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Existence of periodic solutions for a nonlinear system of neutral differential equations

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Dedication

I dedicate this work to my family , my best friends and my loved ones.

Acknowledgments

*First of all, i would like to thank my supervisor **Dr. Benhaderi Mimia** for training me throughout the achivement of this master graduation thesis and for their advices and instructions.*

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I extend my sincere thanks to all the proffessors who supported and encouraged me throughout my academic career.

Abstract

The aim of this work is to study the existence of periodic solutions for a nonlinear system of neutral differential equations with variable delays.

In the process, we transform the differential system into an equivalent integral system. Then, we construct appropriate mappings on a bounded closed and convex set of a Banach space. Under specific conditions, we apply *the Krasnoselskii fixed point theorem* for obtaining the existence result. Whereas, the existence of a unique periodic solution is determined by *the contraction mapping principle.* Our results are considered a generalization of some conclusions proposed in recent literature. The main results are illustrated with an appropriate example.

Keywords: Functional differential equations, delay differential equations, neutral differential equations, fixed point theorems, Krasnoselskii fixed point theorem, contraction mapping principle, periodic solution.

Résumé

L'objectif de ce travail est d'étudier l'existence des solutions périodiques pour un système non linéaire des équations différentielles de type neutre avec retards variables.

Dans le processus, on transforme le système différentielles à un système intégrale équivalent. Puis, on construit des applications appropriées sur un ensemble convexe fermé et borné d'un espace de Banach. Sous des conditions spécifiques, on applique *le théorème du point fixe de Krasnoselskii* pour obtenir le résultat d'existence. Alors que, l'existence d'une seule solution périodique est déterminée par *le principe d'application de contraction.* Nos résultats sont considérés comme une généralisation de certaines conclusions proposées dans la littérature récente. Les principaux résultats sont illustrés par un exemple approprié.

Les mots clés: équations différentielles fonctionnelles, équations différentielles retardées, équations différentielles de type neutre, théorèmes du point fixe, théorème du point fixe de Krasnoselskii, le principe d'application de contraction, solution périodique.

الملخص

الهدف من هذا العمل هو دراسة وجود حلول دورية لجملة غير خطية من المعادلات التفاضلية الحيادية مع تأخرات متغيرة. في هذه العملية، نحول الجملة التفاضلية إلى جملة تكاملية مكافئة. ثم ننشئ تطبيقات ملائمة على مجموعة محدبة مغلقة ومحدودة من فضاء بناخ. تحت شروط محددة، نطبق نظرية النقطة الثابتة لكراسنوسلسكي للحصول على نتيجة الوجود. بينما وجود حل دوري وحيد تم تحديده بواسطة مبدأ التطبيق المقلص. تعتبر نتائجنا بمثابة تعميم للنتائج المقترحة في الأدبيات الحديثة. النتائج الرئيسية موضحة بمثال ملائم.

الكلمات المفتاحية: معادلات تفاضلية دالية، معادلات تفاضلية متأخرة، معادلات تفاضلية حيادية، نظريات النقطة الثابتة، نظرية النقطة الثابتة لكراسنوسلسكي، مبدأ التطبيق المقلص، حل دوري.

Contents

Introduction

Delay differential equations (abbreviated: DDEs) are a class of functional differential equations (FDEs) which involve time delays or memory effects in their formulations. This class is widely used in modeling phenomena that exhibit time delays in their systems and processes creating more realistic models. DDEs have a rich history and have found numerous applications in various field of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, etc. Therefore, the study of this class of dynamical systems knew many activities concerning quantitative and qualitative properties of solutions, such as existence and uniqueness, oscillation, stability, periodicity, positivity and asymptotic behavior (see [12], [29], [37], [42], [45], [33] for example). For the general theory of such differential equations, we refer to the monographs ([19], [22], [23], [27], [28], [32]).

Recently, the researchers have given a special interest to the study of a kind of delay differential equation in which the delay argument occurs in the highest derivative of the state variable as well as in the independent variables, so called neutral differential equations. This category of equations is not only an extension of delay differential equations, but also provides more useful models in many fields and describes actual problems, including biology, mechanics, economics, electronics, and so on. Due to their practical importance in numerous applications, the analysis of the qualitative properties for neutral differential equations has taken great interest because of the importance of understanding continuous phenomena in applications of mathematics. We refer as examples to [4], [18], [21], [25].

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics, as well as one of the strongest areas of nonlinear analysis. Its origins, which date to the late nineteenth century, can be traced to successive approximations which establish the existence and uniqueness of solutions to differential equations. In recent years a number of excellent books, monographs and surveys by distinguished authors about fixed point theory have appeared. A number of mathematicians have contributed to the development of this theory, such as Banach, Brouwer, Schauder, Krasnoselskii, and Burton. See [2], [8], [10], [11], [30], [39], [44] and the references cited therein.

Over last years, the study of the existence and qualitative properties of periodic solutions for various kinds of delay functional equations, especially for differential, difference and dynamic equations with delays has attracted much attention. For related results, we refer the reader to [1], [3], [5], [43], [36], and the references cited therein. There are many methods for obtaining the existence and uniqueness of periodic and positive periodic solutions. For example, Fourier analysis method, Lyapunov method, fixed point theory. Particular, we are interested in the use of the fixed point theory to problems of periodicity, positivity and stability for delay functional equations. Its advantege is that it can prove the existence, uniqueness, multiplication of periodic or positive periodic solutions for each type of functional differential equations (ordinary, partial, delay). All we need to apply fixed point method, a complete metric space, a suitable fixed point theorem and an elementary integral methods to solve the problems.

The present dissertation consists of three principal chapters.

In chapter 1, we give the necessary definitions and theorems as well as some useful functional analysis tools, which are needed in this work. In addition, we demonstrate how some fixed point theorems can be applied to prove periodic solutions to ODEs.

The second chapter is devoted to present some useful preliminaries and essential definitions needed to better understand the manuscript. Precisely, it contains a background of delay differential equations as well as an overview of significant and interesting models of equations with delay emanating from biology, economy and chemistry are given in this part. Additionally, the steps method is presented that explains how to generate explicit solutions to DDEs, using an example to illustrate the process.

The third chapter exposes results published in [33]. The main bulk of the present work is to study the existence and uniqueness of periodic solutions of the following neutral differential system

$$\frac{d}{dt}y(t) = A(t)h(y(t-\tau_1(t))) + \frac{d}{dt}Q(t, y(t-\tau_2(t))) + G(t, y(t), y(t-\tau_2(t))),$$

with two variable delays and non-linear term $\boldsymbol{h(y(t-\tau_1(t)))}$, where $\boldsymbol{A(.)}$ is a non-singular $\boldsymbol{n \times n}$ matrix with continuous real valued functions as entries. $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ are variable delays with $\boldsymbol{\tau_1}$ is continuously differentiable. The functions $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{Q} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $\boldsymbol{G} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments.

Motivated by the works of M.N.Islam and Y.N.Raffoul [24], and M.B.Mesmoulli, A.Ardjouni and A.Djoudi [35], we will use the fundamental matrix solution of the system $\boldsymbol{\frac{d}{dt}y(t) = A(t)y(t)}$ coupled with Floquet theory to transform the system into integral equation. By means of Krasnoselskii fixed point theorem, and under suitable conditions, we establish the existence of periodic solutions and by using the contraction mapping principle, we prove the uniqueness.

Chapter 1

Preliminary Notions

In this chapter, we mention the necessary elements of functional analysis and some necessary notations, definitions, and preliminary results that will be used throughout this work, as well as introduce results relating to fixed point theory, contributed by as an example an application of some mentioned fixed point theorems for discussing the existence, and uniqueness of periodic solutions for ODEs.

1.1 Functional analysis elements

In this section, we mention more general spaces which are metric spaces and normed spaces, and mappings of these spaces with some of their properties, theorems and results.

1.1.1 Banach space

Definition 1.1.1. *(**Metric space**)[17] Suppose X is a set. A map $d : X \times X \to \mathbb{R}$ is called a metric (or distance function) on X if the following properties hold:*

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$,

(ii) $d(x, y) = 0$ *if and only if* $x = y$,

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$ *(called the triangle inequality).*

We call (X, d) *a metric space.*

Definition 1.1.2. *(**Cauchy Sequence**)[17] Suppose* (x_n) *is a sequence in a metric space* (X, d)*. We call* (x_n) *a Cauchy sequence if for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$ *for all* $n, m \geq n_0$.

Definition 1.1.3. *[17] The metric space* (X, d) *is complete if every Cauchy sequence in that space has a limit.*

Definition 1.1.4. *(**Relatively compact sets**)[17] We call a subset of a metric relatively compact if its closure is compact.*

Proposition 1.1.1. *[17] Let* K *be a closed subset of a complete metric space. Then* K *is compact if and only if it is relatively compact.*

Definition 1.1.5. *(**Normed space**)[17] Let* X *be a vector space. A map* $\|.\| : X \to \mathbb{R}$ *is called a norm on* X *if*

1. $\|x\| \geq 0$ *for all* $x \in X$,
2. $\|x\| = 0$ *if and only if* $x = 0$,
3. $\|\alpha x\| = |\alpha| \|x\|$ *for all* $x \in X$ *and* $\alpha \in \mathbb{R}$ *or* $\mathbb{C}$,
4. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$ *(called the triangle inequality).*

We call the pair $(X, \|.\|)$ *a normed space.*

Remark 1.1.1. *([31], p59) A norm on* X *defines a metric on* X *which is given by*

$$d(x, y) = \|x - y\|,$$

for all $x, y \in X$*, we call the metric induced by the norm. Then, the normed space is a metric space. But the metric space is not always a normed space.*

Definition 1.1.6. *(**Banach space**)[17] A Banach space is a complete normed space with respect to the metric induced by the norm.*

Corollary 1.1.1. *[13] Every finite-dimensional normed vector space is complete (Banach space).*

Corollary 1.1.2. *(**General Heine-Borel Theorem**)[17] Let* X *be a finite dimensional normed space and let* $K \subset X$*. Then* K *is compact if and only if* K *is closed and bounded.*

1.1.2 Compact, Lipschitz, Bounded, Continuous Linear Operator

In the case of normed spaces, a mapping is called an operator, which preserve the two algebraic operations of vector space. The purpose of this part is to introduce bounded linear operators in normed spaces, their boundedness and continuousness, as well as the concept of compactness for nonlinear operators.

