi(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_\varepsilon^*} \phi(\mathbf{x}, t) d\mathbf{x} - \int_{\Lambda_\varepsilon^*} \phi(\mathbf{x}, t) d\mathbf{x},$
- (ii) $\|\mathcal{T}_\varepsilon^*(\mathbf{w})\|_{L^p(\Omega \times Y^*)} = |Y|^{\frac{1}{p}} \|\mathbf{w}\|_{L^p(\widetilde{\Omega_\varepsilon^*})} \leq |Y|^{\frac{1}{p}} \|\mathbf{w}\|_{L^p(\Omega_\varepsilon^*)}.$

Proposition 2.2 For $q \in [1, +\infty]$, let ϕ_ε be in $L^q(0, T; L^1(\Omega_\varepsilon^*))$ and satisfy

$$\int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| d\mathbf{x} dt \longrightarrow 0,$$

then

$$\int_0^T \int_{\Lambda_\varepsilon^*} \phi_\varepsilon d\mathbf{x} dt - \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) d\mathbf{x} d\mathbf{y} dt \longrightarrow 0.$$

As usual, this convergence is denoted by

$$\int_0^T \int_{\Lambda_\varepsilon^*} \phi_\varepsilon d\mathbf{x} dt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) d\mathbf{x} d\mathbf{y} dt.$$

Moreover, we have the following convergences:

Proposition 2.3 (i) For $p, q \in (1, +\infty]$, let $\phi_\varepsilon \in L^q(0, T; L^p(\Omega_\varepsilon^*))$ and $\psi \in L^q(0, T; L^{p'}(\Omega_\varepsilon^*))$ $\left(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1\right)$ such that

$$\|\phi_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C, \quad \text{and} \quad \|\psi\|_{L^{q'}(0, T; L^{p'}(\Omega_\varepsilon^*))} \leq C, \quad (2.2)$$

then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi \, dx \, dt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi) \, dx \, dy \, dt.$$

(ii) For $p, q \in [1, +\infty[$, let $\phi_\varepsilon \in L^q(0, T; L^{p_0}(\Omega_\varepsilon^*))$ and $\psi_\varepsilon \in L^{q'}(0, T; L^{p_0}(\Omega_\varepsilon^*))$ $\left(\frac{1}{p} + \frac{1}{p_0} < 1, \frac{1}{q} + \frac{1}{q'} = 1\right)$ such that

$$\|\phi_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C, \quad \text{and} \quad \|\psi_\varepsilon\|_{L^{q'}(0, T; L^{p_0}(\Omega_\varepsilon^*))} \leq C, \quad (2.3)$$

then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi_\varepsilon \, dx \, dt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \, dx \, dy \, dt.$$

Proof

(i) For a.e $\mathbf{x} \in \Omega$, notice that $\mathbf{1}_{\Lambda_\varepsilon^*} \rightarrow \mathbf{0}$ holds true. For a.e. $t \in [0, T]$, in view of (2.1), it follows from the Lebesgue dominated convergence theorem that

$$\int_{\Lambda_\varepsilon^*} |\psi|^{p'} \rightarrow 0$$

and

$$\int_0^T \left(\int_{\Lambda_\varepsilon^*} |\psi|^{p'} \, dx \right)^{\frac{q'}{p'}} dt \rightarrow 0$$

This implies

$$\int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon \psi| \, dx \, dt \rightarrow 0$$

hence statement (i) holds true due to Proposition 2.2.

(ii) Let $\frac{1}{p'} + \frac{1}{p} = 1$. By (2.2) and the Hölder inequality, we have

$$\begin{aligned} \int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon \psi| \, dx \, dt &\leq C \left(\int_0^T \left[\int_{\Lambda_\varepsilon^*} |\psi_\varepsilon|^{p'} \, dx \right]^{\frac{q'}{p'}} dt \right)^{\frac{1}{q'}} \\ &\leq C \left(\int_0^T \left[\int_{\Lambda_\varepsilon^*} |\psi_\varepsilon|^{p_0} \, dx \right]^{\frac{q'}{p_0}} dt \right)^{\frac{1}{q'}} \cdot |\Lambda_\varepsilon^*|^{1 - \left(\frac{1}{p} + \frac{1}{p_0}\right)} \\ &\leq C |\Lambda_\varepsilon^*|^{1 - \left(\frac{1}{p} + \frac{1}{p_0}\right)}. \end{aligned}$$

As ε tends to zero, we get

$$\int_0^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon \psi_\varepsilon| \, dx dt \longrightarrow 0$$

Which implies that statement (ii) holds true, due to Proposition 2.5

Proposition 2.4 (Some convergence properties)

(i) For $p, q \in [1, +\infty)$, let $\omega \in L^q(0, T; L^p(\Omega))$. Then

$$\mathcal{T}_\varepsilon^*(\omega) \longrightarrow \omega, \text{ strongly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

(ii) For $p, q \in [1, +\infty)$, let $\{\omega_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega))$ such that

$$\omega_\varepsilon \longrightarrow \omega, \text{ strongly in } L^q(0, T; L^p(\Omega)),$$

then

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \longrightarrow \omega, \text{ strongly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

(iii) For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let $\{\omega_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega_\varepsilon^*))$ such that

$$\|\omega_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C.$$

If

$$\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \rightharpoonup \tilde{\omega} \text{ weakly in } L^q(0, T; L^p(\Omega \times Y^*)),$$

then we have

$$\tilde{\omega}_\varepsilon \rightharpoonup \theta \mathcal{M}_{Y^*}(\tilde{\omega}) \text{ weakly in } L^q(0, T; L^p(\Omega)).$$

For $q = \infty$, the weak convergences above are replaced by the weak* convergence, respectively.

Proof

(i) using Proposition 2.4 and (2.1), we have

$$\begin{aligned} & \|\mathcal{T}_\varepsilon^*(\omega) - \omega\|_{L^q(0, T; L^p(\Omega \times Y^*))} \\ &= \|\mathcal{T}_\varepsilon^*(\omega - \phi\varphi) + \mathcal{T}_\varepsilon^*(\phi\varphi) - \phi\varphi + \phi\varphi - \omega\|_{L^q(0, T; L^p(\Omega \times Y^*))} \\ &\leq 2 \|Y\|^{\frac{1}{p}} \|\phi\varphi - \omega\|_{L^q(0, T; L^p(\Omega))} + (\mathcal{T}_\varepsilon^*(\phi) - \phi)\varphi \|_{L^q(0, T; L^p(\Omega \times Y^*))} \end{aligned}$$

for any $\phi \in D(\Omega)$ and $\varphi \in D(0, T)$. Since

$\mathcal{T}_\varepsilon^*(\phi) \longrightarrow \phi$ strongly in $L^p(\Omega \times Y^*)$, then

$$\left(\mathcal{T}_\varepsilon^*(\phi) - \phi \right) \varphi \left\|_{L^q(0,T;L^p(\Omega \times Y^*))} \longrightarrow 0$$

Consequently, we have

$$\limsup_{\varepsilon \rightarrow 0} \left\| \mathcal{T}_\varepsilon^*(\omega) \right\|_{L^q(0,T;L^p(\Omega \times Y^*))} \leq 2 |Y|^{\frac{1}{p}} \left\| \phi \varphi - \omega \right\|_{L^q(0,T;L^p(\Omega))},$$

which implies statement (i) due to the density of $D(0, T) \otimes D(\Omega)$ in $L^q(0, T; L^p(\Omega))$.

(ii) From 2.3, we have

$$\left\| \mathcal{T}_\varepsilon^*(\omega_\varepsilon - \omega) \right\|_{L^q(0,T;L^p(\Omega \times Y^*))} \leq |Y|^{\frac{1}{p}} \left\| \omega_\varepsilon - \omega \right\|_{L^q(0,T;L^p(\Omega))}.$$

Hence statement (ii) follows from statement (i) (iii) Let $\psi \in L^{q'}(0, T; L^{p'}(\Omega))$ $\left(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1 \right)$. From 2.4 (i)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \tilde{\omega}_\varepsilon \psi dx dt &= \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} \omega_\varepsilon \psi dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\omega_\varepsilon) \mathcal{T}_\varepsilon^*(\psi) dx dy dt. \end{aligned}$$

Since statement (i) gives

$$\mathcal{T}_\varepsilon^*(\psi) \longrightarrow \psi \text{ strongly in } L^{q'}(0, T; L^{p'}(\Omega \times Y^*)),$$

then we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \tilde{\omega}_\varepsilon \psi dx dt &= \int_0^T \int_\Omega \left\{ \frac{1}{|Y|} \int_{Y^*} \tilde{\omega} dy \right\} \psi dx dt \\ &= \theta \int_0^T \int_\Omega \mu_{Y^*}(\tilde{\omega}) \psi dx dt \end{aligned}$$

which implies the desired result.

Now, we complete this section with a convergence result related to the space $L^q(0, T; W^{1,p}(\Omega_\varepsilon^*))$.

This proof can be obtained from that of Theorem 2.1 in [9]

Proposition 2.5 For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let ϕ_ε belong to $L^q(0, T, W^{1,p}(\Omega_\varepsilon^*))$ and satisfy

$$\left\| \phi_\varepsilon \right\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} + \varepsilon \left\| \nabla \phi_\varepsilon \right\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq C$$

Then, there exists $\phi \in L^q(0, T; L^p(\Omega; W^{1,p}(Y^*)))$, such that, up to a subsequence, $\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightharpoonup \phi$ weakly in $L^q(0, T; L^p(\Omega; W^{1,p}(Y^*)))$, $\varepsilon \mathcal{T}_\varepsilon^*(\nabla \phi_\varepsilon) \rightharpoonup \nabla_y \phi$ weakly in $L^q(0, T; L^p(\Omega \times Y^*))$. For $q = \infty$, the weak convergences above are replaced by the weak* convergences, respectively.

2.2 The time-dependent averaging operator in perforated domains

In this section, we adapt the averaging operator in [9] to time-dependent functions.

Then we show some properties needed in this paper.

definition 2.3 For $p \in [1, +\infty)$ and $q \in [1, +\infty]$, let φ be in $L^q(0, T; L^p(\Omega \times Y^*))$. The averaging operator $\mathcal{U}_\varepsilon^* : L^q(0, T; L^p(\Omega \times Y^*)) \mapsto L^q(0, T; L^p(\Omega_\varepsilon^*))$ is defined as follows:

$$\mathcal{U}_\varepsilon^*(\phi)(x, t) = \begin{cases} \frac{1}{|Y|} \int_Y \phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\}_Y, t \right) dz & \text{a.e. for } (x, t) \in \tilde{\Omega}_\varepsilon^* \times (0, T), \\ 0 & \text{a.e. for } (x, t) \in \Lambda_\varepsilon^* \times (0, T). \end{cases}$$

Remark 2.2 (i) If $\psi \in L^q(0, T; L^p(\Omega \times Y^*))$ and $\omega \in L^q(0, T; L^p(\Omega \times Y))$, then we have

$$\begin{cases} \mathcal{U}_\varepsilon^*(\psi) = \mathcal{U}_\varepsilon(\tilde{\psi}) \mid_{\Omega_\varepsilon \times 0, T}, \\ \mathcal{U}_\varepsilon(\omega \mid_{\Omega \times Y^* \times (0, T)}) = \mathcal{U}_\varepsilon(\omega) \mid_{\Omega_\varepsilon^* \times Y^* \times (0, T)}, \end{cases} \quad (2.4)$$

where \mathcal{U}_ε is the averaging operator for the fixed domain $\Omega \times Y \times (0, T)$ in [11].

