وزارة التعليم العالي والبحث العلمي

Université 20 Aout 1955 de Skikda

Faculté des Sciences

Département de Mathématiques

جامعة 20 أوت 1955 ، سكيكدة

كلـــــــية العلــــوم

قسم الرياضيات

N[0] : U.S/F.S/D.M/……/2022.

Faculté des Sciences
Département de Mathématiques

Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

Résolution numérique des équations différentielles non linéaires d'ordre fractionnaire

Option : ***Analyse Numérique des équations aux dérivées partielles***

Présenté par : ***CHIDOUH Samra***

Encadré par : ***BOULARES Salah*** **M.C.B** **E N S. AZZABA**

Devant le jury :

Président : ***KHEMIS Rabah*** **M.C.A** **U.SKIKDA**

Examinateur : ***KAREK Chafia*** **M.C.B** **U.SKIKDA**

Année universitaire : 2021 / 2022

Acknowledg

Above all, I thank God for giving me the strength and will to complete this work.

First of all, I would like to thank my thesis supervisor BOULARES Salah for this advice, guidance and support.

I would also like to thank the members of the jury for accepting to evaluate this work and for all their comments and criticisms.

I address my sincere thanks to all who have contributed directly or indirectly to the development of this work.

Dedicate

I dedicate this work to;

My father, may God have mercy on him, and may he rest in peace.

My mother, may God prolong her life and preserve her.

My brothers and sisters and their children each in his name.

To Linda and Rayan, may God have mercy on them.

To all my friends.

To all of them I dedicate this humble research.

Contents

The objective of this work is to study some differential problems with fractional orders,in Banach space by applying some fixed point theorems, where we dealt with two problems:

The first problem is a CaputoKatugampola fractional Cauchy problem, We used the fixed point theorem of Banach to show the existence of the solution.

And the second problem is a Caputo derivative problem, wher we relied in prove the existence of the solution on the fixed point theorem of Banach and then schauder.

Then we proposed an approximate solution to the two problems and showed that the proposed numerical solution converges towards the exact solution of each problem.

Keywords: Caputo -Katugampola fractional derivatives , Caputo fractional derivative , Fractional differential equation, Existence of solution, fixed point theorems, the exact solution, the approximate solution.

L'objectif de ce mémoire est d'étudier quelques problèmes différentièls ordres fractionnaire, dans espace de Banach en appliquant quelques théorèmes de point fixe, ou nous avons traité deux problèmes:

Premier problème est un problème de Cauchy fractionnaire de type Caputo-Katugampola.Nous avons utilisé le théorème du point fixe de Banach pour montré lexistence de la solution.

et le deuxième problème est un problème fractionnaire de la dérivée de caputo ,ou l'on s'est appuyé pour prouver l'existence de la solution sur le théorème du point fixe de Banach puis de Schauder.

puis on proposé une solution approchée aux deux problèmes,et on a montré que la solution numérique proposée converge vers la solution exacte de chaque problème.

Mots-clés: dérivée fractionnaire au sens de Caputo-Katugampola , dérivée fractionnaire au sens de Caputo, equation differentielle fractionnaire, existence de la solution, théorèmes du point fixe, la solution exacte, la solution approchée.

ملخص

الهدف من هذه المذكرة هو دراسة بعض المسائل التفاضلية ذات الرتب الكسرية، في فضاء بناخ بتطبيق بعض نظريات النقطة الثابتة، حيث تناولنا مسألتين:

الاولى: مسألة كابوتو-كاتوجامبولا واعتمدنا في اثبات وجود الحل على نظرية النقطة الثابتة لبناخ.

الثانية: مسألة كابوتو العادي حيث اعتمدنا في اثبات وجود الحل على نظرية النقطة الثابتة لبناخ ثم لشودر.

ثم اقترحنا حلا تقريبيا للمسألتين وبينا ان الحل العددي المقترح يتقارب نحو الحل الدقيق لكل مسألة.

الكلمات المفتاحية: المشتق الكسري لكابوتو-كاتوجامبولا، المشتق الكسري لكابوتو، المعادلة التفاضلية الكسرية ، وجود الحل ، نظرية النقطة الثابتة، الحل الدقيق، الحل التقريبي.

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibnizs letter to lHospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, GrĹunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. A lot of them proposed original approaches, which can be found chronologically in [16]. The theory of fractional calculus includes even complex orders of differintegrals and left and right differintegrals (analogously to left and right derivatives).

The fact, that the differintegral is an operator which includes both integer-order derivatives and integrals as special cases, is the reason why in present fractional calculus becomes very popular and many applications arise. The fractional integral may be used e.g. for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubnt'y presents in [18]. Analogously the fractional derivative is sometimes used for describing damping.

Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc.

In this memoir we consider only the most common definitions named after

Riemann and Liouville, Caputo, katugampola which will be introduced in chapter 1.

In this memoir, we study analytically and numerically some differential problems with fractional orders.

The first chapter contains some definitions of basic functions (the function Gamma, Beta, Mittag-Leffler) ,the integration and derivation fractional of Riemann-Liouville and caputo, their properties, finely some fixed point theorems such as Banach, Schauder, Leary Schauder .

In the second and third chapter we prove the existence and numerical solution of the following problems :

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\rho}x\left(t\right)=f\left(t,x\left(t\right)\right), \quad t\in(a,b), \\ x\left(a\right)=x_0. \end{cases}$$

Where $\alpha,\rho \in (0,1)$ and ${}^{C}D_{a^+}^{\alpha,\rho}$ denotes the left Caputo-Katugampola fractional derivatives, and $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given function.

$$\begin{cases} {}^{C}D_{a^+}^{\beta}y\left(t\right)=f\left(t,y\left(t\right)\right), \quad t\in(a,b), \\ y\left(a\right)=y_0, \qquad y'\left(a\right)=y_1. \end{cases}$$

Where $1<\beta\leq 2$ and ${}^{C}D_{a^+}^{\beta}$ denotes the left Caputo fractional derivative, and $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given function.

CHAPTER 1

Preliminaries

Preliminaries

We introduced some useful definitions, properties and lemmas that will be used in the remainder of this thesis, for more details see [9, 11, 17, 18, 19, 21, 22, 24]

1.1 Functional spaces

1.1.1 Space for continuous and absolutely continuous functions

Definition 1.1.1. Let $\Omega = [a;b], (-\infty < a < b < \infty)$, a finite interval, we denote by $AC([a,b])$the space of primitive functions of integrable functions, that is to say

$$f \in AC([a,b]) = \left\{ f : \exists \varphi \in L^1([a,b]) : f(x) = c + \int_a^x \varphi(t)dt \right\}.$$

and we call, $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$.

Definition 1.1.2. for $n \in \mathbb{N}$ we denote by $AC^n([a,b])$ the space of functions f having derivatives up to order $(n-1)$ absolutely continuous on $[a,b]$ such as

$$AC^n([a,b]) = \left\{ f : [a,b] \to \mathbb{C} \text{ such that } f^{(n-1)} \in AC([a,b]) \right\}$$

In particular $AC^1([a,b]) = AC([a,b])$.

1.1.2 C^n space

Definition 1.1.3. Let $\Omega = [a,b], (-\infty \leq a < b \leq \infty)$, and $n \in \mathbb{N}$, we designate by $C^n(\Omega)$ the space of functions f which have their derivatives of a lower order or equal to n continue on Ω, provided with the standard:

$$\|f\|_{C^n} = \sum_{i=0}^{n} \left\| f^{(i)} \right\|_C = \sum_{i=0}^{n} \max_{x\in\Omega} \left| f^{(i)}(x) \right|.$$

In particular
if $n = 0$, $C^0(\Omega) = C(\Omega)$ the space of continuous functions $f$on provided with the standard: $\|f\|_c = \max_{x\in\Omega} |f(x)|$

1.2 Special functions

The more important functions used in fractional calculus is the Gamma function of Euler. Its interpretation is simply the generalization of the factorial $n!$ and it allows n to take non-integer values.

1.2.1 The Gamma function

Definition 1.2.1. For every $x > 0$,the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

which converges for $x > 0$.

Lemma 1.2.1. *For every $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$ we have:*
1 .$\Gamma(x+1) = x.\Gamma(x)$
2 .$\Gamma(n+1) = n!$
3 .*The Euler's reflextion:* $\forall\, x \notin \mathbb{Z}$, $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{sin(\pi x)}$

4 . $\sum_{i=0}^{n-1} \frac{\Gamma(i-\alpha)}{i!} = -\frac{\Gamma(n-\alpha)}{\alpha\,\Gamma(n)}$

For examples: $\Gamma(2) = 1 \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

1.2.2 The Beta function

Definition 1.2.2. For every $x, y \in \mathbb{R}_+^*$,the beta function $B(x, y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Lemma 1.2.2. *For every* $x, y \in \mathbb{R}_+^*$ *we have:*

1 .$B(x,y) = B(y,x)$

2 .$B(x,y+1) = \frac{y}{x} B(x+1,y)$

3 .$B(x+1,y) = \frac{x}{x+y} B(x,y)$

An interesting formula relating the gamma and beta functions is

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

1.2.3 Mittag-Leffler functions

Definition 1.2.3. The one-parametric Mittag-Leffler function (M-L for short) $E_\alpha(z)$ is a special function of $z \in \mathbb{C}$ which depends on the complex parameter α and is defined by the power series

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \tag{1.2.1}$$

One can see that the series 1.2.1 converges in the whole complex plane for all $\Re(\alpha) > 0$.