Definition 1.1.7. *(**Linear operator**)[31] A linear operator $\boldsymbol{A}$ is an operator such that*

(i) the domain $\mathfrak{D}(\boldsymbol{a})$ of $\boldsymbol{A}$ is a vector space and the range $\mathfrak{R}(\boldsymbol{A})$ lies in a vector space over the same field,

(ii) for all $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{D}(\boldsymbol{A})$ and scalar $\boldsymbol{\alpha}$,

$$\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y},$$

$$\boldsymbol{A}(\boldsymbol{\alpha}\boldsymbol{x}) = \boldsymbol{\alpha}\boldsymbol{A}\boldsymbol{x}.$$

Notation: *we write $\boldsymbol{A}\boldsymbol{x}$ instead of $\boldsymbol{A}(\boldsymbol{x})$.*
· *$\mathfrak{D}(\boldsymbol{A})$ denotes the domain of $\boldsymbol{A}$.*
· *$\mathfrak{R}(\boldsymbol{A})$ denotes the range of $\boldsymbol{A}$.*
· *$\mathcal{N}(\boldsymbol{A})$ denotes the null space of $\boldsymbol{A}$ (which is the set of all $\boldsymbol{x} \in \mathfrak{D}(\boldsymbol{A})$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, is also called kernel).*

Definition 1.1.8. *(**Bounded linear operator**)[13] Let $(\boldsymbol{X}, \|.\|_X)$ and $(\boldsymbol{Y}, \|.\|_Y)$ be real normed vector spaces. A linear operator $\boldsymbol{A} : \boldsymbol{X} \to \boldsymbol{Y}$ is called bounded if there exists a constant $\boldsymbol{c} > \boldsymbol{0}$ such that*

$$\|\boldsymbol{A}\boldsymbol{x}\|_Y \leq \boldsymbol{c}\|\boldsymbol{x}\|_X, \quad \textit{for all } \boldsymbol{x} \in \mathfrak{D}(\boldsymbol{A}).$$

The space of bounded linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is denoted by $\boldsymbol{\mathcal{L}}(\boldsymbol{X}, \boldsymbol{Y})$

$$\boldsymbol{\mathcal{L}}(\boldsymbol{X}, \boldsymbol{Y}) = \{\boldsymbol{A} : \boldsymbol{X} \to \boldsymbol{Y} / \ \boldsymbol{A} \textit{ is linear and bounded } \}.$$

Definition 1.1.9. *(**Operator norm**)[17] For $\boldsymbol{A} \in \boldsymbol{\mathcal{L}}(\boldsymbol{X}, \boldsymbol{Y})$, we define*

$$\|\boldsymbol{A}\|_{\mathcal{L}(X,Y)} = \inf\{\boldsymbol{c} > \boldsymbol{0} : \|\boldsymbol{A}\boldsymbol{x}\|_Y \leq \boldsymbol{c}\|\boldsymbol{x}\|_X \textit{ for all } \boldsymbol{x} \in \mathfrak{D}(\boldsymbol{A})\}.$$

We call $\|.\|_{\mathcal{L}(X,Y)}$ *the operator norm of* A*. Then* $(\mathcal{L}(X,Y), \|.\|_{\mathcal{L}(X,Y)})$ *is a normed vector space.*

Proposition 1.1.2. *[17] Suppose that* $A \in \mathcal{L}(X,Y)$*. Then*

$$\|Ax\|_Y \leq \|A\|_{\mathcal{L}(X,Y)} \|x\|_X,$$

for all $x \in \mathfrak{D}(A)$*. Moreover,*

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y = \sup_{\|x\|_X \leq 1} \|Ax\|_Y.$$

Theorem 1.1.1. *[17],[31] Let* X *and* Y *be normed vector spaces and suppose* $\dim(X) < \infty$*. Then every linear operator* $A : X \to Y$ *is bounded.*

Theorem 1.1.2. *(**Continuity and boundedness**)[13] Let* X *and* Y *be real normed spaces and* $A : X \to Y$ *be a linear operator. Then, the following are equivalents:*

(i) A *is bounded.*

(ii) A *is continuous.*

(iii) A *is continuous at* $x = 0$.

Definition 1.1.10. *(**Lipschitz continuous operator**)[39] Let* X *and* Y *be normed spaces. An operator* $A : X \to Y$ *is called Lipschitz continuous if there is* $c > 0$ *such that*

$$\|Ax - Ay\|_Y \leq k\|x - y\|_X, \quad \text{for all } x, y \in \mathfrak{D}(A).$$

If $k \in [0, 1[$*,* A *is called a contraction operator.*

Remark 1.1.2. A *is a contraction* $\implies$ A *is Lipschitz continuous* $\implies$ A *is continuous on* X*.*

Definition 1.1.11. *(**Compact operator**) An operator* $A : X \to Y$ *is called compact if and only if* A *maps bounded sets into relatively compact set.*
Equivalently, A *is compact if and only if for every bounded sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X*, the sequence* $(Ax_n)_{n\in\mathbb{N}}$ *has a convergent subsequence in* Y*.*

1.1.3 Arzelà-Ascoli theorem

Let K be a compact subset of a normed vector space $(X, \|.\|)$ and let $C(K, \mathbb{R})$ be the normed vector space of real valued continuous functions on K with the sup-norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

Let $\mathcal{F}$ be a collection of functions in $C(K, \mathbb{R})$.

Definition 1.1.12. *[9] The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in K$ with $\|x - y\| < \delta$, we have $|f(x) - f(y)| < \varepsilon$, for every $f \in \mathcal{F}$.*

Definition 1.1.13. *[9] The collection $\mathcal{F}$ is said to be uniformly bounded if there exists $M > 0$ such that*

$$\|f\|_\infty = \sup_{x \in K} |f(x)| \leq M, \text{ for all } f \in \mathcal{F}.$$

Theorem 1.1.3. *[9] If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $C(K, \mathbb{R})$, then $\mathcal{F}$ is relatively compact in $C(K, \mathbb{R})$.*

1.2 Periodic functions

Let $\omega > 0$, and let h ba a function defined on a set Ω.

Definition 1.2.1. *The function h is said to be ω-periodic function if*

$$h(t + \omega) = h(t)$$

for all $t \in \Omega$.

Corollary 1.2.1. *The derivative of ω-periodic function is also a ω-periodic function.*

Proof. Let h be a periodic function of period ω and derivable at t. Then

$$\frac{d}{dt}h(t_0) = \lim_{t \to t_0} \frac{h(t) - h(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{h(t + \omega) - h(t_0 + \omega)}{t - t_0} = \frac{d}{dt}h(t_0 + \omega)$$

□

Remark 1.2.1. *The antiderivative of a ω-periodic function is not necessarily a ω-periodic function for $t \in \mathbb{R}$. For example, $h(t) = 1$ is periodic but $\int 1dt = t$ is not.*

Remark 1.2.2. *Let h be ω-periodic function, then*

$$\int_t^{t+\omega} h(s)ds = \int_0^{\omega} h(s)ds.$$

fot all $t \in \mathbb{R}$ (we obtain this result by variable changement, we put $v = s - t$).

Definition 1.2.2. *A solution $x = \zeta(t)$ of a system is called a periodic solution if there exists a positive number ω such that $\zeta(t + \omega) = \zeta(t)$ for all $t \in \mathbb{R}$.*

1.3 Fixed point theory

Fixed point theory becomes a strong tool for solving various problems arising in different fields of pure and applied mathematics, especially in nonlinear functional analysis. Historically, *Poincare* in 1886 was the first to work in this field. However, the major classical result in fixed point theory is due to *Brouwer*, given in 1912, which states that a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point. An important generalization of this result was discovered in 1930 by *Schauder*, which states that a continuous map on a convex compact subset of a Banach space has a fixed point. Subsequently, Schauder extended Brouwer's theorem to the case of infinite dimensional subsets of some function spaces. Before that, *Banach* in 1922 proved that a contraction mapping on a complete metric space has a unique fixed point. Thereafter, *Krasnoselskii* in 1955 studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name.

Definition 1.3.1. *(**Fixed point**)[39] Let $(X, \|.\|_X)$ be a normed vector space over $\mathbb{K}$. A fixed point of a mapping $S : X \to X$ is an $x \in X$ which is mapped to itself by S:*

$$S(x) = x.$$

1.3.1 Banach fixed point

In 1922, the Polish mathematician *Stefan Banach* formulated and proved a geometrical theorem which concerns, under appropriate conditions, the existence and uniqueness of a fixed point in a complete metric space. His result is known as Banach's fixed point theorem or the contraction mapping principle. Due to its simplicity and generality, Banach principle was considered as one of the fundamental principles in the field of functional analysis.

Definition 1.3.2. *Let* (X, ρ) *a complete metric space and let* $P : X \to X$. *We say that the operator* P *is a contraction, if there exists a positive constant* $\alpha < 1$ *such that for* $x, y \in X$ *we have:*

$$\rho(Px, Py) \leq \alpha\rho(x, y).$$

Theorem 1.3.1. *[39] Let* (X, ρ) *a complete metric space and* $P : X \to X$ *a contraction operator. Then there exists a unique point* $z \in X$ *with* $Pz = z$.

1.3.2 Schauder fixed point theorem

Theorem 1.3.2. *(Version 1)[39] Let* $(X, \|.\|)$ *be a Banach space and* $M \subset X$ *be a closed bounded convex non-empty subset. Any compact operator* $P : M \to M$ *has at least one fixed point.*

Since a continuous operator on a compact set is always a compact operator, the Schauder fixed point theorem -version 1 yields the Schauder fixed point theorem-version2.

Theorem 1.3.3. *(Version 2)[39] Let* $(X, \|.\|)$ *be a Banach space and* $M \subset X$ *be a compact, convex and non-empty subset. Any continuous operator* $P : M \to M$ *has at least one fixed point.*

1.3.3 Krasnoselskii's fixed point theorem

In 1955, *Krasnoselskii* combined two main results of fixed point theory, which are Banach's fixed point theorem and Schauder's fixed point theorem, into his theorem. Krasnoselskii's fixed point theorem is applicable to a mapping that can be expressed as a sum of two maps: a contraction and a continuous mapping in a Banach space. In 1996, *Burton* has investigated Krasnoselskii's fixed point theorem and has established, what we have called, Krasnoselskii-Burton theorem.

Theorem 1.3.4. *[39] Let M be a closed convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map M into X such that the following conditions hold:*

- $Ax + By \in M$ *for all* $x, y \in M$,
- A *is continuous and* AM *is contained in a compact set,*
- B *is a contraction.*

Then there is a $z \in M$ *with* $z = Az + Bz$.

Proof. According to the third condition, we have

$$\begin{aligned} \|(I-B)x-(I-B)y\| &= \|(x-y)-(Bx-By)\| \\ &\leq \|x-y\|+\|Bx-By\| \\ &\leq \|x-y\|+\lambda\|x-y\| \\ &= (1+\lambda)\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} \|(I-B)x-(I-B)y\| &= \|(x-y)-(Bx-By)\| \\ &\geq \|x-y\|-\|Bx-By\| \\ &\geq \|x-y\|-\lambda\|x-y\| \\ &= (1-\lambda)\|x-y\|. \end{aligned}$$

In short

$$(1-\lambda)\|x-y\| \leq \|(I-B)x-(I-B)y\| \leq (1+\lambda)\|x-y\|.$$

This inequality shows that $(I-B) : M \to (I-B)M$ is continuous and bijective. Thus, $(I-B)^{-1}$ exists and is continuous.
Let us pose $U := (I-B)^{-1}A$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in M \text{ such that } (I-B)^{-1}Az = z.$$

This is equivalent to $\boldsymbol{z} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{z}$. □

Remark 1.3.1. *Note that, if* $\boldsymbol{A} = \mathbf{0}$ *then the theorem becomes Banach theorem, and if* $\boldsymbol{B} = \mathbf{0}$ *then it becomes Schauder theorem.*

Remark 1.3.2. *If* $\boldsymbol{A}$ *is continuous and* $\boldsymbol{A}(\boldsymbol{M})$ *is contained in a compact subset of* $\boldsymbol{X}$*, then we say that* $\boldsymbol{A}$ *is a compact mapping (we also say that* $\boldsymbol{A}$ *is completely continuous).*

1.3.4 Krasnoselskii-Burton fixed point theorem

Definition 1.3.3. *(**Large contraction mapping**) Let* $(\boldsymbol{X}, \boldsymbol{\rho})$ *be a metric space and* $\boldsymbol{B} : \boldsymbol{X} \to \boldsymbol{X}$*.* $\boldsymbol{B}$ *is said to be a large contraction if for each pair* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{X}$ *with* $\boldsymbol{\phi} \neq \boldsymbol{\psi}$*, then*

$$\boldsymbol{\rho}(\boldsymbol{B}\boldsymbol{\phi}, \boldsymbol{B}\boldsymbol{\psi}) < \boldsymbol{\rho}(\boldsymbol{\phi}, \boldsymbol{\psi})$$

and if for each $\boldsymbol{\epsilon} > \mathbf{0}$ *there exists* $\boldsymbol{\delta} < \mathbf{1}$ *such that*

$$[\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{X}, \boldsymbol{\rho}(\boldsymbol{\phi}, \boldsymbol{\psi}) \geq \boldsymbol{\epsilon}] \Rightarrow \boldsymbol{\rho}(\boldsymbol{B}\boldsymbol{\phi}, \boldsymbol{B}\boldsymbol{\psi}) < \boldsymbol{\delta}\boldsymbol{\rho}(\boldsymbol{\phi}, \boldsymbol{\psi}).$$

Theorem 1.3.5. *(**Krasnoselskii-Burton**)[10] Let* $(\boldsymbol{X}, \|.\|)$ *be a Banach space,* $\boldsymbol{M}$ *a closed bounded convex non empty subset of* $\boldsymbol{X}$*. Suppose that* $\boldsymbol{A}, \boldsymbol{B} : \boldsymbol{M} \to \boldsymbol{M}$ *and that*

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{M}$, $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \in \boldsymbol{M}$,

(ii) $\boldsymbol{A}$ *is continuous and* $\boldsymbol{A}\boldsymbol{M}$ *is contained in a compact set,*

(iii) $\boldsymbol{B}$ *is a large contraction.*

Then there is a $\boldsymbol{z} \in \boldsymbol{M}$ *with* $\boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{z}$*.*

1.3.5 Fixed point theorem in cone

The fixed-point index in cone spaces is a concept in topology, particularly in fixed-point theory, that helps determine the "multiplicity" of fixed points for certain types of mappings. It extends the notion of the fixed-point index from Euclidean spaces to spaces equipped with a cone, which introduces an ordering.