(ii) The operator $\mathcal{U}_\varepsilon^*$ is almost a left-inverse of $\mathcal{T}_\varepsilon^*$. That is to say, any $\phi \in L^q(0, T; L^p(\Omega_\varepsilon^*))$,

$$\mathcal{U}_\varepsilon^*(\mathcal{T}_\varepsilon^*(\phi))(x, t) = \begin{cases} \phi(x, t) & \text{a.e. for } (x, t) \in \tilde{\Omega}_\varepsilon^* \times (0, T), \\ 0 & \text{a.e. for } (x, t) \in \Lambda_\varepsilon^* \times (0, T). \end{cases} \quad (2.5)$$

By remark 2.2 (i) and the fact that \mathcal{U}_ε is the adjoint of \mathcal{T}_ε , we get that $\mathcal{U}_\varepsilon^*$ is actually the

adjoint of $\mathcal{T}_\varepsilon^*$. namely,

$$\int_0^T \int_{\Omega_\varepsilon^*} \mathcal{U}_\varepsilon^*(v) u dx dt = \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y} \mathcal{T}_\varepsilon^*(u) v dx dy dt,$$

where $u \in L^q(0, T; L^p(\Omega_\varepsilon^*))$ and $v \in L^{q'}(0, T; L^{p'}(\Omega \times Y^*))$ $\left(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1\right)$.

This implies the following estimate:

Proposition 2.6 For $p \in [1, +\infty)$ $q \in [1, \infty]$, the operator $\mathcal{U}_\varepsilon^*$ is linear and continuous from $L^q(0, T; L^{p'}(\Omega \times Y^*))$ to $L^q(0, T; L^p(\Omega_\varepsilon^*))$ and

$$\|\mathcal{U}_\varepsilon^*(\phi)\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq |Y|^{-\frac{1}{p}} \|\phi\|_{L^q(0, T; L^{p'}(\Omega \times Y^*))}.$$

Next we show some convergence properties related to $\mathcal{U}_\varepsilon^*$ whose proofs are essentially same as those in [9] and [4]

Proposition 2.7 Let $p, q \in [1, +\infty)$.

(i) For $\omega \in L^q(0, T; L^{p'}(\Omega))$,

$$\|\mathcal{U}_\varepsilon^*(\omega) - \omega\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \longrightarrow 0.$$

(ii) Let $\{\Phi_\varepsilon\}$ be a sequence such that $\Phi_\varepsilon \longrightarrow \Phi$ strongly in $L^q(0, T; L^p(\Omega \times Y^*))$. Then

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)) \longrightarrow \Phi \text{ strongly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

(iii) If $\omega_\varepsilon \in L^q(0, T; L^p(\Omega_\varepsilon^*))$ and $\omega \in L^q(0, T; L^p(\Omega \times Y^*))$, then the following two assertions are equivalent:

- (a) $\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \longrightarrow \omega$ strongly in $L^q(0, T; L^p(\Omega \times Y^*))$ and $\|\omega_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \longrightarrow 0$,
- (b) $\|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \longrightarrow 0$.

(iv) If $\omega_\varepsilon \in L^q(0, T; L^p(\Omega_\varepsilon^*))$ and $\omega \in L^q(0, T; L^p(\Omega \times Y^*))$, then the following two assertions are equivalent:

- (a) $\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \longrightarrow \omega$ strongly in $L^q(0, T; L^p(\Omega \times Y^*))$.
- (b) $\|\omega_\varepsilon - \omega\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \longrightarrow 0$.

Furthermore, (a) together with $\|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \rightarrow 0$ is equivalent to

$$\|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \rightarrow 0.$$

Proof

(i) From (2.5) and Proposition 2.6,

$$\begin{aligned} \|\mathcal{U}_\varepsilon^*(\omega) - \omega\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} &\leq \|\mathcal{U}_\varepsilon^*(\omega) - \mathcal{U}_\varepsilon^*(\mathcal{T}_\varepsilon^*(\omega))\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} + \|\omega\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \\ &\leq |Y|^{-\frac{1}{p}} \|\mathcal{T}_\varepsilon^*(\omega) - \omega\|_{L^q(0,T;L^p(\Omega \times Y^*))} + \|\omega\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))}. \end{aligned}$$

Combining this with Proposition 2.4 (i) and the convergence

$$\|\omega\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \rightarrow 0,$$

we get the desired result.

(ii) In view of Proposition 2.2 (ii) and Proposition 2.6, we have

$$\begin{aligned} \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)) - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} &\leq \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)) - \mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi))\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\quad + \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi)) - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\leq \|\Phi_\varepsilon - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\quad + \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi)) - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))}. \end{aligned}$$

Hence it is sufficient to prove that

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi)) \rightarrow \Phi \text{ strongly in } L^q(0,T;L^p(\Omega \times Y^*)). \quad (2.6)$$

To do that, we first prove that, for any $\psi \in \mathcal{D}(\Omega \times Y^*)$ and $\varphi \in \mathcal{D}(0,T)$,

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi\varphi)) \rightarrow \psi\varphi \text{ strongly in } L^q(0,T;L^p(\Omega \times Y^*)). \quad (2.7)$$

In fact, we deduce from (2.1)(ii) and Definition 2.3 that

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi\varphi)) = \varphi \mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi)).$$

This, together with the fact (see [9]) that

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi)) \longrightarrow \psi \text{ strongly in } L^p(\Omega \times Y^*),$$

give (2.7).

By virtue of Proposition 2.1(ii) and Proposition 2.6,

$$\begin{aligned} \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi)) - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} &\leq \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi)) - \mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi\varphi))\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\quad + \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi\varphi)) - \psi\varphi\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\quad + \|\psi\varphi - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\leq 2 \|\psi\varphi - \Phi\|_{L^q(0,T;L^p(\Omega \times Y^*))} \\ &\quad + \|\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\psi\varphi)) - \psi\varphi\|_{L^q(0,T;L^p(\Omega \times Y^*))}. \end{aligned}$$

Hence (2.6) follows from (2.7), due to the density of $\mathcal{D}(0, T) \otimes \mathcal{D}(\Omega \times Y^*)$

in $L^q(0, T; L^p(\Omega \times Y^*))$.

(iii) From (2.5) and Proposition 2.6, we have

$$\begin{aligned} \|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} &\leq \|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} + \|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \\ &= \|\mathcal{U}_\varepsilon^*(\mathcal{T}_\varepsilon^*(\omega_\varepsilon)) - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} + \|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \\ &\leq C \|\mathcal{T}_\varepsilon^*(\omega_\varepsilon) - \omega\|_{L^q(0,T;L^p(\Omega \times Y^*))} + \|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))}, \end{aligned}$$

which gives (a) \implies (b).

Now we prove the converse (b) \implies (a). In fact, (b) implies that

$$\|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} \longrightarrow 0 \text{ and } \|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_\varepsilon^*))} \longrightarrow 0.$$

Moreover, the first convergence, due to Proposition 2.1(ii), gives

$$\|\mathcal{T}_\varepsilon(\omega_\varepsilon) - \mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\omega))\|_{L^q(0,T;L^p(\Omega \times Y^*))} \longrightarrow 0.$$

Thanks to statement **(ii)**, we get the first convergence in **(a)**.

(iv) Making use of (2.5) and Proposition 2.6, we have

$$\begin{aligned} \|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} &\leq \|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} + \|\mathcal{U}_\varepsilon^*(\omega) - \omega\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} \\ &\leq C \|\mathcal{T}_\varepsilon^*(\omega_\varepsilon) - \omega\|_{L^q(0,T;L^p(\Omega \times Y^*))} + \|\mathcal{U}_\varepsilon^*(\omega) - \omega\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))}. \end{aligned}$$

Hence **(b)** follows from statement **(i)** and **(a)**.

Now we prove the converse **(b)** \implies **(a)**. In fact, **(b)** and Proposition 2.1 **(ii)** give

$$\|\mathcal{T}_\varepsilon^*(\omega_\varepsilon) - \mathcal{T}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\Omega \times Y^*))} \longrightarrow 0.$$

This together with Proposition 2.4 **(i)**, implies **(a)**.

Owing to

$$\left| \|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} - \|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \right| \leq \|\mathcal{U}_\varepsilon^*(\omega) - \omega\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))}$$

and statement **(i)**, we obtain that

$$\|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \longrightarrow 0 \Leftrightarrow \|\omega_\varepsilon - \mathcal{U}_\varepsilon^*(\omega)\|_{L^q(0,T;L^p(\tilde{\Omega}_\varepsilon^*))} \longrightarrow 0$$

Combining this with statement **(iii)**, we get the last statement.

We end this section with a convergence result related to the averaging operator $\mathcal{U}_\varepsilon^*$, which will be used to get the corrector result in the the last section. The proof of this convergence result is a direct consequence of proposition 1.7 in [9]

Proposition 2.8 For $p, q \in [1, \infty)$, let $f \in L^q(0, T; L^p(\Omega))$ and $g \in L^\infty(\Omega; L^p(Y^*))$.

Then we have

$$\|\mathcal{U}_\varepsilon^*(fg) - \mathcal{U}_\varepsilon(f)\mathcal{U}_\varepsilon^*(g)\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \longrightarrow 0. \quad (2.8)$$

Proof

From the proof of Proposition 1.7 in [9], we have

$$\|\mathcal{U}_\varepsilon^*(fg) - \mathcal{U}_\varepsilon(f)\mathcal{U}_\varepsilon^*(g)\|_{L^p(\Omega_\varepsilon^*)}^p \leq \frac{1}{|Y|} \|f - \mathcal{U}_\varepsilon(f)\|_{L^p(\Omega_\varepsilon^*)}^p \|g\|_{L^\infty(\Omega, L^p(Y^*))}^p,$$

which gives

$$\| \mathcal{U}_\varepsilon^*(fg) - \mathcal{U}_\varepsilon(f)\mathcal{U}_\varepsilon^*(g) \|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq \frac{1}{|\mathbf{Y}|} \| f - \mathcal{U}_\varepsilon(f) \|_{L^q(0,T;L^p(\Omega))} \| g \|_{L^\infty(\Omega, L^p(Y^*))}.$$

Combining this with the convergence

$$\| f - \mathcal{U}_\varepsilon(f) \|_{L^q(0,T;L^p(\Omega))} \longrightarrow 0$$

given by proposition 1.6 of [9], we get (2.8).

2.3 The time-dependent macro-micro decomposition and some convergence results

To treat time-dependent functions defined in the perforated domains, we adapt the scale-splitting operators $\mathcal{Q}_\varepsilon^*$ and $\mathcal{R}_\varepsilon^*$ for the perforated domains, originally introduced in [5] and [9]. Some estimates are directly listed since the proofs are the same as those in [9].

moreover, we give some specific convergence results to the time-dependent case, which will play a crucial role in the study of the homogenization result.

To do that, we first recall some notations in [9]. Let

$$\begin{aligned} \mathcal{K} &= \text{the set of vertices of } Y, \\ \mathcal{Y} &= \text{interior} \left\{ \bigcup_{l \in \mathcal{K}} (l + \bar{Y}) \right\}, \Xi_\varepsilon^\mathcal{Y} = \xi \in \Xi_\varepsilon \mid \varepsilon(\xi + \bar{\mathcal{Y}}) \subset \Omega, \\ \hat{\Omega}_\varepsilon^\mathcal{Y} &= \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon^\mathcal{Y}} \varepsilon(\Xi + \bar{Y}) \right\}, \Lambda_\varepsilon^\mathcal{Y} = \Omega \setminus \hat{\Omega}_\varepsilon^\mathcal{Y}. \end{aligned}$$

definition 2.4 For $p, q \in [1, +\infty]$, let ϕ be in $L^q(0, T; L^p(\Omega_\varepsilon^*))$.