A first generalization of $E_\alpha(z)$ is the two-parametric M-L function of $z \in \mathbb{C}$ defined by the series

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \qquad \alpha, \beta \in \mathbb{C} \text{ with } \Re(\alpha) > 0. \tag{1.2.2}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function in (1.2.1)

$E_{\alpha,1}(z) = E_\alpha(z)$.

When $\alpha = \beta = 1 : E_{1,1}(z) = E_1(z) = e^z$.

Lemma 1.2.3. *(Integration of the Mittag-Leffler function)*

Let $\Omega = [a; b], (-\infty < a < b < \infty)$, a finite interval, integrating (1.2.2) term-by-term, we obtain:

$$\int_a^x E_{\alpha,\beta}(\lambda s^\alpha) s^{\beta-1} ds = (x-a)^\beta E_{\alpha,\beta+1}\left(\lambda (x-a)^\alpha\right) \quad , \quad (\alpha > 0\ ,\ \beta > 0) \tag{1.2.3}$$

Relationship (1.2.3) is a particular case of the following more general relationship obtained by the fractional order term-by-term integration of the series (1.2.2), for $(\alpha > 0, \beta > 0, \gamma > 0)$:

$$\frac{1}{\Gamma(\gamma)} \int_a^x (x-s)^{\gamma-1} E_{\alpha,\beta}(\lambda s^\alpha) s^{\beta-1} ds = (x-a)^{\beta+\gamma-1} E_{\alpha,\beta+\gamma}\left(\lambda (x-a)^\alpha\right)$$

When $\alpha = \beta = 1$ *:*

$$\frac{1}{\Gamma(\gamma)} \int_a^x (x-s)^{\gamma-1} e^{\lambda s} ds = (x-a)^{\gamma} E_{1,\gamma+1}(\lambda\ (x-a)) \quad , \qquad (\gamma > 0) \tag{1.2.4}$$

1.3 Fractional integrals and Fractional derivatives

1.3.1 Fractional integrals

Let $\Omega = [a,b], (-\infty < a < b < +\infty)$, a finite interval on the real axis $\mathbb{R}$ and $f \in L^1([a,b])$

Definition 1.3.1. The fractional (arbitrary) order integral of a function f of order $\alpha \in \mathbb{R}_+$ is defined by:

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\, ds \qquad (x > a) \tag{1.3.5}$$

When $\alpha = n \in \mathbb{N}$ the definition (1.3.5) coincide with the n th integrals for the forme

$$\begin{aligned}\left(I_{a^+}^{n} f\right)(x) &= \int_a^x ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} f(s_n)\, ds_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s)\, ds\end{aligned}$$

Example 1.3.1. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(\ x > a\ )$

$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} (s-a)^{\beta} ds$

By using the following change of variable $s = a + (x-a)y$ we find

$$\begin{aligned}
\left(I_{a^+}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-(x-a)\,y-a)^{\alpha-1}((x-a)\,y)^{\beta}\,(x-a)\,dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\alpha-1}(1-y)^{\alpha-1}(x-a)^{\beta+1}y^{\beta}dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\alpha+\beta}(1-y)^{\alpha-1}y^{\beta}dy \\
&= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-y)^{\alpha-1}y^{\beta}dy \\
&= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} B\,(\beta+1,\alpha)
\end{aligned}$$

And by applying Lemma 1.2.2 we find

$$I_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta} \tag{1.3.6}$$

Lemma 1.3.1. *Let* $f \in L^1([a,b])$ *and* $\alpha, \beta \in \mathbb{R}_+^*$ *:* $\left(I_{a^+}^{\alpha} \circ I_{a^+}^{\beta} f\right)(x) = \left(I_{a^+}^{\alpha+\beta} f\right)(x)$

1.3.2 Fractional derivative

Definition 1.3.2. The Riemann Liouville fractional derivative $D_{a^+}^{\alpha}$ of a function f of order $\alpha \in \mathbb{R}_+$ is defined by

$$\begin{aligned}\left(D_{a^+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-s)^{n-\alpha-1} f(s)\, ds \qquad (n = [\alpha]+1 \ , \quad x > a)\end{aligned}$$

where $[\alpha]$ mean the right portion for α
In particular, when $\alpha = m \in \mathbb{N}$, $\left(D_{a^+}^{m} f\right)(x) = f^{(m)}(x)$ where $f^{(m)}(x)$ is the usuel derivative of $f(x)$ of order m

Example 1.3.2. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(x > a)$, we have $\left(D_{a^+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha} (x-a)^{\beta}$. According to equality (1.3.6) we find

$$\begin{aligned}\left(D_{a^+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^n \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} (x-a)^{n-\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)(n-\alpha+\beta)(n-1-\alpha+\beta)\ldots(n-(n-1)-\alpha+\beta)}{\Gamma(n-\alpha+\beta+1)} (x-a)^{n-n-\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)}{(\beta-\alpha)!} (x-a)^{\beta-\alpha}\end{aligned}$$

that's mean

$$D_{a^+}^{\alpha} (x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha} \tag{1.3.7}$$

In particular , if $\beta = 0$ then the Riemann-Liouville fractional derivative of a constant is, in general, not equal to zero:

$$D_{a^+}^{\alpha}1=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$$

Lemma 1.3.2. *Let $\alpha>0$, and $n=[\Re(\alpha)]+1$ the equality $D_{a^+}^{\alpha}f(x)=0$ is valid, if and only if*

$$f(x)=\sum_{j=1}^{n}c_j(x-a)^{\alpha-j},$$

where $c_j\in\mathbb{R}$, $\quad j=1,2,...,n$ is arbitrary constant.

Lemma 1.3.3. ***(semigroup property)*** *Let $f\in L^1([a,b])$ and $\alpha,\beta\in\mathbb{R}_+$ then, the equation*

$$\left(I_{a^+}^{\alpha}I_{a^+}^{\beta}f\right)(x)=\left(I_{a^+}^{\alpha+\beta}f\right)(x)$$

hold almost everywhere on $[a,b]$.

Lemma 1.3.4. *Let $f\in L^1([a,b])$ and $\alpha\in\mathbb{R}_+$ then, the following equality*

$$\left(D_{a^+}^{\alpha}I_{a^+}^{\alpha}f\right)(x)=f(x)$$

hold almost everywhere on $[a,b]$.
In general case for $\alpha>\beta$ we have $\left(D_{a^+}^{\beta}I_{a^+}^{\alpha}f\right)(x)=\left(I_{a^+}^{\alpha-\beta}f\right)(x)$.

Lemma 1.3.5. *Let $\alpha\in\mathbb{R}_+$, $f\in L^1([a,b])$ and $f_{n-\alpha}(x)\in AC^n[a,b]$ then, the equality*

$$\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha}f\right)(x)=f(x)+\sum_{j=1}^{n}\frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j} \tag{1.3.8}$$

hold almost everywhere on $[a,b]$, where $f_{n-\alpha}=I_{a^+}^{n-\alpha}f$ be the fractional integral of f of order $n-\alpha$.

Definition 1.3.3. For a function $f \in L^1([a,b],\mathbb{R})$, the Caputo fractional derivative of order $\alpha \in \mathbb{R}_+$ of f is defined by

$$\left({}^C D^{\alpha}_{a^+} f\right)(x) = \left(D^{\alpha}_{a^+} \left[f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(x-a)^k \right] \right), \qquad (n = [\alpha]+1\ ,\ \ x > a)$$

Lemma 1.3.6. *For a function $f \in AC^n([a,b],\mathbb{R})$, then the Caputo fractional derivative $\left({}^C D^{\alpha}_{a^+} f\right)(x)$ exist almost everywhere on $[a,b]$, with*

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+} f\right)(x) &= \left(I^{n-\alpha}_{a^+} f^{(n)} \right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \qquad (n = [\alpha]+1\ ,\ \ x > a)\end{aligned}$$

In particular, when $\alpha = m \in \mathbb{N}$: $\left({}^C D^{\alpha}_{a^+} f\right)(x) = f^{(m)}(x)$

Example 1.3.3. if $\alpha > 0$, $\beta > -1$ and $f(x) = (x-a)^{\beta}$, $(x > a)$

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+} f\right)(x) &= \left(I^{n-\alpha}_{a^+} f^{(n)} \right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds \\ &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1}(s-a)^{\beta-n} ds\end{aligned}$$

By using the following change of variable $s = a + (x-a)\,y$ we obtain

$$\begin{aligned}\left({}^{C}D_{a^+}^{\alpha} f\right)(x) &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)} \int_0^1 (x-a-(x-a)\,y)^{n-\alpha-1} \\ &\times ((x-a)\,y)^{\beta-n}(x-a)\,dy \\ &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)} \int_0^1 (x-a)^{n-\alpha-1}(1-y)^{n-\alpha-1} \\ &\times (x-a)^{\beta-n+1} y^{\beta-n} dy \\ &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)}(x-a)^{\beta-\alpha} \int_0^1 y^{\beta-n}(1-y)^{n-\alpha-1} dy \\ &= \frac{\beta(\beta-1)\ldots(\beta-(n-1))}{\Gamma(n-\alpha)}(x-a)^{\beta-\alpha} B(\beta-n+1, n-\alpha)\end{aligned}$$

And by applying Lemma 1.2.2 we find

$$ {}^{C}D_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} \tag{1.3.9}$$

in particular, ${}^{C}D_{a^+}^{\alpha} 1 = 0$

Lemma 1.3.7. *Let $\alpha > 0$, and $n = [\Re(\alpha)] + 1$ the equality ${}^{C}D_{a^+}^{\alpha} f(x) = 0$ is valid, if and only if*

$$f(x) = \sum_{j=0}^{n-1} c_j (x-a)^j,$$

where $c_j \in \mathbb{R}$, $\quad j = 1, 2, \ldots, n$ is arbitrary constant.