Definition 1.3.4. *(**Cone**) Let* $\boldsymbol{X}$ *be a Banach space and let* $\boldsymbol{\Lambda}$ *be a closed, non-empty subset of* $\boldsymbol{X}$*.* $\boldsymbol{\Lambda}$ *is said to be a cone if* *i)* $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Lambda}, \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \geq \mathbf{0}$: $\boldsymbol{\alpha x} + \boldsymbol{\beta y} \in \boldsymbol{\Lambda}$,

ii) $\boldsymbol{y}, -\boldsymbol{y} \in \boldsymbol{\Lambda}$ *imply* $\boldsymbol{y} = \mathbf{0}$.

Theorem 1.3.6. *[30] Let* $\boldsymbol{X}$ *be a Banach space and let* $\boldsymbol{\Lambda} \subset \boldsymbol{X}$ *be a cone. Assume that* $\boldsymbol{\Omega_1}$ *and* $\boldsymbol{\Omega_2}$ *are bounded open balls of* $\boldsymbol{X}$ *such that* $\mathbf{0} \in \boldsymbol{\Omega_1} \subset \overline{\boldsymbol{\Omega_1}} \subset \boldsymbol{\Omega_2}$. *Suppose that* $\boldsymbol{P} : \boldsymbol{\Lambda} \cap (\overline{\boldsymbol{\Omega_2}} \setminus \boldsymbol{\Omega_1}) \to \boldsymbol{\Lambda}$ *is a completely continuous operator such that one of the following conditions holds:*

i) $\|\boldsymbol{Px}\| \leq \|\boldsymbol{x}\|$ *for* $\boldsymbol{x} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_1}$ *and* $\|\boldsymbol{Px}\| \geq \|\boldsymbol{x}\|$ *for* $\boldsymbol{x} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_2}$,

ii) $\|\boldsymbol{Px}\| \geq \|\boldsymbol{x}\|$ *for* $\boldsymbol{x} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_1}$ *and* $\|\boldsymbol{Px}\| \leq \|\boldsymbol{x}\|$ *for* $\boldsymbol{x} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_2}$.

Then $\boldsymbol{P}$ *has a fixed point in* $\boldsymbol{\Lambda} \cap (\overline{\boldsymbol{\Omega_2}} \setminus \boldsymbol{\Omega_1})$.

1.4 Fundamental matrix solution and Floquet theory

In this section, we give the definition of fundamental matrix solution of a linear system of first order equations and some properties, also the role of Floquet theory on studying the fundamental matrix solution.

1.4.1 Fundamental matrix solution

Let $\boldsymbol{I}$ be an interval of $\mathbb{R}$, $\boldsymbol{E} = \mathbb{K}^{\boldsymbol{n}}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) a normed vector space over a field $\mathbb{K}$, and $\boldsymbol{\mathcal{L}(E)}$ the space of continuous linear maps from $\boldsymbol{E}$ to $\boldsymbol{E}$. Let $\| \quad \|_{\boldsymbol{E}}$ be a norm on $\boldsymbol{E}$, and $\| \quad \|$ be the induced supremum normon $\boldsymbol{\mathcal{L}}$. Consider $\boldsymbol{A} \in \boldsymbol{\mathcal{M}_n}(\mathbb{K})$, $\boldsymbol{n} \times \boldsymbol{n}$ matrices over the field $\mathbb{K}$. A linear system of first order equations is defined by

$$\boldsymbol{x}'(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}). \tag{1.1}$$

Definition 1.4.1. *[6] An* $\boldsymbol{n} \times \boldsymbol{n}$ *matrix function* $\boldsymbol{t} \to \boldsymbol{\Phi(t)}$, *defined on an open interval* $\boldsymbol{I}$, *is called a matrix solution of the homogeneous linear system* (3.2) *if each of its columns is a (vector) solution.*

- *A matrix solution* $\boldsymbol{\Phi}$ *is called a fundamental matrix solution if its columns form a fundamental set of solutions.*
- *If in addition* $\boldsymbol{\Phi(t_0)} = \boldsymbol{I_n}$ *for some* $\boldsymbol{t_0}$, *a fundamental matrix solution is called the principal fundamental matrix solution.*

Theorem 1.4.1. *[6] A solution matrix* Φ *of* (1.1) *is a fundamental solution matrix on* I *if and only if* $\det \Phi(t) \neq 0$ *for all* $t \in I$.

Proof. Let Φ be a fundamental matrix with column vectors $\phi_i (i = 1..n)$, and suppose that ϕ is any nontrivial solution of (1.1). Then there exists $c_1, ..., c_n$, not all zero, such that

$$\phi = \sum_{j=1}^{n} c_j \phi_j,$$

or, writing this equation in terms of Φ, $\phi = \Phi c$, if $c = (c_1, ..., c_n)^T$. At any point $t_0 \in I$, this is a system of n linear equations with n unknowns $c_1, ..., c_n$. This system has a unique solution for any choice of $\phi(t_0)$. Thus $\det \Phi(t_0) \neq 0$, and by the remark ([6] page 39), $\det \Phi(t) \neq 0$ for all $t \in I$.
Reciproqually, let Φ be a solution matrix of (1.1), and suppose that $\det \Phi(t) \neq 0$ for $t \in I$. Then the column vectors are linearly independent at every $t \in I$. □

1.4.2 Floquet theory

A significant theoretical tool for studying the fundamental matrix solution is provided by Floquet theory, which assures that the function $\Phi(.)$ solving the system (1.1) can be decomposed into the product $\Phi(t) = Q(t)e^{Rt}$, where $R \in M_n(\mathbb{R})$ and $Q(.) \in M_n(\mathbb{R})$ is a nonsingular, 2ω-periodic matrix valued function. It identifies the non periodic function $\Phi(.)$ with the couple $(R, Q(t))$, where R a constant matrix and $Q(.)$ a periodic function [14]. Floquet's theorem gives a canonical form for fundamental matrix solution.
Consider the linear system (1.1), with $x \in \mathbb{R}^n$, where $t \to A(t)$ is ω-periodic continuous matrix-valued function.
Floquet's theorem is a corollary of the following result about the range of the exponential map.

Theorem 1.4.2. *[15]* • *If* C *is a nonsingular* $n \times n$ *matrix, then there is an* $n \times n$ *matrix* B *(which may be complexe) such that* $e^B = C$.

• *If* C *is a nonsingular real* $n \times n$ *matrix, then there is a real* $n \times n$ *matrix* B *such that* $e^B = C^2$.

Theorem 1.4.3. *[15](**Floquet's theorem**) If* $\Phi(t)$ *is a fundamental matrix solution of the*

ω-periodic system (1.1)*, then, for all $t \in \mathbb{R}$,*

$$\Phi(t+\omega) = \Phi(t)\Phi^{-1}(0)\Phi(\omega).$$

In addition, there is a matrix B (which may be complex) such that

$$e^{\omega B} = \Phi^{-1}(0)\Phi(\omega),$$

and a ω-periodic matrix function $t \to P(t)$ (which may be complex valued) such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R and a real 2ω-periodic matrix function $t \to Q(t)$ such that $\Phi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Proof. Since the function $t \to A(t)$ is periodic, it is defined for all $t \in \mathbb{R}$. Thus, by Theorem 2.4 (see [15] page 130), all solutions of the system are defined for $t \in \mathbb{R}$.
If $\Psi(t) := \Phi(t+\omega)$, then $\Psi(t)$ is a matrix solution. Indeed, we have that

$$\dot{\Psi}(t) = \dot{\Phi}(t) = A(t+\omega)\Phi(t+\omega) = A(t)\Psi(t),$$

as required.
Define

$$C := \Phi^{-1}(0)\Phi(\omega) = \Phi^{-1}(0)\Psi(0),$$

and note that C is nonsingular. The matrix function $t \to \Phi(t)C$ is clearly a matrix solution of the linear system with initial value $\Phi(0)C = \Psi(0)$. By the uniqueness of solution, $\Psi(t) = \Phi(t)C$ for all $t \in \mathbb{R}$. In particular, we have that

$$\Phi(t+\omega) = \Phi(t)C = \Phi(t)\Phi^{-1}(0)\Phi(\omega),$$

$$\Phi(t+2\omega) = \Phi((t+\omega)+\omega) = \Phi(t+\omega)C = \Phi(t)C^2.$$

By Theorem 1.4.2, there is a matrix B, possibly complex, such that $e^{\omega B} = C$.
Also, there is a real matrix R such that $e^{2\omega R} = C^2$.
If $P(t) := \Phi(t)e^{-tB}$ and $Q(t) = \Phi(t)e^{-tR}$, then

$$P(t+\omega) = \Phi(t+\omega)e^{-\omega B}e^{-tB} = \Phi(t)Ce^{-\omega B}e^{-tB} = \Phi(t)e^{-tB} = P(t),$$

$$Q(t+2\omega) = \Phi(t+2\omega)e^{-2\omega R}e^{-tR} = \Phi(t)C^2e^{-2\omega R}e^{-tR} = \Phi(t)e^{-tR} = Q(t).$$

Thus, we have $P(t+\omega) = P(t)$, $Q(t+2\omega) = Q(t)$ and

$$\Phi(t) = P(t)e^{tB} = Q(t)e^{tR},$$

as required. □

The representation $\Phi(t) = P(t)e^{tB}$ in Floquet's theorem is called a *Floquet normal form* for the fundamental matrix $\Phi(t)$.

1.5 Application of fixed point theorems on ODE

In this section, we will apply three different famous fixed point theorems to establish several existence results and state a set of new periodicity conditions for the following ODEs with a parameter $\lambda \in \mathbb{R}$:

$$x'(t) = a(t)x(t) - \lambda f(t, x(t)), \quad t \in \mathbb{R}.$$

We will resort to the contraction mapping principle to obtain first results concerning the uniqueness of periodic solutions for the considered ODEs. Next, by using Schauder's fixed point theorem, the second existence result is presented for the existence of at least one periodic solution. The assumptions given in both results do not ask for fixed sign conditions on the coefficient functions, of the above ODEs. For the third results, existence theorem for two positive periodic solutions of ODEs are established by using a well-known fixed-point index theorem due to Krasnoselskii under the conditions that $\dfrac{f(x)}{x}$ have the some limits, where the positivity of the coefficient functions of the considred ODEs are needed. Let us begin to explore this issue of periodicity.

Example 1.5.1. *We study the existence of periodic solutions for the ODE follow:*

$$x'(t) = a(t)x(t) - \lambda f(t, x(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $\lambda \in \mathbb{R}$, the functions $a : \mathbb{R} \to (0, \infty)$ and $f : \mathbb{R} \times (0, \infty) \to (0, \infty)$ are continuous.

