وزارة التعليم العالي والبحث العلمي

Université 20 Aout 1955 de Skikda
Faculté des Sciences
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جامعة 20 أوت 1955 ، سكيكدة
كلـــــــية العلــــوم
قسم الرياضيات

N⁰ : U.S/F.S/D.M/…./2024.

Faculté des Sciences
Département de Mathématiques

Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

The 2D Fourier transform and its generalizations

Option : AN-EDP

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Année 2023 – 2024

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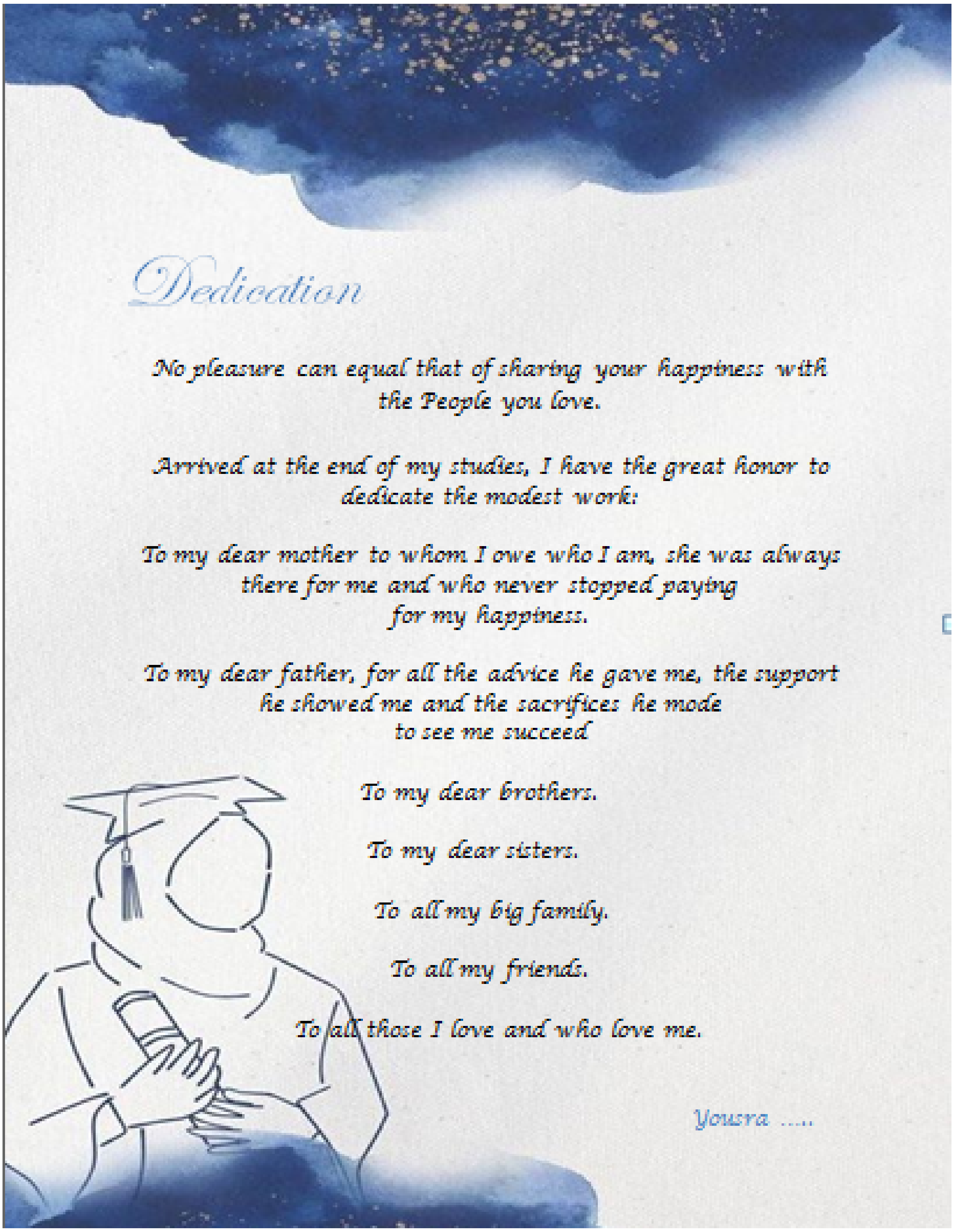

Dedication

No pleasure can equal that of sharing your happiness with the People you love.

Arrived at the end of my studies, I have the great honor to dedicate the modest work:

To my dear mother to whom I owe who I am, she was always there for me and who never stopped paying for my happiness.

To my dear father, for all the advice he gave me, the support he showed me and the sacrifices he mode to see me succeed

To my dear brothers.

To my dear sisters.

To all my big family.

To all my friends.

To all those I love and who love me.

Yousra .....

Acknowledgments

A dissertation is not an end in itself, but a special moment ina student's journey. We have always waited for the writing ofthis page which would mark the end of it.

First of all, we thank almighty ALLAH who gave us the courage, the will and for having blessed us until the realizationof this work.

Many thanks DR. KIMOUCHE Karima, she guided me to dothis work, following her guidance and precious help. I will not forget to thank my teachers from the Mathematic Department, for the efforts the have made during my coursesin order to bring me to the end of the training.

I take this opportunity to thank all those who have collaboratedirectly or indirectly in the realization of this thesis.
I am thankful to my thesis defence committee for their remarkable comments and suggestions that led to improve thisdissertation.

Sebbagh Yousra

Abstract

This graduation note is essentially devoted to the study of 2D Fourier transform and its generalization. In fact, it is a natural generalization of the idea of Fourier transform. This idea makes the 2D Fourier transform very natural in the study.

In the first part, we defined the 2D Fourier transform and we established some properties of this transform. Moreover, some examples and applications across various fields are illustrated . Then, we gave definitions, properties and some applications for 2D windowed Fourier transform. Finally, the definition of 2D fractional Fourier transform with some applications are presented.

Keywords:

- 2D Fourier transform
- 2D windowed Fourier transform
- 2D fractional Fourier

Résumé

Ce mémoire est consacré essentiellement à l'étude de la transformée de Fourier de deux dimension et à sa généralisation. En fait, il s'agit d'une généralisation naturelle de l'idée de transformée de Fourier. Cette idée rend la transformée de Fourier 2D très naturelle dans l'étude.

Dans la première partie, nous avons défini la transformée de Fourier de deux dimension et nous avons établi quelques propriétés de cette transformée. De plus, quelques exemples et applications dans divers domaines sont illustrés. Ensuite, nous avons donné des définitions, des propriétés et quelques applications pour la transformée de Fourier fenêtrée de deux dimension. Enfin, la définition de la transformée de Fourier fractionnaire de deux dimension avec quelques applications est présentée.

Mots clés:

- Transformée de Fourier de deux dimension

- Transformée de Fourier fenêtrée de deux dimension

- Transformée de Fourier fractionnaire de deux dimension

ملخص

كرست هذه المذكرة بشكل أساسي لدراسة تحويل فورييه ثنائي الأبعاد وتعميمه. في الواقع، إنه تعميم طبيعي لفكرة تحويل فورييه، هذه الفكرة تجعل تحويل فورييه ثنائي الأبعاد أمرًا طبيعيًا جدًا في الدراسة.

في الجزء الأول قمنا بتعريف تحويل فورييه ثنائي الأبعاد وحددنا بعض خصائص هذا التحويل. علاوة على ذلك، تم توضيح بعض الأمثلة والتطبيقات في مختلف المجالات. ثم قدمنا التعاريف والخصائص وبعض التطبيقات لتحويل فورييه السريع ثنائي الأبعاد. وأخيرا، تم عرض تعريف تحويل فورييه الكسري ثنائي الأبعاد مع بعض التطبيقات.

الكلمات المفتاحية:

- تحويل فورييه ثنائي الأبعاد
- تحويل فورييه السريع ثنائي الأبعاد
- تحويل فورييه الكسري ثنائي الأبعاد

List of Symbols

$L^2(R^2)$: is space of square integrable functions in $\mathbb{R}^2$

$\mathcal{F}[f(x_1, x_2)]$ or $\widehat{f}(\xi_1, \xi_2)$: is Fourier transform of function $f(x_1, x_2)$

$\mathcal{F}^{-1}(\mathcal{F}[f(x, y)])$: is the inversion formula of Fourier transform

$\overline{f(x_1, x_2)}$: is the conjugate of $f(x_1, x_2)$

$f * g$ the convolution of the function $f(x_1, x_2)$ and $g(x_1, x_2)$

$\nabla_{\mathbf{t}} f(x_1, x_2) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]^T$: is the gradient vector of the function $f(x_1, x_2)$

$\mathcal{V}_{\Psi}\left[f(x_1, x_2)\right]$: is the 2D window Fourier transform

$\Psi\left(.,.\right)$: is the 2D window function in $L^2\left(\mathbb{R}^2\right)$

$\mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1, x_2)\right]$: is the 2D fractional Fourier transform

$K_{\boldsymbol{\alpha}}\left(\left(x_1, x_2\right), \left(\zeta_1, \zeta_2\right)\right)$: is the kernel of the 2D fractional Fourier transform

$K_{\alpha_i}\left(x_i, \zeta_i\right)$: is the kernel of the 1D fractional Fourier transform

General introduction

The Fourier transform (FT) is a mathematical technique used in signal processing and many areas of scientific study, starting from solving linear differential equation to developing computer models to data processing and analysis, image processing, noise removal, as well as medical fields and many other fields. The primary objective of the Fourier transform is to convert a signal, such as a sound or light wave, from its original representation in the time or spatial domain into a different representation in the frequency domain and vice versa.

The Fourier transform was named in honor of the French Mathematician Jean-Baptiste Fourier,since he used it in (1768-1830) in a manuscript submitted to the Institute of France in 1807, in which he said that the Fourier transform is a mathematical procedure that converts a function from the time domain to the frequency domain.

The idea of extending the (FT) to 2DFT is motivated by several factors, each highlighting its significance in various fields of study and practical applications. Indeed, there has been some compelling reasons to introduce the 2D Fourier transform, we mention:

- Analysis of Spatially Varying Phenomena: Many natural phenomena and signals, such as images, wavefronts, and spatially varying fields, exhibit complex variations in two dimensions. The 2D Fourier transform provides a systematic way to analyze and understand these variations by decomposing them into their frequency components.

- Extension of 1D Fourier Transform: Just as the 1D Fourier transform is indispensable for analyzing one-dimensional signals, the 2D Fourier transform extends this capability to two-dimensional signals. It allows for a deeper understanding of spatially distributed phenomena and enables the application of Fourier analysis techniques to a broader range of problems.

- Image Processing and Computer Vision: In image processing and computer vision, the 2D Fourier transform is a fundamental tool for tasks such as image enhancement, filtering, compression, and feature extraction. Introducing the 2D Fourier transform equips students and practitioners with the necessary tools to analyze and manipulate images effectively.

- Optics and Wave Phenomena: In optics and wave physics, understanding the behavior of light and wave phenomena in two dimensions is crucial. The 2D Fourier transform plays a central role in optics for analyzing optical systems, diffraction patterns, holography, and image formation. Introducing the 2D Fourier transform lays the foundation for understanding advanced optical concepts and applications.