Lemma 1.3.8. *Let $\alpha \in \mathbb{R}_+$ and $f \in AC^n[a,b]$ or $f \in C^n[a,b]$ then, the equality*

$$\left(I_{a^+}^{\alpha}\ {}^{C}D_{a^+}^{\alpha} f\right)(x) = f(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(x-a)^j \tag{1.3.10}$$

hold almost everywhere on $[a,b]$.

1.3.3 Generalized fractional integrals and fractional derivatives

In this section we introduced new type of fractional derivative called Katugampola fractional derivative

Definition 1.3.4. (Katugampola fractional integrals) Let a, b be two real and $f : [a,b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional integrals of order $\alpha \in (0,1)$, parameter $\rho > 0$, of f is defined as

$$I_{a^+}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{s^{\rho-1}}{(x^\rho - s^\rho)^{1-\alpha}} f(s)\, ds.$$

$$I_{b^-}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{s^{\rho-1}}{(s^\rho - x^\rho)^{1-\alpha}} f(s)\, ds.$$

Definition 1.3.5. (Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real $f : [a,b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional derivative of order $\alpha > 0$, and parameter $\rho > 0$, is defined as

$$D_{a^+}^{\alpha,\rho} f(x) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} x^{1-\rho} \frac{d}{dx} \int_a^x \frac{s^{\rho-1}}{(x^\rho - s^\rho)^{\alpha}} f(s)\, ds.$$

and

$$D_{b^-}^{\alpha,\rho} f(x) = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)} x^{1-\rho} \frac{d}{dx} \int_x^b \frac{s^{\rho-1}}{(s^\rho - x^\rho)^{\alpha}} f(s)\, ds.$$

The relation between these two fractional operators is the following:

$$D_{a^+}^{\alpha,\rho} f(x) = x^{1-\rho} \frac{d}{dx} I_{a^+}^{1-\alpha,\rho} f(x)$$

and

$$D_{b^-}^{\alpha,\rho} f(x) = -x^{1-\rho} \frac{d}{dx} I_{b^-}^{1-\alpha,\rho} f(x)$$

Also,when $\alpha > 0$ is an integer, these fractional integrals are generalizations of the n-fold integrals

$$\int_a^x s_1^{\rho-1} ds_1 \int_a^{s_1} s_2^{\rho-1} ds_2 ... \int_a^{s_{n-1}} s_n^{\rho-1} f(s_n) ds_n$$

and

$$\int_x^b s_1^{\rho-1} ds_1 \int_{s_1}^b s_2^{\rho-1} ds_2 ... \int_{s_{n-1}}^b s_n^{\rho-1} f(s_n) ds_n$$

respectively. Motivated by those definitions, we introduce a new fractional derivative as follows.

Proposition 1.3.1. *We have the following properties for Katugompola fractional integral and derivative.*

$$D_{a^+}^{\alpha,\rho} \left(I_{a^+}^{\alpha,\rho} \right) f(x) = f(x),$$

$$I_{a^+}^{\alpha,\rho} \left(I_{a^+}^{\beta,\rho} \right) f(x) = I_{a^+}^{\alpha+\beta,\rho} f(x),$$

$$\lim_{\rho \to 1} I_{a^+}^{\alpha,\rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\, ds,$$

$$\lim_{\rho\to 1} D_{a^+}^{\alpha,\rho} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-s)^{n-\alpha-1} f(s)\, ds.$$

1.3.4 Caputo-Katugampola fractional derivative

Definition 1.3.6. (Caputo-Katugampola fractional derivative) Let $0 < a < b < \infty$,$f : [a,b] \to \mathbb{R}$ be an integrable function, and $\alpha \in (0,1)$ and $\rho > 0$ two fixed reals. The left and right Caputo-Katugampola fractional derivative of order α are defined by

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\rho} f(x) &= D_{a^+}^{\alpha,\rho} [f(x) - f(a)] \\ &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} x^{1-\rho} \frac{d}{dx} \int_a^x \frac{s^{\rho-1}}{(x^\rho - s^\rho)^\alpha} [f(s) - f(a)]\, ds, \end{aligned}$$

and

$$\begin{aligned} {}^C D_{b^-}^{\alpha,\rho} f(t) &= D_{b^-}^{\alpha,\rho} [f(x) - f(b)] \\ &= \frac{-\rho^\alpha}{\Gamma(1-\alpha)} x^{1-\rho} \frac{d}{dx} \int_x^b \frac{s^{\rho-1}}{(s^\rho - x^\rho)^\alpha} [f(s) - f(b)]\, ds, \end{aligned}$$

respectively

Theorem 1.3.1. *[1]. Let $f \in C^1([a,b])$ Then,*

$${}^C D_{a^+}^{\alpha,\rho} f(x) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^x (x^\rho - s^\rho)^{-\alpha} f'(s) ds,$$

and

$${}^C D_{b^-}^{\alpha,\rho} f(x) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \int_x^b (s^\rho - x^\rho)^{-\alpha} f'(s) ds$$

Theorem 1.3.2. *Let $f \in C([a,b])$. Then,*

$$ {}^{C}D^{\alpha,\rho}_{a^+} I^{\alpha,\rho}_{a^+} f(x) = f(x) $$

1.4 Fixed point theorems

In this section we present some important fixed point theorems which we use for proving the existence and uniqueness of solutions of our problems

Theorem 1.4.1. ***Ascoli-Arzela theorem***

Let $\mathcal{F} \subseteq C([a,b])$, for certain $a < b$, then $\mathcal{F}$ is relatively compact in $C([a,b])$ if and only if $\mathcal{F}$ is equicotinuous (that's mean for all $\varepsilon > 0$, their exist $\delta > 0$ such that for all $f \in \mathcal{F}$ and x , $x' \in [a,b]$ with $|x - x'| < \delta$ we have $|f(x) - f(x')| < \varepsilon$) , and uniformly bounded (that's mean their exist $C > 0$ such that $\|f\|_\infty \leq C$ for all $f \in \mathcal{F}$).

Theorem 1.4.2. ***Banach fixed point theorem***

Let $\mathcal{N}$ be a contraction on a Banach space E , then $\mathcal{N}$ has a unique fixed point in E.

Theorem 1.4.3. ***Schauders fixed point theorem***

Let E be a Banach space. U be a closed, convex and nonempty subset of E. Let $\mathcal{N} : U \to U$ be a continuous mapping such that $\mathcal{N}(U)$ is a relatively compact subset of E. Then $\mathcal{N}$ has at least one fixed point in U.

Chapter 2

Numerical approach to the Caputo–Katugampola fractional Cauchy problem

Numerical approach to the Caputo–Katugampola fractional Cauchy problem

2.1 Existence of solutions

In this section we study the existence of solutions for the following initial value problem:

$$^{C}D_{a^{+}}^{\alpha,\rho}x\left(t\right)=f\left(t,x\left(t\right)\right),\quad t\in\left(a,b\right), \tag{2.1.1}$$

$$x\left(a\right)=x_{0}. \tag{2.1.2}$$

Where $\alpha,\rho\in\left(0,1\right)$ and $^{C}D_{a^{+}}^{\alpha,\rho}$ denotes the left Caputo-Katugampola fractional derivatives, and $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is given function.
firstly we solve the following linear problem:

$$^{C}D_{a^{+}}^{\alpha,\rho}x\left(t\right)=h\left(t\right),\quad t\in\left(a,b\right), \tag{2.1.3}$$

$$x\left(a\right)=x_{0}. \tag{2.1.4}$$

Lemma 2.1.1. *Assume that $x\in AC^{1}\left(\left[a,b\right],\mathbb{R}\right)$, then x is a solution to the linear boundary value problem (2.1.3)-(2.1.4) if and only if x satisfies the integral equation*

$$x\left(t\right)=x_{a}+\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{a}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\alpha}}h(s)ds \tag{2.1.5}$$

Define the Banach space $E=C\left(\left[a,b\right],\mathbb{R}\right)$. We introduce the following hypothesis
(H1) $f:\left[a,b\right]\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function
(H2) There exist $p,q\in C\left(\left[a,b\right],\mathbb{R}_{+}\right)$ such that

$$\left|f\left(t,x\right)\right|\leq p\left(t\right)\left|x\right|+q\left(t\right),\text{for each } t\in\left[a,b\right],\text{ and } x\in\mathbb{R}.$$

(H3) There exists $L_f > 0$ such that

$$\left|f(t,x) - f(t,y)\right| \leq L_f \left|x-y\right|, \quad \text{for each } t \in [a,b] \quad \text{and all } x, y \in \mathbb{R}.$$

Note that if (H1) and (H3) are satisfied then (H2) is satisfied.