For $\omega > 0$*, let* $(P_\omega, \|.\|)$ *a Banach space of* ω*-periodic continuous functions,*

$$P_\omega = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R})/x(t+\omega) = x(t) \text{ for all } t \in \mathbb{R}\},$$

equipped with the supremum norm:

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)|.$$

We assume that the functions $a(.)$ *and* $f(.,.)$ *are* ω*-periodic on* t*, that is*

$$f(t+\omega, v) = f(t, v), \text{ and } a(t+\omega) = a(t) \text{ with } \int_0^\omega a(s)ds > 0. \tag{1.3}$$

for all $t \in \mathbb{R}$ *and* $v \in (0, \infty)$*. For positive constants* μ_1 *and* μ_2*, the function* f *is assumed Lipschitz in their composents:*

$$|f(t, v) - f(t, w)| \leq \mu_1 |v - w|, \tag{1.4}$$

$$|f(t_2, v) - f(t_1, v)| \leq \mu_2 |t_2 - t_1|. \tag{1.5}$$

Consider the mapping defined on $(P_\omega, \|.\|)$ *by*

$$(Tx)(t) := \lambda \int_t^{t+\omega} g(t, u) f(u, x(u)) du, \tag{1.6}$$

where

$$g(t, u) := \frac{\exp\Big(\int_u^t a(s)ds\Big)}{1 - \exp\Big(-\int_0^\omega a(s)ds\Big)}. \tag{1.7}$$

Lemma 1.5.1. *[29] Assume* (1.3) *holds. Then* $x \in P_\omega$ *satisfies* $(Tx)(t) = x(t)$ *for all* $t \in \mathbb{R}$ *if and only if* x *is a* ω*-periodic solution of* (1.2)*.*

Suppose (1.3) *holds and let* $x \in P_\omega$ *be a solution of* (1.2)*, by multiplying both sides of* (1.2) *by* $\exp\Big(-\int_0^t a(s)ds\Big)$ *and taking integral from* t *to* $t+\omega$*, we get*

$$x(t+\omega)e^{-\int_0^{t+\omega} a(s)ds} - x(t)e^{-\int_0^t a(s)ds} = \lambda \int_t^{t+\omega} e^{-\int_0^u a(s)ds} f(u, x(u)) du,$$

and by employing ω-periodicity of x and a we deduce

$$x(t)e^{-\int_0^t a(s)ds} = \frac{1}{1-e^{-\int_0^T a(s)ds}}\lambda \int_t^{t+\omega} e^{-\int_0^u a(s)ds} f(u,x(u))du,$$

which gives

$$x(t) = \lambda \int_t^{t+\omega} g(t,u)f(u,x(u))du = (Tx)(t),$$

where g is as in (1.7)*. Conversely, one may easily verify that*

$$x(t) = \lambda \int_t^{t+\omega} g(t,u)f(u,x(u))du,$$

solve (1.2)*. We have $g(t+\omega, u+\omega) = g(t,u)$ and*

$$0 < \beta_1 = \frac{\theta}{1-\theta} \leq g(t,u) \leq \frac{1}{1-\theta} = \beta_2,\ \forall u \in [t, t+\omega], \tag{1.8}$$

where

$$\theta := \exp\Big(-\int_0^\omega a(s)ds\Big). \tag{1.9}$$

Now, we will apply the Banach fixed point theorem to prove that the operator (1.6) *has one fixed point.*

We will show that $T : P_\omega \to P_\omega$ is a contraction, let $\varphi, \psi \in P_\omega$

$$\begin{aligned} |(T\varphi)(t) - (T\psi)(t)| &= \Big|\lambda \int_t^{t+\omega} g(t,u)\Big(f(t,\varphi(t)) - f(t,\psi(t))\Big)du\Big| \\ &\leq |\lambda| \int_t^{t+\omega} |g(t,u)|\Big|f(t,\varphi(t)) - f(t,\psi(t))\Big|du \\ &\leq |\lambda|\beta_2 \int_t^{t+\omega} \Big|f(t,\varphi(t)) - f(t,\psi(t))\Big|du. \end{aligned}$$

By the assumption (1.4)*, we have*

$$|(T\varphi)(t) - (T\psi)(t)| \leq |\lambda|\beta_2\omega\mu\|\varphi - \psi\|.$$

For $|\lambda|\beta_2\omega\mu < 1$, H is a contraction. Then, it has one fixed point.

Lemma 1.5.2. *If* (1.3)-(1.9) *and $|\lambda|\beta_2\omega\mu < 1$ hold, then the equation* (1.2) *has one periodic solution.*

For applying the Schauder fixed point theorem, we need to construct a bounded, closed *and convex set from the Banach space* $\boldsymbol{P_\omega}$ *as follow:*

For $\boldsymbol{L, N > 0}$

$$\boldsymbol{P_\omega(L,N) = \{x \in P_\omega,\ \|x\| \le L,\ |x(t_2) - x(t_1)| \le N|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}\}.}$$

We have to show that the operator $\boldsymbol{T}$ *is well defined, i.e.* $\boldsymbol{T(P_\omega(L,N)) \subset P_\omega(L,N)}$. *Hence, we make sur that for* $\boldsymbol{\varphi \in P_\omega}$, *we have* $\boldsymbol{(T\varphi) \in P_\omega}$, $\boldsymbol{\|T\varphi\| \le L}$, *and*

$$\boldsymbol{|(T\varphi)(t_2) - (T\varphi)(t_1)| \le N|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}}$$

.

It is easy to verify that $\boldsymbol{(T\varphi)(t+\omega) = (T\varphi)(t)}$. *By* (1.4), *we ontain*

$$\begin{aligned} \boldsymbol{|f(t,\varphi)|} &= \boldsymbol{|f(t,v) - f(t,0) + f(t,0)|} \\ &\le \boldsymbol{|f(t,v) - f(t,0)| + |f(t,0)|} \\ &\le \boldsymbol{\mu\|v\| + |f(t,0)|.} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{|f(t,\varphi)|} &= \boldsymbol{|f(t,\varphi) - f(0,\varphi) + f(0,\varphi)|} \\ &\le \boldsymbol{|f(t,\varphi) - f(0,\varphi)| + |f(0,\varphi)|} \\ &\le \boldsymbol{\mu_2|t| + |f(0,\varphi)|.} \end{aligned}$$

We note $\boldsymbol{\eta = \sup_{t\in[0,\omega]} |f(t,0)|}$ *and* $\boldsymbol{\delta = \sup_{t\in[0,\omega]} |f(0,\varphi)|}$
. *Then*

$$\begin{aligned} \boldsymbol{|(T\varphi)|} &\le \boldsymbol{|\lambda| \int_t^{t+\omega} |g(t,u)||f(t,\varphi(t))|du} \\ &\le \boldsymbol{|\lambda|\beta_2\omega(\mu\|\varphi\| + \eta)} \\ &\le \boldsymbol{|\lambda|\beta_2\omega(\mu L + \eta)} \\ &\le \boldsymbol{L.} \end{aligned}$$

Therefore, $\boldsymbol{\|(T\varphi)\| = \sup_{t\in[0,\omega]} |(T\varphi)(t)| \le L}$.

Let $t_1, t_2 \in \mathbb{R}$ *with* $t_1 < t_2$, *we have*

$$\begin{aligned}
|(T\varphi)(t_2) - (T\varphi)(t_1)| &= \Big|\lambda \int_{t_2}^{t_2+\omega} g(t_2,u) f(u\varphi(u))du - \lambda \int_{t_1}^{t_1+\omega} g(t_1,u) f(u,\varphi(u))\Big| \\
&= |\lambda|\Big|\int_{t_2}^{t_1} g(t_2,u) f(u,\varphi(u))du + \int_{t_1+\omega}^{t_2+\omega} g(t_2,u) f(u,\varphi(u))du \\
&\quad + \int_{t_1}^{t_1+\omega} (g(t_2,u) - h(t_1,u)) f(u,\varphi(u))du\Big| \\
&\leq |\lambda|\Big|\int_{t_2}^{t_1} g(t_2,u) f(u,\varphi(u))du + \int_{t_1+\omega}^{t_2+\omega} g(t_2,u) f(u,\varphi(u))du\Big| \\
&\quad + |\lambda|\Big|\int_{t_1}^{t_1+\omega} (g(t_2,u) - g(t_1,u)) f(u,\varphi(u))du\Big|.
\end{aligned}$$

Since (1.3) *holds, we have by letting* $v = u - \omega$

$$\begin{aligned}
&|\lambda|\Big|\int_{t_2}^{t_1} g(t_2,u) f(u,\varphi(u))du + \int_{t_1+\omega}^{t_2+\omega} g(t_2,u) f(u,\varphi(u))du\Big| \\
&= |\lambda|\Big|\int_{t_2}^{t_1} g(t_2,u) f(u,\varphi(u))du + \int_{t_1}^{t_2} g(t_2,v+\omega) f(v+\omega,\varphi(v+\omega))dv\Big| \\
&\leq |\lambda|| - \int_{t_1}^{t_2} g(t_2,u) f(u,\varphi(u))du| + |\lambda|| \int_{t_1}^{t_2} g(t_2,u) f(u,\varphi(u))du| \\
&\leq 2|\lambda|\beta_2 \int_{t_1}^{t_2} (\mu_2|u| + \delta)du \\
&= 2|\lambda|\beta_2(\mu_2|t_2 - t_1| + \delta),
\end{aligned}$$

and

$$\begin{aligned}
&|\lambda|\Big|\int_{t_1}^{t_1+\omega} (g(t_2,u) - g(t_1,u)) f(u,\varphi(u))du\Big| \\
&\leq |\lambda| \int_{t_1}^{t_1+\omega} \left| \frac{\exp\left(\int_u^{t_2} a(s)ds\right)}{1 - \exp\left(-\int_0^{\omega} a(s)ds\right)} - \frac{\exp\left(\int_u^{t_1} a(s)ds\right)}{1 - \exp\left(-\int_0^{\omega} a(s)ds\right)} \right| |f(u,\varphi(u))|du \\
&\leq \frac{|\lambda|}{|1-\theta|} \int_{t_1}^{t_1+\omega} \Big|\exp(\int_u^{t_1} a(s)ds)\Big|\Big|\exp(\int_{t_1}^{t_2} a(s)ds) - 1\Big||f(u,\varphi(u))|du.
\end{aligned}$$

We will employ finite increment theorem on the function $q(t) = e^{\int_{t_1}^{t} a(s)ds}$ *in the interval* $[t_1, t_2]$. *It is clear that the function* q *is well-defined and continuous in* $[t_1, t_2]$ *and derivable in* $]t_1, t_2[$, *hence*

$$q(t_2) - q(t_1) = e^{\int_{t_1}^{t_2} a(s)ds} - 1 = a(\alpha) e^{\int_{t_1}^{\alpha} a(s)ds}(t_2 - t_1), \quad \alpha \in]t_1, t_2[.$$

Then

$$\begin{aligned}
|\lambda|\Big|\int_{t_1}^{t_1+\omega}(g(t_2,u)-g(t_1,u))f(u,\varphi(u))du\Big| &\leq \frac{|\lambda|}{|1-\theta|}\int_{t_1}^{t_1+\omega}\Big|\exp(\int_u^{t_1}a(s)ds)\Big| \\
&\quad\times\Big|a(\alpha)e^{\int_{t_1}^{\alpha}a(s)ds}(t_2-t_1)\Big||f(u,\varphi(u))|du \\
&\leq |\lambda|\int_{t_1}^{t_1+\omega}\left|\frac{e^{\int_u^{\alpha}a(s)ds}}{1-e^{-\int_0^{\omega}a(s)ds}}\right||a(\alpha)||t_2-t_1||f(u,\varphi(u))|du \\
&\leq |\lambda|\|a\||t_2-t_1|\int_{t_1}^{t_1+\omega}|g(\alpha,u)||f(u,\varphi(u))|du \\
&\leq |\lambda|\|a\|\omega\beta_2(\mu L+\eta)|t_2-t_1|.
\end{aligned}$$