(i) The operator $\mathcal{Q}_\varepsilon^* : L^q(0, T; L^p(\Omega_\varepsilon^*)) \mapsto L^q(0, T; W^{1,\infty}(\hat{\Omega}_\varepsilon^\mathcal{Y}))$ is defined as follows:

$$\mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi, t) = \frac{1}{|\mathbf{Y}^*|} \int_{Y^*} \phi(\varepsilon\xi + \varepsilon z, t) dz = \mathcal{M}_{\varepsilon\xi + \varepsilon Y^*}(\phi), \forall \xi \in \Xi_\varepsilon^\mathcal{Y} + \mathcal{K}, \forall t \in (0, T),$$

and for a.e. $t \in (0, T)$, $\mathcal{Q}_\varepsilon^*(\phi)(x, t)$ is the \mathcal{Q}_1 -interpolate of the values of $\mathcal{Q}_\varepsilon^*(\phi)(x, t)$ at

the vertices of the celle $\varepsilon \begin{bmatrix} \mathbf{x} \\ - \\ \varepsilon \end{bmatrix}_Y + \varepsilon \mathbf{Y}$ for every $\mathbf{x} \in \widehat{\Omega}_\varepsilon^{\mathcal{Y}}$.

(ii) Let $\widehat{\Omega}_\varepsilon^{**} = \widehat{\Omega}_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^{\mathcal{Y}}$ Define $\mathcal{R}_\varepsilon^*(\phi)$ by

$$\mathcal{R}_\varepsilon^*(\phi) = \phi - \mathcal{Q}_\varepsilon^*(\phi) \text{ a.e. in } \widehat{\Omega}_\varepsilon^{**} \times (0, T).$$

Proposition 2.9 Let $q \in [1, +\infty]$.

(i) For $p \in [1, +\infty]$, let $\phi \in L^q(0, T; L^p(\Omega_\varepsilon^*))$. Then

$$\begin{cases} \|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^q(0, T; L^\infty(\widehat{\Omega}_\varepsilon^{\mathcal{Y}}))} \leq C \varepsilon^{-\frac{n}{p}} \|\phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}, \\ \|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^q(0, T; L^p(\widehat{\Omega}_\varepsilon^{\mathcal{Y}}))} \leq C \|\phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}, \end{cases}$$

where C only depend on n, \mathbf{Y} and \mathbf{Y}^* .

(ii) Let $p \in [1, +\infty)$. Then there exists a constant C independent of ε and q such that for any $\phi \in L^q(0, T; W^{1,p}(\Omega_\varepsilon^*))$,

$$\begin{cases} \|\nabla \mathcal{Q}_\varepsilon^*(\phi)\|_{L^q(0, T; L^p(\widehat{\Omega}_\varepsilon^{\mathcal{Y}}))} \leq C \|\nabla \phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}, \\ \|\mathcal{R}_\varepsilon^*(\phi)\|_{L^q(0, T; L^p(\widehat{\Omega}_\varepsilon^{**}))} \leq \varepsilon C \|\nabla \phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}, \\ \|\nabla \mathcal{R}_\varepsilon^*(\phi)\|_{L^q(0, T; L^p(\widehat{\Omega}_\varepsilon^{**}))} \leq C \|\nabla \phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}. \end{cases}$$

(iii) Let $p \in [1, +\infty)$. Then there exists a constant C independent of ε such that

$$\|\phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C \|\nabla \phi\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))}, \quad \forall \phi \in L^q(0, T; W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega)).$$

Now we focus on the action of the unfolding operator on some sequences in $L^q(0, T; W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega))$, where $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega)$ is defined by

$$W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega) = \{\phi \text{ in } W_0^{1,p}(\Omega_\varepsilon^*) \mid \phi = 0 \text{ on } \partial\Omega\}.$$

To do that, we first prove the following convergences.

Lemma 2.1 For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let $\{\mathbf{w}_\varepsilon\}$ be a sequence in $L^q(0, T; W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega))$.

(i) If

$$\| \mathbf{w}_\varepsilon \|_{L^q(0,T;W^{1,p}(\Omega_\varepsilon^*))} \leq C,$$

then there exists $\mathbf{w} \in L^q(0, T; W_0^{1,p}(\Omega_\varepsilon^*))$ such that, up to subsequence,

$$\begin{aligned} (i) \quad & \mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon)) \rightharpoonup \mathbf{w} \text{ weakly in } L^q(0, T; L^p(\Omega; W^{1,p}(Y))), \\ (ii) \quad & \mathcal{T}_\varepsilon([\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)]) \rightharpoonup \nabla \mathbf{w} \text{ weakly in } L^q(0, T; L^p(\Omega \times Y)). \end{aligned} \quad (2.9)$$

For $q = \infty$, the weak convergence above are replaced by the weak* convergences, respectively.

(ii) If q belongs to $(1, +\infty)$ and

$$\| \mathbf{w}_\varepsilon \|_{L^q(0,T;W^{1,p}(\Omega_\varepsilon^*))} \leq C, \quad \left\| \frac{\partial \mathbf{w}_\varepsilon}{\partial t} \right\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \leq C,$$

then (2.9) holds and

$$\mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon)) \longrightarrow \mathbf{w} \text{ strongly in } L^q(0, T; L_{loc}^p(\Omega; W^{1,p}(Y))). \quad (2.10)$$

Proof

(i) From Proposition 2.9, we get that $\{\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon)\}$ and $\{[\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)]\}$ are bounded in $L^q(0, T; L^p(\Omega))$. Then there exist $\mathbf{w} \in L^q(0, T; L^p(\Omega))$ and $\mathbf{g} \in L^q(0, T; L^p(\Omega))$ such that, up to a subsequence,

$$\begin{aligned} (i) \quad & \widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon) \rightharpoonup \mathbf{w} \text{ weakly in } L^q(0, T; L^p(\Omega)) \text{ (weakly* for } q = \infty) \\ (ii) \quad & [\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)] \rightharpoonup \mathbf{g} \text{ weakly in } L^q(0, T; L^p(\Omega)) \text{ (weakly* for } q = \infty) \end{aligned} \quad (2.11)$$

Let $\bar{\omega}$ be a relatively compact open subset in Ω and let Φ be in $(\mathcal{D}(\Omega) \times \mathcal{D}(0, T))^n$.

For ε sufficiently small, we have $\bar{\omega} \subset \widetilde{\Omega}_\varepsilon^*$. Hence,

$$\begin{aligned} \int_0^T \int_{\Omega} [\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)] \cdot \Phi \, dx \, dt &= \int_0^T \int_{\bar{\omega}} [\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)] \cdot \Phi \, dx \, dt, \\ &= - \int_0^T \int_{\Omega} \widetilde{\Omega}_\varepsilon^*(\mathbf{w}_\varepsilon) \operatorname{div}_x \Phi \, dx \, dt. \end{aligned}$$

Passing to the limit and getting that

$$\int_0^T \int_{\Omega} \mathbf{g} \cdot \Phi \, dx \, dt = - \int_0^T \int_{\Omega} \mathbf{w} \operatorname{div}_x \Phi \, dx \, dt,$$

which implies $\mathbf{g} = \nabla \mathbf{w}$. This shows that \mathbf{w} belongs to $L^q(\mathbf{0}, T; \mathbf{W}^{1,p}(\Omega))$.

Notice that $\nabla_{\mathbf{y}}[\mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon))]$ is of order ε . Since \mathbf{w} does not depend on \mathbf{y} , then convergence (2.11)(i) implies convergence (2.9)(i).

To prove (2.11) (ii), we consider the sequence $\mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon) |_{\bar{\omega}}$. Since it is bounded in $L^q(\mathbf{0}, T; \mathbf{W}^{1,p}(\bar{\omega}))$, there exists $\widehat{\mathbf{w}}_{\bar{\omega}} \in L^q(\mathbf{0}, T; L^p(\bar{\Omega}; \mathbf{W}_{per}^{1,p}(\mathbf{Y})))$ such that, up to a subsequence,

$$\mathcal{T}_{\varepsilon, \bar{\omega}}(\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon) |_{\bar{\omega}}) \rightharpoonup \nabla \mathbf{w} + \nabla_{\mathbf{y}} \widehat{\mathbf{w}}_{\bar{\omega}} \text{ weakly in } L^q(\mathbf{0}, T; L^p(\bar{\Omega} \times \mathbf{Y})) \text{ (weakly* for } q = \infty).$$

where $\mathcal{T}_{\varepsilon, \bar{\omega}}$ is the unfolding operator associated with $\bar{\omega} \times (\mathbf{0}, T)$.

For a.e. $\mathbf{x} \in \bar{\omega}$ and $t \in (\mathbf{0}, T)$, the definition of $\mathcal{Q}_\varepsilon^*$ implies that $\widehat{\mathbf{w}}_{\bar{\omega}}(\mathbf{x}, \mathbf{y}, t)$ is a polynomial with degree less or equal to one with respect to $\mathbf{y}_1, \dots, \mathbf{y}_n$. Notice also that $\widehat{\mathbf{w}}_{\bar{\omega}}$ is \mathbf{Y} -periodic, we get that $\widehat{\mathbf{w}}_{\bar{\omega}}$ does not depend on \mathbf{y} . Together with the boundedness of $\mathcal{T}_\varepsilon([\nabla \mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)])$ $L^q(\mathbf{0}, T; L^p(\Omega \times \mathbf{Y}))$, convergence (2.9)(ii) follows from convergence (2.11)(ii).

It remains to prove that \mathbf{w} vanishes on $\partial\Omega \times (\mathbf{0}, T)$. To do that, we consider an open set Ω' containing Ω . \mathbf{w}_ε . Observe that the regularity of $\partial\Omega$ implies that \mathbf{w}_ε belongs to $L^q(\mathbf{0}, T; \mathbf{W}^{1,p}((\Omega')_\varepsilon^*))$. Consequently, arguing as we did for getting (2.11), we obtain that there exists $\mathbf{w}_0 \in L^q(\mathbf{0}, T; \mathbf{W}^{1,p}(\Omega'))$ such that, up to a subsequence,

$$\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon) \rightharpoonup \mathbf{w}_0 \text{ weakly in } L^q(\mathbf{0}, T; \mathbf{W}^{1,p}(\Omega')) \text{ (weakly* for } q = \infty).$$

Moreover, we get $\mathbf{w}_0 |_{\Omega \times (\mathbf{0}, T)} = \mathbf{w}$ and $\mathbf{w}_0 |_{\Omega' \setminus \bar{\Omega} \times (\mathbf{0}, T)} = \mathbf{0}$ which imply that \mathbf{w} vanishes on $\partial\Omega \times (\mathbf{0}, T)$.

(ii) From proposition 2.9(i), we have

$$\left\| \frac{\partial}{\partial t} [\mathcal{Q}_\varepsilon^*(\mathbf{w}_\varepsilon)] \right\|_{L^q(\mathbf{0}, T; L^p(\widehat{\Omega}_\varepsilon^{\mathbf{y}}))} \leq C.$$

Then for ε small enough, we get

$$\left\| \widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon) \right\|_{L^q(\mathbf{0}, T; \mathbf{W}^{1,p}(\bar{\omega}))} \leq C \text{ and } \left\| \frac{\partial}{\partial t} [\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon)] \right\|_{L^q(\mathbf{0}, T; L^p(\bar{\omega}))} \leq C. \quad (2.12)$$

By the classical compactness result (see [13] and the references therein), $\widetilde{\mathcal{Q}}_\varepsilon^*(\mathbf{w}_\varepsilon)$ is relatively

compact in $L^q(0, T; W^{1,p}(\bar{\omega}))$ and $C^0(0, T; L^p(\bar{\omega}))$. Combining this with (2.11), we have

$$\begin{cases} \widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon) \longrightarrow w \text{ strongly in } L^q(0, T; W_{loc}^{1,p}(\Omega)), \\ \widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon) \longrightarrow w \text{ strongly in } C^0(0, T; L_{loc}^p(\Omega)). \end{cases} \quad (2.13)$$

Observe that $\mathcal{T}_{\varepsilon, \bar{\omega}}(\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon))$ is bounded in $L^q(0, T; L^p(\bar{\omega}, W^{1,p}(Y)))$ and $\nabla_y[\mathcal{T}_{\varepsilon, \bar{\omega}}(\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon))]$ is of order ε . Hence convergence (2.10) follows from (2.13).