Theorem 2.1.1. *Assume that $(H1)$ and $(H3)$ hold, with*

$$\frac{\rho^{-\alpha} L_f}{\Gamma(\alpha+1)} \left(b^\rho - a^\rho\right)^\alpha < 1. \tag{2.1.6}$$

then there exists a unique solution of the problem (2.1.1)-(2.1.2).

Proof. We consider the operator $F : E \to E$ defined by

$$F(x(t)) = x_a + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{\left(t^\rho - s^\rho\right)^{1-\alpha}} f(s, x(s)) ds$$

we have

$$\begin{aligned}
&\left|F(x(t)) - F(y(t))\right| \\
&\quad = \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{\left(t^\rho - s^\rho\right)^{1-\alpha}} f(s, x(s)) ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{\left(t^\rho - s^\rho\right)^{1-\alpha}} f(s, y(s)) ds \right| \\
&\quad \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{\left(t^\rho - s^\rho\right)^{1-\alpha}} \left|f(s, x(s)) - f(s, y(s))\right| ds \\
&\quad \leq \frac{\rho^{1-\alpha} L_f}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{\left(t^\rho - s^\rho\right)^{1-\alpha}} \left|x(s) - y(s)\right| ds \\
&\quad \leq \frac{\rho^{-\alpha} L_f}{\Gamma(\alpha+1)} \left(b^\rho - a^\rho\right)^\alpha \left\|x - y\right\|_\infty
\end{aligned}$$

it implies that

$$\|F(x) - F(y)\|_\infty \leq \frac{\rho^{-\alpha} L_f}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha \|x-y\|_\infty$$

By 2.1.6, the operator F is a contraction. Hence, by Banach's contraction principle, F has a unique fixed point which is a unique solution of the problem (2.1.1)-(2.1.2).

□

2.2 Numerical solutions

Theorem 2.2.1. *Let $x : [a,b] \longrightarrow \mathbb{R}$ be a function of classe C^2, and $N \geq 1$ an integer. Define the quantities*

$$\begin{aligned} A_N &= \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{k=0}^{N} \frac{\Gamma(k-1+\alpha)}{\Gamma(\alpha-1)k!} \\ B_{N,k} &= \frac{\rho^\alpha \Gamma(k-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(k-1)!}, \qquad k = 1, \ldots, N; \end{aligned}$$

and function $V_k; [a,b] \rightarrow \mathbb{R}$ by

$$V_k(t) = \int_a^t (s^\rho - a^\rho)^{k-1} x'(s) ds, \qquad k = 1, \ldots, N;$$

Then

$${}^C D_{a^+}^{\alpha,\rho} x(t) = A_N (t^\rho - a^\rho)^{1-\alpha} t^{1-\rho} x'(t) - \sum_{k=1}^{N} B_{N,k} (t^\rho - a^\rho)^{1-\alpha-k} V_k(t) + E_N(t)$$

with

$$\lim_{N\to\infty} E_N(t) = 0, \qquad \forall t \in [a,b]$$

Proof. set

$$u'(s) = (t^\rho - s^\rho)^{-\alpha} s^{\rho-1}, \text{and} v(s) = s^{1-\rho} x'(s).$$

Then, integrating by parts we obtain

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\rho} x(t) &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\rho - s^\rho)^{-\alpha} x'(s) ds \\ &= \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} (t^\rho - a^\rho)^{1-\alpha} a^{1-\rho} x'(a) + \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \int_a^t (t^\rho - s^\rho)^{1-\alpha} \frac{d}{ds} (s^{1-\rho} x'(s)) ds \end{aligned}$$

Using,the generalized binomial theorem, we get the decomposition

$$\begin{aligned} (t^\rho - s^\rho)^{1-\alpha} &= (t^\rho - a^\rho)^{1-\alpha} \left(1 - \frac{s^\rho - a^\rho}{t^\rho - a^\rho}\right)^{1-\alpha} \\ &= (t^\rho - a^\rho)^{1-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(k-1+\alpha)}{\Gamma(\alpha-1)k!} \left(\frac{s^\rho - a^\rho}{t^\rho - a^\rho}\right)^k \end{aligned}$$

Substituting this series into the expression of the fractional derivative,we obtain

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\rho} x(t) &= \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} (t^\rho - a^\rho)^{1-\alpha} a^{1-\rho} x'(a) + \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \int_a^t (t^\rho - a^\rho)^{1-\alpha} \\ &\quad \times \sum_{k=0}^{N} \frac{\Gamma(k-1+\alpha)}{\Gamma(\alpha-1)k!} \left(\frac{s^\rho - a^\rho}{t^\rho - a^\rho}\right)^k \frac{d}{ds}\left(s^{1-\rho} x'(s)\right) ds + E_N(t), \end{aligned}$$

where

$$E_N(t) = \frac{\rho^\alpha - 1}{\Gamma(2-\alpha)} \int_a^t (t^\rho - a^\rho)^{1-\alpha} \sum_{k=N+1}^{\infty} \frac{\Gamma(k-1+\alpha)}{\Gamma(\alpha-1)k!} \left(\frac{s^\rho - a^\rho}{t^\rho - a^\rho}\right)^k \frac{d}{ds}(s^{1-\rho} x'(s)) ds$$

Now, let us split the sum into $k=0$ and k=1 ,...,N, that is,

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha,\rho}x\left(t\right) &= \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\left(t^{\rho}-a^{\rho}\right)^{1-\alpha}t^{1-\rho}x'\left(t\right)+\frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\left(t^{\rho}-a^{\rho}\right)^{1-\alpha} \\ &\times\sum_{k=1}^{N}\frac{\Gamma\left(k-1+\alpha\right)}{\Gamma\left(\alpha-1\right)k!\left(t^{\rho}-a^{\rho}\right)^{k}}\int_a^t\left(s^{\rho}-a^{\rho}\right)^{k}\frac{d}{ds}\left(s^{1-\rho}x'\left(s\right)\right)ds+E_N\left(t\right). \end{aligned}$$

Setting

$$u(s)=(s^{\rho}-a^{\rho})^{k}, \text{and} v'(s)=\frac{d}{ds}(s^{1-\rho}x'(s))$$

and then integrating by parts, we obtain

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha,\rho}x\left(t\right) &= \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\left(t^{\rho}-a^{\rho}\right)^{1-\alpha}t^{1-\rho}x'\left(t\right)\left[1+\sum_{k=1}^{N}\frac{\Gamma\left(k-1+\alpha\right)}{\Gamma\left(\alpha-1\right)k!}\right] \\ &-\frac{\rho^{\alpha}}{\Gamma\left(2-\alpha\right)}\sum_{k=1}^{N}\frac{\Gamma\left(k-1+\alpha\right)}{\Gamma\left(\alpha-1\right)\left(k-1\right)!}\left(t^{\rho}-a^{\rho}\right)^{1-\alpha-k}\int_a^t\left(s^{\rho}-a^{\rho}\right)^{k-1}x'\left(s\right)ds+E_N\left(t\right). \end{aligned}$$

It remains to show that

$$\lim_{N\to\infty}E_N(t)=0, \qquad \forall t\in[a,b]$$

and to do this we determine an upper bound for the error. Since $s\in[a,t]$,we have

$$\sum_{k=N+1}^{\infty}\left|\frac{\Gamma\left(k-1+\alpha\right)}{\Gamma\left(\alpha-1\right)k!}\left(\frac{s^{\rho}-a^{\rho}}{t^{\rho}-a^{\rho}}\right)^{k}\right|\leq\sum_{k=N+1}^{\infty}\frac{\exp\left(\left(1-\alpha\right)^2+1-\alpha\right)}{k^{2-\alpha}}$$

$$\leq\int_N^{\infty}\frac{\exp\left(\left(1-\alpha\right)^2+1-\alpha\right)}{k^{2-\alpha}}dk=\frac{\exp\left(\left(1-\alpha\right)^2+1-\alpha\right)}{N^{1-\alpha}\left(1-\alpha\right)}.$$

Taking

$$M(t) := \max_{s\in[a,t]} \left| \frac{d}{ds}\left(s^{1-\rho}x'(s)\right)\right|,$$

we get the following upper bound:

$$\begin{aligned}
|E_N(t)| &\leq M(t)\frac{\exp\left((1-\alpha)^2+1-\alpha\right)}{N^{1-\alpha}(1-\alpha)}\frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)}\int_a^t (t^\rho - a^\rho)^{1-\alpha} ds \\
&= M(t)\frac{\exp\left((1-\alpha)^2+1-\alpha\right)\rho^{\alpha-1}}{N^{1-\alpha}(1-\alpha)\Gamma(2-\alpha)}(t^\rho - a^\rho)^{1-\alpha}(t-a),
\end{aligned}$$

which converges to zero as N goes to ∞, for al $t \in [a,b]$. □

Theorem 2.2.2. *Let $x : [a,b] \longrightarrow \mathbb{R}$ be a function of classe C^2, and $N \geq 1$ an integer, Define the quantities*

$$\begin{aligned}
A_N &= \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)}\sum_{k=0}^{N}\frac{\Gamma(k-1+\alpha)}{\Gamma(\alpha-1)k!} \\
B_{N,k} &= \frac{\rho^\alpha\Gamma(k-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(k-1)!}, \qquad k = 1,\ldots,N;
\end{aligned}$$

and functions $W_k(t) : [a,b] \longrightarrow \mathbb{R}$ by

$$W_k(t) = \int_t^b (b^\rho - s^\rho)^{k-1} x'(s)ds, \qquad k = 1,\ldots,N;$$

Then

$${}^CD_{b^-}^{\alpha,\rho}x(t) = -A_N(b^\rho - t^\rho)^{1-\alpha}t^{1-\rho}x'(t) + \sum_{k=1}^{N} B_{N,k}(b^\rho - t^\rho)^{1-\alpha-k}W_k(t) + E_N(t)$$