- Mathematical Foundations: From a mathematical perspective, the 2D Fourier transform provides a powerful framework for solving partial differential equations, studying boundary value problems, and analyzing functions defined over two-dimensional domains. Introducing the 2D Fourier transform enhances students' mathematical understanding and problem-solving skills.

- Interdisciplinary Applications: The 2D Fourier transform finds applications across a wide range of disciplines, including physics, engineering, biology, medical imaging, geophysics, and more. By introducing the 2D Fourier transform, educators enable students to apply their knowledge and skills to diverse real-world problems and interdisciplinary research areas.

- Technological Advancements: With the increasing use of digital imaging systems, multimedia technologies, and complex data analysis techniques, the demand for expertise in 2D signal processing and analysis is growing. Introducing the 2D Fourier transform prepares students for careers in fields where these skills are essential for innovation and technological advancement.

Overall, introducing the 2D Fourier transform provides a powerful mathematical toolset for understanding and analyzing two-dimensional signals and phenomena.

In this work, we will give a study of 2D Fourier transform and its generalization. So, we give definitions of 2D Fourier transform and we establish some properties of this transform. Some examples are illustrated. Also, we give definitions and properties for some generalization of 2D Fourier transform as: 2D windowed Fourier transform, 2D fractional Fourier transform. In fact, it is a natural generalization of the idea of Fourier transform. This idea makes the 2D Fourier transform very natural in the study, for more see Bahat (2023).

The work is organized as. In Chapter 1, we introduced the definition of the 2D Fourier transform and we give some examples and basic properties from a mathematical point of view, in addition we give some application of 2DFT in many fields. Chapter 2, studied the 2D windowed Fourier transform (STFT), and its properties with some applications. Finally, we introduced the notion of 2D fractional Fourier transform and its applications.

CHAPTER 1

2D Fourier transform

Contents

The Fourier transform is considered of the most valuable integral transforms because of its ubiquitous applications in applied mathematics, physics, and engineering.

Recall that the Fourier transform, denoted by $\widehat{f}$ or $\mathcal{F}$ of a function f, is defined as

$$\mathcal{F}[f(x)](\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)dx$$

and

$$\mathcal{F}^{-1}(\mathcal{F}[f(x)](\xi))(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \mathcal{F}[f(x)](\xi)dx.$$

If $f(x_1, x_2)$ is a function of two variables then the two-dimensional Fourier transform is simply obtained by repeating the one dimensional Fourier transform in both dimensions. The two-dimensional Fourier transform (2D Fourier transform) is a fundamental tool in signal processing, image processing, and various other fields of science and engineering. It extends the concept of the one-dimensional Fourier transform to two-dimensional signals or functions.

1.1 Definition

Definition 1.1 The Fourier transform for any function $f(x_1, x_2) \in L^2(\mathbb{R}^2)$ is defined as

$$\mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2) = \widehat{f}(\xi_1, \xi_2) = \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} e^{-i(\xi_1 x_1 + \xi_2 x_2)} f(x_1, x_2) dx_1 dx_2 \tag{1.1}$$

where

- $\mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2)$ or $\widehat{f}(\xi_1, \xi_2)$ is the transformed function in frequency domain,
- $f(x_1, x_2)$ is the original function in spatial domain,
- ξ_1 and ξ_2 are the frequency variables, representing spatial frequencies in the x_1- and x_2-directions, respectively,
- i is the imaginary unit,
- The integral is taken over all real values of x_1 and x_2,

and corresponding inversion formula is given by

$$\mathcal{F}^{-1}(\mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2))(x_1, x_2) = \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} e^{i(\xi_1 x_1 + \xi_2 x_2)} \mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2) dx_1 dx_2. \tag{1.2}$$

It is perfectly possible to apply the transform to one dimension leaving the other one untouched and this is often useful for data that is a function of one spatial dimension and time. For example, you might be interesting in seeing how the frequency of a time series varied with spatial position.

1.2 Examples

Example 1.1 Consider a function $rect(x_1, x_2) = \begin{cases} 1 \;\; if \;\; |x_1| \leq 0.5 \text{ and } |x_2| \leq 0.5 \\ 0 \qquad\qquad otherwise. \end{cases}$

Then the 2D-Fourier transform of $rect(x_1, x_2)$ is obtained as

$$\begin{aligned} \mathcal{F}[rect(x_1,x_2)](\xi_1,\xi_2) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} rect(x_1,x_2)e^{-i(\xi_1 x+\xi_2 y)}dx_1dx_2 \\ &= \frac{1}{2\pi}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5} e^{-i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2 \\ &= \frac{1}{2\pi}\int_{-0.5}^{0.5} e^{-i\xi_1 x_1}dx_1 \int_{-0.5}^{0.5} e^{-i\xi_2 x_2}dx_2 \\ &= \frac{1}{2\pi}\left[\frac{e^{-i\xi_1 x_1}}{-i\xi_1}\right]_{-0.5}^{0.5}\left[\frac{e^{-i\xi_2 x_2}}{-i\xi_2}\right]_{-0.5}^{0.5} \\ &= [\sin c(\xi_1)\sin c(\xi_2)] \end{aligned}$$

where $\sin c(x) = \frac{\sin(\pi x)}{\pi x}$ is the sinus cardinal function.

Remark 1.1 Note that the $rect(x_1, x_2)$ is separable (i.e. $rect(x_1, x_2) = rect(x_1)rect\,(x_2)$), the resulting 2D Fourier transform is the product of the corresponding 1D Fourier transform.

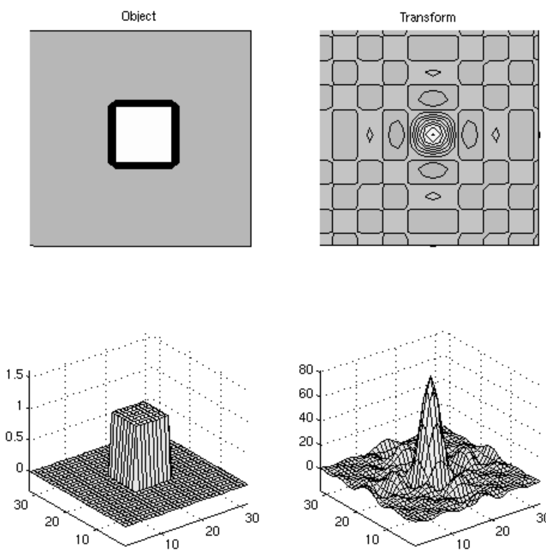

Figure (1.1): Fourier transform for rect function

The function $rect(x_1)rect(x_2)$ is shown on the left. Its transform is the function $\sin c(\xi_1)\sin c(\xi_2)$ shown on the right. (Ignore the units in the axes, they are the units of the discrete FT used to make the figure.)

Example 1.2 Two-dimensional Fourier transform of some functions are given in the following table

$f(x_1, x_2)$	$\mathcal{F}[f(x_1, x_2)](\zeta_1, \zeta_2)$
$rect(x_1, x_2)$	$\frac{\sin(\pi\zeta_1)}{\pi\zeta_1}\frac{\sin(\pi\zeta_2)}{\pi\zeta_2}$
$crect(ax_1, bx_2)$	$\frac{c}{ab}\sin c\left(\frac{\zeta_1}{a}\right) j_1 \sin c\left(\frac{\zeta_2}{b}\right)$
$\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\}$	$2\pi\exp\left\{-2\pi^2\left(\zeta_1^2+\zeta_2^2\right)\right\}$
$\exp\left\{-2\pi\sqrt{x_1^2+x_2^2}\right\}$	$\frac{1}{2\pi^2}\frac{1}{\left(1+\zeta_1^2+\zeta_2^2\right)^{3/2}}$
$\cos\left(\pi\left(x_1^2+x_2^2\right)\right)$	$\sin\left(\pi\left(\zeta_1^2+\zeta_2^2\right)\right)$
$\exp\left\{i\pi\left(x_1^2+x_2^2\right)\right\}$	$i\exp\left\{-i\pi\left(\zeta_1^2+\zeta_2^2\right)\right\}$
$\delta(x_1, x_2)$	1

1.3 Properties

The properties of the 2-D Fourier transform are analogous to the one-dimensional transform. Indeed

Theorem 1.1 *(Linearity) Let $f(x_1, x_2)$ and $g(x_1, x_2)$in $L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}[Af(x_1,x_2)+Bg(x_1,x_2)](\xi_1,\xi_2) = A\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)+B\mathcal{F}[g(x_1,x_2)](\xi_1,\xi_2). \quad (1.3)$$

Proof. Let $S(x_1, x_2) = Af(x_1, x_2) + Bg(x_1, x_2)$, then

$$\begin{aligned}
\mathcal{F}[S(x_1,x_2)](\xi_1,\xi_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}S(x_1,x_2)dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}(Af(x_1,x_2)+Bg(x_1,x_2))dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}Af(x_1,x_2)dx_1dx_2 \\
&\quad +\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}Bg(x_1,x_2)dx_1dx_2 \\
&= A\left[\frac{1}{\left(\sqrt{2\pi}\right)^2}e^{-i(\xi_1x_1+\xi_2x_2)}f(x_1,x_2)dx_1dx_2\right] \\
&\quad +B\left[\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}g(x_1,x_2))dx_1dx_2\right] \\
&= A\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)+B\mathcal{F}[g(x_1,x_2)](\xi_1,\xi_2).
\end{aligned}$$

This completes the proof. ■

Theorem 1.2 *(Translation) The 2D Fourier transform of any function* $f(x_1-k_1, x_2-k_2)$ *is given by*

$$\mathcal{F}[f(x_1-k_1,x_2-k_2)](\xi_1,\xi_2) = e^{-i(\xi_1k_1+\xi_2k_2)}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2). \tag{1.4}$$

Proof. From definition (1.1), we have

$$\begin{aligned}
&\mathcal{F}[f(x_1-k_1,x_2-k_2)(\xi_1,\xi_2) \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{R^2} e^{-i(\xi_1x_1+\xi_2x_2)}f(x_1-k_1,x_2-k_2)dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i[\xi_1(u_1+k_1)+\xi_2(u_2+k_2)]}f(u_1,u_2)du_1du_2\right\}, \\
\text{where } u_1 &= x_1-k_1, u_2 = x_2-k_2. \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1u_1+\xi_2u_2)}e^{-i(\xi_1k_1+\xi_2k_2)}f(u_1,u_2)du_1du_2\right\} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}e^{-i(\xi_1k_1+\xi_2k_2)}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1u+\xi_2v)}f(u_1,u_2)du_1du_2\right\} \\
&= e^{-i(\xi_1k_1+\xi_2k_2)}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2).
\end{aligned}$$

This completes the proof. ■

Theorem 1.3 *(Modulation) The 2D Fourier transform of any function* $e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} f(x_1, x_2)$ *is given by*

$$\mathcal{F}[e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} f(x_1, x_2)](\xi_1, \xi_2) = \mathcal{F}[f(x_1, x_2)](\xi_1 - \zeta_1^0, \xi_2 - \zeta_2^0). \tag{1.5}$$