For the convergences related to the sequence $\{\mathcal{R}_\varepsilon^*(w_\varepsilon)\}$, using Proposition 2.5 and Proposition 2.9(ii), we follow the proof of Lemma 2.1 in [9] to get the following result.

Lemma 2.2 *For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let $\{w_\varepsilon\}$ be a sequence in $L^q(0, T; W^{1,p}(\Omega_\varepsilon^*))$ such that*

$$\|\nabla w_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C.$$

Then there exists $\widehat{w} \in L^q(0, T; L^p(\Omega, W_{per}^{1,p}(Y^)))$ such that, up to subsequence,*

$$\begin{aligned} (i) \quad & \varepsilon^{-1} \mathcal{T}_\varepsilon^*(\widetilde{\mathcal{R}}_\varepsilon^*(w_\varepsilon)) \rightharpoonup \widehat{w} \text{ weakly in } L^q(0, T; L^p(\Omega, W^{1,p}(Y^*))), \\ (ii) \quad & \mathcal{T}_\varepsilon^*([\nabla \mathcal{R}_\varepsilon^*(w_\varepsilon)]^\top) \rightharpoonup \nabla_y \widehat{w} \text{ weakly in } L^q(0, T; L^p(\Omega \times Y^*)), \\ (iii) \quad & \mathcal{T}_\varepsilon^*(\widetilde{\mathcal{R}}_\varepsilon^*(w_\varepsilon)) \longrightarrow 0 \text{ strongly in } L^q(0, T; L^p(\Omega, W^{1,p}(Y^*))). \end{aligned} \quad (2.14)$$

For $q = \infty$, the weak convergences in (2.14) are replaced by the weak convergences respectively.*

Remark 2.3 *For $p \in (1, +\infty)$ and $q \in (1, +\infty]$, let $\{\phi_\varepsilon\}$ be a sequence in $L^q(0, T; W^{1,p}(\Omega_\varepsilon^*; \partial\Omega_\varepsilon^*))$ such that*

$$\|\nabla \phi_\varepsilon\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \leq C.$$

If there exists $\phi \in L^q(0, T; W_0^{1,p}(\Omega))$ such that

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightharpoonup \text{ weakly in } L^q(0, T; L^p(\Omega \times Y^*)),$$

then by Remark 2.1, Lemma 2.1(i) and Lemma 2.2(iii), we have

$$\mathcal{T}_\varepsilon^*([\nabla \mathcal{Q}_\varepsilon^*(\phi_\varepsilon)]^\top |_{\Omega_\varepsilon^*}) \rightharpoonup \nabla \phi \text{ weakly in } L^q(0, T; L^p(\Omega \times Y^*)).$$

For $q = \infty$, the weak convergences above are replaced by the weak* convergences, respectively. Now we are in a position to state the following convergence theorem which is crucial to obtaining our homogenization result.

Theoreme 2.1 For $p \in (1, \infty)$, let $\{w_\varepsilon\}$ be a sequence in $L^\infty(0, T; W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega))$ such that

$$\|\nabla w_\varepsilon\|_{L^\infty(0, T; L^p(\Omega_\varepsilon^*))} \leq C \quad \text{and} \quad \left\| \frac{\partial w_\varepsilon}{\partial t} \right\|_{L^\infty(0, T; L^p(\Omega_\varepsilon^*))} \leq C. \quad (2.15)$$

Then there exist $w \in L^\infty(0, T; W_0^{1,p}(\Omega))$ with $\frac{\partial w}{\partial t} \in L^\infty(0, T; L^p(\Omega))$ and $\widehat{w} \in L^\infty(0, T; L^p(\Omega; W_{per}^{1,p}(Y^*)))$ with $M_{y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,

- (i) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p}(Y^*)))$,
- (ii) $\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w}$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p} \times Y^*))$,
- (iii) $\mathcal{T}_\varepsilon^*\left(\frac{\partial w_\varepsilon}{\partial t}\right) \rightharpoonup \frac{\partial w}{\partial t}$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p} \times Y^*))$,
- (iv) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w$ strongly in $L^q(0, T; L^p(\Omega; W^{1,p}(Y^*)))$,
- (v) $\|w_\varepsilon - w\|_{L^q(0, T; L^p(\Omega_\varepsilon^*))} \rightarrow 0$,

where q is any number in $(1, +\infty)$.

Proof

By Proposition 2.9 and Lemmas 2.1-2.2, we argue as that in the proof of there exist $w \in L^\infty(0, T; W_0^{1,p}(\Omega))$ and $\widehat{w} \in L^\infty(0, T; L^p(\Omega; W_{per}^{1,p}(Y^*)))$ with $M_{y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,

- (i) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p}(Y^*)))$,
- (ii) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w$ strongly* in $L^q(0, T; L_{loc}^p(\Omega; W^{1,p}(Y^*)))$,
- (iii) $\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w}$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p} \times Y^*))$.

This gives convergences (2.16)(i)(ii). Moreover, convergence (2.16)(iii) holds true due to (2.15).

Let Ω_1 and Ω_2 be open sets of \mathbb{R}^n such that

$$\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$$

to get the other convergences in (2.16), we extend w_ε by zero to $(\Omega_2)_\varepsilon^* \times (0, T)$ and this extension is denoted by w_ε . Arguing as we did for getting (2.17), we get that there exists $w_1 \in L^q(0, T; W^{1,p}(\Omega_2))$ such that, up to subsequence,

$$\mathcal{T}_{\varepsilon, \Omega_2}^*(w_\varepsilon) \longrightarrow w_1 \text{ strongly in } L^q(0, T; L_{loc}^p(\Omega_2; W^{1,p}(Y^*))), \quad (2.18)$$

where $\mathcal{T}_{\varepsilon, \Omega_2}^*$ is the unfolding operator associated with $(\Omega_2)_\varepsilon^* \times (0, T)$. By the fact that $\mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_{\varepsilon, \Omega_2}^*(w_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon}) |_{\Omega \times Y^* \times (0, T)}$, convergence (2.17)(i) implies $w = w_1 |_{\Omega \times (0, T)}$. Hence convergence (2.16)(iv) follows from (2.18).

Making use of Proposition 2.7(iv) and (2.18), we obtain

$$\| w_\varepsilon - w_1 \|_{L^q(0, T; L^p(\widehat{\Omega}_1)_\varepsilon^*)} \longrightarrow 0.$$

For ε small enough, notice that Ω_ε^* is contained in $(\widehat{\Omega}_1)_\varepsilon^*$, we get convergence (2.16) (v).

Remark 2.4 *If the functions considered in this chapter are independent of time parameter t , all results coincide with those in [9], respectively.*

2.4 Application of hyperbolic problems

In this section, we use the adapted unfolding method presented in Chapter 2 to study the asymptotic behavior of a class of hyperbolic problems in perforated domains.

To introduce the coefficient matrix, we define, for $\alpha, \beta \in \mathcal{R}$ with $0 \leq \alpha \leq \beta$, the set $M(\alpha, \beta, \mathcal{O})$ of the $n \times n$ matrix-valued functions in $L^\infty(\mathcal{O})$ such that

$$(A(x)\lambda, \lambda) \geq \alpha |\lambda|^2, \quad |A(x)\lambda| \leq \beta |\lambda|$$

for any $\lambda \in \mathbb{R}^n$ and a.e. an \mathcal{O} . For any ε , we suppose that

$$\begin{cases} A^\varepsilon \in M(\alpha, \beta, \Omega), \\ A^\varepsilon \text{ symmetric.} \end{cases} \quad (2.19)$$

Consider the following hyperbolic problem with homogeneous Dirichlet-Neumann boundary

$$\begin{cases} \mathbf{u}_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla \mathbf{u}_\varepsilon) = \mathbf{f}_\varepsilon & \text{in } \Omega_\varepsilon^* \times (0, T), \\ \mathbf{u}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega \times (0, T), \\ A^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon = 0 & \text{on } \partial S_\varepsilon \times (0, T), \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \mathbf{u}_\varepsilon^0, \mathbf{u}_\varepsilon'(\mathbf{x}, 0) = \mathbf{u}_\varepsilon^1 & \text{in } \Omega_\varepsilon^*, \end{cases} \quad (2.20)$$

where \mathbf{n}_ε is the outward unit normal vector field defined on ∂S_ε .

We suppose that

$$\begin{cases} \mathbf{u}_\varepsilon^0 \in V^\varepsilon, \\ \mathbf{u}_\varepsilon^1 \in L^2(\Omega_\varepsilon^*), \\ \mathbf{f}_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon^*)). \end{cases} \quad (2.21)$$

Set

$$\mathcal{W}_\varepsilon = \left\{ \mathbf{v}_\varepsilon \mid \mathbf{v}_\varepsilon \in L^2(0, T; V^\varepsilon), \mathbf{v}_\varepsilon' \in L^2(0, T; L^2(\Omega_\varepsilon^*)) \right\} \quad (2.22)$$

with

$$V^\varepsilon = \left\{ \mathbf{v} \in H^1(\Omega_\varepsilon^*) \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \right\}$$

with the norm defined by

$$\| \mathbf{v}_\varepsilon \|_{\mathcal{W}_\varepsilon} = \| \mathbf{v}_\varepsilon \|_{L^2(0, T; V^\varepsilon)} + \| \mathbf{v}_\varepsilon' \|_{L^2(0, T; L^2(\Omega_\varepsilon^*))}.$$

2.4.1 Variational Formulation

We multiply (2.20) by a test function \mathbf{v} where \mathbf{v} in $\mathcal{D}(\Omega)$, we get

$$\mathbf{u}_\varepsilon'' \mathbf{v} - \operatorname{div}(A^\varepsilon \nabla \mathbf{u}_\varepsilon) \mathbf{v} = \mathbf{f}_\varepsilon \mathbf{v},$$

we integrate on $\Omega_\varepsilon^* \times (0, T)$ we get

$$\int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v} dx - \int_{\Omega_\varepsilon^*} \operatorname{div}(A^\varepsilon \nabla \mathbf{u}_\varepsilon) \mathbf{v} dx = \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v} dx, \quad (2.23)$$

we apply the Green's formula on the second integral in above equation we get

$$-\int_{\Omega_\varepsilon^*} \Delta u_\varepsilon v dx = \int_{\Omega_\varepsilon^*} \nabla u_\varepsilon \nabla v dx - \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial \eta} v d\Gamma,$$

substituting this formula into (2.23) we get

$$\langle u_\varepsilon'', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \int_{\Omega_\varepsilon^*} \nabla u_\varepsilon \nabla v dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v dx.$$

The variational formulation of problem (2.20) is to find $u_\varepsilon \in \mathcal{W}_\varepsilon$ such that

$$\begin{cases} \langle u_\varepsilon'', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \cdot \nabla v dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v dx \\ \text{in } \mathcal{D}'(0, T) \text{ for all } v \in V^\varepsilon \\ u_\varepsilon(x, 0) = u_\varepsilon^0, u_\varepsilon'(x, 0) = u_\varepsilon^1 \text{ in } \Omega_\varepsilon^*. \end{cases} \quad (2.24)$$

For every fixed ε , classical results provide that problem (2.24) has a unique solution u_ε such that

$$u_\varepsilon \in \mathbb{C}^0([0, T]; V^\varepsilon) \cap \mathbb{C}^1([0, T]; L^2(\Omega_\varepsilon^*)).$$

In order to study the homogenization of problem (2.20), we suppose that there exists a matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ such that

$$\mathcal{T}_\varepsilon^*(A^\varepsilon) \longrightarrow A \text{ strongly in } (L^1(\Omega \times Y^*))^{n \times n}, \quad (2.25)$$

which implies $A \in M(\alpha, \beta, \Omega \times Y^*)$ (see also [9] and [7]).