Such that

$$\lim_{N\to\infty} E_N(t) = 0, \qquad \forall t \in [a,b]$$

The solution of the problem 2.1.1-2.1.2 can be rewritten as:

$$x(t) = x(a) +^C D_{a^+}^{1-\alpha,\rho} \left[\int_a^t f(s, x(s)) s^{\rho-1} ds \right].$$

Applying the decomposition formula given in Theorem, we obtain

$$x(t) = x(a) + A_N (t^\rho - a^\rho)^\alpha f(t, x(t)) - \sum_{k=1}^{N} B_{N,k} (t^\rho - a^\rho)^{\alpha-k} V_k(t) + E_N(t), \quad (2.2.7)$$

where

$$A_N = \frac{\rho^{-\alpha}}{\Gamma(1+\alpha)} \sum_{k=1}^{N} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)k!}$$
$$B_{N,k} = \frac{\rho^{1-\alpha}\Gamma(k-\alpha)}{\Gamma(1+\alpha)\Gamma(-\alpha)(k-1)!}, fork \in \{1, \cdots, N\}$$

$$V_k(t) = \int_a^t (s^\rho - a^\rho)^{k-1} f(s, x(s)) s^{\rho-1} ds, fork \in \{1, \cdots, N\},$$

and the error is bounded by the formula

$$|E_N(t)| \leq M(t) \frac{\exp(\alpha^2 + \alpha)\rho^{-\alpha}}{\alpha N^\alpha \Gamma(1-\alpha)} (t^\rho - a^\rho)^\alpha (t-a)$$

with

$$M(t) := \max_{s \in [a,t]} \left| \frac{d}{ds} f(s, x(s)) \right|$$

In order to an approximated solution,x_N, we truncate the formula up to order N,receiving

$$x_N(t) = x(a) + A_N (t^\rho - a^\rho)^\alpha f(t, x_N(t)) - \sum_{k=1}^{N} B_{N,k} (t^\rho - a^\rho)^{\alpha-k} V_{k,N}(t) \quad (2.2.8)$$

where

$$V_{k,N}(t) = \int_a^t (s^\rho - a^\rho)^{k-1} f(s, x_N(s)) s^{\rho-1} ds, fork \in \{1, \cdots, N\}$$

(we remark that $x_N(a) = x(a)$)

Theorem 2.2.3. *Let $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and Lipschitz with respect to the second variable. For $N \in \mathbb{N}$, let x and x_N be given by conditions 2.2.7 and 2.2.8, respectively. Also, let $T \in \mathbb{R}$ be a real in the open interval*

$$a < T < \left(a^\rho + \rho \left(\frac{\Gamma(1+\alpha)}{L} \right)^{\frac{1}{\alpha}} \right)^{\frac{1}{\rho}}$$

Then, for all $t \in [a, T]$,$x_N(t) \to x(t)$ as $N \to \infty$.

Proof. Starting with relations 2.2.7 and 2.2.8 we have

$$\begin{aligned} |x_N(t) - x(t)| \leq & \; |A_N| (t^\rho - a^\rho)^\alpha |f(t, x_N(t)) - f(t, x(t))| \\ & + \sum_{k=0}^{N} |B_{N,k}| (t^\rho - a^\rho)^{\alpha-k} |V_{k,N}(t) - V_k(t)| + |E_N(t)| \end{aligned}$$

for all $t \in [a, T]$. Define

$$\delta x_N := \max_{t \in [a,T]} |x_N(t) - x(t)|.$$

As a consequence of the following relations:

$$|f(t, x_N(t)) - f(t, x(t))| \leq L |x_N(t) - x(t)| \leq L \delta x_N$$

$$\begin{aligned}\left|V_{k,N}(t)-V_k(t)\right| &= \left|\int_a^t (s^\rho-a^\rho)^{k-1}\left|f(s,x_N(s))-f(s,x(s))\right| s^{\rho-1}ds\right| \\ &\leq L\delta x_N \int_a^t (s^\rho-a^\rho)^{k-1} s^{\rho-1}ds = \frac{L\delta x_N}{k\rho}(t^\rho-a^\rho)^k\end{aligned}$$

$$\begin{aligned}|A_N| &= \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left|\sum_{k=0}^{N}\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,k!}\right| \\ &= \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)\left|\Gamma(-\alpha)\right|}.\frac{\Gamma(N+1-\alpha)}{\alpha\Gamma(N+1)} \\ &= \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)\Gamma(1-\alpha)}.\frac{\Gamma(N+1-\alpha)}{\Gamma(N+1)} \\ &\leq \frac{1}{\rho^\alpha\alpha\pi}.\frac{\Gamma(N+1-\alpha)}{\Gamma(N+1)}\end{aligned}$$

by the Euler′s reflection formula and equation (3) in [11]

$$\begin{aligned}&\sum_{k=1}^{N}\left|B_{N,k}\right|(t^\rho-a^\rho)^{\alpha-k}\left|V_{k,N}(t)-V_k(t)\right| \\ &\quad\leq \frac{L\delta x_N(t^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(1+\alpha)\left|\Gamma(-\alpha)\right|}\sum_{k=1}^{N}\frac{\Gamma(k-\alpha)}{k!} \\ &\quad\leq \frac{L\delta x_N(t^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(1+\alpha)\left|\Gamma(-\alpha)\right|}\left[\frac{\Gamma(N+1-\alpha)}{\alpha\Gamma(N+1)}+\left|\Gamma(-\alpha)\right|\right] \\ &\quad\leq \frac{L\delta x_N}{\rho^\alpha}(t^\rho-a^\rho)^\alpha\left[\frac{1}{\alpha\pi}.\frac{\Gamma(N+1-\alpha)}{\Gamma(N+1)}+\frac{1}{\Gamma(1+\alpha)}\right],\end{aligned}$$

we conclude that

$$\left|x_N(t)-x(t)\right| \leq \frac{L\delta x_N}{\rho^\alpha}(t^\rho-a^\rho)^\alpha\left[\frac{2}{\alpha\pi}.\frac{\Gamma(N+1-\alpha)}{\Gamma(N+1)}+\frac{1}{\Gamma(1+\alpha)}\right]+\left|E_N(t)\right|.$$

for all $t\in[a,T]$. Taking the maximum, over $t\in[a,T]$, on both sides of the inequality,we get

$$\delta x_N \leq \frac{L\delta x_N}{\rho^\alpha}\left(T^\rho - a^\rho\right)^\alpha \left[\frac{2}{\alpha\pi}.\frac{\Gamma\left(N+1-\alpha\right)}{\Gamma\left(N+1\right)} + \frac{1}{\Gamma\left(1+\alpha\right)}\right] + \max_{t\in[a,T]}\left|E_N\left(t\right)\right|. \tag{2.2.9}$$

It is obvious that

$$\lim_{N\to\infty}\left|E_N\left(t\right)\right| = 0.$$

Moreover, as a consequence of the Stirling's formula (see e.g.[23]),we have

$$\lim_{N\to\infty}\frac{\Gamma\left(N+1-\alpha\right)}{\Gamma\left(N+1\right)} = 0.$$

Therefore, setting $N \longrightarrow \infty$ in 2.2.9 we obtain

$$\lim_{N\to\infty}\delta x_N\left[1 - \frac{L}{\rho^\alpha\Gamma\left(\alpha+1\right)}\left(T^\rho - a^\rho\right)^\alpha\right] \leq 0,$$

and by the definition of T , we must have $\delta x_N \to 0$.
From Eq. 2.2.9 we obtain that

$$\delta x_N\left[1 - \frac{L}{\rho^\alpha}\left(T^\rho - a^\rho\right)^\alpha\left[\frac{2}{\alpha\pi}.\frac{\Gamma\left(N+1-\alpha\right)}{\Gamma\left(N+1\right)} + \frac{1}{\Gamma\left(\alpha+1\right)}\right]\right] \leq \max_{t\in[a,T]}\left|E_N\left(t\right)\right|.$$

since

$$0 < \frac{L}{\rho^\alpha}\left(T^\rho - a^\rho\right)^\alpha < \Gamma\left(1+\alpha\right),$$

we have

$$-\frac{2\Gamma\left(1+\alpha\right)\Gamma\left(N+1-\alpha\right)}{\alpha\pi\Gamma\left(N+1\right)} \leq 1 - \frac{L}{\rho^\alpha}\left(T^\rho - a^\rho\right)^\alpha\left[\frac{2}{\alpha\pi}.\frac{\Gamma\left(N+1-\alpha\right)}{\Gamma\left(N+1\right)} + \frac{1}{\Gamma\left(\alpha+1\right)}\right] < 1.$$