Proof. From definition (1.1), we have

$$\begin{aligned}
& \mathcal{F}[e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} f(x_1, x_2)](\xi_1, \xi_2) \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi_1 x_1+\xi_2 x_2)} e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} f(x_1, x_2) dx_1 dx_2 \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\left[(\xi_1-\zeta_1^0)x_1+(\xi_2-\zeta_2^0)x_2\right]} f(x_1, x_2) dx_1 dx_2 \\
= & \mathcal{F}[f(x_1, x_2)](\xi_1 - \zeta_1^0, \xi_2 - \zeta_2^0).
\end{aligned}$$

This completes the proof. ∎

Theorem 1.4 *(Orthogonality relation) The 2D Fourier transform of the functions* $f(x_1, x_2)$ *and* $g(u_1, u_2)$ *in* $\mathrm{L}^2(\mathbb{R}^2)$ *satisfies the following orthogonality relation*

$$\langle \mathcal{F}[f(x_1, x_2)], \mathcal{F}[g(u_1, u_2]\rangle = \langle f(x_1, x_2), g(u_1, u_2)\rangle. \tag{1.6}$$

Proof. We have

$$\langle \mathcal{F}[f(x_1, x_2)], \mathcal{F}[g(u_1, u_2]\rangle = \int_{\mathbb{R}^2} \mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2)\overline{\mathcal{F}[g(u_1, u_2)](\xi_1, \xi_2)} d\xi_1 d\xi_2$$

and where

$$\overline{\mathcal{F}[g(u_1, u_2)](\xi_1, \xi_2)} = \overline{\left(\frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} e^{-i(\xi_1 u_1+\xi_2 u_2)} g(u_1, u_2) du_1 du_2\right)}$$

we get

$$\begin{aligned}
\langle \mathcal{F}[f(x_1,x_2)], \mathcal{F}[g(u_1,u_2]\rangle &= \int_{\mathbb{R}^2}\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)} f(x_1,x_2)dx_1dx_2\right) \\
&\quad \times \left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{i(\xi_1u_1+\xi_2u_2)}\overline{g(u_1,u_2)}du_1du_2\right)d\xi_1d\xi_2 \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(u_1,u_2)} \\
&\quad \times \left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(\xi_1(u_1-x_1)+\xi_2(u_2-x_2))}d\xi_1d\xi_2\right)dx_1dx_2du_1du_2 \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(u_1,u_2)}\delta(u_1-x_1,u_2-x_2)dx_1dx_2du_1du_2 \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)\overline{g(u_1,u_2)}dx_1dx_2 \\
&= \langle f(x_1,x_2), g(u_1,u_2)\rangle.
\end{aligned}$$

This completes the proof. ■

Remark 1.2 If we take $f(x_1,x_2)=g(u_1,u_2)$, the orthogonality relation yields Plancherels theorem for the Fourier transforms that states the energy of a signal in the time domain, is the same as the energy in the frequency domain given as

$$\left\|\mathcal{F}[f(x_1,x_2)]\right\| = \left\|f(x_1,x_2)\right\|.$$

Next, we show that the inverse Fourier operator is the adjoint of the Fourier operator.

Theorem 1.5 *Let $f(x_1,x_2)$ and $g(x_1,x_2)$ in $\mathbf{L}^2(\mathbb{R}^2)$, then*

$$\langle \mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2), g(\xi_1,\xi_2)\rangle = \langle f(x_1,x_2), \mathcal{F}^{-1}[g](x_1,x_2)\rangle. \tag{1.7}$$

Proof. We have

$$\langle \mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2), g(\xi_1,\xi_2)\rangle = \int_{\mathbb{R}^2}\mathcal{F}\left[f(x_1,x_2)\right](\xi_1,\xi_2)\overline{g(\xi_1,\xi_2)}d\xi_1d\xi_2$$

and where

$$\mathcal{F}\left[f(x_1,x_2)\right](\xi_1,\xi_2) = \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}f(x_1,x_2)dx_1dx_2$$

we get

$$
\begin{aligned}
& \langle \mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2), g(\xi_1,\xi_2)\rangle \\
= & \int_{\mathbb{R}^2} f(x_1,x_2)\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}\overline{g(\xi_1,\xi_2)}d\xi_1 d\xi_2\right)dx_1dx_2 \\
= & \int_{\mathbb{R}^2} f(x_1,x_2)\overline{\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{i(\xi_1x_1+\xi_2x_2)}g(\xi_1,\xi_2)d\xi_1 d\xi_2\right)}dx_1dx_2 \\
= & \int_{\mathbb{R}^2} f(x_1,x_2)\overline{\mathcal{F}^{-1}[g]}dx_1dx_2 \\
= & \langle f(x_1,x_2), \mathcal{F}^{-1}[g](x_1,x_2)\rangle.
\end{aligned}
$$

This completes the proof. ∎

Theorem 1.6 *(Convolution) Let $f(x_1,x_2)$ and $g(x_1,x_2)$ in $\mathbf{L}^2\left(\mathbb{R}^2\right)$, then*

$$
\mathcal{F}[(f*g)](\xi_1,\xi_2) = \left(\sqrt{2\pi}\right)^2 \mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)\mathcal{F}[g(x_1,x_2)](\xi_1,\xi_2), \tag{1.8}
$$

where $f*g$ denotes the convolution of the function $f(x_1,x_2)$ and $g(x_1,x_2)$ and is given by

$$
(f*g)(x_1,x_2) = \int_{\mathbb{R}^2} f(x_1,x_2)g(u_1-x_1,u_2-x_2)dx_1dx_2,
$$

Proof. By applying definition of Fourier transform to the convolution of the functions $f(x,y)$ and $g(x,y)$, we obtain

$$\begin{aligned}
\mathcal{F}[(f*g)](\xi_1,\xi_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}(f*g)(u_1,u_2)e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}f(u_1,u_2)g(x_1-u_1,x_2-u_2)du_1du_2\right) \\
&\quad \times e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(u_1,u_2)g(w_1,w_2) \\
&\quad \times e^{-i(\xi_1(w_1+u_1)+\xi_2(w_2+u_2)}dw_1dw_2du_1du_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(u_1,u_2)e^{-i(\xi_1u_1+\xi_2u_2)} \\
&\quad \times g(w_1,w_2)e^{-i(\xi_1w_1+\xi_2w_2)}dw_1dw_2du_1du_2 \\
&= \left(\sqrt{2\pi}\right)^2\left\{\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}f(u_1,u_2)e^{-i(\xi_1u_1+\xi_2u_2)}du_1du_2\right\} \\
&\quad \times\left\{\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}g(w_1,w_2)e^{-(\xi_1w_1+\xi_2w_2)}dw_1dw_2\right\} \\
&= \left(\sqrt{2\pi}\right)^2\mathcal{F}[f(u_1,u_2)](\xi_1,\xi_2)\mathcal{F}[g(w_1,w_2)](\xi_1,\xi_2).
\end{aligned}$$

This completes the proof. ■

Theorem 1.7 *(Multiplication) Let $f(x_1,x_2)$ and $g(x_1,x_2)$ in $L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}[f(x_1,x_2)g(x_1,x_2)](\xi_1,\xi_2) = \frac{1}{\left(\sqrt{2\pi}\right)^2}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)*\mathcal{F}[g(x_1,x_2)](\xi_1,\xi_2). \quad (1.9)$$

Proof. Let $S(x_1, x_2) = f(x_1, x_2)g(x_1, x_2)$, then

$$\begin{aligned}
\mathcal{F}[f(x_1,x_2)g(x_1,x_2)](\xi_1,\xi_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} f(x_1,x_2)g(x_1,x_2)e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2,\\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\left(\frac{1}{\sqrt{(2\pi)}^2}\int_{\mathbb{R}^2}\widehat{f}(\zeta_1,\zeta_2)e^{i(\zeta_1x_1+\zeta_2x_2)}d\zeta_1d\zeta_2\right)\\
&\quad\times g(x_1,x_2)e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2\\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\widehat{f}(\zeta_1,\zeta_2)\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\right.\\
&\quad\left.\times\int_{\mathbb{R}^2} g(x_1,x_2)e^{-i[(\xi_1-\zeta_1)x_1+(\xi_2-\zeta_2)x_2])}dx_1dx_2\right)d\zeta_1d\zeta_2\\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\widehat{f}(\zeta_1,\zeta_2)\widehat{g}(\xi_1-\zeta_1,\xi_2-\zeta_2)d\zeta_1d\zeta_2\\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)*\mathcal{F}[g(x_1,x_2)](\xi_1,\xi_2).
\end{aligned}$$

This completes the proof. ■

Theorem 1.8 *(Linear transformation of the domain $\mathbb{R}^2$) Let $f(\mathbf{x}) \in L^2(\mathbb{R}^2)$, then we have*

$$\mathcal{F}[f(A\mathbf{x})](\boldsymbol{\xi}) = \frac{1}{|\det(A)|}\mathcal{F}[f(\mathbf{x})](A^{-T}\boldsymbol{\xi}), \tag{1.10}$$

where A is a nonsingular 2×2 matrix defined as $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$, A^{-T} denotes $(A^{-T})^{-1} = (A^{-1})^T$, $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$.

Proof. If we denote $g(\mathbf{x}) = f(A\mathbf{x})$, then

$$\mathcal{F}[g(\mathbf{x})](\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)}^2}\int_{\mathbb{R}^2} e^{-i\boldsymbol{\xi}.\mathbf{x}}f(A\mathbf{x})d\mathbf{x}$$

with the change of variables $\mathbf{s} = A\mathbf{x}$, the Jacobian is $\frac{\partial(\mathbf{s})}{\partial(\mathbf{x})} = \det(A)$, and using standard techniques for change of variables in an integral, we obtain where $\mathbf{s} = (s_1, s_2)$

$$\begin{aligned}
\mathcal{F}[g(\mathbf{x})](\boldsymbol{\xi}) &= \frac{1}{|\det(A)|\sqrt{(2\pi)}^2}\int_{\mathbb{R}^2} e^{-i\boldsymbol{\xi}.(A^{-1}\mathbf{s})}f(\mathbf{s})d\mathbf{s}\\
&= \frac{1}{|\det(A)|\sqrt{(2\pi)}^2}\int_{\mathbb{R}^2} e^{-i.(A^{-1}\boldsymbol{\xi}).\mathbf{s}}f(\mathbf{s})d\mathbf{s}\\
&= \frac{1}{|\det(A)|}\mathcal{F}[f(\mathbf{x})](A^{-T}\boldsymbol{\xi}),
\end{aligned}$$

where we use the identity $a.(Cb) = (C^Ta).b$ or equivalently $a^T.Cb = (C^Ta)^Tb$. ■

Theorem 1.9 *(Differentiation) Let $f(x_1,x_2)\in L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}\left[\nabla_{\mathbf{x}} f(x_1,x_2)\right](\xi_1,\xi_2) = i\,(\xi_1,\xi_2)\,\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2). \tag{1.11}$$