Concerning the initial data, we assume that

$$\begin{cases} \|u_\varepsilon^0\|_{V^\varepsilon} \leq C, \\ \widetilde{u}_\varepsilon^0 \rightharpoonup \theta u^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{u}_\varepsilon^1 \rightharpoonup \theta u^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{f}_\varepsilon \rightharpoonup \theta f \text{ weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \quad (2.26)$$

where C is a constant independent of ε . Under these assumptions, classical results show that problem (2.24) has a unique solution \mathbf{u}_ε with the following uniform estimate:

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;H^1(\Omega_\varepsilon^*))} + \|\mathbf{u}'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon^*))} \leq C, \quad (2.27)$$

where the constant C does not depend on ε .

Now we state the main theorem of this section.

Theoreme 2.2 *Let \mathbf{A}^ε satisfy (2.19) and (2.24). Suppose that \mathbf{u}_ε is the solution of problem (2.20) with (2.21) and (2.25). Then there exist $\mathbf{u} \in L^\infty(0,T;L^2(\omega, H^1_{per}(Y^*)))$ with $\mathcal{M}_{Y^*}(\hat{\mathbf{u}} = \mathbf{0})$, such that*

$$\left\{ \begin{array}{l} \mathcal{T}_\varepsilon^*(\mathbf{u}_\varepsilon) \rightharpoonup \mathbf{u} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega, H^1(Y^*))), \\ \mathcal{T}_\varepsilon^*(\mathbf{u}'_\varepsilon) \rightharpoonup \mathbf{u}' \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega \times Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla \mathbf{u}_\varepsilon) \rightharpoonup \nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}} \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega \times Y^*)), \\ \mathcal{T}_\varepsilon^*(\mathbf{u}_\varepsilon) \longrightarrow \mathbf{u} \text{ strongly}^* \text{ in } L^\infty(0,T;L^2(\Omega, H^1(Y^*))), \\ \|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^q(0,T;L^2(\Omega_\varepsilon^*))} \longrightarrow 0, \end{array} \right. \quad (2.28)$$

where q is any number in $(1, +\infty)$. The pair $(\mathbf{u}, \hat{\mathbf{u}})$ with $\mathcal{M}_{Y^*}(\hat{\mathbf{u}}) = \mathbf{0}$ is the unique solution of the following problem :

$$\left\{ \begin{array}{l} \theta \int_0^T \int_\Omega \mathbf{u} \Psi \varphi'' dx dt + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}})(\nabla \Psi + \nabla_y \Phi) \varphi dx dy dt \\ = \theta \int_0^T \int_\Omega f \Psi \varphi dx dy dt, \\ \text{for any } \Psi \in H_0^1, \Phi \in L^2(\Omega, H^1(Y^*)) \text{ and } \varphi \in \mathcal{D}(0,T), \mathbf{u} = \mathbf{0} \text{ on } \Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}^0, \mathbf{u}'(x,0) = \mathbf{u}^1 \text{ in } \Omega. \end{array} \right. \quad (2.29)$$

We also have

$$\hat{\mathbf{u}} = \sum_{j=1}^n \frac{\partial \mathbf{u}}{\partial x_j} \chi_j, \quad (2.30)$$

with $\mathcal{X}_j \in L^\infty(\Omega; H_{per}^1(Y^*))$ ($j = 1, \dots, n$) being the solution of the cell problem

$$\begin{cases} -\operatorname{div}_y(A\nabla_y(\mathcal{X}_j + y_j)) = 0 & \text{in } Y^*, \\ A\nabla_y(\mathcal{X}_j + y_j) \cdot n_1 = 0 & \text{on } \partial S, \\ \mathcal{M}_{Y^*}(\mathcal{X}_j)(x, \cdot), \quad \mathcal{X}_j(w, \cdot) & Y\text{-periodic.} \end{cases} \quad (2.31)$$

Moreover,

$$\begin{aligned} (i) \quad & \tilde{u}_\varepsilon \rightharpoonup \theta u \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \\ (ii) \quad & A^\varepsilon \widetilde{\nabla u}_\varepsilon \rightharpoonup \theta A^0 \nabla u \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \end{aligned} \quad (2.32)$$

and u is the unique solution of the following homogenized wave equation

$$\begin{cases} u'' - \operatorname{div}(A^0 \nabla u) = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u^0, u'(x, 0) = u^1 & \text{in } \Omega, \end{cases} \quad (2.33)$$

where the homogenized matrix $A^0 = (a_{ij}^0)_{1 \leq i, j \leq n}$ is defined by

$$a_{ij}^0(x) = \mathcal{M}_{Y^*} \left(a_{ij} + \sum_{k=1}^n a_{ik} \frac{\partial \mathcal{X}_j}{\partial y_k} \right). \quad (2.34)$$

Remark 2.5 Observe that the homogenized matrix A^0 here, as defined in [9], differs from the classical one (see for instance [1]) by a factor of θ^{-1} . We also notice that here A^0 depends upon x while the one in [1] does not.

Proof of Theorem 2.2

In view of (2.27) and Theorem 2.1, we get that there exist $u \in L^\infty(0, T; H_0^1(\Omega))$ with $u' \in L^\infty(0, T; L^2(\Omega))$ and $\hat{u} \in L^\infty(0, T; L^2(\Omega, H_{per}^1(Y^*)))$ with $\mathcal{M}_{Y^*}(\hat{u}) = 0$, such that, up to subsequence (still denoted by ε), (2.28) holds. From Proposition 2.4(iii), we further get that

$$\begin{aligned} \hat{u} & \rightharpoonup \theta \mathcal{M}_{Y^*}(u) \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ A^\varepsilon \widetilde{\nabla u}_\varepsilon & \rightharpoonup \theta \mathcal{M}_{Y^*}[A(\nabla u + \nabla_y \hat{u})] \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (2.35)$$

Since \mathbf{u} is independent of \mathbf{y} , then convergence (2.32) (i) holds true for the above subsequence.

Let $\Psi, \Phi \in \mathcal{D}(\Omega)$ and $\psi \in H_{per}^1(Y^*)$. Set

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \Psi(\mathbf{x}) + \varepsilon\phi(\mathbf{x})\psi^\varepsilon(\mathbf{x}) \quad \text{with} \quad \psi^\varepsilon(\mathbf{x}) = \psi\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (2.36)$$

Then

$$\nabla \mathbf{v}_\varepsilon = \nabla \Psi + \varepsilon\psi^\varepsilon \nabla \phi + \phi(\nabla_{\mathbf{y}} \psi) \left(\frac{\cdot}{\varepsilon}\right). \quad (2.37)$$

By Proposition 2.4 (ii), we have

$$\begin{aligned} \mathcal{T}_\varepsilon^*(\mathbf{v}_\varepsilon) &\longrightarrow \Psi \quad \text{strongly in } L^2(\Omega \times Y), \\ \mathcal{T}_\varepsilon^*(\phi\psi^\varepsilon) &\longrightarrow \Phi \quad \text{strongly in } L^2(\Omega \times Y) \quad \text{with} \quad \Phi = \phi(\mathbf{x})\psi(\mathbf{y}), \\ \mathcal{T}_\varepsilon^*(\nabla \mathbf{v}_\varepsilon) &\longrightarrow \nabla \Psi + \nabla_{\mathbf{y}} \Phi \quad \text{strongly in } L^2(\Omega \times Y). \end{aligned} \quad (2.38)$$

Let $\varphi(\mathbf{t}) \in \mathcal{D}(0, T)$, and $\mathbf{v}_\varepsilon \varphi$ a test function in the problem (2.24)

$$\int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla (\mathbf{v}_\varepsilon \varphi) \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt, \quad (2.39)$$

then

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \left(\nabla \Psi + \varepsilon\psi^\varepsilon \nabla \phi + \phi \nabla_{\mathbf{y}} \psi^\varepsilon \left(\frac{\cdot}{\varepsilon}\right) \varphi \right) \, d\mathbf{x} \, dt \\ = \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt, \end{aligned}$$

then

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon (\nabla \Psi + \phi + \phi \nabla_{\mathbf{y}} \psi^\varepsilon \varphi) \, d\mathbf{x} \, dt \\ + \varepsilon \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \phi \psi^\varepsilon \left(\frac{\mathbf{x}}{\varepsilon}\right) \varphi \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt \end{aligned} \quad (2.40)$$

We pass at the limit in Problem (2.40) term by term.

We start by the first term

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt = \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon \mathbf{v}_\varepsilon \varphi'' \, d\mathbf{x} \, dt.$$

By the unfolding operator $\mathcal{T}_\varepsilon^*$ and we use the Definition 2.2(i)(ii) we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(u_\varepsilon) \mathcal{T}_\varepsilon^*(v_\varepsilon) \mathcal{T}_\varepsilon^*(\varphi'') \, dx \, dt,$$

On the other hand for $\varepsilon \rightarrow 0$ we have $v_\varepsilon \rightarrow \Psi$, now by Proposition 2.4 we have the following convergence

$$\mathcal{T}_\varepsilon^*(v_\varepsilon) \rightarrow \Psi. \quad (2.41)$$

By this convergence and the convergence (2.28) we get

$$\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon v_\varepsilon \varphi'' \, dx \, dt = \frac{1}{|y|} \int_0^T \int_{\Omega \times Y^*} u \psi \varphi'' \, dx \, dt. \quad (2.42)$$

Now, we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} u \psi \varphi'' \, dx \, dt. \quad (2.43)$$

For the second term we use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by definition 2.2(i)(ii) we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon (\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi \, dx \, dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi \, dx \, dt. \end{aligned}$$

By (2.25) we have

$$\mathcal{T}_\varepsilon^*(A^\varepsilon) \rightarrow A \text{ strongly in } (L^1(\Omega \times Y^*))^{n \times n}, \quad (2.44)$$

and by convergence (2.28) we get

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightarrow \nabla u + \nabla_y \hat{u} \text{ strongly in } L^\infty(0, T; L^2(\Omega \times Y)). \quad (2.45)$$

using proposition 1.1 (1, 4) we get

$$\begin{aligned} \mathcal{T}_\varepsilon^*(\nabla \Psi + \phi \nabla_y \psi^\varepsilon) &= \mathcal{T}_\varepsilon^*(\nabla \Psi) + \mathcal{T}_\varepsilon^*(\phi \nabla_y \psi^\varepsilon) \\ &= \nabla \Psi + \phi \nabla_y \psi(y). \end{aligned}$$

For simplify the notation we denote $\Phi = \phi(x)\psi(y)$. This imply $\nabla_y \Phi = \phi(x)\nabla\psi(y)$.

Now, we have

$$\mathcal{T}_\varepsilon(\nabla\Psi + \phi\nabla_y\psi^\varepsilon) = \nabla\Psi + \nabla_y\Phi. \quad (2.46)$$

Finally by (2.28), (2.17) and (2.44) we get the following limit

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega \times Y^*} A^\varepsilon \nabla u_\varepsilon (\nabla\Psi + \phi\nabla_y\psi^\varepsilon) \varphi \, dx \, dt \\ &= \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u}) (\nabla\Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt. \end{aligned} \quad (2.47)$$

For the last integral we also use the unfolding operator \mathcal{T}_ε and by Definition 2.2 (i), (ii).