Therefore,

$$|(T\varphi)(t_2)-(T\varphi)(t_1)| \leq 2|\lambda|\beta_2(\mu_2|t_2-t_1|+\delta)+|\lambda|\|a\|\omega\beta_2(\mu L+\eta)|t_2-t_1|.$$

Consequently, $T(P_\omega(L,N))\subset P_\omega(L,N)$.
By the definition of the set $P_\omega(L,N)$*, it is uniformly bounded. For* $t_2\in[t_1,t_1+\omega], t_2>t_1$*, we have* $t_2-t_1\leq\omega$*. Let* $\varphi\in P_\omega(L,N)$*, then*

$$\begin{aligned}
|(T\varphi)(t_2)-(T\varphi)(t_1)| &\leq N|t_2-t_1| \\
&\leq N\omega=\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$*,* $P_\omega(L,N)$ *is equicontinuous. By Arzela-Ascoli theorem,* $P_\omega(L,N)$ *is relatively compact, and by the proposition 1.1.1,* $P_\omega(L,N)$ *is compact.*
Now, we show that the operator T *is continuous. Let* $\varphi,\psi\in P_\omega(L,N)$

$$\begin{aligned}
|(T\varphi)(t)-(T\psi)(t)| &\leq |\lambda|\int_t^{t+\omega}|g(t,u)||f(u,\varphi(u))-f(u,\psi(u))|du \\
&\leq |\lambda|\beta_2\omega\mu_1\|\varphi-\psi\|.
\end{aligned}$$

$\delta=|\lambda|\beta_2\omega\mu_1>0$.
Then, T *is continuous. By the Schauder fixed point theorem-version 2,* T *has at least one fixed point.*

Lemma 1.5.3. *If* (1.3)-(1.9) *hold, then the equation* (1.2) *has at least one periodic solution.*

Next, we introduce the following limits:

$$\underline{f_0} = \lim_{v\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,v)}{v}, \qquad \underline{f_\infty} = \lim_{v\to\infty} \min_{t\in[0,\omega]} \frac{f(t,v)}{v},$$

$$\overline{f_0} = \lim_{v\to 0^+} \max_{t\in[0,\omega]} \frac{f(t,v)}{v}, \qquad \overline{f_\infty} = \lim_{v\to\infty} \max_{t\in[0,\omega]} \frac{f(t,v)}{v},$$

and we define a cone Λ *as a subset of* $P_\omega(L,N)$ *by*

$$\Lambda = \{x \in P_\omega(L,N) : x(t) \geq \theta\|x\| \text{ for all } t \in [0,\omega]\}, \tag{1.10}$$

where θ *is as in* (1.9).

Definition 1.5.1. *Let* K *be a subset of a Banach space* X *and* $A : K \to X$ *a map. If* A *is continuous and* $A(K)$ *is contained in a compact subset of* X*, then we say that* A *is a compact mapping (we also say that* A *is completely continuous).*

We have to prove that T *is continuous on* Λ *and* $T(\Lambda) \subset \Lambda$*.*
Let $\varphi \in P_\omega(L,N)$*, we have before* $T\varphi \in P_\omega(L,N)$*,*

$$\begin{aligned}
(T\varphi(t)) &= \lambda \int_t^{t+\omega} g(t,u) f(u,\varphi(u))du \\
&\geq \lambda\beta_1 \int_t^{t+\omega} f(u,\varphi(u))du \\
&= \theta\Big(\lambda \int_t^{t+\omega} \frac{1}{1-\theta} f(u,\varphi(u))du\Big) \\
&\geq \theta\Big(\lambda \int_t^{t+\omega} g(t,u) f(u,\varphi(u))du\Big) \\
&= \theta(T\varphi)(t).
\end{aligned}$$

We arrive to $(T\varphi)(t) \geq \theta(T\varphi)(t), \forall t \in \mathbb{R}$*.*
Then

$$(T\varphi)(t) \geq \sup_{t\in[0,\omega]} |(T\varphi)(t)| = \|T\varphi\|.$$

Hence, $T(\Lambda) \subset \Lambda$*.*
T *is continuous on* $P_\omega(L,N)$ *and the cone* Λ *is a subset of* $P_\omega(L,N)$*. Then* T *is continuous on* Λ*.*

Theorem 1.5.1. *[29] Let* (1.8) *and* (1.9) *hold. Assume* $\underline{f_0} = \underline{f_\infty} = \infty$, *and there exists a constant* $r > 0$ *such that*

$$f(t, v) \leq r\theta \tag{1.11}$$

for $v \in [0, W]$ *and* $t \in [0, \omega]$. *Then, for* $\lambda > 0$ *the equation* (1.2) *has two positive solutions.*

If $\underline{f_0} = \infty$, *then there exists* $0 < \underline{r} < r$ *such that* $f(t, v) \geq c_1 v$ *for* $0 < v \leq \underline{r}$. *And if* $\underline{f_\infty} = \infty$, *then there exists* $\overline{r} > r$, *such that* $f(t, v) \geq c_2 v$ *for* $v \geq \overline{r}$.
Then, we introduce three bounded open balls as follow:

$$\Omega_1 = \{x \in P_\omega(L, N) : \|x\| < \underline{r}\},$$

$$\Omega_2 = \{x \in P_\omega(L, N) : \|x\| < r\},$$

$$\Omega_3 = \{x \in P_\omega(L, N) : \|x\| < \overline{r}\}.$$

Since (1.8) *and* (1.11) *hold, and for* $\varphi \in \Lambda \cap \partial\Omega_1$, *we get*

$$\begin{aligned} |(T\varphi)(t)| &\geq \lambda\beta_1 \int_0^\omega |f(u, \varphi(u))| du \\ &\geq \lambda\beta_1 \int_0^\omega c_1 \varphi(u) du \\ &\geq \lambda\beta_1 c_1 \omega\theta \|\varphi\|. \end{aligned}$$

For $c_1 \geq \lambda\beta_1\omega\theta$, *we get* $|(T\varphi)(t)| \geq \|\varphi\|$.
Therefore,

$$\|T\varphi\| \geq \|\varphi\|, \text{ for } \varphi \in \Lambda \cap \partial\Omega_1.$$

For $\varphi \in \Lambda \cap \partial\Omega_2$, *since* (1.8) *and* (1.11) *hold, we have*

$$\begin{aligned} |(T\varphi)(t)| &\leq \lambda\beta_2 \int_0^\omega |f(u, \varphi(u))| du \\ &\leq \lambda\beta_2 \int_0^\omega r\theta du \\ &\leq \lambda\beta_2 \omega\theta r. \end{aligned}$$

We set $\lambda \leq \dfrac{1}{\beta_2\omega\theta}$, *then* $|(T\varphi)(t)| \leq r = \|\varphi\|$.
Therefore,

$$\|T\varphi\| \geq \|\varphi\|, \text{ for } \varphi \in \Lambda \cap \partial\Omega_2.$$

The case of $\boldsymbol{\varphi} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_3}$ *has the same procedure of* $\boldsymbol{\varphi} \in \boldsymbol{\Lambda} \cap \boldsymbol{\partial\Omega_1}$ *for* $\boldsymbol{c_2} \geq \boldsymbol{\lambda\beta_1\omega\theta}$*. Consequently, since* $\underline{\boldsymbol{r}} < \boldsymbol{r} < \overline{\boldsymbol{r}}$*, by the fixed point theorem in cone, the mapping* $\boldsymbol{T}$ *has two fixed points; one in* $\boldsymbol{\Lambda} \cap (\overline{\boldsymbol{\Omega_2}} \setminus \boldsymbol{\Omega_1})$ *and second in* $\boldsymbol{\Lambda} \cap (\overline{\boldsymbol{\Omega_3}} \setminus \boldsymbol{\Omega_2})$*. This is equivalent to the existence of at least two positive periodic solutions of* (1.2)*.*

Chapter 2

Delay differential equations

This chapter provides background materials necessary for the rest of the dissertation. Some preliminaries and basic defininitions are given for delay differential equations. Additionally, we illustrate a method for resolving conflicts with an example.

2.1 Basic Concepts of Delay differential Equations

Delay differential equations (DDEs) are a class of differential equations that depend on previous states or have memory effects in their formulations. Mathematically speaking, DDEs are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. Many real life phenomena such in physics, biology, medicine, economics, engineering, etc, are modeled by initial value problem or Cauchy problem, for ordinary differential equations of the type

$$\begin{cases} \boldsymbol{x'(t) = f(t, x(t)), \quad t > t_0} \\ \boldsymbol{x(t_0) = x_0,} \end{cases} \tag{2.1}$$

where the function $\boldsymbol{x(.)}$, called the state variable, represents some physical quantity that evolves over time.

Nevertheless, models including differential equations with time delay terms exist. The delay differs from one model to other, it can represnts incubation periods of an infectious disease in epidemiology, the transit time or the duration of a cellular transformation in the dynamics of cell populations or transport delays. In many cases, it can be related to the time of gestation, forestation, deforestation, life cycle or the period of maturation in population dynamics or little lags such as acceleration and deceleration in physical processes.

Thus, it seems clear that DDEs are more suitable to describe a wide class of phenomena in many branches of science, such as life science, physical science and economical science, then the ordinary differential models which can, at best, be approximations of real world problems. Therefore, it has be necessity to include time delay terms into the differential equations, in order to make models more consistent with real phenomena.

For that, we need to modify (2.1) by including the dependence of $\boldsymbol{x'}$ on the past time values of the state variable $\boldsymbol{x}$. By doing so, we obtain:

$$\begin{cases} \boldsymbol{x'(t) = f(t, x_t)}, & \boldsymbol{t > t_0}, \\ \boldsymbol{x_{t_0}(s) = x(t_0 + s) = \Phi(s)}, & \boldsymbol{s \in [-\tau, t_0]}, \end{cases} \tag{2.2}$$

where $\boldsymbol{x_t}$ and $\boldsymbol{\Phi} \in \boldsymbol{C([-\tau, t_0]}, \mathbb{R}^n\boldsymbol{)}$ and $\boldsymbol{f} : \boldsymbol{\Omega} \to \mathbb{R}^n$ is a given function of the set $\boldsymbol{\Omega} \subset \mathbb{R} \times \boldsymbol{C([-\tau, t_0]}, \mathbb{R}^n\boldsymbol{)}$ into $\mathbb{R}^n$, $\boldsymbol{\Phi}$ represents the initial point or the initial data.

Functional differential equations with delays have an ancient history coming back to the early of 18th century. The first attempts have been made by *J.Bernoulli, L.Euler, J.L.Lagrange, P.Laplace, S.Poisson*, but unfortunately, equations of this kind have been ignored at that time. In many cases, the delay was estimated as non significant and ignored to simplify the studies. In 1728, J.Bernolli was who put the first retarted differential equation

$$\boldsymbol{x'(t) = x(t-1)},$$

when he considered a weighted stretched vibrating cord with distributed masses on it, but he ignored it since he thought that there were several mistakes in deducing the equation. Furthermore, due to many factors including limitation of the theory and mathematical tools to deal with this equations, also the fact that the introduction of delays may further complicate the dynamics or may effect the stability of the systems.