Proof. We define the gradient vector $G(x_1,x_2)=\nabla_{\mathbf{x}} f(x_1,x_2)=\left[\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\right]^T$, taking the derivative of synthesis equation for f to get the synthesis equation for

$$\begin{aligned}
\mathcal{F}\left[\nabla_{\mathbf{x}} f(x_1,x_2)\right](\xi_1,\xi_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\nabla_{\mathbf{x}} f(x_1,x_2)e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2 \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\left[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right]e^{-i(\xi_1x_1+\xi_2x_2)}dx_1dx_2
\end{aligned}$$

$$\begin{aligned}
g_1(x_1,x_2) &= \frac{\partial f}{\partial x_1}(x_1,x_2)=\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\widehat{f}(x_1,x_2)i\xi_1e^{i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2 \\
g_2(x_1,x_2) &= \frac{\partial f}{\partial x_2}(x_1,x_2)=\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\widehat{f}(x_1,x_2)i\xi_2e^{i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2
\end{aligned}$$

thus

$$\begin{aligned}
\mathcal{F}[g_1(x_1,x_2)](\xi_1,\xi_2) &= i\xi_1\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2) \\
\mathcal{F}[g_2(x_1,x_2)](\xi_1,\xi_2) &= i\xi_2\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2)
\end{aligned}$$

and in matrix form

$$\mathcal{F}[G(x_1,x_2)](\xi_1,\xi_2)=i(\xi_1,\xi_2)\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2).$$

This completes the proof. ■

Theorem 1.10 *(Differentiation in frequency) Let $f(x_1,x_2)\in L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}\left[(x_1,x_2)\,f(x_1,x_2)\right](\xi_1,\xi_2) = i\nabla_{\xi}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2). \tag{1.12}$$

Proof. Let $G(x_1, x_2) = (x_1, x_2)\, f(x_1, x_2)$, taking the derivative of analysis equation for f with respect to (ξ_1, ξ_2)

$$
\begin{aligned}
\frac{\partial \mathcal{F}[f(x_1,x_2)]}{\partial x_1}(\xi_1,\xi_2) &= \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} f(\xi_1 x_1+\xi_2 x_2)(-ix)e^{-i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2\\
&= -i\frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} g_1(x_1,x_2)e^{-i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2\\
&= -i\mathcal{F}[g_1(x_1,x_2)](\xi_1,\xi_2).\\
\frac{\partial \mathcal{F}[f(x_1,x_2)]}{\partial x_2}(\xi_1,\xi_2) &= \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} f(\xi_1 x_1+\xi_2 x_2)(-iy)e^{-i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2\\
&= -i\frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} g_2(x_1,x_2)e^{-i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2\\
&= -i\mathcal{F}[g_2(x_1,x_2)](\xi_1,\xi_2).
\end{aligned}
$$

Thus

$$
\mathcal{F}[G(x_1,x_2)](\xi_1,\xi_2) = i\nabla_{\boldsymbol{\xi}}\mathcal{F}[f(x_1,x_2)](\xi_1,\xi_2).
$$

This completes the proof. ■

Theorem 1.11 *(Complex conjugation) Let $f(x_1, x_2) \in L^2(\mathbb{R}^2)$, then*

$$
\mathcal{F}\left[\overline{f(x_1,x_2)}\right](\xi_1,\xi_2) = \overline{\mathcal{F}\left[f(x_1,x_2)\right](-\xi_1,-\xi_2)}. \tag{1.13}
$$

Proof. Let $g(x_1, x_2) = \overline{f(x_1, x_2)}$, then where.

$$
\begin{aligned}
\mathcal{F}\left[\overline{f(x_1,x_2)}\right](\xi_1,\xi_2) &= \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2}\overline{f(x_1,x_2)}e^{-i(\xi_1 x_1+\xi_2 x_1)}dx_1dx_2,\\
&= \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2}\overline{f(x_1,x_2)e^{i(\xi_1 x_1+\xi_2 x_2)}}dx_1dx_2,\\
&= \overline{\frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} f(x_1,x_2)e^{i(\xi_1 x_1+\xi_2 x_2)}dx_1dx_2},\\
&= \overline{\mathcal{F}\left[f(x_1,x_2)\right](-\xi_1,-\xi_2)}.
\end{aligned}
$$

This completes the proof. ■

Theorem 1.12 *(Duality) Let $f(x_1, x_2) \in L^2(\mathbb{R}^2)$, then*

$$
\widehat{f}(x_1,x_2) = f(-\xi_1,-\xi_2). \tag{1.14}
$$

Proof. Let $g(x_1, x_2) = \widehat{f}(x_1, x_2)$, then

$$\begin{aligned}
\mathcal{F}\left[g((x_1, x_2))\right](\xi_1, \xi_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} g(x_1, x_2) e^{-i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2, \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \widehat{f}(x_1, x_2) e^{-i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2, \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \widehat{f}(x_1, x_2) e^{i[(-\xi_1)x_1 + (-\xi_2)x_2]} dx_1 dx_2, \\
&= f(-\xi_1, -\xi_2).
\end{aligned}$$

This completes the proof. ■

Theorem 1.13 *(Separability) Let $f_1(\mathbf{x}_1), \ldots, f_n(\mathbf{x}_n) \in L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}[f_1(\mathbf{x}_1) \ldots f_n(\mathbf{x}_n)](\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) = \mathcal{F}[f_1(\mathbf{x}_1)](\boldsymbol{\xi}_1) \ldots \mathcal{F}[f_n(\mathbf{x}_n)](\boldsymbol{\xi}_n) \tag{1.15}$$

where $\mathbf{x} = (x_1, x_2)$,and $\boldsymbol{\xi} = (\xi_1, \xi_2)$.

Proof. This follow from the separability of the complex exponentials

$$\begin{aligned}
& \mathcal{F}[f_1(\mathbf{x}_1) \ldots f_n(\mathbf{x}_n)](\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) \\
= & \frac{1}{\left(\sqrt{2\pi}\right)} \int_{\mathbb{R}} \cdots \frac{1}{\left(\sqrt{2\pi}\right)} \int_{\mathbb{R}} f_1(\mathbf{x}_1) \ldots f_n(\mathbf{x}_2) e^{-i(\boldsymbol{\xi}_1 \mathbf{x}_1 + \boldsymbol{\xi}_2 \mathbf{x}_2 + \ldots \boldsymbol{\xi}_n \mathbf{x}_n)} d\mathbf{x}_1 \ldots d\mathbf{x}_n \\
= & \left(\frac{1}{\left(\sqrt{2\pi}\right)} \int_{\mathbb{R}} f_1(\mathbf{x}_1) e^{-i\boldsymbol{\xi}_1 \mathbf{t}_1} d\mathbf{x}_1 \right) \cdots \left(\frac{1}{\left(\sqrt{2\pi}\right)} \int_{\mathbb{R}} f_n(\mathbf{x}_n) e^{-i\boldsymbol{\xi}_n \mathbf{t}_n} d\mathbf{x}_n \right) \\
= & \mathcal{F}[f_1(\mathbf{x}_1)](\boldsymbol{\xi}_1) \ldots \mathcal{F}[f_n(\mathbf{x}_n)](\boldsymbol{\xi}_n).
\end{aligned}$$

This completes the proof. ■

Theorem 1.14 *(Parseval relation) Let $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2 = \int_{\mathbb{R}^2} (\mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2)) \left(\overline{\mathcal{F}[g(x_1, x_2)](\xi_1, \xi_2)} \right) d\xi_1 d\xi_2. \tag{1.16}$$

Proof. Let $h(x_1, x_2) = \overline{g(x_1, x_2)}$ and $S(x_1, x_2) = f(x_1, x_2) h(x_1, x_2)$, then

$$\begin{aligned}
\mathcal{F}\left[S(x_1, x_2)\right](\xi_1, \xi_2) &= \int_{\mathbb{R}^2} (\mathcal{F}[f(x_1, x_2)](\zeta_1, \zeta_2)) (\mathcal{F}[h(x_1, x_2)](\xi_1 - \zeta 1, \xi_2 - \zeta_2) d\zeta_1 d\zeta_2 \\
&= \int_{\mathbb{R}^2} (\mathcal{F}[f(x_1, x_2)](\zeta_1, \zeta_2)) \left(\overline{\mathcal{F}[g(x_1, x_2)](\xi_1 - \zeta 1, \xi_2 - \zeta_2)} \right) d\zeta_1 d\zeta_2
\end{aligned}$$

Evaluating at $(\xi_1, \xi_2) = (0, 0)$,

$$\begin{aligned}\mathcal{F}\left[S(x_1, x_2)\right](0,0) &= \int_{\mathbb{R}^2} f(x_1, x_2)\overline{g(x_1, x_2)}dx_1 dx_2 \\ &= \int_{\mathbb{R}^2} \left(\mathcal{F}\left[f(x_1, x_2)\right](\zeta_1, \zeta_2)\right)\left(\overline{\mathcal{F}\left[g(x_1, x_2)\right](\zeta_1, \zeta_2)}\right) d\zeta_1, \zeta_2.\end{aligned}$$

If $f(x_1, x_2) = g(x_1, x_2)$,we obtain $\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \, dx_1 dx_2 = \int_{\mathbb{R}^2} |\mathcal{F}[f(x_1, x_2)](\xi_1, \xi_2)|^2$. This completes the proof. ■

1.4 Applications

The 2D Fourier Transform (2D FT) is a powerful mathematical tool used to analyze two-dimensional signals in the frequency domain. Here are some applications of the 2D Fourier transform:

1. Image Processing:

   - Image Filtering: The 2D FT is used for spatial frequency analysis in images, enabling techniques such as low-pass, high-pass, and band-pass filtering for tasks such as image enhancement, noise reduction, and feature extraction.
   - Image Reconstruction: In computed tomography (CT) and magnetic resonance imaging (MRI), the 2D FT is utilized for image reconstruction from projection data acquired in the frequency domain, providing detailed anatomical information.
   - Image Registration: The 2D FT aids in image registration by analyzing frequency content in images to align them spatially, facilitating applications such as medical image fusion and remote sensing.

2. Optics and Photonics:

   - Fourier Optics: In optical systems, the 2D FT helps in the design and analysis of optical components such as lenses, mirrors, and diffraction gratings, allowing for precise control of light propagation, beam shaping, and imaging.

- Holography: The 2D FT plays a crucial role in holography, where it is used to encode and decode holographic information, enabling applications such as 3D imaging, holographic data storage, and optical security.

3. Signal Processing:

   - Radar Signal Processing: In radar systems, the 2D FT is applied to analyze radar signals in the range-Doppler domain, facilitating target detection, localization, and tracking in applications such as air traffic control and weather monitoring.
   - Sonar Signal Processing: Similar to radar, the 2D FT aids in analyzing sonar signals to detect underwater objects, map seabed features, and study oceanographic phenomena in marine environments.

4. Geophysics and Remote Sensing:

   - Seismic Data Analysis: The 2D FT assists in analyzing seismic data to image subsurface structures, locate hydrocarbon reservoirs, and assess earthquake hazards in geophysical exploration and earthquake engineering.
   - Remote Sensing: In remote sensing applications, the 2D FT is used to analyze satellite images and aerial photographs for land cover classification, environmental monitoring, and natural resource management.

5. Biomedical Imaging:

   - Electroencephalography (EEG) Analysis: The 2D FT helps in analyzing EEG signals to study brain activity, identify neural oscillations, and diagnose neurological disorders such as epilepsy and sleep disorders.
   - Functional MRI (fMRI): In functional MRI studies, the 2D FT is used to analyze temporal fluctuations in blood oxygenation level-dependent (BOLD) signals, revealing brain activation patterns associated with cognitive tasks and neurological conditions.