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon(f_\varepsilon) \mathcal{T}_\varepsilon(v_\varepsilon) \varphi \, dx \, dy \, dt.$$

By (2.28) and (2.32) we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon(f_\varepsilon) \mathcal{T}_\varepsilon \varphi \, dx \, dy \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} f \psi \varphi \, dx \, dy \, dt.$$

by Fubini theorem we obtaine:

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon \varphi \, dx \, dt = \theta \int_0^T \int_{\Omega} f \psi \varphi \, dx \, dt. \quad (2.48)$$

Now, we can pass at the limit in expretion (2.39) using (2.42) , (2.46) and (2.47) we get

$$\begin{aligned} \theta \int_0^T \int_{\Omega} u \psi \varphi'' \, dx \, dt + \frac{1}{|Y|} \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u}) (\nabla\Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt \\ = \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt, \end{aligned}$$

setting $\Psi = 0$ we get

$$\frac{1}{|y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u}) (\nabla_y \Phi) \varphi \, dx \, dy \, dt = 0,$$

we obtain

$$\operatorname{div}_y A(\nabla u + \nabla_y \hat{u}) = 0.$$

Since \mathbf{u} is independent of \mathbf{y} and $\mathcal{M}_{Y^*}(\hat{\mathbf{u}}) = \mathbf{0}$ we obtain (2.30) Moreover, we have the following identity

$$\begin{aligned} \int_{Y^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_{\mathbf{y}} \hat{\mathbf{u}}) \nabla \Psi \, dx \, dy &= \int_{Y^*} \mathbf{A} \nabla \mathbf{u} \nabla \Psi \, dy + \int_{Y^*} \mathbf{A} \nabla_{\mathbf{y}} \hat{\mathbf{u}} \nabla \Psi \, dy \\ &= \nabla \mathbf{u} \nabla \Psi \int_{Y^*} \mathbf{A} \, dy \\ &= \nabla \mathbf{u} \nabla \Psi \frac{|Y^*|}{|Y^*|} \int_{Y^*} \mathbf{A} \, dy \\ &= \nabla \mathbf{u} \nabla \Psi |Y^*| \frac{1}{|Y^*|} \int_{Y^*} \mathbf{A} \, dy \end{aligned}$$

from (2.34) we obtain

$$\int_{Y^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_{\mathbf{y}} \hat{\mathbf{u}}) \nabla \Psi \, dx \, dy = |Y^*| \mathbf{A}^0 \nabla \mathbf{u} \nabla \Psi. \quad (2.49)$$

Substituting (2.30) and (2.49) into (2.29), we get that

$$\mathbf{u}'' - \operatorname{div}(\mathbf{A}^0 \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, T).$$

Wich gives the equation in (2.33) and $\mathbf{u}'' \in L^2(0, T; H^{-1}(\Omega))$. Hence from classical results, we have

$$\mathbf{u} \in \mathbb{C}^0([0, T]; L^2(\Omega)) \quad \text{and} \quad \mathbf{u}' \in \mathbb{C}^0([0, T]; H^{-1}(\Omega)).$$

Now, in order to check the initial conditions, let \mathbf{v}_ε be given by (2.36) and $\varphi \in \mathbb{C}^\infty([0, T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Choosing $\mathbf{v}_\varepsilon \varphi$ as test function in the variational formulation (2.24) we have

$$\int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, dx \, dt - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v}_\varepsilon \varphi \, dx \, dt = \int_0^T \langle \mathbf{u}'' , \mathbf{v}_\varepsilon \varphi \rangle_{V'_\varepsilon, V_\varepsilon} \, dx \, dt \quad (2.50)$$

we integrate by parts we give

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi' \, dx \, dt &= \int_{\Omega_\varepsilon^*} [\varphi \mathbf{u}' \mathbf{v}_\varepsilon]_0^T - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, dx \, dt \\ &= \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon(x, 0) \mathbf{v}_\varepsilon \varphi(0) \, dx - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, dx \, dt \end{aligned}$$

substituting this formula into (2.50) we get

$$\begin{aligned} & \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, dx \, dt - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v}_\varepsilon \varphi \, dx \, dt \\ &= \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon(x, 0) \mathbf{v}_\varepsilon \, dx - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, dx \, dt. \end{aligned} \quad (2.51)$$

By similar method above passing to the limit into expression (2.51), using (2.28), (2.17) and (2.44)

$$\begin{aligned} & -\frac{1}{|\mathbf{Y}|} \int_{\Omega \times Y^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt \\ & + \theta \int_0^T \int_{\Omega} \mathbf{f} \Psi \varphi \, dx \, dt = -\theta \int_{\Omega} \mathbf{u}^1 \Psi \, dx - \theta \int_0^T \int_{\Omega} \mathbf{u}' \Psi \varphi' \, dx \, dt, \end{aligned}$$

by integration by parts for the last integral we get

$$\begin{aligned} & \frac{1}{|\mathbf{Y}|} \int_{\Omega \times Y^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_y \hat{\mathbf{u}})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt + \theta \int_0^T \int_{\Omega} \mathbf{f} \Psi \varphi \, dx \, dt \\ & - \theta \int_{\Omega} \mathbf{u}^1 \Psi \, dx + \theta \int_{\Omega} \mathbf{u}'(x, 0) \Psi \, dx + \theta \int_{\Omega} \langle \mathbf{u}'', \Psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \varphi \, dt, \end{aligned}$$

Combining this with (2.24) we have

$$\mathbf{u}'(x, 0) = \mathbf{u}^1.$$

Choosing $\varphi \in \mathbb{C}$ with $\varphi(0) = \varphi(T) = \varphi'(T) = 0$, $\varphi'(0) = 1$ and taking $\mathbf{v}_\varepsilon \varphi$ as test function in the variational formulation (2.24) we have

$$\int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, dx \, dt - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v}_\varepsilon \varphi \, dx \, dt = \int_0^T \langle \mathbf{u}'', \mathbf{v}_\varepsilon \varphi \rangle_{V'_\varepsilon, V_\varepsilon} \, dx \, dt, \quad (2.52)$$

we integrate by parts we give

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}''_\varepsilon \mathbf{v}_\varepsilon \varphi \, dx \, dt &= \int_{\Omega_\varepsilon^*} [\varphi \mathbf{u}'_\varepsilon]_0^T - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, dx \, dt \\ &= - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, dx \, dt, \end{aligned}$$

by integration by parts we give

$$-\int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi' dx dt = -\int_{\Omega_\varepsilon^*} u(x, 0) + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' dx dt,$$

by similar method we give

$$\begin{aligned} & \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx dt - \frac{1}{|Y|} \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx dt \\ &= \theta \int_{\Omega_\varepsilon^*} u'_\varepsilon(x, 0) v_\varepsilon(x) dx - \int_0^T \int_{\Omega_\varepsilon^*} u'_\varepsilon v_\varepsilon \varphi' dx dt. \end{aligned} \quad (2.53)$$

By similar method above passing to the limit into expretion (2.19), using (2.24), (2.25) and (2.48)

$$\begin{aligned} & -\frac{1}{|Y|} \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi dx dy dt + \theta \int_0^T \int_{\Omega} f \Psi \varphi dx dt \\ &= -\theta \int_{\Omega} u^0 \Psi dx - \theta \int_0^T \int_{\Omega} u \Psi \varphi'' dx dt. \end{aligned}$$

by integration by parts for the forth integral we get

$$\begin{aligned} & -\frac{1}{|Y|} \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi dx dy dt + \theta \int_0^T \int_{\Omega} f \Psi \varphi dx dt \\ & - \theta \int_{\Omega} u^0 \Psi dx + \theta \int_{\Omega} u(x, 0) \Psi dx + \theta \int_0^T \int_{\Omega} u'' \Psi \varphi dt. \end{aligned}$$

Combining this equation with (2.53) we have

$$u(x, 0) = u^0.$$

CHAPTER

3

HOMOGENIZATION OF THE HYPERBOLIC-PARABOLIC EQUATIONS IN DOMAINS WITH SMALL HOLES

In this chapter, we study a class of hyperbolic-parabolic problems in periodically perforated domains with a homogeneous Neuman condition on the boundary of holes. We focus on the homogenization of equations (3.1). The proof is based on the periodic unfolding method in perforated domains.

we consider the hyperbolic-parabolic equation with homogenous Dirichlet-Neumann boundary in the perforated domain, namely:

$$\begin{cases} \mathbf{u}_\varepsilon'' + \mathbf{u}_\varepsilon' - \operatorname{div}(\mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon) = \mathbf{f}_\varepsilon, & \text{on } \Omega_\varepsilon^* \times (0, T), \\ \mathbf{u}_\varepsilon = \mathbf{0}, & \text{in } \partial\Omega \times (0, T), \\ \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon = 0, & \text{on } \partial S_\varepsilon \times (0, T), \\ \mathbf{u}_\varepsilon(x, 0) = \mathbf{u}_\varepsilon^0, \mathbf{u}_\varepsilon'(x, 0) = \mathbf{u}_\varepsilon^1, & \text{in } \Omega_\varepsilon^*, \end{cases} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz continuous boundary, S_ε is a set of ε -periodic holes of size ε and $\Omega_\varepsilon^* = \Omega \setminus S_\varepsilon$, \mathbf{n}_ε is the outward unit normal vector field defined on ∂S_ε .

For the initial data, we always assume $\mathbf{u}_\varepsilon^0 \in H_0^1(\Omega)$, $\mathbf{u}_\varepsilon^1 \in L^2(\Omega_\varepsilon^*)$ and $\mathbf{f}_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon^*))$.

The coefficient matrix \mathbf{A}^ε satisfies the following assumptions:

$$\begin{cases} \mathbf{A}^\varepsilon \in M(\alpha, \beta, \Omega), \\ \mathbf{A}^\varepsilon \text{ symmetric}, \end{cases} \quad (3.2)$$

where $M(\alpha, \beta, \Omega)$ is the classical set of the $n \times n$ matrix-valued functions (see [2]).

We need to introduce some necessary assumptions. In what follows, we suppose that the coefficient matrix \mathbf{A}^ε satisfies

$$\mathcal{T}_\varepsilon(\mathbf{A}^\varepsilon) \longrightarrow \mathbf{A} \text{ strongly in } (L^1(\Omega \times Y))^{n \times n}, \quad (3.3)$$

where \mathcal{T}_ε is the unfolding operator in fixed domains [4]. These assumptions recover the classical periodic coefficient case mentioned above.

For the initial data, we make the following assumptions:

$$\begin{cases} (i) \mathbf{u}_\varepsilon^0 \rightharpoonup \mathbf{u}^0 \text{ weakly in } H_0^1(\Omega), \\ (ii) \widetilde{\mathbf{u}}_\varepsilon^1 \rightharpoonup \mathbf{u}^1 \text{ strongly in } L^2(\Omega), \\ (iii) \widetilde{\mathbf{f}}_\varepsilon \rightharpoonup \mathbf{f} \text{ weakly in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (3.4)$$

Set

$$\mathcal{W}_\varepsilon = \left\{ \mathbf{v}_\varepsilon \mid \mathbf{v}_\varepsilon \in L^2(0, T; V^\varepsilon), \mathbf{v}_\varepsilon' \in L^2(0, T; L^2(\Omega_\varepsilon^*)) \right\}, \quad (3.5)$$

with

$$V^\varepsilon = \{v \in H^1(\Omega_\varepsilon^*) \mid v = 0 \text{ on } \partial\Omega\},$$

with the norm defined by

$$\|v_\varepsilon\|_{\mathcal{W}_\varepsilon} = \|v_\varepsilon\|_{L^2(0,T;V^\varepsilon)} + \|v'_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))}.$$

3.1 Variational formulation

we multiply the equation of problem (3.1) by a test function v where $v \in \mathcal{D}(\Omega)$ and we integrate on Ω_ε^* , we get

$$\int_{\Omega_\varepsilon^*} u_\varepsilon'' v \, dx + \int_{\Omega_\varepsilon^*} u_\varepsilon' v \, dx - \int_{\Omega_\varepsilon^*} \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) v \, dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v \, dx, \quad (3.6)$$

Now, we apply the Green's formula on third integral in above equation we get:

$$- \int_{\Omega_\varepsilon^*} A^\varepsilon \Delta u_\varepsilon v \, dx = \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx - \int_{\partial\Omega \cup \partial S_\varepsilon} A^\varepsilon \frac{\partial u_\varepsilon}{\partial \eta} v \, d\Gamma, \quad (3.7)$$

the last integral equal to zero because $\int_{\partial S_\varepsilon} A^\varepsilon \frac{\partial u_\varepsilon}{\partial \eta} v \, d\Gamma = 0$ and $v = 0$ on $\partial\Omega$ are essential conditions.