They had their moment of glory in the 20th century which has been marked by an explosion of scientific researches on this topic at that time. In 1908, during the international conference of mathematicians, Picard has revealed the importance of taking into account the effect of delays in modelling physical phenomena. He emphasized the significance of accounting for hereditary effects when constructing models of physical system. Volterra seemed to be the first to applay this idea. In 1931, he wrote a fundamental book on the role of hereditary effects on models for the interaction of species [41], which have contributed in developing both the theoretical ecology and the theory of delay differential equations and their applications. They gained much momentum after 1940, especially in the Soviet Union, due to the consideration of meaningful models of engineering systems and control. During the 50s and 60s, the theory of delay differential equation were widely developped and several books about FDEs with delayed argument were appeared by the authors *Myshkis* (1951), *Krasovskii* (1959), Bellman and Cooke (1963), Halanay (1966), and others. Such as [7], [19], [22], [23], [27], [28], [32].

Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, physics, population dynamics, industrial robotics, neuronal networks, and even economics and finance. They continue to be an active area of research, with new applications and theoretical results being discovered all the time.

2.2 Classification

In this part, we are interested about DDEs classification according to the types of delays cited in the literature. We distinguish two main classes which are "delay differential equations" and "neutral differential equations".

2.2.1 Delay differential equations

In this class, we find three categories of delays: constant, variable, and distribued.

Constant delay differential equations

(also called discrete delay) In their simplest form, it is written as follows:

$$\boldsymbol{x'(t) = f(t, x(t), x(t - \tau))},$$

where $\boldsymbol{f}$ is a given function and $\boldsymbol{\tau}$ is a positive constant. the presence of the term $\boldsymbol{x(t-\tau)}$ indicates that the state of the system at time $\boldsymbol{t}$ depends on its state at a previous time $\boldsymbol{t-\tau}$. We can find this type of equations in Nicholson's blowflies model (see[28]).

Variable delay differential equations

The delay in this case is time varying, depending on the state, or time and state-depending.

Time-varying delay differential equtions: (also called time-dependent delay) In their simplest form, this kind of equations is written as follows:

$$\boldsymbol{x'(t) = f(t, x(t), x(t - \tau(t)))},$$

where $\boldsymbol{f}$ is a given function.

State-dependent variable delay differential equations: In their simplest form, this kind of equations is written as follows:

$$\boldsymbol{x'(t) = f(t, x(t), x(t - \tau(x(t))))},$$

where $\boldsymbol{f}$ is a given function.

Time and state-dependent delay equations: In their simplest form, this kind of equations is written as follow:

$$\boldsymbol{x'(t) = f(t, x(t), x(t - \tau(t, x(t))))},$$

where $\boldsymbol{f}$ is a given function. If we take $\boldsymbol{\tau(t) = t - x(t)}$, then we obtain the iterative functional differential equations of the form

$$\boldsymbol{x(t) = f(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t))},$$

where $\boldsymbol{x^{[0]}(t) = t}$, $\boldsymbol{x^{[1]}(t) = x(t)}$ and $\boldsymbol{x^{[2]}(t) = x(x(t))}$ are the iterates of the state $\boldsymbol{x}$. For examples, we refer to the works [20], [26], [38], [40], [46] for more informations on the subject.

All these equations are **arbitrary variable delay differential equations** where the delay and its derivative are not bounded. In some cases, we encounter conditions on the delay or its derivative , as follow:

Differential equations with increased delay: This type of equations requires knowing the maximum value of the delay.

$$\boldsymbol{0 \leq \tau(t) \leq \tau_{max}}.$$

Bounded delay differential equations (bi-bounded): This kind of equations which is often less studied than the previous case, contains a delay verifying the following constraint:

$$\tau_{min} \leq \tau(t) \leq \tau_{max}.$$

Delay differential equations varying slowly over time: $\tau(t)$ is an almost everywhere derivable function such that

$$\dot{\tau}(t) \leq \lambda < 1,$$

which then indicates a limitation on the speed of variation of the delay and that the latter varies slowly over time, in other words that the delayed information arrives in chronological order.

Moderately time-varying delay equations: $\tau(t)$ is an almost everywhere derivable function such that

$$\dot{\tau}(t) \leq \lambda \text{ with } \lambda \geq 1.$$

Delay equations varying rapidly in time: In this type, there are no constraints on the delay and its derivative.

Distributed delay differential equations

In their simplest form, these equations are written as follows:

$$x'(t) = -\alpha x(t) - \beta \int_{-\tau}^{0} g(t, x(t+\theta))d(\theta).$$

For example, we find this type of equation in the population dynamics model presented by *Volterra* in 1934 where he used a distributed delay term to examine a cumulative effect on the mortality rate of a species.

Differential equation with unknown delay

In this case, no assumption about the delay is considered whether it is constant, variable or distributed.

2.2.2 Neutral differential equations

Delay differential equations, in which the delay occurs in the derivative of the state variable as well as in the independent variable, are called neutral differential equations. Also, the derivative of the state at the current time depends not only on the values of the past state

but even on the highest order derivative intervening in the equation of the past time.
In their simplest form, these equations are written as follows:

$$\boldsymbol{\frac{d}{dt}[Dx(t-\tau(t))] = f(t, x(t), x(t-\tau(t)))},$$

where $\boldsymbol{f}$ is given and $\boldsymbol{D}$ is an operator. For example, we find this type of equations in the distributed network model [28], such equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise in high speed computers where lossless transmission lines are used to interconnect switching circuits. Moreover, they arise in the study of two or more simple oscillatory systems with some interconnections between them ([37], [16]), and in modeling physical problems such as vibration of masses attached to an elastic bar [37].

2.3 Difference between DDEs and ODEs

Delay differential equations (DDEs) and ordinary differential equations (ODEs) differ in how systems behavior is modeled over time. In ODEs, the derivative at a given time depends only on the current state of the system at that time. In DDEs, the derivative also depends on the state of the system at previous times, introducing a "memory" or "delay" effect. Among these differences, we mention the following:

ODE	DDE
The rate of variability change is determined by its current value and perhaps the values of other variables at the same time point.	The rate of variability depends on its current value and also on the previous value with a time delay.
used to model dynamic systems where the system's behavior depends on the current state.	used to model systems in which the current situation depends on previous cases and therefore the effect of time delay enters the dynamics of the system.
have an instantaneous effect, and generate finite dimensional systems.	have not an instantaneous effect, and generate infinite dimensional system.
The initial problem value of the form $x'(t) = f(t, x(t)), \quad t \geq t_0,$ the initial condition at initial time point $x(t_0) = x_0$.	The initial problem value of the form $x'(t) = f(t, x_t), \quad t \geq t_0,$ the **history function** $\forall t \in [t_0 - \tau, t_0], x(t) = \varphi(t)$.

Dynamical strutures: DDEs already show a richer range of dynamics structures compared to ODEs. One of the main reasons for this richness is the presence of delays, which introduce memory effects into the dynamics of the system. These memory effects can lead to a variety of complex behaviors such as oscillations and even chaotic dynamics.

Propagated discontinuities: In both of DDEs and ODEs, discontinuity can occur when there is a difference between the left and right side boundaries of the solution function derivative at a given point. This can be expressed as follows:

$$\lim_{t \to t_0^-} x_0'(t) \neq \lim_{t \to t_0^+} x_0'(t).$$

There exists a jump derivative discontinuity at t_0 where:
$x_0(t)$ is the solution function.
t_0 is the point at which the discontinuity occurs.
$x_0'(t)$ represents the derivative of the solution function with respect to time.
This applies to the both of ODEs and DDEs. However, in the case of DDEs, the existence of delays leads to additional complications, and the disruptions may arise due to the reliance of the system's status on previous values.

2.4 Existence and uniqueness results

In this section, we will see theorems on the existence and uniqueness of solutions for the both of FDDEs and NDEs.

2.4.1 Functional differential equations with delays

Given a number $\tau > 0$. $\mathcal{C}([a,b],\mathbb{R}^n)$, the Banach space of continuous functions defined on $[a,b]$ with values in $\mathbb{R}^n$ is provided with the norm of uniform convergence.
Let $[a,b] = [-\tau,0]$, we put $C = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and we denote the norm of an element $\varphi \in C$ by

$$\|\varphi\| = \sup_{-\tau\leq t\leq 0} |\varphi(t)|.$$

If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in \mathcal{C}([t_0-\tau, t_0+A],\mathbb{R}^n)$, then for $t \in [t_0, t_0+A]$, we define $x_t \in C$ by

$$x_t(s) = x(t+s),$$

for all $s \in [-\tau, 0]$.

Definition 2.4.1. *[22] If $\mathfrak{D}$ is a subset of $\mathbb{R} \times C$, $f : \mathfrak{D} \to \mathbb{R}^n$ is a given function and " $\cdot$ " here represents the derivative on the right, the equation*

$$\dot{x}(t) := f(t, x_t), \tag{2.3}$$

where

$$x_t(s) := x(t+s), s \in [-\tau, 0], \tag{2.4}$$

is a delay functional differential equation on $\mathfrak{D}$ denoted and the number τ is called the delay. It is clear that the case $\tau = 0$ corresponds to the case of ordinary differential equations.

Example 2.4.1. *The following equations are delay differential equations*

$$x'(t) = 2x(t) + x(t-3),$$

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-\tau(t)) + \gamma(t),$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)ds,$$

where α, β, τ *are continuous functions.*

The first equation represents an autonomous linear differential equation with constant delay $\tau = 3$.

The second is a linear differential equation with non-homogenuous non-autonomous functional delay.

The last represents a linear delay integro-differential equation.

It is obvious that an initial condition appropriate to the time $t = t_0$ requires the determination of the function x over the entire interval $[t_0 - \tau, t_0]$.

$$x(t) = \psi(t), t \in [t_0 - \tau, t_0], \tag{2.5}$$

where $\psi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a given function assumed to be continuous called the initial condition of the delay equation (2.3). Thus, the equation (2.3) can be written in the form

$$\begin{cases} x'(t) := f(t, x_t), & t \geq t_0 \\ x(t) = \psi(t), & t \in [t_0 - \tau, t_0], \end{cases}$$

where ψ is a given continuous function on the interval $[t_0 - \tau, t_0]$.

Definition 2.4.2. *[22] Given* $\psi \in C$ *and* $t_0 \in \mathbb{R}$. *If* $t \in [t_0, t_0 + A]$ *with* $A > 0$, *then the function* $x(.)$ *such that* $x(t) = \psi(t)$ *if* $t \in [t_0 - \tau, t_0]$ *is a solution of the equation* (2.3) *and called the solution of* (2.3) *through* (t_0, ψ) *and it is often denoted by*

$$x(t) = x(t_0, \psi, f). \tag{2.6}$$

Lemma 2.4.1. *[22] Let* $\psi \in C, t_0 \in \mathbb{R}$ *and* $f(t, \psi)$ *a continuous function. Finding solutions of the equation* (2.3) *through* (t_0, ψ) *is equivalent to solving*

$$\begin{cases} x_{t_0} = \psi, \\ x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u) du, & t \geq t_0. \end{cases} \tag{2.7}$$

Proof. **Necessary condition**. Let (t, t_0, ψ) a solution of (2.3), then

$$\begin{cases} x'(t) = f(t, x_t), & t \geq t_0 \\ x_t = \psi. \end{cases}$$

By integration, we get

$$\int_{t_0}^{t} x'(s)ds = x(t) - x(t_0) = \int_{t_0}^{t} f(s, x_s)ds, \quad t \geq t_0.$$

As $x(t_0) = x(t_0 + 0) = x_{t_0}(0) = \psi(0)$, we obtain

$$\begin{cases} x_{t_0} = \psi, \\ x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u)du, & t \geq t_0. \end{cases}$$

Sufficient condition. Let $x(t, t_0, \psi)$ a solution of the integral equation (2.7). So,

$$x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(s, x_s)ds.$$

Since $f(t, x_t)$ is continuous in t, by applying the mean theorem, we have

$$lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(s, x_s)ds = f(t, x_t),$$

hence the result. □

Theorem 2.4.1. *(**Existence**)[22] for the equation* (2.3)*, suppose that* Ω *is an open subset of* $\mathbb{R} \times C$ *and* $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$ *is a continuous mapping on* Ω*. If* $(t_0, \psi) \in \Omega$*, then there exists a solution of the equation* (2.3) *passing through* (t_0, ψ)*.*

Definition 2.4.3. *[22] The function* $f(t, \varphi)$ *is said to be a Lipschitz mapping with respect to* φ *on a compact* K *of* $\mathbb{R} \times C$ *if there is a constant* $k > 0$ *such that for all* $(t, \psi_i) \in K, i = 1, 2, ...$*, we have*

$$|f(t, \psi_1) - f(t, \psi_2)| \leq K|\psi_1 - \psi_2|. \tag{2.8}$$

Theorem 2.4.2. *(**Uniqueness**)[22] Suppose that* Ω *is an open subset of* $\mathbb{R} \times C$*,* $f : \Omega \to \mathbb{R}^n$ *is continuous and* $f(t, \psi)$ *is a Lipschitz mapping with respect to* ψ *on any compact subset* *of* $\boldsymbol{\Omega}$. *If* $(\boldsymbol{t_0}, \boldsymbol{\psi}) \in \boldsymbol{\Omega}$, *then there is a unique solution of the equation* (2.3) *passing through* $(\boldsymbol{t_0}, \boldsymbol{\psi})$.