These applications demonstrate the wide-ranging utility of the 2D Fourier transform across various fields, enabling the analysis and manipulation of two-dimensional signals in the frequency domain for diverse applications in science, engineering, and technology. (see Dubois (2019), Healy, Kutay, Ozaktas and Sheridan(2016))

CHAPTER 2

2D Windowed Fourier transform (STFT)

Contents

The Fourier transform provides a signal representation in terms of global, periodic functions. To allow for localized descriptions, one needs to focus Fourier analysis only in a portion of the signal. This can be accomplished using a window function, which retains only a piece of the signal under analysis, and that can then be successively shifted to other locations, to cover all the time domain of interest. This approach is known as the windowed Fourier transform (WFT) or short time Fourier transform, and was proposed by Gabor, who adopted a Gaussian function for this purpose (hence the name also used for the implementation of WFT, known as the Gabor transform), given its optimal properties in terms of the localization obtained both in the time and frequency domains. in this chapter, we introduce the 2D windowed Fourier transform, also known as the short-time Fourier transform (STFT), which is a technique used in signal processing to analyze non-stationary signals, where the frequency content of the signal changes over time.

2.1 Definition

Definition 2.1 Let Ψ be a given 2D window function in $L^2(\mathbb{R}^2)$, then the 2D window Fourier transform (2DWFT) of any function $f(x_1, x_2) \in L^2(\mathbb{R}^2)$ is defined and denoted as

$$\mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)) = \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)} f(x_1,x_2) \times\overline{\Psi(x_1-b_1,x_2-b_2)}dx_1dx_2, \quad b_1,b_2,\zeta_1,\zeta_2 \in \mathbb{R}. \tag{2.1}$$

where

- $f(x_1, x_2)$ is the original signal in the time domain, which is a function of the continuous variables x_1 and x_2.
- $\Psi(x_1 - b_1, x_2 - b_2)$ is a window function, which is typically a short-duration function that is used to window the signal in both time dimensions. The window function is usually chosen to control the time-frequency resolution trade-off.

Further, the WFT (2.1) can be rewritten as

$$\mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)) = \mathcal{F}\left[f(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)}\right]. \tag{2.2}$$

Applying inverse FT (1.2), (2.2) yields

$$\begin{aligned} & f(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)} \\ = \; & \mathcal{F}^{-1}\left[\mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2))\right] \\ = \; & \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}\mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2))\,d\zeta_1 d\zeta_2 \end{aligned} \tag{2.3}$$

Multiplying (2.3) both sides by $\Psi(x_1 - b_1, x_2 - b_2)$ and then integrating with respect to $db_1 db_2$, we get

$$\begin{aligned} f(x_1,x_2)\|\Psi\|^2 = \; & \frac{1}{(\sqrt{2\pi})^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}\mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)) \\ & \times\Psi(x_1-b_1,x_2-b_2)\,d\zeta_1 d\zeta_2 db_1 db_2. \end{aligned}$$

Equivalently, we have

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{\left(\sqrt{2\pi}\right)^2 \|\Psi\|^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)} \mathcal{V}_\Psi \left[f(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right) \\ &\quad \times \Psi\left(x_1 - b_1, x_2 - b_2\right) d\zeta_1 d\zeta_2 db_1 db_2. \end{aligned} \tag{2.4}$$

equation (2.4) gives the inversion formula corresponding to 2DWFT (2.1).

Remark 2.1 The 2DWFT or STFT essentially applies the Fourier transform to small, overlapping windows of the signal, allowing for analysis of the signal's frequency content at different points in time. By varying the window function and the amount of overlap between adjacent windows, you can control the time and frequency resolution of the STFT.

2.2 Properties

The two dimensional windowed Fourier transform is a power tool used in signal processing for analyzing non-stationary signals. It extends the concept of the windowed Fourier transform to two dimensions, allowing for time-frenquency analysis of 2D signals, such as images. In this section we give some properties of the 2D windowed Fourier transform along with their proofs.

Theorem 2.1 *(Linearity) For any two functions $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2\left(\mathbb{R}^2\right)$, we have following*

$$\begin{aligned} &\mathcal{V}_\Psi \left[Af(x_1, x_2) + Bg(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right) \\ = \; &A\mathcal{V}_\Psi \left[f(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right) + B\mathcal{V}_\Psi \left[g(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right). \end{aligned} \tag{2.5}$$

Proof. we have from definition (2.1)

$$
\begin{aligned}
& \mathcal{V}_\Psi \left[Af(x_1,x_2)+Bg(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}\left[Af(x_1,x_2)+Bg(x_1,x_2)\right]\times\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2 \\
= & A\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}f(x_1,x_2)\times\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2 \\
& +B\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}g(x_1,x_2)\times\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2 \\
= & A\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
& +B\mathcal{V}_\Psi\left[g(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right).
\end{aligned}
$$

This completes the proof. ∎

Theorem 2.2 *(Translation) For any two functions $f(x_1,x_2)$ and $g(x_1,x_2)$ in $L^2\left(\mathbb{R}^2\right)$, we have following*

$$
\mathcal{V}_\Psi\left[f(x_1-k_1,x_2-k_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)=e^{i(\zeta_1 k_1+\zeta_2 k_2)}\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right). \tag{2.6}
$$

Proof. we have from definition (2.1)

$$
\begin{aligned}
& \mathcal{V}_\Psi\left[f(x_1-k_1,x_2-k_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i[\zeta_1(x_1-k_1)+\zeta_2(x_2-k_2)]}f(x_1-k_1,x_2-k_2) \\
& \times\overline{\Psi\left((x_1-k_1)-b_1,(x_2-k_2)-b_2\right)}dx_1dx_2 \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{i(\zeta_1 k_1+\zeta_2 k_2)}e^{-i(\zeta_1 x_1+\zeta_2 x_2)}f(u_1,u_2) \\
& \times\overline{\Psi\left(u_1-b_1,u_2-b_2\right)}du_1du_2,
\end{aligned}
$$

where $u_1=x_1-k_1, u_2=x_2-k_2$

$$
\begin{aligned}
= & e^{i(\zeta_1 k_1+\zeta_2 k_2)}\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}f(u_1,u_2) \\
& \times\overline{\Psi\left(u_1-b_1,u_2-b_2\right)}du_1du_2 \\
= & e^{i(\zeta_1 k_1+\zeta_2 k_2)}\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right).
\end{aligned}
$$

This completes proof. ∎

Remark 2.2 The time shifting property of the 2D windowed Fourier transform states that shifting the input signal in the time domain results in a corresponding shift in the WFT coefficients.

Proof.

Remark 2.3 Mathematically, if we have a signal $f(x_1, x_2)$ and we shift it by Δk_1 units in the time domain, the resulting 2D-WFT $\mathcal{V}_\Psi\left[f(x_1, x_2)\right]\left(\left(b_1, b_2\right), \left(\zeta_1, \zeta_2\right)\right)$ will be shifted by Δk_1 units in the time axis. Indeed

$$\begin{aligned}
& \mathcal{V}_\Psi\left[f(x_1 - k_1, x_2)\right]\left(\left(b_1, b_2\right), \left(\zeta_1, \zeta_2\right)\right) \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{-i[\zeta_1 x_1 + \zeta_2 x_2]} f(x_1 - k_1, x_2) \\
& \times \overline{\Psi\left((x_1 - k_1) - b_1, x_2 - b_2\right)} dx_1 dx_2 \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{i(\zeta_1 k_1)} e^{-i(\zeta_1 u_1 + \zeta_2 x_2)} f(u_1, x_2) \\
& \times \overline{\Psi\left(u_1 - b_1, x_2 - b_2\right)} du_1 dx_2,
\end{aligned}$$

where $u_1 = x_1 - k_1$,

$$\begin{aligned}
= \; & e^{i(\zeta_1 k_1)} \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{-i(\zeta_1 u_1 + \zeta_2 x_2)} f(u_1, x_2) \\
& \times \overline{\Psi\left(u_1 - b_1, x_2 - b_2\right)} du_1 dx_2 \\
= \; & e^{i(\zeta_1 k_1)} \mathcal{V}_\Psi\left[f(u_1, x_2)\right]\left(\left(b_1, b_2\right), \left(\zeta_1, \zeta_2\right)\right).
\end{aligned}$$

Hence, the shift in the time domain results in a corresponding shift in the time axis of the 2D windowed Fourier transform coefficients.

■

Theorem 2.3 *(Modulation) The 2D Fourier transform of any function* $e^{-i(\zeta_1^0 x_1 + \zeta_2^0 x_2)} f(x_1, x_2)$ *is given by*

$$\mathcal{V}_\Psi[e^{-i(\zeta_1^0 x_1 + \zeta_2^0 x_2)} f(x_1, x_2)]\left(\left(b_1, b_2\right), \left(\zeta_1, \zeta_2\right)\right) = \mathcal{V}_\Psi[f(x_1, x_2)]\left(\left(b_1, b_2\right), \left(\zeta_1 - \zeta_1^0, \zeta_2 - \zeta_2^0\right)\right). \tag{2.7}$$

Proof. From definition (2.1), we have

$$\begin{aligned}
& \mathcal{V}_\Psi[e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)) \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta_1 x_1+\zeta_2 x_2)} e^{-i(\zeta_1^0 x_1+\zeta_2^0 x_2)} \\
& \times f(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)}dx_1dx_2, \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left[(\xi_1-\zeta_1^0)x_1+(\xi_2-\zeta_2^0)x_2\right]} \\
& \times f(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)}dx_1dx_2, \\
= & \mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\xi_1-\zeta_1^0,\xi_2-\zeta_2^0)).
\end{aligned}$$

This completes the proof. ■

Theorem 2.4 *(Orthogonality) The 2D windowed Fourier transform of the functions $f(x_1,x_2)$ and $g(u_1,u_2)$ in $\mathrm{L}^2(\mathbb{R}^2)$ satisfies the following orthogonality relation*

$$\begin{aligned}
& \langle \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right), \mathcal{V}_\Psi\left[g(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)\rangle \\
= & \|\Psi\|^2 \langle f(x_1,x_2), g(x_1,x_2)\rangle .
\end{aligned} \tag{2.8}$$

Proof. By definition (2.1), we have

$$\begin{aligned}
& \langle \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right), \mathcal{V}_\Psi\left[g(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)\rangle \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
& \times \overline{\mathcal{V}_\Psi\left[g(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)}d\zeta_1d\zeta_2db_1db_2 \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
& \times \overline{\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)} g(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)}dx_1dx_2\right)}d\zeta_1d\zeta_2db_1db_2 \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1+\zeta_2 x_2)}\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)d\zeta_1d\zeta_2\right) \\
& \overline{g(x_1,x_2)}\Psi\left(x_1-b_1,x_2-b_2\right)dx_1dx_2db_1db_2.
\end{aligned}$$

By virtue of Equation (2.3), we have

$$\begin{aligned}
& \langle \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right), \mathcal{V}_\Psi\left[g(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right)\rangle \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x_1,x_2)\overline{\Psi(x_1-b_1,x_2-b_2)}\Psi\left(x_1-b_1,x_2-b_2\right)\overline{g(x_1,x_2)}dt_1dt_2db_1db_2
\end{aligned}$$

therefore Fubini's theorem allows us to interchange the order of integration. We continue as follows:

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(x_1,x_2)}dt_1dt_2 \int_{\mathbb{R}^n} \overline{\Psi\left(x_1-b_1,x_2-b_2\right)}\Psi\left(x_1-b_1,x_2-b_2\right)db_1db_2 \\
&= \left\|\Psi\right\|^2 \left\langle f(x_1,x_2), g(x_1,x_2)\right\rangle .
\end{aligned}
$$

This completes the proof. ■

Theorem 2.5 *Let $f(x_1,x_2)$ and $g(x_1,x_2)$ in $\boldsymbol{L}^2\left(\mathbb{R}^2\right)$, then*

$$
\langle \mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)), g(\zeta_1,\zeta_2)\rangle = \langle f(x_1,x_2), \mathcal{V}_\Psi[\overline{g}](x_1,x_2)\rangle. \tag{2.9}
$$

Proof. We have

$$
\langle \mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)), g(\zeta_1,\zeta_2)\rangle = \int_{\mathbb{R}^2} \mathcal{V}_\Psi\left[f(x_1,x_2)\right](\zeta_1,\zeta_2)\overline{g(\zeta_1,\zeta_2)}d\zeta_1 d\zeta_2
$$

and where

$$
\begin{aligned}
& \mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\zeta_1,\zeta_2)\right) \\
=\; & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1x_1+\zeta_2x_2)} f(x_1,x_2)\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2
\end{aligned}
$$

we get

$$
\begin{aligned}
& \langle \mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)), g(\zeta_1,\zeta_2)\rangle \\
=\; & \int_{\mathbb{R}^2} f(x_1,x_2)\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{-i(\zeta_1x_1+\zeta_2x_2)}\overline{g(\zeta_1,\zeta_2)}d\zeta_1d\zeta_2\right)dx_1dx_2 \\
=\; & \int_{\mathbb{R}^2} f(x_1,x)\overline{\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} e^{i(\zeta_1x_1+\zeta_2x_2)}g(\zeta_1,\zeta_2)\Psi\left(x_1-b_1,x_2-b_2\right)d\zeta_1d\zeta_2\right)}dx_1dx_2 \\
=\; & \int_{\mathbb{R}^2} f(x_1,x_2)\mathcal{V}_\Psi[\overline{g}]dx_1dx_2 \\
=\; & \langle f(x_1,x_2), \mathcal{V}_\Psi[\overline{g}](x_1,x_2)\rangle.
\end{aligned}
$$

This completes the proof. ■

Theorem 2.6 *(Convolution) Let $f(x_1,x_2)$ and $g(x_1,x_2)$ in $\boldsymbol{L}^2\left(\mathbb{R}^2\right)$, then*

$$
\begin{aligned}
& \mathcal{V}_\Psi[(f*g)]((b_1,b_2),(\zeta_1,\zeta_2)) \\
=\; & \left(\sqrt{2\pi}\right)^2 \mathcal{V}_\Psi[f(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2))\mathcal{V}_\Psi[g(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)),
\end{aligned} \tag{2.10}
$$

where $f * g$ denotes the convolution of the function $f(x_1, x_2)$ and $g(x_1, x_2)$ and is given by

$$(f * g)(x_1, x_2) = \int_{\mathbb{R}^2} f(x_1, x_2) g(u_1 - x_1, u_2 - x_2) dx_1 dx_2.$$

Proof. By applying definition of Fourier transform to the convolution of the functions $f(x, y)$ and $g(x, y)$, we obtain

$$\begin{aligned}
& \mathcal{V}_\Psi[(f * g)]((b_1, b_2), (\zeta_1, \zeta_2)) \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} (f * g)(u_1, u_2) e^{-i(\zeta_1 x_1 + \zeta_2 x_2)} \overline{\Psi(x_1 - b_1, x_2 - b_2)} dx_1 dx_2 \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(u_1, u_2) g(x_1 - u_1, x_2 - u_2) du_1 du_2 \right) \\
& \times e^{-i(\zeta_1 x_1 + \zeta_2 x_2)} \overline{\Psi(x_1 - b_1, x_2 - b_2)} dx_1 dx_2 \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u_1, u_2) g(w_1, w_2) \\
& \times e^{-i(\zeta_1 (w_1 + u_1) + \zeta_2 (w_2 + u_2)} \overline{\Psi((w_1 + u_1) - b_1, (w_2 + u_2) - b_2)} \\
& dw_1 dw_2 du_1 du_2 \\
= & \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u_1, u_2) e^{-i(\zeta_1 u_1 + \zeta_2 u_2)} \overline{\Psi(u_1 - b_1, u_2 - b_2)} \\
& \times g(w_1, w_2) e^{-i(\zeta_1 w_1 + \zeta_2 w_2)} \overline{\Psi(w_1 - b_1, w_2 - b_2)} dw_1 dw_2 du_1 du_2 \\
= & \left(\sqrt{2\pi}\right)^2 \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} f(u_1, u_2) e^{-i(\zeta_1 u_1 + \zeta_2 u_2)} \overline{\Psi(u_1 - b_1, u_2 - b_2)} du_1 du_2 \\
& \times \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbb{R}^2} g(w_1, w_2) e^{-(\zeta_1 w_1 + \zeta_2 w_2)} \overline{\Psi(w_1 - b_1, w_2 - b_2)} dw_1 dw_2 \\
= & \left(\sqrt{2\pi}\right)^2 \mathcal{V}_\Psi[f(u_1, u_2)](\zeta_1, \zeta_2) \mathcal{V}_\Psi[g(w_1, w_2)](\zeta_1, \zeta_2).
\end{aligned}$$

This completes the proof. ∎

Theorem 2.7 *(Multiplication) Let $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2(\mathbb{R}^2)$, then*

$$\begin{aligned}
& \mathcal{V}_\Psi[f(x_1, x_2) g(x_1, x_2)]((b_1, b_2), (\zeta_1, \zeta_2)) \qquad (2.11) \\
= & \frac{1}{(\sqrt{2\pi})^2} \mathcal{V}_\Psi[f(x_1, x_2)]((b_1, b_2), (\xi_1, \xi_2)) * \mathcal{V}_\Psi[g(x_1, x_2)]((b_1, b_2), (\xi_1, \xi_2)).
\end{aligned}$$

Proof. Let $S(x_1, x_2) = f(x_1, x_2)g(x_1, x_2)$, then

$$
\begin{aligned}
& \mathcal{V}_\Psi[f(x_1,x_2)g(x_1,x_2)]((b_1,b_2),(\zeta_1,\zeta_2)) \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} f(x_1,x_2)g(x_1,x_2)e^{-i(\zeta_1x_1+\zeta_2x_2)} \times \overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2, \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\left(\frac{1}{\left(\sqrt{2\pi}\right)^2\|\Psi\|^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(\xi_1x_1+\xi_2x_2)}\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\right. \\
& \times\Psi\left(x_1-b_1,x_2-b_2\right)dx_1dx_2) \\
& \times g(x_1,x_2)e^{-i(\zeta_1x_1+\zeta_2x_2)}\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2, \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2\|\Psi\|^2}\int_{\mathbb{R}^2}\mathcal{V}_\Psi\left[f(x_1,x_2)\right]\left((b_1,b_2),(\xi_1,\xi_2)\right)\times\Psi\left(x_1-b_1,x_2-b_2\right)dx_1dx_2 \\
& \times\left(\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} g(x_1,x_2)e^{-i[(\zeta_1-\xi_1)x_1+(\zeta_2-\xi_2)x_2])}\times\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2\right)d\xi_1d\xi_2 \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2\|\Psi\|}\mathcal{V}_\Psi[f(x_1,x_2)](\xi_1,\xi_2)*\mathcal{V}_\Psi[g(x_1,x_2)](\zeta_1,\zeta_2).
\end{aligned}
$$

This completes the proof. ■

Theorem 2.8 *(Complex conjugation) Let $f(x_1, x_2) \in L^2(\mathbb{R}^2)$, then*

$$
\mathcal{V}_\Psi\left[\overline{f((x_1,x_2))}\right]((b_1,b_2),(\zeta_1,\zeta_2)) = \overline{\mathcal{V}_\Psi\left[f(x_1,x_2)\right](b_1,b_2),(-\zeta_1,-\zeta_2))}. \tag{2.12}
$$

Proof. Let $g(x_1, x_2) = \overline{f(x_1, x_2)}$, then

$$
\begin{aligned}
& \mathcal{V}_\Psi\left[\overline{f(x_1,x_2)}\right]((b_1,b_2),(\zeta_1,\zeta_2)) \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\overline{f(x_1,x_2)}e^{-i(\zeta_1x_1+\zeta_2x_1)}\overline{\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2, \\
= \; & \frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2}\overline{f(x_1,x_2)e^{i(\zeta_1x_1+\zeta_2x_2)}\Psi\left(x_1-b_1,x_2-b_2\right)}dx_1dx_2, \\
= \; & \overline{\frac{1}{\left(\sqrt{2\pi}\right)^2}\int_{\mathbb{R}^2} f(x_1,x_2)e^{i(\zeta_1x_1+\zeta_2x_2)}\Psi\left(x_1-b_1,x_2-b_2\right)dx_1dx_2}, \\
= \; & \overline{\mathcal{V}_\Psi\left[f(x_1,x_2)\right](b_1,b_2),(-\zeta_1,-\zeta_2))}.
\end{aligned}
$$

This completes the proof. ■

Theorem 2.9 *(Duality) Let $f(x_1, x_2) \in L^2(\mathbb{R}^2)$, then*

$$
\mathcal{V}_\Psi[\mathcal{V}_\Psi[f(x_1,x_2)]]((b_1,b_2),(\zeta_1,\zeta_2)) = f((-b_1,-b_2)(-\xi_1,-\xi_2)). \tag{2.13}
$$

Proof. Let $g(x_1, x_2) = \mathcal{V}_\Psi[f(x_1, x_2)]$, then

$$\begin{aligned}
& \mathcal{V}_\Psi \left[g(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right) \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} g(x_1, x_2) e^{-i(\xi_1 x_1 + \xi_2 x_2)} \overline{\Psi\left(x_1 - b_1, x_2 - b_2\right)} dx_1 dx_2, \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \mathcal{V}_\Psi[f\left(x_1, x_2\right)] e^{-i(\xi_1 x_1 + \xi_2 x_2)} \overline{\Psi\left(x_1 - b_1, x_2 - b_2\right)} dx_1 dx_2, \\
= & \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \mathcal{V}_\Psi[f(x_1, x_2)] e^{i[(-\xi_1) x_1 + (-\xi_2) x_2]} \Psi\left(x_1 + b_1, x_2 + b_2\right) dx_1 dx_2, \\
= & f((-b_1, -b_2)(-\xi_1, -\xi_2)).
\end{aligned}$$

This completes the proof. ■

Theorem 2.10 *(Parseval relation) Let $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2\left(\mathbb{R}^2\right)$, then*

$$\begin{aligned}
& \int_{\mathbb{R}^2} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2 \qquad (2.14) \\
= & \int_{\mathbb{R}^2} \mathcal{V}_\Psi \left[f(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right) \left(\overline{\mathcal{V}_\Psi \left[g(x_1, x_2)\right] \left((b_1, b_2), (\zeta_1, \zeta_2)\right)}\right) d\zeta_1 d\zeta_2.
\end{aligned}$$