Substituting the formula (3.7) into (3.5) we get:

$$\langle u_\varepsilon'', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u_\varepsilon', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \int_{\Omega_\varepsilon^*} \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v \, dx.$$

The variational formulation of problem (3.1) is to find $u_\varepsilon \in \mathcal{W}_\varepsilon$ such that

$$\begin{cases} \langle u_\varepsilon'', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u_\varepsilon', v \rangle_{(V^\varepsilon)', V^\varepsilon} + \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v \, dx, \\ \text{in } \mathcal{D}'(0, T) \text{ for all } v \in V^\varepsilon, \\ u_\varepsilon(x, 0) = u_\varepsilon^0, \quad u_\varepsilon'(x, 0) = u_\varepsilon^1 \text{ in } \Omega_\varepsilon^*. \end{cases} \quad (3.8)$$

For every fixed ε , classical results provide that problem (3.6) has a unique solution \mathbf{u}_ε such that :

$$\mathbf{u}_\varepsilon \in \mathbb{C}^0([0, T]; V^\varepsilon) \cap \mathbb{C}^1([0, T]; L^2(\Omega_\varepsilon^*)).$$

3.2 Homogenization of hyperbolic-parabolic equation

In this section we state our homogenization result of problem (3.1), the main result of the homogenization give by the next theorem.

Theoreme 3.1 *Let A^ε be a matrix satisfying (3.2) and (3.3). Suppose that u_ε is the solution of problem (3.1) with (3.4). Then there exist $u \in L^2(0, T; H_0^1(\Omega))$ with $u' \in L^2(0, T; L^2(\Omega))$ and $\hat{u} \in L^2(\Omega, H_{per}^1(\times Y^*))$ with $\mathcal{M}_{Y^*}(\hat{u}) = 0$, such that*

$$\left\{ \begin{array}{l} \mathcal{T}_\varepsilon^*(u_\varepsilon) \longrightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega, H^1(Y^*))), \\ \mathcal{T}_\varepsilon^*(u'_\varepsilon) \rightharpoonup u' \text{ weakly in } L^2(0, T; L^2(\Omega \times Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \hat{u} \text{ weakly in } L^2(0, T; L^2(\Omega \times Y^*)), \\ \| u_\varepsilon - u \|_{L^2(0, T; L^2(\Omega_\varepsilon^*))} \longrightarrow 0. \end{array} \right. \quad (3.9)$$

The paire (u, \hat{u}) with $\mathcal{M}_Y^*(\hat{u}) = 0$ is the unique solution of the following problem:

$$\left\{ \begin{array}{l} \theta \int_0^T \int_\Omega u \Psi \varphi'' dx dt + \theta \int_0^T \int_\Omega u \Psi \varphi' dx dt \\ + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi dx dy dt \\ = \theta \int_0^T \int_\Omega f \Psi \varphi dx dt \\ \text{for any } \Psi \in H_0^1(\Omega), \Phi \text{ in } L^2(\Omega; H_{per}^1(Y^*)) \text{ and } \varphi \in \mathcal{D}(0, T), \\ u = 0 \text{ on } \Omega \times (0, T), \\ u(x, 0) = u^0, \quad u'(x, 0) = u^1 \text{ in } \Omega. \end{array} \right. \quad (3.10)$$

We also have

$$\hat{u} = \sum_{j=1}^n \frac{\partial u}{\partial x_j} \mathcal{X}_j, \quad (3.11)$$

with $\mathcal{X}_j \in L^\infty(\Omega; H_{per}^1(Y^*)) (j = 1, \dots, n)$ being the solution of the cell problem

$$\left\{ \begin{array}{ll} -\text{div}_y(A \nabla_y(\mathcal{X}_j + y_j)) = 0, & \text{in } Y^*, \\ A \nabla_y(\mathcal{X}_j + y_j) \cdot n_1, & \text{on } \partial S, \\ \mathcal{M}_Y^*(\mathcal{X}_j)(x, \cdot) = 0, & \mathcal{X}_j(x, \cdot) \text{ Y - periodic.} \end{array} \right.$$

Moreover, \mathbf{u} is the unique solution of the following homogenized hyperbolic-parabolic equation

$$\begin{cases} \mathbf{u}'' + \mathbf{u}' - \operatorname{div}(\mathbf{A}^0 \nabla \mathbf{u}) = \theta^{-1} \mathbf{f}, & \text{in } \Omega \times (0, T), \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}^0, \quad \mathbf{u}'(x, 0) = \mathbf{u}^1, & \text{in } \Omega, \end{cases} \quad (3.12)$$

where the homogenized matrix $\mathbf{A}^0 = (\mathbf{a}_{ij}^0)_{1 \leq i, j \leq n}$ is defined by

$$\mathbf{a}_{ij}^0(x) = \mathcal{M}_{Y^*} \left(\mathbf{a}_{ij} + \sum_{k=1}^n \mathbf{a}_{ik} \frac{\partial \mathcal{X}_j}{\partial \mathbf{y}_k} \right). \quad (3.13)$$

In addition, we have the following convergences:

$$\begin{cases} (i) \widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \theta \mathbf{u} \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ (ii) \mathbf{A}^\varepsilon \widehat{\nabla} \widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \theta \mathbf{A}^0 \nabla \mathbf{u} \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)). \end{cases} \quad (3.14)$$

We would like to mention that the effective matrix filed \mathbf{A}^0 depends on \mathbf{x} , while the classical matrix is constant

Proof

The proof of theorem 3.1 is fundamentally based on the periodic unfolding method .Our starting point is following variational formulation of problem (3.1).

Find $\mathbf{u}_\varepsilon \in L^2(0, T; V^\varepsilon)$ with $\mathbf{u}'_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon^*))$ and $\mathbf{u}''_\varepsilon \in L^2(0, T; (V^\varepsilon)')$ such that (3.8).

For every ε , we have the following uniforme estimates:

$$\| \mathbf{u}_\varepsilon \|_{L^\infty(0, T; V^\varepsilon)} + \| \mathbf{u}'_\varepsilon \|_{L^2(0, T; L^2(\Omega_\varepsilon^*))} \leq C. \quad (3.15)$$

In view of (3.15) and Theorem 2.1, we get that there exist $\mathbf{u} \in L^\infty(0, T; \mathbf{H}_0^1(\Omega))$ with $\mathbf{u}' \in L^\infty(0, T; L^2(\Omega))$ and $\widehat{\mathbf{u}} \in L^\infty(0, T; L^2(\Omega, H_{per}^1(Y^*)))$ with $\mathcal{M}_{Y^*}(\widehat{\mathbf{u}}) = \mathbf{0}$, such that, up to subsequence (still denoted by ε), the convergence in (3.9) hold. From Proposition

2.4(iii) , we further get that

$$\begin{aligned} \hat{u} &\rightharpoonup \theta \mathcal{M}_{Y^*}(u) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ A^\varepsilon \widehat{\nabla u_\varepsilon} &\rightharpoonup \theta \mathcal{M}_{Y^*}[A(\nabla u + \nabla_y \hat{u})] \text{ weakly* in } L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (3.16)$$

Now let $\Psi, \Phi \in \mathcal{D}(\Omega)$ and $\psi \in H_{per}^1(Y^*)$. Set

$$v_\varepsilon(x) = \Psi(x) + \varepsilon \phi(x) \psi^\varepsilon(x) \quad \text{with} \quad \psi^\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right). \quad (3.17)$$

Then

$$\nabla v_\varepsilon = \nabla \Psi + \varepsilon \psi^\varepsilon \nabla \phi + \phi(\nabla_y \psi) \left(\frac{\cdot}{\varepsilon}\right). \quad (3.18)$$

By Proposition 2.4 (ii), we have

$$\begin{aligned} \mathcal{T}_\varepsilon^*(v_\varepsilon) &\longrightarrow \Psi \text{ strongly in } L^2(\Omega \times Y), \\ \mathcal{T}_\varepsilon^*(\phi \psi^\varepsilon) &\longrightarrow \Phi \text{ strongly in } L^2(\Omega \times Y) \text{ with } \Phi = \phi(x) \psi(y), \\ \mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) &\longrightarrow \nabla \Psi + \nabla_y \Phi \text{ strongly in } L^2(\Omega \times Y). \end{aligned} \quad (3.19)$$

Let $\varphi(t) \in \mathcal{D}(0, T)$, we take $v_\varepsilon \varphi$ a test function in the problem (3.1) we get

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx dt + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi dx dt + \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla (v_\varepsilon \varphi) dx dt \\ = \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx dt, \end{aligned} \quad (3.20)$$

then

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx dt + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi dx dt \\ + \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \left(\nabla \Psi + \varepsilon \psi^\varepsilon \nabla \phi + \phi \nabla_y \psi^\varepsilon \left(\frac{\cdot}{\varepsilon}\right) \varphi \right) dx dt = \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx dt, \end{aligned}$$

then

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx dt + \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi dx dt + \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon (\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi dx dt \\ + \varepsilon \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla \phi \psi^\varepsilon \left(\frac{x}{\varepsilon}\right) \varphi dx dt = \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx dt. \end{aligned} \quad (3.21)$$

We pass at the limit in Problem (3.21) term by term.

We start by the first term

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' \, dx \, dt. \quad (3.22)$$

By the unfolding operator $\mathcal{T}_\varepsilon^*$ and we use the Definition 2.2(i)(ii) we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon'' v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbf{y}|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(u_\varepsilon) \mathcal{T}_\varepsilon^*(v_\varepsilon) \mathcal{T}_\varepsilon^*(\varphi'') \, dx \, dy \, dt,$$

by convergence (3.19) and the convergence (3.9) we get

$$\int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' \, dx \, dt = \frac{1}{|\mathbf{y}|} \int_0^T \int_{\Omega \times Y^*} u \Psi \varphi'' \, dx \, dy \, dt,$$

by fubini theorem we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi \, dx \, dt = \theta \int_0^T \int_{\Omega} u \Psi \varphi'' \, dx \, dt. \quad (3.23)$$

For the second integral, we use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by Definition 2.2(i)(ii) we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbf{y}|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(u_\varepsilon) \mathcal{T}_\varepsilon^*(v_\varepsilon) \mathcal{T}_\varepsilon^*(\varphi') \, dx \, dy \, dt,$$

by convergence (3.19) and the convergence (3.9) we get

$$\int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi' \, dx \, dt = \frac{1}{|\mathbf{y}|} \int_0^T \int_{\Omega \times Y^*} u \Psi \varphi' \, dx \, dy \, dt.$$

Now, by Fubini theorem we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} u_\varepsilon' v_\varepsilon \varphi \, dx \, dt = \theta \int_0^T \int_{\Omega} u \Psi \varphi' \, dx \, dt. \quad (3.24)$$

For the third term we use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by definition 2.2(i)(ii) we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon (\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi \, dx \, dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi \, dx \, dt. \end{aligned}$$

By (3.3) we have

$$\mathcal{T}_\varepsilon^*(A^\varepsilon) \longrightarrow A \text{ strongly in } (L^1(\Omega \times Y^*))^{n \times n}, \quad (3.25)$$

and by convergence (3.9) we get

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \longrightarrow \nabla u + \nabla_y \hat{u} \text{ strongly in } L^\infty(0, T; L^2(\Omega \times Y)), \quad (3.26)$$

using proposition 1.1 (1, 4) we get

$$\begin{aligned} \mathcal{T}_\varepsilon^*(\nabla \Psi + \phi \nabla_y \psi^\varepsilon) &= \mathcal{T}_\varepsilon^*(\nabla \Psi) + \mathcal{T}_\varepsilon^*(\phi \nabla_y \psi^\varepsilon) \\ &= \nabla \psi + \phi \nabla_y \psi(y). \end{aligned}$$

For simplify the notation we denote $\Phi = \phi(x)\psi(y)$ This imply $\nabla_y \Phi = \phi(x)\nabla \psi(y)$.