2.4.2 Neutral differential equation

Definition 2.4.4. *[23] Suppose that* $\boldsymbol{\Omega}$ *is an open subset of* $\mathbb{R} \times \boldsymbol{C}$ *of elements* $(\boldsymbol{t}, \boldsymbol{\psi})$. *A Function* $\boldsymbol{g} : \boldsymbol{\Omega} \to \mathbb{R}^n$ *is said to be atomic at a point* γ *of* $\boldsymbol{\Omega}$ *if* $\boldsymbol{g}$ *and its first and second derivatives are continuous in the sense of Fréchet with respect to* $\boldsymbol{\psi}$ *and* $\boldsymbol{g}_{\psi}$, *the derivative with respect to* $\boldsymbol{\psi}$, *is atomic at* γ *of* $\boldsymbol{\Omega}$.

Definition 2.4.5. *[23] Suppose that* $\boldsymbol{\Omega}$ *is an open subset of* $\mathbb{R} \times \boldsymbol{C}$, $\boldsymbol{g} : \boldsymbol{\Omega} \to \mathbb{R}^n$ *and* $\boldsymbol{f} : \boldsymbol{\Omega} \to \mathbb{R}^n$ *are two given continuous functions with* $\boldsymbol{g}$ *is atomic in zero. The relation*

$$\frac{d}{dt}\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{x_t}) = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x_t}), \tag{2.9}$$

is called a differential equation of neutral type.

Example 2.4.2.

$$x'(t) = 3x'(t-5),$$

$$x'(t) = x(t) + x'(t-2) - x'(t-7),$$

$$x'(t) = x(t-1) + [x'(t-3) + 1]^2.$$

Example 2.4.3. *If* $\boldsymbol{\tau} > \boldsymbol{0}$, $\boldsymbol{x}$ *is a scalar,* $\boldsymbol{g}(\varphi) = \varphi(\mathbf{0}) - \sin(-\boldsymbol{\tau})$, *and* $\boldsymbol{f} : \overline{\boldsymbol{\Omega}} \to \mathbb{R}^n$ *is continuous, then the pair* $(\boldsymbol{g}, \boldsymbol{f})$ *defines an neutral differential equation,*

$$\frac{d}{dt}[x(t) - \sin x(t-\tau)] = f(t, x_t).$$

Theorem 2.4.3. *(**Existence**)[23] If* $\boldsymbol{\Omega}$ *is an open subset of* $\mathbb{R} \times \boldsymbol{C}$ *and* $(\boldsymbol{t_0}, \boldsymbol{\psi}) \in \boldsymbol{\Omega}$, *then there is a solution of the equation* (2.9) *passing through* $(\boldsymbol{t_0}, \boldsymbol{\psi})$.

Theorem 2.4.4. *(**Uniqueness**)[23] If* $\boldsymbol{\Omega}$ *is an open subset of* $\mathbb{R} \times \boldsymbol{C}$ *and* $\boldsymbol{f}(\boldsymbol{t}, \boldsymbol{\psi})$ *is a Lipschitz mapping with respect to* $\boldsymbol{\psi}$ *on any compact subset of* $\boldsymbol{\Omega}$, *so for all* $(\boldsymbol{t_0}, \boldsymbol{\psi}) \in \boldsymbol{\Omega}$, *there is a unique solution for the equation* (2.9) *passing through* $(\boldsymbol{t_0}, \boldsymbol{\psi})$.

2.5 Method of steps

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solving the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solutions such as the variation of constants method, the separation of variables method and others. However, these methods are inapplicable to dealing with delay differential equations. Hence, qualitative and numerical analysis equations gather great importance.

The method of steps (also called method of successive integrations) is an elementary method that makes it possible to solve DDEs and NDEs digitally (analytically) and at the same time to establish the existence and uniqueness of the solution. It was presented in 1965, by *R.Bellman* for constant delay. In 1973, *El'sgol'ts*, Norkin and others have shown that it also remains valid for variable delays, provided that the delay never cancels. The principle of this method is to look for solutions on successive intervals of the type $[t_0+k\tau, t_0+(k+1)\tau]$ where $k \in \mathbb{N}$ by solving ordinary differential equations without delays in each interval. To fix the ideas, we explain the steps as follow:

Consider the following delay differential equation

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), & t \geq t_0, \\ x(t) = \varphi(t), & \forall t \in [t_0 - \tau, t_0], \end{cases}$$

For simplicity, we put $t_0 = 0$.

First step: In the interval $[-\tau, 0]$, the function $x(t)$ is the given function $\varphi(t)$, so the equation is solved in the interval $[-\tau, 0]$ and we denote $x_0(t)$ this solution. It should be pointed out here that if $t \in [0, \tau]$, so $t - \tau$ will reside in $[-\tau, 0]$.

Second step: In the interval $[0, \tau]$,we have $x(t-\tau) = x_0(t-\tau)$, we obtain the solution by solving the system:

$$\begin{cases} x'(t) = f(t, x(t), x_0(t-\tau)) = g_0(t, x(t)), & \forall t \in [0, \tau], \\ x(0) = \varphi(0), \end{cases}$$

which is an initial valued problem for an ordinary differential equation where $x_0(t-\tau) = \varphi(t-\tau)$ is known. Under suitable hypotheses on g_0, the existence and uniqueness of the solution of this system can be established and we denote by x_1 this solution in $[0,\tau]$.

Third step: In the interval $[\tau, 2\tau]$, the system becomes

$$\begin{cases} x'(t) = f(t, x(t), x_1(t-\tau)) = g_1(t, x(t)) \quad \forall t \in [\tau, 2\tau], \\ x(\tau) = x_1(\tau), \end{cases}$$

for which we can again find the solution x_2.

In general, by assuming that $x_{k-1}(t), \forall k = 1, 2, ...$ is defined on the interval $[(k-2)\tau, (k-1)\tau]$, then, we can solve the system

$$\begin{cases} x'(t) = f(t, x(t), x_{k-1}(t)) = g_{k-1}(t, x(t)), \quad t \in [(k-2)\tau, (k-1)\tau], \\ x((k-1)\tau) = x_{k-1}((k-1)\tau). \end{cases}$$

We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[-\tau, +\infty]$.

Example 2.5.1. *Consider the problem*

$$\begin{cases} x'(t) = -x(t-1), \quad t > 0 \\ x(t) = \varphi(t) = 2, \qquad \forall t \in [-1, 0] \end{cases}$$

For $t \in [0,1]$, we integrate the both sides of the equation, we obtain

$$x(t) - x(0) = -\int_0^t \varphi(s-1)ds,$$

then, the solution in the interval $[0,1]$ is

$$x_1(t) = 2 - 2t.$$

Now, we can find the solution in the interval $[1,2]$

$$\begin{aligned} x(t) &= x(1) - \int_1^t x_1(s-1)ds \\ &= -\int_1^t [2-2(s-1)]ds \\ &= t^2-4t-3. \end{aligned}$$

then, $x_2(t) = t^2 - 4t - 3$.

In the interval $[2,3]$*, the solution* x_3 *is given by*

$$\begin{aligned} x(t) &= x(2) - \int_2^t x_2(s-1)ds \\ &= -7 - \int_2^t [(s-1)^2 - 4(s-1) - 3]ds \\ &= \frac{-t^3}{3} + 3t^2 - 2t - 15 + \frac{8}{3}. \end{aligned}$$

In the interval $[3,4]$*, the solution* x_4 *is given by*

$$\begin{aligned} x(t) &= x(3) - \int_3^t x_3(s-1)ds \\ &= -\frac{1}{3} - \int_3^t \Big[\frac{-(s-1)^3}{3} + 3(s-1)^2 - 2(s-1) - 15 + \frac{8}{3}\Big]ds \\ &= \frac{-t^4}{12} - \frac{4t^3}{3} + \frac{9t^2}{2} + 7t + \frac{152}{12}. \end{aligned}$$

2.6 Delayed models

2.6.1 Economic model

Cooke and Yorke (1973) also briefly consider a related economic model. Let $x(t)$ be the value of capital stock. Assume that productionof new capital depends only on $x(t)$ and that the rate of production is $g(x(t))$. Also, assume that the lifetime of equipment is L and that depreciation is independent of the type of equipment; in particular, at time s after production the value of a unit of capital equipment has decreased in value to $P(s)$ times its original value. Here, $P(0) = 1$, $P(L) = 0$.

Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t-L, t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)g(x(t-s))ds + c \\ &= \int_{t-L}^{t} P(t-u)g(x(u))du + c. \end{aligned} \tag{2.10}$$

Cooke and Yorke obtain certain boundedness results for (2.10) and they pose the problem of determining conditions under which (2.10) has a periodic solution.

2.6.2 Biology model

Early in this century A.J.Lotka (1907) formulated basic principles for the mathematical theory of population growth. He assumed that

(a) individuals belong to different classes and the relative proportion of members of each class during any time period is constant,

(b) the life span of an individual in a class is independent of the number of members of that class and independent of the age distribution in the class, and

(c) the life support conditions remain constant.

On the basis of these assumptions Lotka derived the equation

$$x' = B_t - D_t, \tag{2.11}$$

where $x(t)$ is the population, B_t is the number of births per unit time and D_t the deaths. He generalized the model to the renewal equation (cf. Bellman and Cooke, 1963; Feller, 1941)

$$B(t) = G(t) + \int_0^t B(t-s)P(s)m(s)ds, \tag{2.12}$$

where $B(t)$ is the number of births between 0 and t, G is the number of births at time t to parents surviving from an initial population at $t = 0$, $P(s)$ is the probability density of survival to age at least s, and $m(s)$ is the probability density for a parent of age s giving birth.

2.6.3 Mixing of liquids

Consider a tank containing B gallons of salt water brine. Fresh water flows in at the top of the tank at a rate of q gallons per minute (see [19]). The brine in the tank is continually stirred, and the mixed solution flows out through a hole at the bottom, also at the rate of q gallons per minute.

Let $x(t)$ be the amount (in pounds) of salt in the brine in the tank at time t. If we assume continual, instantaneous, perfect mixing throughout the tank, then the brine leaving the tank contains $x(t)/B$ lbs. of salt per gallon, and hence

$$x'(t) = -qx(t)/B.$$

But, more realistically, let us agree that mixing cannot occur instantaneously throughout the tank. Thus, the concentration of the brine leaving the tank at time t will equal the average concentration at some earlier instant, say $t - \tau$. We shall assume that τ is a positive constant, although this assumption may also be subject to improvement. The differential equation for x then becomes a delay differential equation,

$$x'(t) = -qx(t - \tau)/B,$$

or setting $c = q/B$,

$$x'(t) = -cx(t - \tau),$$

where τ is the delay or time lag.