Therefore, for N sufficiently large, by

$$\lim_{N\to\infty} \frac{\Gamma\left(N+1-\alpha\right)}{\Gamma\left(N+1\right)} = 0,$$

we get that there exists a function C (depending on α, ρ, t, but independent of N) such that

$$\delta x_N \leq C N^{\alpha-1}.$$

$\square$

CHAPTER 3

Analysis and numerical solution to the Caputo fractional Cauchy problem

Analysis and numerical solution to the Caputo fractional Cauchy problem

3.1 Existence of solutions

In this section we study the existence of solutions for the following nonlinear boundary value problem:

$$^{C}D_{a^{+}}^{\beta}y\left(t\right)=f\left(t,y\left(t\right)\right),\quad t\in\left(a,b\right),\tag{3.1.1}$$

$$y\left(a\right)=y_{0},\qquad y'\left(a\right)=y_{1}.\tag{3.1.2}$$

Where $1<\beta\leq 2$ and $^{C}D_{a^{+}}^{\beta}$ denotes the left Caputo fractional derivative, and $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is given function.
firstly we solve the following linear problem:

$$^{C}D_{a^{+}}^{\beta}y\left(t\right)=h\left(t\right),\quad t\in\left(a,b\right),\tag{3.1.3}$$

$$y\left(a\right)=y_{0},\qquad y'\left(a\right)=y_{1}.\tag{3.1.4}$$

Lemma 3.1.1. *Assume that $y\in AC^{2}\left([a,b],\mathbb{R}\right)$, then y is a solution to the linear boundary value problem (3.1.3)-(3.1.4) if and only if y satisfies the integral equation*

$$y\left(t\right)=\frac{1}{\Gamma\left(\beta\right)}\int_{a}^{t}\left(t-s\right)^{\beta-1}h(s)ds+\sum_{k=0}^{1}y_{k}(t-a)^{k},$$

Define the Banach space $E=C\left(\left[a,b\right],\mathbb{R}\right)$. We introduce the following hypothesis

(H1) $f:[a,b]\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function

(H2) There exist $p, q \in C([a,b], \mathbb{R}_+)$ such that

$$|f(t,y)| \leq p(t)|y| + q(t), \text{for each } t \in [a,b], \text{ and } y \in \mathbb{R}.$$

(H3) There exists $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f |x-y|, \quad \text{for each } t \in [a,b] \quad \text{and all } x, y \in \mathbb{R}.$$

Note that if (H1) and (H3) are satisfied then (H2) is satisfied.

Theorem 3.1.1. *Assume that* $(H1)$ *and* $(H3)$ *hold, with*

$$\frac{L_f(b-a)^\beta}{\Gamma(\beta+1)} < 1. \tag{3.1.5}$$

then there exists a unique solution of the problem (3.1.1)-(3.1.2).

Proof. We consider the operator $N : E \to E$ defined by

$$N(y(t)) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, y(s))ds + \sum_{k=0}^{1} y_k (t-a)^k, \tag{3.1.6}$$

We have

$$\begin{aligned}
&|N(x(t)) - N(y(t))| \\
&\quad = \left|\frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1} f(s,x(s))ds - \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1} f(s,y(s))ds\right| \\
&\quad \leq \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1} |f(s,x(s)) - f(s,y(s))|\, ds \\
&\quad \leq \frac{L_f}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1} |x(s) - y(s)|\, ds \\
&\quad \leq \frac{L_f(b-a)^\beta}{\Gamma(\beta+1)} \|x-y\|_\infty ,
\end{aligned}$$

it implies that

$$\|N(x) - N(y)\|_\infty \leq \frac{L_f(b-a)^\beta}{\Gamma(\beta+1)} \|x-y\|_\infty$$

By 3.1.5, the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (3.1.1)-(3.1.2).

□

Theorem 3.1.2. *Assume that $(H1)$ and $(H2)$ hold, then the problem (3.1.1)-(3.1.2) has a fixed point in B_R, where*

$$B_R = \{y \in E, \text{ such that } \|y\|_\infty \leq R\}.$$

with $1 \geq \frac{(b-a)^\beta}{\Gamma(\beta+1)} \|p\|_\infty$ *and*

$$R \geq \left(\frac{(b-a)^\beta}{\Gamma(\beta+1)} \|q\|_\infty + \sum_{k=0}^{1} |y_k| (b-a)^k\right)\left(1 - \frac{(b-a)^\beta}{\Gamma(\beta+1)} \|p\|_\infty\right)^{-1}. \tag{3.1.7}$$

Proof. Let the operator N defined in (3.1.6). We shall show that N satisfies the assumption of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1, We prove that N is a continuous operator.

Let (y_n) be a sequence converges to y in E. Then, for every $t \in [a,b]$ we have

$$\begin{aligned}
&\left|N\left(y_n\left(t\right)\right) - N\left(y\left(t\right)\right)\right| \\
&\quad = \left|\frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1} f(s,y_n(s))ds - \frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1} f(s,y(s))ds\right| \\
&\quad \leq \frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1}\left|f(s,y_n(s)) - f(s,y(s))\right| ds \\
&\quad \leq \left\|f\left(.,y_n\left(.\right)\right) - f\left(.,y\left(.\right)\right)\right\|_\infty \frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1} ds \\
&\quad \leq \frac{(b-a)^\beta}{\Gamma\left(\beta+1\right)}\left\|f\left(.,y_n\left(.\right)\right) - f\left(.,y\left(.\right)\right)\right\|_\infty
\end{aligned}$$

It implies that, $\|Ny_n - Ny\|_\infty \to 0$, when $n \to \infty$, since f is a continuous function

Step 2, We prove that $N(B_R) \subset B_R$.

For any $y \in B_R$ and $t \in [a,b]$ we have

$$\begin{aligned}
\left|N\left(y\left(t\right)\right)\right| &= \left|\frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1} f(s,y(s))ds + \sum_{k=0}^{1} y_k (t-a)^k\right| \\
&\leq \frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1}\left|f\left(s,y\left(s\right)\right)\right| ds + \sum_{k=0}^{1}\left|y_k\right|(t-a)^k \\
&\leq \frac{1}{\Gamma\left(\beta\right)}\int_a^t (t-s)^{\beta-1}\left(p(s)\left|y(s)\right| + q(s)\right) ds + \sum_{k=0}^{1}\left|y_k\right|(b-a)^k \\
&\leq \frac{(b-a)^\beta}{\Gamma\left(\beta+1\right)}\left(\|p\|_\infty R + \|q\|_\infty\right) + \sum_{k=0}^{1}\left|y_k\right|(b-a)^k \\
&\leq R
\end{aligned}$$

Step 3, We prove that $N(B_R)$ is equicontinuous sets of E.
Let $t_1, t_2 \in (a,b)$ such that $t_1 < t_2$ and $y \in B_R$, then, we have

$$\begin{aligned}
&\left|N\left(y\left(t_2\right)\right)-N\left(y\left(t_1\right)\right)\right| \\
&\quad = \frac{1}{\Gamma\left(\beta\right)}\int_a^{t_1}\left(\left(t_2-s\right)^{\beta-1}-\left(t_1-s\right)^{\beta-1}\right)\left|f(s,y(s))\right| ds \\
&\quad + \frac{1}{\Gamma\left(\beta\right)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\beta-1}\left|f(s,y(s))\right| ds + \sum_{k=1}^{1}\left|y_k\right|\left[\left(t_2-a\right)^k-\left(t_1-a\right)^k\right] \\
&\quad \leq \frac{\left(\|p\|_\infty R+\|q\|_\infty\right)}{\Gamma\left(\beta+1\right)}\left(\left(t_2-a\right)^\beta-\left(t_1-a\right)^\beta\right)+\sum_{k=1}^{1}\left|y_k\right|\left[\left(t_2-a\right)^k-\left(t_1-a\right)^k\right]
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the last above inequality tends to zero. By Arzela-Ascoli Theorem we conclude that $N(B_R)$ is relatively compact. As a consequence of Schauders fixed point Theorem, we deduce that, N has a fixed point $y \in B_R$ which is a solution of the problem (3.1.1)-(3.1.2). □

3.1.1 Ulam stability

We consider the following inequality

$$\left|{}^C D_{a^+}^{\beta} x\left(t\right)-f\left(t,x\left(t\right)\right)\right| \leq \epsilon, \text{ for } t \in [a,b] \text{ and } \epsilon > 0. \tag{3.1.8}$$

Definition 3.1.1. Equation (3.1.1) is $\mathbb{E}_\beta$-Ulam-Hyers stable if there exists $c > 0$ such that for each $\epsilon > 0$, and for each solution $x \in C\left([a,b],\mathbb{R}\right)$ of (3.1.8) there exists a solution $y \in C\left([a,b],\mathbb{R}\right)$ of equation (3.1.1) with

$$\left|x\left(t\right)-y\left(t\right)\right| \leq c\mathbb{E}_\beta\left(L_f\left(t-a\right)^\beta\right)\epsilon. \tag{3.1.9}$$

Remark 3.1.1. Let x be a solution of the inequality (3.1.8) then x is a solution of the following integral inequality

$$\left| x\left(t\right) - \left(\frac{1}{\Gamma\left(\beta\right)} \int_a^t \left(t-s\right)^{\beta-1} f(s,x(s))ds + \sum_{k=0}^{1} x_k (t-a)^k \right) \right| \leq \frac{\epsilon (b-a)^{\beta}}{\Gamma\left(\beta+1\right)}.$$