Proof. Let $r(x_1, x_2) = \overline{g(x_1, x_2)}$ and $s(x_1, x_2) = f(x_1, x_2) r(x_1, x_2)$, then

$$\begin{aligned}
= & \int_{\mathbb{R}^2} \left[\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} f(x_1, x_2) e^{-i(\zeta_1 x_1 + \zeta_2 x_1)} \overline{\Psi\left(x_1 - b_1, x_2 - b_2\right)} dx_1 dx_2\right] \\
& \times \left[\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \overline{g(x_1, x_2) e^{-i(\zeta_1 x_1 + \zeta_2 x_1)} \Psi\left(x_1 - b_1, x_2 - b_2\right)} dx_1 dx_2\right] d\zeta_1 d\zeta_2 \\
= & \int_{\mathbb{R}^2} f(x_1, x_2) \overline{g(x_1, x_2)} \left(\overline{\Psi\left(x_1 - b_1, x_2 - b_2\right)}\right)^2 dx_1 dx_2 \\
& \times \left[\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{-i(\zeta_1 x_1 + \zeta_2 x_1)}\right] \left[\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} \overline{e^{-i(\zeta_1 x_1 + \zeta_2 x_1)}}\right] d\zeta_1 d\zeta_2 \\
= & \int_{\mathbb{R}^2} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2.
\end{aligned}$$

This completes the proof. ■

2.3 Applications

The 2D continuous Short-Time Fourier Transform (2D continuous STFT) extends the concept of the continuous STFT to two-dimensional signals, enabling analysis of time-varying frequency content in images or other two-dimensional data. Here are some applications of the 2D continuous STFT:

1. Image Processing:

    - Texture Analysis: By applying the 2D continuous STFT to images, texture features can be extracted, aiding in tasks such as texture classification, segmentation, and synthesis.
    - Edge Detection: Analysis of frequency content in different image regions using the 2D continuous STFT can help in detecting edges and boundaries in images.
    - Image Compression: The 2D continuous STFT can be utilized in image compression algorithms for efficient representation of image features in the frequency domain.

2. Remote Sensing:

    - Satellite Image Analysis: The 2D continuous STFT can be applied to analyze satellite images for various purposes including land cover classification, environmental monitoring, and disaster management.
    - Agricultural Monitoring: Monitoring crop health and vegetation dynamics through the analysis of multispectral satellite imagery using the 2D continuous STFT can aid in precision agriculture practices.

3. Biomedical Imaging:

    - MRI Analysis: In magnetic resonance imaging (MRI), the 2D continuous STFT can be used for analyzing time-varying frequency characteristics of tissue signals, providing insights into tissue properties and abnormalities.
    - Ultrasound Imaging: Analysis of ultrasound images using the 2D continuous STFT can assist in the detection and characterization of tissue structures and pathologies.

4. Sonar and Radar Imaging:

- Underwater Sonar Imaging: The 2D continuous STFT can aid in analyzing sonar data for underwater imaging applications such as underwater navigation, object detection, and marine exploration.
- Radar Imaging: In radar applications, the 2D continuous STFT can be employed for analyzing radar signals to detect and track moving targets, and for forming high-resolution radar images.

5. Material Characterization:
   - Material Inspection: Analysis of material properties using techniques such as terahertz imaging and microwave imaging with the 2D continuous STFT can assist in non-destructive testing and material inspection in industries such as manufacturing and aerospace.

6. Environmental Monitoring:
   - Seismic Analysis: In seismology, the 2D continuous STFT can be used for analyzing seismic data to study earthquakes, monitor subsurface structures, and detect underground resources.
   - Oceanography: Analyzing oceanographic data with the 2D continuous STFT can provide insights into ocean currents, wave patterns, and marine life distribution.

These applications highlight the versatility of the 2D continuous STFT in various fields (see Bahri and Ashino (2011), Bahat and Dar (2023), Mawardi, Hitzer, Ashino and Vaillancourt (2010)), where it enables the analysis of time-varying frequency content in two-dimensional signals for a wide range of purposes, from image processing to environmental monitoring and beyond.

CHAPTER 3

2D fractional Fourier transform

Contents

The Fourier transform has been used for more than a century in a wide range of applications However,more recently it was shown that the Fourier transform is inadequate for describing some physical applications or dealing with their underlying mathematical problems. As a result some off-shoots of the Fourier transform,such as the windowed Fourier transform,the wavelet transform and the fractional Fourier transform (FrFT) introduced to address the shortcoming of the Fourier transform. The (FrFT), which is a generalization of the Fourier transform,has gained considerable attention in the last 20 years or so because of its important applications in signal analysis, obtics and signal recovery and also because of its ability to treat some mathematical problems that could not otherwise be handled by the standard Fourier transform.

In this chapter, we introduce the 2D fractional Fourier transform as a generalization of the classical 2D Fourier transform. The fractional Fourier transform was introduced by Namias (1980) and a rigorous mathematical framework of the properties of fractional Fourier transform the Schwartz space of rapidly decreasing functions was given by McBride and Kerr (1987). Let us define 2D fractional Fourier transform.

3.1 Definition

Definition 3.1 Let $f(x_1, x_2)$ be a signal in $L^2(\mathbb{R}^2)$.The 2D fractional Fourier transform with order $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (-\pi, \pi)^2$on $L^1(\mathbb{R}^2)$ of $f(x_1, x_2)$ is defined by

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)](\zeta_1, \zeta_2) = \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) f(x_1, x_2) dx_1 dx_2, \tag{3.1}$$

where $K_{\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) = \prod_{i=1}^{2} K_{\alpha_i}(x_i, \zeta_i)$ and $K_{\alpha_i}(x_i, \zeta_i)$ the kernel of the FrFT and is given by

$$K_{\alpha_i}(x_i, \zeta_i) = \begin{cases} \frac{c(\alpha_i)}{\sqrt{2\pi}} e^{i\left\{a(\alpha_i)\left[x_i^2 + \zeta_i^2 - 2b(\alpha_i)x_i\zeta_i\right]\right\}} & \text{for } \alpha_i \neq k\pi, \\ \delta(x_i - \zeta_i) & \text{for } \alpha_i = 2k\pi, \\ \delta(x_i + \zeta_i) & \text{for } \alpha_i = (2k \pm 1)\pi, k \in \mathbb{Z} \end{cases}$$

where $a(\alpha_i) = \cot \alpha_i / 2, b(\alpha_i) = \sec \alpha_i, c(\alpha_i) = \sqrt{1 - i \cot \alpha_i}$, and the corresponding inversion formula is also a 2DFrFT is given by

$$\begin{aligned} f(x_1, x_2) &= \mathcal{F}_{-\boldsymbol{\alpha}}\{\mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)](\zeta_1, \zeta_2)\}(x_1, x_2) \\ &= \int_{\mathbb{R}^2} \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)](\zeta_1, \zeta_2) K_{-\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) d\zeta_1 d\zeta_2 \end{aligned} \tag{3.2}$$

It is easy to see that, when $\alpha_i = 0, \pi/2, \pi$ and and $3\pi/2$ for $i = 1, 2$, the 2DFrFT is reduced to the identity operation, the 2DFT, time-reverse operation, and the 2DIFT, respectively.

For each $\lambda \in \mathbb{R} \setminus \{0\}$, we define

$$e_{\boldsymbol{\alpha},\lambda}(x_1, x_2) = e^{i\lambda \sum_{i=1}^{2} a(\alpha_i) x_i^2}, \forall (x_1, x_2) \in \mathbb{R}^2.$$

It is easy to observe that $e_{\boldsymbol{\alpha},-\lambda}(x_1, x_2) = e_{-\boldsymbol{\alpha},\lambda}(x_1, x_2)$ and the 2D fractional Fourier transform of $f(x_1, x_2)$ (3.1) can be rewritten as

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)](\boldsymbol{\zeta}) = c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha},1}(\zeta_1, \zeta_2) \mathcal{F}(e_{\boldsymbol{\alpha},1}(x_1, x_2))(\zeta_1 \csc \alpha_1, \zeta_2 \csc \alpha_2), \tag{3.3}$$

where $c(\boldsymbol{\alpha}) = c(\alpha_1)\, c(\alpha_2)$ and $\mathcal{F}(e_{\boldsymbol{\alpha},1}x)$ is the Fourier transform of $e_{\boldsymbol{\alpha},1}x$. Using this notation, we can rewrite

$$K_{\boldsymbol{\alpha}}\left(\left(x_1, x_2\right), \left(\zeta_1, \zeta_2\right)\right) = \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^2} e_{\boldsymbol{\alpha},1}\left(x_1, x_2\right) e_{\boldsymbol{\alpha},1}\left(\zeta_1, \zeta_2\right) e^{-i\sum_{i=1}^{2} x_i \zeta_i \csc\alpha_i}.$$

From (3.3), it is clear that $\mathcal{F}_{\boldsymbol{\alpha}}\left[f\left(x_1, x_2\right)\right]\left(\zeta_1, \zeta_2\right) \in C_0\left(\mathbb{R}^2\right)$ for all $f\left(x_1, x_2\right) \in L^2\left(\mathbb{R}^2\right)$.

Next, we highlight some properties of 2DFrFT.

3.2 Properties

Theorem 3.1 *(Linearity) Let* $f(x_1, x_2), g(x_1, x_2)$ *in* $L^2(\mathbb{R}^2)$ *and* $\left(k_1, k_2\right), \left(\zeta_1, \zeta_2\right) \in \mathbb{R}^2$.

$$\mathcal{F}_{\boldsymbol{\alpha}}[(\lambda f(x_1, x_2) + \mu g(x_1, x_2))]\left(\zeta_1, \zeta_2\right) = \lambda \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)]\left(\zeta_1, \zeta_2\right) + \mu \mathcal{F}_{\boldsymbol{\alpha}}[g(x_1, x_2)]\left(\zeta_1, \zeta_2\right) \tag{3.4}$$

Proof. We have form definition (3.1)

$$\begin{aligned}
& \mathcal{F}_{\boldsymbol{\alpha}}[(\lambda f(x_1, x_2) + \mu g(x_1, x_2))]\left(\zeta_1, \zeta_2\right) \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left(\left(x_1, x_2\right), \left(\zeta_1, \zeta_2\right)\right)(\lambda f(x_1, x_2) + \mu g(x_1, x_2)) dx_1 dx_2 \\
= & \lambda \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left(\left(x_1, x_2\right), \left(\zeta_1, \zeta_2\right)\right) f(x_1, x_2) dx_1 dx_2 \\
& + \mu \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left(\left(x_1, x_2\right), \left(\zeta_1, \zeta_2\right)\right) g(x_1, x_2) dx_1 dx_2 \\
= & \lambda \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)]\left(\zeta_1, \zeta_2\right) + \mu \mathcal{F}_{\boldsymbol{\alpha}}[g(x_1, x_2)]\left(\zeta_1, \zeta_2\right).
\end{aligned}$$

This completes the proof. ∎

Theorem 3.2 *(Translation) The 2D fractional Fourier transform of any function* $f(x_1 - k_1, x_2 - k_2)$ *is given by*

$$\mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1 - k_1, x_2 - k_2)\right]\left(\zeta_1, \zeta_2\right) = K_{\boldsymbol{\alpha}}(\left(k_1, k_2\right), \left(\zeta_1, \zeta_2\right)) \mathcal{F}_{\boldsymbol{\alpha}}[f(u_1, u_2)]((\zeta_1, \zeta_2)). \tag{3.5}$$