Now we have

$$\mathcal{T}_\varepsilon^*(\nabla \Psi + \phi \nabla_y \psi^\varepsilon) = \nabla \Psi + \nabla_y \Phi. \quad (3.27)$$

Finally by (3.25), (3.26) and (3.27) we get the following limit

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon \times Y^*} A^\varepsilon \nabla u_\varepsilon (\nabla \Psi + \phi \nabla_y \psi^\varepsilon) \varphi \, dx \, dt \\ &= \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega_\varepsilon \times Y^*} A(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt. \end{aligned} \quad (3.28)$$

The fourth integral go to 0 as ε go to 0.

For the last integral in equation (3.21) we also use the unfolding operator $\mathcal{T}_\varepsilon^*$ and by Definition 2.2 (i), (ii)

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f_\varepsilon) \mathcal{T}_\varepsilon^*(v_\varepsilon) \varphi \, dx \, dy \, dt.$$

By (3.4) and (3.19) we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f_\varepsilon) \mathcal{T}_\varepsilon^* \varphi \, dx \, dy \, dt = \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega \times Y^*} f \Psi \varphi \, dx \, dy \, dt,$$

by Fubini theorem we obtaine:

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \, dx \, dt = \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt. \quad (3.29)$$

Now, we can pass at the limit in expretion (3.19) using (3.23), (3.24), (3.28) and (3.29) we get

$$\begin{aligned} & \theta \int_0^T \int_{\Omega} u \Psi \varphi'' \, dx \, dt + \theta \int_0^T \int_{\Omega} u \Psi \varphi' \, dx \, dt \\ & + \frac{1}{|\mathbf{Y}|} \int_{\Omega \times Y^*} \mathbf{A}(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt = \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt, \end{aligned} \quad (3.30)$$

setting $\Psi = 0$ we get

$$\frac{1}{|\mathbf{y}|} \int_0^T \int_{\Omega \times Y^*} \mathbf{A}(\nabla u + \nabla_y \hat{u})(\nabla_y \Phi) \varphi \, dx \, dy \, dt = 0,$$

we obtain

$$\operatorname{div}_y \mathbf{A}(\nabla u + \nabla_y \hat{u}) = 0.$$

Since \mathbf{u} is independent of \mathbf{y} and $\mathcal{M}_{Y^*}(\hat{u}) = 0$ we obtain (3.11) Moreover, we have the following identity

$$\int_{Y^*} \mathbf{A}(\nabla u + \nabla_y \hat{u}) \nabla \Psi \, dx \, dy = |Y^*| \mathbf{A}^0 \nabla u \nabla \Psi. \quad (3.31)$$

Substituting (3.11) and (3.31) into (3.10), we get that

$$u'' + u' - \operatorname{div}(\mathbf{A}^0 \nabla u) = f \quad \text{in } \Omega \times (0, T).$$

Wich gives the equation (3.12) and $u'' \in L^2(0, T; H^{-1}(\Omega))$. Hence from classical results,

we have

$$\mathbf{u} \in \mathbb{C}^0([0, T]; L^2(\Omega)) \quad \text{and} \quad \mathbf{u}' \in \mathbb{C}^0([0, T]; H^{-1}(\Omega)).$$

Now, in order to check the initial conditions, let \mathbf{v}_ε be given by (3.17) and $\varphi \in \mathbb{C}^\infty([0, T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Choosing $\mathbf{v}_\varepsilon \varphi$ as test function in the variational formulation (3.8) we have

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt &= \int_0^T \langle \mathbf{u}'' , \mathbf{v}_\varepsilon \varphi \rangle_{V_\varepsilon', V_\varepsilon} \, d\mathbf{x} \, dt \\ &+ \int_0^T \langle \mathbf{u}' , \mathbf{v}_\varepsilon \varphi \rangle_{V_\varepsilon', V_\varepsilon} \, d\mathbf{x} \, dt, \end{aligned} \quad (3.32)$$

we integrate by parts (for the variable \mathbf{t}) the first integral in the other side of the equation (3.32) we get

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}_\varepsilon'' \mathbf{v}_\varepsilon \varphi' \, d\mathbf{x} \, dt &= \int_{\Omega_\varepsilon^*} [\varphi \mathbf{u}' \mathbf{v}_\varepsilon]_0^T - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, d\mathbf{x} \, dt \\ &= \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon(\mathbf{x}, 0) \mathbf{v}_\varepsilon(\mathbf{x}) \varphi(0) \, d\mathbf{x} - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, d\mathbf{x} \, dt, \end{aligned}$$

substituting this formula into (3.32) we get

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon^*} \mathbf{f}_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{A}^\varepsilon \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt \\ &= \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon(\mathbf{x}, 0) \mathbf{v}_\varepsilon(\mathbf{x}) \, d\mathbf{x} - \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi' \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_\varepsilon^*} \mathbf{u}'_\varepsilon \mathbf{v}_\varepsilon \varphi \, d\mathbf{x} \, dt. \end{aligned} \quad (3.33)$$

By similar method above, passing to the limit in the expretion (3.32), using (3.9) ,(3.24) , (3.28) and (3.29) we get

$$\begin{aligned} &-\frac{1}{|\mathbf{Y}|} \int_{\Omega \times \mathbf{Y}^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_{\mathbf{y}} \hat{\mathbf{u}})(\nabla \Psi + \nabla_{\mathbf{y}} \Phi) \varphi \, d\mathbf{x} \, d\mathbf{y} \, dt + \theta \int_0^T \int_{\Omega} \mathbf{f} \Psi \varphi \, d\mathbf{x} \, dt \\ &= -\theta \int_{\Omega} \mathbf{u}^1 \Psi \, d\mathbf{x} - \theta \int_0^T \int_{\Omega} \mathbf{u}' \Psi \varphi' \, d\mathbf{x} \, dt + \theta \int_0^T \int_{\Omega} \mathbf{u}' \Psi \varphi \, d\mathbf{x} \, dt, \end{aligned}$$

also, using the integration by parts for the forth integral we get

$$\begin{aligned}
& - \frac{1}{|\mathbf{Y}|} \int_{\Omega \times \mathbf{Y}^*} \mathbf{A}(\nabla \mathbf{u} + \nabla_{\mathbf{y}} \hat{\mathbf{u}})(\nabla \Psi + \nabla_{\mathbf{y}} \Phi) \varphi \, dx \, d\mathbf{y} \, dt + \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt \\
& - \theta \int_{\Omega} \mathbf{u}^1 \Psi \, dx + \theta \int_{\Omega} \mathbf{u}'(x, 0) \Psi \, dx + \theta \int_{\Omega} \langle \mathbf{u}'', \Psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \varphi \, dt \\
& + \theta \int_0^T \int_{\Omega} \mathbf{u}' \Psi \varphi \, dx \, dt.
\end{aligned}$$

Combining this equation with (3.30) we have

$$\mathbf{u}'(x, 0) = \mathbf{u}^1.$$

Now for the second intial condition Choosing $\varphi \in \mathbb{C}$

with $\varphi(0) = \varphi(T) = \varphi'(T) = 0, \varphi'(0) = 1$ and taking $v_{\varepsilon} \varphi$ as test function in the variational formulation (3.8) we have

$$\begin{aligned}
\int_0^T \int_{\Omega} f_{\varepsilon} v_{\varepsilon} \varphi \, dx \, dt - \int_0^T \int_{\Omega_{\varepsilon}^*} A^{\varepsilon} \nabla \mathbf{u}_{\varepsilon} \nabla v_{\varepsilon} \varphi \, dx \, dt \\
= \int_0^T \langle \mathbf{u}'', v_{\varepsilon} \varphi \rangle_{V_{\varepsilon}', V_{\varepsilon}} \, dx \, dt + \int_0^T \langle \mathbf{u}', v_{\varepsilon} \varphi \rangle_{V_{\varepsilon}', V_{\varepsilon}} \, dx \, dt.
\end{aligned} \tag{3.34}$$

The integration by parts gives

$$\begin{aligned}
\int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}_{\varepsilon}'' v_{\varepsilon} \varphi \, dx \, dt &= \int_{\Omega_{\varepsilon}^*} [\varphi \mathbf{u}' v_{\varepsilon}]_0^T - \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}'_{\varepsilon} v_{\varepsilon} \varphi' \, dx \, dt \\
&= - \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}'_{\varepsilon} v_{\varepsilon} \varphi' \, dx \, dt.
\end{aligned}$$

We use it for the second time integration by parts we get

$$- \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}'_{\varepsilon} v_{\varepsilon} \varphi' \, dx \, dt = - \int_{\Omega_{\varepsilon}^*} \mathbf{u}_{\varepsilon}(x, 0) v_{\varepsilon} \, dx + \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}_{\varepsilon} v_{\varepsilon} \varphi'' \, dx \, dt,$$

by substituting this expretion into (3.34) we get

$$\begin{aligned}
& \int_0^T \int_{\Omega_{\varepsilon}^*} f_{\varepsilon} v_{\varepsilon} \varphi \, dx \, dt - \frac{1}{|\mathbf{Y}|} \int_0^T \int_{\Omega_{\varepsilon}^*} A^{\varepsilon} \nabla \mathbf{u}_{\varepsilon} \nabla v_{\varepsilon} \varphi \, dx \, dt \\
& = \theta \int_{\Omega_{\varepsilon}^*} \mathbf{u}_{\varepsilon}(x, 0) v_{\varepsilon}(x) \, dx - \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}_{\varepsilon} v_{\varepsilon} \varphi'' \, dx \, dt + \int_0^T \int_{\Omega_{\varepsilon}^*} \mathbf{u}'_{\varepsilon} v_{\varepsilon} \varphi \, dx \, dt.
\end{aligned} \tag{3.35}$$

By similar method above passing to the limit into expretion (3.2), using (3.9), (3.3) and (3.25)

$$\begin{aligned} & -\frac{1}{|\mathbf{Y}|} \int_{\Omega \times Y^*} \mathbf{A}(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt + \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt \\ & = \theta \int_{\Omega} u^0 \Psi \, dx - \theta \int_0^T \int_{\Omega} u \Psi \varphi'' \, dx \, dt + \theta \int_0^T \int_{\Omega} u' \Psi \varphi \, dx \, dt. \end{aligned}$$

by integration by parts two time for the forth integral we get

$$\begin{aligned} & -\frac{1}{|\mathbf{Y}|} \int_{\Omega \times Y^*} \mathbf{A}(\nabla u + \nabla_y \hat{u})(\nabla \Psi + \nabla_y \Phi) \varphi \, dx \, dy \, dt + \theta \int_0^T \int_{\Omega} f \Psi \varphi \, dx \, dt \\ & - \theta \int_{\Omega} u^0 \Psi \, dx + \theta \int_{\Omega} u(x, 0) \Psi \, dx = \int_{\Omega} u(x, 0) \, dx + \theta \int_0^T \int_{\Omega} u'' \Psi \varphi \, dt \\ & + \theta \int_0^T \int_{\Omega} u' \Psi \varphi \, dx \, dt. \end{aligned}$$

Combining this equation with (3.35) we have

$$u(x, 0) = u^0.$$

CONCLUSION

In this memory, we have treated asymptotic behavior of hyperbolic-parabolic problem with boundary condition in perforated domain in small holes, to the so, we used the periodic unfolding method we get that the pair $(\mathbf{u}, \hat{\mathbf{u}})$ with $\mathcal{M}(\hat{\mathbf{u}}) = \mathbf{0}$ is a unique solution of problem (3.1), this implies that all convergence in theorem (3.1) hold for the hole sequence.

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