Theorem 3.1.3. *Assume that* $(H1)$ *and* $(H3)$ *hold. Then the equation (3.1.1) is* $\mathbb{E}_{\beta}$*-Ulam-Hyers stable.*

Proof. Let x be a solution of inequality (3.1.8) and y the unique solution of the following Cauchy problem

$${}^{C}D^{\beta}_{a^+} y\left(t\right) = f\left(t, y\left(t\right)\right), \quad t \in (a,b),$$

$$y\left(a\right) = x_0, \qquad y'\left(a\right) = x_1.$$

Therefore,

$$y\left(t\right) = \frac{1}{\Gamma\left(\beta\right)} \int_a^t \left(t-s\right)^{\beta-1} f(s,y(s))ds + \sum_{k=0}^{1} x_k (t-a)^k. \tag{3.1.10}$$

By using (H3), Remark 3.1.1 and **Corollary 2** in [25] we obtain

$$
\begin{aligned}
|x(t)-y(t)| &\leq \left| x(t) - \left(\frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s,y(s))ds + \sum_{k=0}^{1} x_k (t-a)^k \right) \right| \\
&\leq \left| x(t) - \left(\frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s,x(s))ds + \sum_{k=0}^{1} x_k (t-a)^k \right) \right| \\
&\quad + \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |f(s,x(s)) - f(s,y(s))|\, ds \\
&\leq \frac{\epsilon (b-a)^\beta}{\Gamma(\beta+1)} + \frac{L_f}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} |x(s)-y(s)|\, ds \\
&\leq \frac{\epsilon (b-a)^\beta}{\Gamma(\beta+1)} \mathbb{E}_\beta \left(L_f (t-a)^\beta \right).
\end{aligned}
$$

Thus, $|x(t)-y(t)| \leq c\mathbb{E}_\beta \left(L_f (t-a)^\beta \right) \epsilon$, for all $t \in [a,b]$, where $c = \dfrac{(b-a)^\beta}{\Gamma(\beta+1)}$.
The proof is complete. □

3.2 Numerical approach

Theorem 3.2.1. *Let $N \geq 1$ an integer and $y : [0,1] \to \mathbb{R}$, be a function of class AC^{n+1}. Let consider*

$$
\mathcal{A}_N = \frac{1}{\Gamma(3-\beta)} \sum_{k=0}^{N} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\, k!},
$$

$$
\mathcal{B}_{N,k} = \frac{\Gamma(k-2+\beta)}{\Gamma(3-\beta)\Gamma(2-\beta)(k-1)!}, \; k = 1,2,...,N,
$$

and functions $\mathcal{V}_k : [a,b] \to \mathbb{R}$ by

$$
\mathcal{V}_k(t) = \int_a^t (s-a)^{k-1} y^{(2)}(s)\, ds,
$$

Then

$$ {}^{C}D_{a^+}^{\beta} y(t) = \mathcal{A}_N (t-a)^{2-\beta} y^{(2)}(t) - \sum_{k=1}^{N} \mathcal{B}_{N,k}(t-a)^{2-\beta-k} \mathcal{V}_k(t) + \mathcal{E}_N(t) $$

where

$$ \lim_{N\to\infty} \mathcal{E}_N(t) = 0. $$

Proof. We have

$$ {}^{C}D_{a^+}^{\beta} y(t) = \frac{1}{\Gamma(2-\beta)} \int_a^t (t-s)^{1-\beta} y^{(2)}(s)\, ds. $$

Taking $u'(s) = (t-s)^{1-\beta}$, $v(s) = y^{(2)}(s)$ we get

$$ {}^{C}D_{a^+}^{\beta} y(t) = \frac{1}{\Gamma(3-\beta)} (t-a)^{2-\beta} y^{(2)}(a) + \frac{1}{\Gamma(3-\beta)} \int_a^t (t-s)^{2-\beta} y^{(3)}(s)\, ds. $$

Using the generalized binomial theorem, we obtain

$$ \begin{aligned} (t-s)^{2-\beta} &= (t-a)^{2-\beta} \left(1 - \frac{s-a}{t-a}\right)^{2-\beta} \\ &= (t-a)^{2-\beta} \sum_{k=0}^{\infty} \frac{\Gamma(k-2+\beta)}{\Gamma(\beta-2)\, k!} \left(\frac{s-a}{t-a}\right)^k . \end{aligned} $$

Substituting the last above equality into the expression of the fractional derivative, we find

$$ \begin{aligned} {}^{C}D_{a^+}^{\beta} y(t) &= \frac{1}{\Gamma(3-\beta)} (t-a)^{2-\beta} y^{(2)}(a) \\ &+ \frac{1}{\Gamma(3-\beta)} \int_a^t (t-a)^{2-\beta} \sum_{k=0}^{N} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\, k!} \left(\frac{s-a}{t-a}\right)^k y^{(3)}(s)\, ds + \mathcal{E}_N(t). \end{aligned} $$

where

$$\mathcal{E}_N(t) = \frac{1}{\Gamma(3-\beta)} \int_a^t (t-a)^{2-\beta} \sum_{k=N+1}^{\infty} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\,k!} \left(\frac{s-a}{t-a}\right)^k y^{(3)}(s)\,ds.$$

It implies that

$$\begin{aligned} {}^C D_{a^+}^{\beta} y(t) &= \frac{1}{\Gamma(3-\beta)} (t-a)^{2-\beta} y^{(2)}(t) \\ &+ \frac{1}{\Gamma(3-\beta)} (t-a)^{2-\beta} \sum_{k=1}^{N} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\,k!\,(t-a)^k} \int_a^t (s-a)^k y^{(3)}(s)\,ds + \mathcal{E}_N(t). \end{aligned}$$

Setting $u(s) = (s-a)^k$, and $v'(s) = y^{(3)}(s)$ and by integrating by parts, we get

$$\begin{aligned} {}^C D_{a^+}^{\beta} y(t) &= \frac{1}{\Gamma(3-\beta)} (t-a)^{2-\beta} y^{(2)}(t) \left(\sum_{k=0}^{N} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\,k!}\right) \\ &- \frac{1}{\Gamma(3-\beta)} \sum_{k=1}^{N} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\,(k-1)!} (t-a)^{2-\beta-k} \int_a^t (s-a)^{k-1} y^{(2)}(s)\,ds + \mathcal{E}_N(t). \end{aligned}$$

We determine an upper bound for the error. From [1], we find that $\forall s \in [a,t]$

$$\sum_{k=N+1}^{\infty} \frac{\Gamma(k+\beta-2)}{\Gamma(\beta-2)\,k!} \left(\frac{s-a}{t-a}\right)^k \leq \frac{\exp\left((2-\beta)^2 + 2 - \beta\right)}{N^{2-\beta}(2-\beta)}$$

Taking $M(t) = \sup_{s\in[a,t]} \left|y^3(s)\right|$. It implies that

$$\begin{aligned} |\mathcal{E}_N(t)| &\leq M(t) \frac{\exp\left((2-\beta)^2 + 2 - \beta\right)}{N^{2-\beta}(2-\beta)\Gamma(3-\beta)} \int_a^t (t-a)^{2-\beta}\,ds \\ &\leq M(t) \frac{\exp\left((2-\beta)^2 + 2 - \beta\right)}{N^{2-\beta}(2-\beta)\Gamma(3-\beta)} (t-a)^{3-\beta} \end{aligned}$$

which converges to zero as N goes to ∞. The proof is finished. $\square$

The problem (3.1.3)-(3.1.4) can be rewritten as

$$y(t) = y_0 + y_1(t-a) + {}^{C}D_{a^+}^{3-\beta} I_{a^+}^{2} \left(\int_a^t f(s, y(s)) ds \right)$$

Using the decomposition formula given in Theorem 3.2.1, we find

$$\begin{aligned} y(t) &= y_0 + y_1(t-a) + \tilde{\mathcal{A}}_N (t-a)^{\beta-1} \left(\int_a^t f(s, y(s)) ds \right) \\ &- \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k} (t-a)^{\beta-k-1} \tilde{\mathcal{V}}_k(t) + \tilde{\mathcal{E}}_N(t), \end{aligned} \tag{3.2.11}$$

where

$$\tilde{\mathcal{A}}_N = \frac{1}{\Gamma(\beta)} \sum_{k=0}^{N} \frac{\Gamma(k-\beta+1)}{\Gamma(1-\beta)\, k!},$$

$$\tilde{\mathcal{B}}_{N,k} = \frac{\Gamma(k-\beta+1)}{\Gamma(\beta)\Gamma(1-\beta)(k-1)!},\ k = 1, 2, ..., N,$$

$$\begin{aligned} \tilde{\mathcal{V}}_k(s) &= \int_a^t (s-a)^{k-1} \left(\int_a^s f(\tau, y(\tau)) d\tau \right) ds \\ &= \frac{1}{k} \int_a^t \left((t-a)^k - (s-a)^k \right) f(s, y(s)) ds, \end{aligned}$$

and the error $\tilde{\mathcal{E}}_N$ satisfies

$$|\mathcal{E}_N(t)| \leq M(t) \frac{\exp\left((\beta-1)^2 + \beta - 1 \right)}{N^{\beta-1}(\beta-1)\Gamma(\beta)} (t-a)^{\beta},$$

where $M(t) = \sup\limits_{s \in [a,t]} |f(s, y(s))|$.