Proof. From definition (3.1), we have

$$
\begin{aligned}
& \mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1-k_1,x_2-k_2)\right](\zeta_1,\zeta_2) \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left((x_1,x_2),(\zeta_1,\zeta_2)\right) f(x_1-k_1,x_2-k_2)dx_1dx_2 \\
= & \int_{R^2} K_{\boldsymbol{\alpha}}((u_1+k_1,u_2+k_2),(\zeta_1,\zeta_2))f(u_1,u_2)\,du_1du_2 \\
\text{where } u_1 = & \; x_1-k_1, u_2 = x_2-k_2 \\
= & \int_{R^2} K_{\boldsymbol{\alpha}}((u_1,u_2),(\zeta_1,\zeta_2))K_{\boldsymbol{\alpha}}((k_1,k_2),(\zeta_1,\zeta_2))f(u_1,u_2)\,du_1du_2 \\
= & \; K_{\boldsymbol{\alpha}}((k_1,k_2),(\zeta_1,\zeta_2))\int_{R^2} K_{\boldsymbol{\alpha}}((u_1,u_2),(\zeta_1,\zeta_2))f(u_1,u_2)\,du_1du_2 \\
= & \; K_{\boldsymbol{\alpha}}((k_1,k_2),(\zeta_1,\zeta_2))\mathcal{F}_{\boldsymbol{\alpha}}[f(u_1,u_2)]((\zeta_1,\zeta_2)).
\end{aligned}
$$

This completes the proof. ∎

Theorem 3.3 *(Orthogonality) Let $f(x_1,x_2)$, $g(x_1,x_2)$ in $L^2(\mathbb{R}^2)$ and $(k_1,k_2),(\zeta_1^0,\zeta_2^0)\in\mathbb{R}^2$, then the 2DFrFT satisfies the following orthogonality Relation*

$$
\left\langle \mathcal{F}_\alpha\left[f(x_1,x_2)\right](\zeta_1,\zeta_2),\mathcal{F}_\alpha\left[g(x_1,x_2)\right](\zeta_1,\zeta_2)\right\rangle = \left\langle f(x_1,x_2),g(x_1,x_2)\right\rangle. \tag{3.6}
$$

Proof. We have

$$
\begin{aligned}
& \left\langle \mathcal{F}_\alpha\left[f(x_1,x_2)\right](\zeta_1,\zeta_2),\mathcal{F}_\alpha\left[g(x_1,x_2)\right](\zeta_1,\zeta_2)\right\rangle \\
& = \int_{\mathbb{R}^2} \mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1,x_2)\right](\zeta_1,\zeta_2)\overline{\mathcal{F}_{\boldsymbol{\alpha}}\left[g(x_1,x_2)\right]((\zeta_1,\zeta_2))}d\zeta_1d\zeta_2 \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left((x_1,x_2),(\zeta_1,\zeta_2)\right)f(x_1,x_2) \\
& \times \overline{K_{\boldsymbol{\alpha}}\left((y_1,y_2),(\zeta_1,\zeta_2)\right)g(y_1,y_2)dy_1dy_2}dx_1dx_2d\zeta_1d\zeta_2 \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(y_1,y_2)} \\
& \left(\int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}\left((x_1,x_2),(\zeta_1,\zeta_2)\right)\overline{K_{\boldsymbol{\alpha}}\left((y_1,y_2),(\zeta_1,\zeta_2)\right)}d\zeta_1d\zeta_2\right)dy_1dy_2dx_1dx_2 \\
& = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(y_1,y_2)}\delta\left(x_1-y_1,x_2-y_2\right)dy_1dy_2dx_1dx_2 \\
& = \int_{\mathbb{R}^2} f(x_1,x_2)\overline{g(y_1,y_2)}dx_1dx_2 \\
& = \left\langle f(x_1,x_2),g(x_1,x_2)\right\rangle.
\end{aligned}
$$

This completes the proof. ∎

Theorem 3.4 *(Convolution) Let $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2) * g(x_1, x_2)]\,(\zeta_1, \zeta_2) = \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)]\,(\zeta_1, \zeta_2)\,\mathcal{F}_{\boldsymbol{\alpha}}[g(x_1, x_2)]\,(\zeta_1, \zeta_2)\,. \tag{3.7}$$

where $f * g$ denotes the convolution of the function $f(t)$ and $g(t)$ and is given by

$$(f * g)(x_1, x_2) = \int_{\mathbb{R}^2} f(x_1, x_2) g(u_1 - x_1, u_2 - x_2) du_1 du_2.$$

Proof. By applying definition of Fourier transform to the convolution of the function $f(x_1, x_2)$ and $g(x_1, x_2)$, we obtain

$$\begin{aligned}
& \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2) * g(x_1, x_2)]\,(\zeta_1, \zeta_2) \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) f(x_1, x_2) * g((x_1, x_2)) dx_1 dx_2 \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) \\
& \times \left(\int_{\mathbb{R}^2} f(x_1, x_2) g(u_1 - x_1, u_2 - x_2) du_1 du_2 \right) dx_1 dx_2 \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((u_1 - v_1, u_2 - v_1), (\zeta_1, \zeta_2)) \\
& \times \left(\int_{\mathbb{R}^2} f(x_1, x_2) g(v_1, v_2) dv_1 dv_2 \right) dx_1 dx_2 \\
\text{where } v_1 = & \; u_1 - x_1, v_2 = u_2 - x_2 \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1, x_2), (\zeta_1, \zeta_2)) f(x_1, x_2) dx_1 dx_2 \\
& \times \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((v_1, v_1), (\zeta_1, \zeta_2)) \left(g(v_1, v_2) dv_1 dv_2\right) \\
= & \; \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)]\,(\zeta_1, \zeta_2)\,\mathcal{F}_{\boldsymbol{\alpha}}[g(x_1, x_2)]\,(\zeta_1, \zeta_2)\,.
\end{aligned}$$

This completes the proof. ■

Theorem 3.5 *(Multiplication) Let $f(x_1, x_2)$ and $g(x_1, x_2)$ in $L^2(\mathbb{R}^2)$, then*

$$\mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2) g(x_1, x_2)]\,(\zeta_1, \zeta_2) = \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1, x_2)]\,(\zeta_1, \zeta_2) * \mathcal{F}_{\boldsymbol{\alpha}}[g(x_1, x_2)]\,(\zeta_1, \zeta_2)\,. \tag{3.8}$$

Proof. We have

$$
\begin{aligned}
& \mathcal{F}_{\boldsymbol{\alpha}}[f(x_1,x_2)g(x_1,x_2)](\zeta_1,\zeta_2) \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1,x_2),(\zeta_1,\zeta_2))f(x_1,x_2)g(x_1,x_2)dx_1dx_2 \\
= & \int_{\mathbb{R}^2} K_{\boldsymbol{\alpha}}((x_1,x_2),(\zeta_1,\zeta_2))g(x_1,x_2) \\
& \times\left(\int_{\mathbb{R}^2} \mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1,x_2)\right](\xi_1,\xi_2)\, K_{-\boldsymbol{\alpha}}\left((x_1,x_2),(\xi_1,\xi_2)\right) d\xi_1 d\xi_2\right) dx_1dx_2 \\
= & \int_{\mathbb{R}^2} \mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1,x_2)\right](\xi_1,\xi_2) \\
& \left(\int_{\mathbb{R}^2} g(x_1,x_2)K_{\boldsymbol{\alpha}}((x_1,x_2),(\zeta_1,\zeta_2))K_{-\boldsymbol{\alpha}}\left((x_1,x_2),(\xi_1,\xi_2)\right)dx_1dx_2\right) d\xi_1 d\xi_2 \\
= & \int_{\mathbb{R}^2} \mathcal{F}_{\boldsymbol{\alpha}}\left[f(x_1,x_2)\right](\zeta_1,\zeta_2)\, \mathcal{F}_{\boldsymbol{\alpha}}\left[g(x_1,x_2)\right](\zeta_1-\xi_1,\zeta_2-\xi_2)d\xi_1 d\xi_2, \\
= & \mathcal{F}[f(x_1,x_2)](\zeta_1,\zeta_2) * \mathcal{F}[g(x_1,x_2)](\zeta_1,\zeta_2).
\end{aligned}
$$

This completes the proof. ■

3.3 Applications

The 2D Fractional Fourier Transform (2D FrFT) is an extension of the fractional Fourier transform (FrFT) to two-dimensional signals. It provides a powerful tool for analyzing time-frequency characteristics and spatial-frequency distribution in images or other two-dimensional data. Here are some applications of the 2D FrFT:

1. Image Processing:

   - Image Compression: The 2D FrFT can be used in image compression algorithms to efficiently represent images in the joint time-frequency domain, leading to improved compression performance.
   - Edge Detection: By analyzing the spatial-frequency distribution of image edges using the 2D FrFT, edge detection algorithms can be developed for applications such as object recognition and scene understanding.
   - Texture Analysis: The 2D FrFT enables the analysis of texture features in images, facilitating tasks such as texture classification, segmentation, and synthesis.

2. Biomedical Imaging:

   - MRI Analysis: Similar to the application in image processing, the 2D FrFT can be applied to analyze MRI data for extracting spatial-frequency features related to tissue properties and abnormalities.

   - Ultrasound Imaging: Analysis of ultrasound images using the 2D FrFT can assist in characterizing tissue structures and detecting pathological changes.

3. Remote Sensing:

   - Satellite Image Analysis: The 2D FrFT can be utilized to analyze multispectral satellite images for land cover classification, environmental monitoring, and disaster assessment based on their spatial-frequency content.

   - Hyperspectral Imaging: Analysis of hyperspectral data using the 2D FrFT allows for the extraction of spectral-spatial features, aiding in tasks such as mineral identification and vegetation mapping.

4. Optical Signal Processing:

   - Optical Imaging: In optical imaging systems, the 2D FrFT can be used to analyze the spatial-frequency distribution of optical wavefronts, enabling aberration correction, beam shaping, and holography.

   - Lens Design: By analyzing the spatial-frequency response of optical components using the 2D FrFT, lens designs can be optimized for various applications including imaging, lithography, and laser beam manipulation.

5. Material Characterization:

   - Material Inspection: Analysis of material properties using techniques such as terahertz imaging and microwave imaging with the 2D FrFT can assist in non-destructive testing and material characterization in industries such as manufacturing and aerospace.

6. Signal Processing:

- Audio Signal Processing: Similar to the FrFT in one dimension, the 2D FrFT can be applied to analyze time-frequency characteristics in audio signals, aiding in tasks such as audio coding, source separation, and musical instrument recognition.

These applications demonstrate the versatility of the 2D FrFT in analyzing spatial-frequency content in two-dimensional signals across various domains, from image processing to signal processing and beyond (see Kamalakkannan and Roopkumar (2019)).

CHAPTER 4

Conclusion

The 2D Fourier transform is a powerful mathematical tool that allows us to analyze the frequency content of two-dimensional signals and functions. Its properties and applications make it indispensable in fields ranging from signal and image processing to optics and mathematics. Understanding and mastering the 2D Fourier transform is essential for researchers and practitioners in these fields.

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