We take the formula of approximate solution, y_N by

$$\begin{aligned} y_N(t) &= y_0 + y_1(t-a) + \tilde{\mathcal{A}}_N (t-a)^{\beta-1} \left(\int_a^t f(s, y_N(s)) ds \right) \\ &\quad - \sum_{k=1}^{N} \tilde{\mathcal{B}}_{N,k} (t-a)^{\beta-k-1} \tilde{\mathcal{V}}_{k,N}(t), \end{aligned} \tag{3.2.12}$$

where

$$\tilde{\mathcal{V}}_{k,N}(t) = \frac{1}{k} \int_a^t \left((t-a)^k - (s-a)^k \right) f(s, y_N(s)) ds, \text{ for } k = 1, 2, ..., N.$$

Theorem 3.2.2. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies (H_3). For $N \in \mathbb{N}$, let y and y_N as in (3.2.11) and (3.2.12) respectively. If*

$$a < b < a + \min \left\{ \left(\frac{\Gamma(\beta)}{L} \right)^{\frac{1}{\beta}}, \left(\frac{\Gamma(\beta)}{L} \right)^{\frac{1}{\beta-1}} \right\}. \tag{3.2.13}$$

Then, $y_N(t)$ tends to $y(t)$ as N tends to ∞.

Proof. It follows from (3.2.11) and (3.2.12) that

$$\begin{aligned} |y_N(t) - y(t)| &\leq \left| \tilde{\mathcal{A}}_N \right| (t-a)^{\beta-1} \left(\int_a^t |f(s, y_N(s)) - f(s, y(s))| \, ds \right) \\ &\quad - \sum_{k=1}^{N} \left| \tilde{\mathcal{B}}_{N,k} \right| (t-a)^{\beta-k-1} \left| \tilde{\mathcal{V}}_{k,N}(t) - \tilde{\mathcal{V}}_k(t) \right| + \left| \tilde{\mathcal{E}}_N(t) \right|, \end{aligned}$$

for all $t \in [a,b]$. Define

$$\delta_{y_N} = \max_{t \in [a,b]} |y_N(t) - y(t)|.$$

Then, we have

$$\left|f\left(t,y_N\left(t\right)\right)-f\left(t,y\left(t\right)\right)\right|\leq L_f\left|y_N\left(t\right)-y\left(t\right)\right|\leq L\delta_{y_N},$$

$$\begin{aligned}\left|\tilde{\mathcal{V}}_{k,N}\left(t\right)-\tilde{\mathcal{V}}_k\left(t\right)\right| &\leq \frac{1}{k}\int_a^t\left((t-a)^k-(s-a)^k\right)\left|f(s,y_N(s))-f(s,y(s))\right|ds\\ &\leq \frac{L\delta_{y_N}}{k}(t-a)^{k+1},\end{aligned}$$

As a similar estimation in [1], we obtain that

$$\begin{aligned}\left|\tilde{\mathcal{A}}_N\right| &= \frac{1}{\Gamma\left(\beta\right)}\left|\sum_{k=0}^{N}\frac{\Gamma\left(k-\beta+1\right)}{\Gamma\left(1-\beta\right)k!}\right|\\ &\leq \frac{1}{(\beta-1)\pi}\frac{\Gamma(N+2-\beta)}{\Gamma(N+1)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{N}\left|\tilde{\mathcal{B}}_{N,k}\right|(t-a)^{\beta-k-1}\left|\tilde{\mathcal{V}}_{k,N}(t)-\tilde{\mathcal{V}}_k(t)\right|\\ &\qquad\leq L\delta_{y_N}(t-a)^{\beta}\left[\frac{1}{(\beta-1)\pi}\frac{\Gamma(N+2-\beta)}{\Gamma(N+1)}+\frac{1}{\Gamma(\beta)}\right].\end{aligned}$$

Thus

$$|y_N(t)-y(t)|\leq \max\left\{(t-a)^{\beta},(t-a)^{\beta-1}\right\}\left[\frac{1}{(\beta-1)\pi}\frac{\Gamma(N+2-\beta)}{\Gamma(N+1)}+\frac{1}{\Gamma(\beta)}\right]L\delta_{y_N}+\left|\tilde{\mathcal{E}}_N\left(t\right)\right|$$

It implies that

$$\delta_{y_N} \leq \max\left\{(b-a)^{\beta},(b-a)^{\beta-1}\right\}\left[\frac{1}{(\beta-1)\pi}\frac{\Gamma(N+2-\beta)}{\Gamma(N+1)}+\frac{1}{\Gamma(\beta)}\right]L\delta_{y_N}+\left|\tilde{\mathcal{E}}_N\,(t)\right|, \tag{3.2.14}$$

where $\left|\tilde{\mathcal{E}}_N\,(t)\right| \to 0$ as $n \to \infty$. By using that fact $\lim\limits_{N\to\infty}\dfrac{\Gamma(N+2-\beta)}{\Gamma(N+1)}=0$, (see e.g. [23]) and setting $N \to \infty$ in 3.2.14 we get

$$\lim_{N\to\infty}\delta_{y_N}\left[1-\frac{L}{\Gamma(\beta)}\max\left\{(b-a)^{\beta},(b-a)^{\beta-1}\right\}\right]\leq 0,$$

from the definition of b , we obtain $\delta_{y_N} \to 0$ as $N \to \infty$. □

In this memory, a fractional boundary value problem with Caputo–Katugampola derivative and fractional Cauchy problem with Caputo derivative are provided. A numerical technique based on a decomposition formula for Caputo–Katugampola derivative and Caputo derivative. Furthermore, the convergence analysis of the method is achived.

The field remains open for further study and research on this topic.

Bibliography

[1] R. Almeida, AB. Malinowska, and Tatiana Odzijewicz. "Fractional differential equations with dependence on the CaputoKatugampola derivative." Journal of Computational and Nonlinear Dynamics 11(6), 2016.

[2] RB. Albadarneh, M. Zerqat, and IM. Batiha. Numerical solutions for linear and non-linear fractional differential equations. International Journal of Pure and Applied Mathematics, 106(3), 859871, 2016.

[3] J. Alzabut, T. Abdeljawad, F. Jarad, W. Sudsutad, A Gronwall inequality via the generalized proportional fractional derivative with applications. Journal of Inequalities and Applications, 101, 1-12, 2019.

[4] M. Benchohra, S. Hamani and S.K. Ntouyas, boundary value problems for differential equations with fractional order, Surv. Math. Appl. 3, 1-12, 2008.

[5] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with non linear fractional diferential equations, Appl.Anal. 87 (7), 851-863, 2008.

[6] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional functional differential inclusions with infinite delay and application to control theory, Fract. Calculus Appl. Anal. 11 (1), 35-56, 2008.

[7] M. Benchohra and S. Hamani, Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative, Topol. Meth. Nonlinear Anal. (à paraître).

[8] M. Benchohra, J.E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional diferential equations with boundary conditions. Rom. J. Math. Comput. Sci, 4(1),60-72, 2014.

[9] D. Boucenna,D. Baleanu, A.B. Makhlouf, A.M. Nagy, Analysis and numerical solution of the generalized proportional fractional Cauchy problem. Applied Numerical Mathematics, 167, 173-186, 2021.

[10] K. Diethelm, The Analysis of fractional differential equations. An application oriented exposition using differential operators of Caputo type. Springer, Hei delberg, 2010.

[11] R. Garrappa, Some formulas for sums of binomial coefficients and gamma functions, Int-Math.From 2, 725-733, 2007.

[12] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[13] F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives. The European Physical Journal Special Topics, 226(16), 3457-3471, 2017.

[14] GW. Leibniz. Letter from hanover, germany, to gfa lhopital, september 30 ; 1695. Mathematische Schriften, 2 :301302, 1849.

[15] GW. Leibniz, Letter from Hanover, and September Germany. 30, 1695 to ga lhospital. leibnizen mathematische schriften, vol. 2, 1962.

[16] K. Oldham, and J. Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier, 1974.

[17] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA,1999.

[18] I. Podlubny and Kenneth V. Thimann (Eds.). Fractional Differential Equations : An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in science and engineering 198. Academic Press, 1st edition, 1999.

[19] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation, fractional calculus and applied analysis, vol.5, number 4, 2002.

[20] S. Pooseh, R. Almeida, and D.F.M. Torres, Numerical approximations of fractional derivatives with applications, Asian J. Control 15, 698-712, 2013.

[21] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional integrals and derivatives,translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[22] H. M. Srivastava, A. A. Kilbas and J. J. Trujillo (Eds.). Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204. Elsevier, 1 edition, 2006.

[23] F.G. Tricomi, and A. Erdélyi, "The asymptotic expansion of a ratio of gamma functions", Pacific J. Math. 1 (1), 133142, 1951.

[24] M. Weilber, Efficient numerical methods for fractional differential equations and their analytical background. Papierflieger, 2005.

[25] H. Ye, J. Gao, Y. Ding. "A generalized Gronwall inequality and its application to a fractional differential equation." Journal of Mathematical Analysis and Applications 328(2), 1075-1081, 2007