Chapter 3

Existence and uniqueness of periodic solutions for nonlinear system of neutral differential equations

In this chapter, we are interesting to the study of existence and uniqness of periodic solution for the nonlinear neutral system

$$\frac{d}{dt}y(t) = A(t)h(y(t-\tau_1(t))) + \frac{d}{dt}Q(t, y(t-\tau_2(t))) + G(t, y(t), y(t-\tau_2(t))), \quad (3.1)$$

with two variable delays and the nonlinear term $A(t)h(y(t - \tau_1(t)))$, where $A(.)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as entries. The functions $h : \mathbb{R}^n \to \mathbb{R}^n$, $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments, and τ_1 is continuously differentiable.

In the process, we use the fundamental matrix solution on $y'(t) = A(t)y(t)$ coupled with Floquet theory to transform the system (3.1) into an integral system. Then, we construct appropriate mappings and employ Krasnoselskii's fixed point theorem to show the existence of a periodic solution of the system (3.1). We also use the contraction mapping principle to show the existence of one periodic solution of the proposed system.

The study of the periodicity of functional differential equations is far from systematic study, especially for higher-dimensional equations. Hence, it is meaningful to study the periodicity of n-dimensional differential systems. We should point out that the form of the system (3.1) is rather general and incorporates as special cases various problems which have been studied extensively in the literature. For example, Mesmouli, Ardjouni and Djoudi [35] have studied a case of this system where $h(x(t)) = x(t)$ as follow

$$\frac{d}{dt}y(t) = A(t)y(t-\tau(t)) + \frac{d}{dt}Q(t, y(t-g(t))) + G(t, y(t), y(t-g(t))),$$

with the linear term $A(t)y(t-\tau(t))$ and two variable delays, motivated by the works of Islam and Raffoul [24] whose study the system

$$\frac{d}{dt}y(t) = A(t)y(t) + \frac{d}{dt}Q(t, y(t-g(t))) + G(t, y(t), y(t-g(t))),$$

with one variable delay, where $\tau(t) = 0$.

Our main purpose is, by using Krasnoselskii's fixed point theorem, to establish the existence of periodic solutions of (3.1). We shall emphasize that we present some existence results for periodic solutions of a very general form of functional differential system, which may cover some special equations already handled, since our system involves the nonlinear term $A(t)h(y(t-\tau_1(t)))$, which is a grueling task in the application of fixed point to the existence of periodic solutions, and consequentially we contribute to the ongoing theory with the outcomes of this work.

3.1 Preliminaries

For $\omega > 0$, let P_ω be the set of all n-vector valued functions $y(t)$, which are continuous and periodic in t to period ω.
Then $(P_\omega, \|.\|)$ is a Banach space with the supremum norm

$$\|y(.)\| = \sup_{t\in\mathbb{R}} |y(t)| = \sup_{t\in[0,T]} |y(t)|,$$

where $|.|$ denotes the norm for $\mathbb{R}^n$. Also, if A is an $n \times n$ real matrix, then we define the norm $|A| = \max_{1\leq i\leq n} \sum_{j=1}^{n} \|a_{ij}\|$.

Definition 3.1.1. *Let matrix $A(.)$ be periodic of period ω. The linear system*

$$y'(t) = A(t)y(t), \tag{3.2}$$

is said to be noncritical with respect to ω, if it has no periodic solution of period ω except for the trivial solution $y = 0$.

In this chapter, we assume that

$$A(t+\omega) = A(t),\ \tau_1(t+\omega) = \tau_1(t) \geq \tau_1^* > 0,\ \tau_2(t+\omega) = \tau_2(t) \geq \tau_2^* > 0, \tag{3.3}$$

with τ_1 is continuously differentiable and τ_1^*, τ_2^* are constants.
For $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, the functions $Q(t, x)$ and $G(t, x, y)$ are periodic in t of period ω. That is

$$Q(t+\omega, x) = Q(t, x), \quad G(t+\omega, x, y) = G(t, x, y). \tag{3.4}$$

The functions Q, G, h are also globally Lipschitz continuous. That is, for $x, y, z, w \in \mathbb{R}^n$, there are positive constants k_1, k_2, k_3, k_4 such that

$$|Q(t, x) - Q(t, y)| \leq k_1 \|x - y\|, \tag{3.5}$$

$$|G(t, x, y) - G(t, z, w)| \leq k_2 \|x - z\| + k_3 \|y - w\|, \tag{3.6}$$

$$|h(x) - h(y)| \leq k_4 \|x - y\|. \tag{3.7}$$

Throughout this chapter, we assume system (3.2) is noncritical. Let $\Phi(t)$ represents the fundamental matrix of (3.2) with $\Phi(0) = I$, where I is the $n \times n$ identity matrix.
Then:

(a) $det(\Phi(t)) \neq 0$,

(b) There exists a costant matrix B such that $\Phi(t+\omega) = \Phi(t)\exp(\omega B)$, by Floquet theory,

(c) System (3.2) is noncritical if and only if $det(I - \Phi(\omega)) \neq 0$.

The following lemma is fundamental for our results.

Lemma 3.1.1. *[33] Suppose* (3.3) *and* (3.4) *hold, then* $\boldsymbol{y}$ *is a solution of* (3.1) *if and only if*

$$\begin{aligned}
&\boldsymbol{y(t)}\\
&= \boldsymbol{Q(t, y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_{t}^{t+\omega} \Phi^{-1}(s)A(s)[Q(s, y(s-\tau_2(s)))}\\
&\quad - \boldsymbol{(y(s)-h(y(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_{t}^{t+\omega} \Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s, y(s), y(s-\tau_2(s)))]ds,}
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
&\boldsymbol{U(\omega) = (\Phi^{-1}(\omega) - I)^{-1},}\\
&\boldsymbol{F(t) = A(t) - (1-\tau_1'(t))A(t-\tau_1(t)),}
\end{aligned} \tag{3.9}$$

where $\boldsymbol{\Phi(.)}$ *is the fundamental matrix solution of* (3.2).

Proof. Let $\boldsymbol{y(t)} \in \boldsymbol{P_\omega}$ be a solution of (3.1). We rewrite (3.1) as

$$\begin{aligned}
\boldsymbol{\frac{d}{dt}y(t)} &= \boldsymbol{A(t-\tau_1(t))h(y(t-\tau_1(t)))(1-\tau_1'(t))}\\
&\quad \boldsymbol{-A(t-\tau_1(t))h(y(t-\tau_1(t)))(1-\tau_1'(t))}\\
&\quad \boldsymbol{+A(t)h(y(t)) - A(t)h(y(t)) + A(t)h(y(t-\tau_1(t)))}\\
&\quad \boldsymbol{+\frac{d}{dt}Q(t, y(t-\tau_2(t))) + G(t, y(t), y(t-\tau_2(t)))}\\
&= \boldsymbol{\frac{d}{dt}Q(t, y(t-\tau_2(t))) - \frac{d}{dt}\int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds + A(t)h(y(t))}\\
&\quad \boldsymbol{+h(y(t-\tau_1(t)))[A(t) - A(t-\tau_1(t))(1-\tau_1'(t))]}\\
&\quad \boldsymbol{+G(t, y(t), y(t-\tau_2(t))).}
\end{aligned}$$

Putting $F(t) = A(t) - A(t-\tau_1(t))(1-\tau_1'(t))$, we have

$$\begin{aligned}
&\frac{d}{dt}[y(t) + \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds - Q(t, y(t-\tau_2(t)))] \\
&\quad = A(t)[y(t) + \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds - Q(t, y(t-\tau_2(t)))] \\
&\quad + F(t)h(y(t-\tau_1(t))) - A(t)[y(t) - h(y(t))] + A(t)Q(t, y(t-\tau_2(t))) \\
&\quad - A(t)\int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds + G(t, y(t), y(t-\tau_2(t))).
\end{aligned}$$

Since $\Phi(t)\Phi^{-1}(t) = I$, it follows that

$$\begin{aligned}
0 &= \frac{d}{dt}[\Phi(t)\Phi^{-1}(t)] \\
&= A(t)\Phi(t)\Phi^{-1}(t) + \Phi(t)\frac{d}{dt}\Phi^{-1}(t) \\
&= A(t) + \Phi(t)\frac{d}{dt}\Phi^{-1}(t).
\end{aligned}$$

This implies

$$\frac{d}{dt}\Phi^{-1}(t) = \Phi^{-1}(t)A(t).$$

If $y(t)$ is a solution of (3.1) with $y(0) = y_0$, then

$$\begin{aligned}
&\frac{d}{dt}[\Phi^{-1}\Big(y(t) + \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds - Q(t, y(t-\tau_2(t)))\Big)] \\
&\quad = \frac{d}{dt}\Phi^{-1}(t)[y(t) + \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds - Q(t, y(t-\tau_2(t)))] \\
&\quad + \Phi^{-1}(t)\frac{d}{dt}[y(t) + \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds - Q(t, y(t-\tau_2(t)))] \\
&\quad = \Phi^{-1}(t)A(t)[Q(t, y(t-\tau_2(t))) - (y(t) - h(y(t))) \\
&\quad - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds] + \Phi^{-1}(t)[F(t)h(y(t-\tau_1(t))) \\
&\quad + G(t, y(t), y(t-\tau_2(t)))].
\end{aligned}$$

Integrating of the above equation from 0 to t yields

$$\begin{aligned}
y(t) &= Q(t, y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds \\
&\quad + \Phi(t)\Big(y(0) + \int_{-\tau_1(0)}^{0} A(s)h(y(s))ds - Q(0, y(-\tau_2(0)))\Big)
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&+\Phi(t)\int_0^t \Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s)))-(y(t)-h(y(s)))\\
&-\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds\\
&+\Phi(t)\int_0^t \Phi^{-1}(s)[F(s)h(y(s-\tau_1(s)))\\
&+G(s,y(s),y(s-\tau_2(s)))]ds.
\end{aligned}
$$

Since $y(\omega)=y_0=y(0)$ and

$$
(I-\Phi(\omega))^{-1}=(\Phi(\omega)(\Phi^{-1}(\omega)-I))^{-1}=(\Phi^{-1}(\omega)-I)^{-1}\Phi^{-1}(\omega),
$$

using (3.10), we obtain

$$
\begin{aligned}
&y(0)+\int_{-\tau_1(0)}^{0} A(s)h(y(s))ds-Q(0,y(-\tau_2(0)))\\
&=(\Phi^{-1}(\omega)-I)^{-1}\int_0^{\omega}\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s)))-(y(t)-h(y(s)))\\
&-\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]\\
&+(\Phi^{-1}(\omega)-I)^{-1}\int_0^{\omega}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s)))\\
&+G(s,y(s),y(s-\tau_2(s)))]ds.
\end{aligned}
\tag{3.11}
$$

Substituting (3.11) into (3.10) yields

$$
\begin{aligned}
y(t)&=Q(t,y(t-\tau_2(t)))-\int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds\\
&+\Phi(t)(\Phi^{-1}(\omega)-I)^{-1}\int_0^{\omega}\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s)))-(y(t)-h(y(s)))\\
&-\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds\\
&+\Phi(t)(\Phi^{-1}(\omega)-I)^{-1}\int_0^{\omega}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s)))\\
&+G(s,y(s),y(s-\tau_2(s)))]ds+\Phi(t)\int_0^t\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s)))\\
&-(y(t)-h(y(s)))-\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds\\
&+\Phi(t)\int_0^t\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s)))+G(s,y(s),y(s-\tau_2(s)))]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
y(t) &= Q(t,y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds \\
&\quad +\Phi(t)(\Phi^{-1}(\omega)-I)^{-1}\Big[\int_0^{\omega}\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s))) \\
&\quad -(y(t)-h(y(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds \\
&\quad +\int_0^{\omega}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s,y(s),y(s-\tau_2(s)))]ds \\
&\quad +(\Phi^{-1}(\omega)-I)\int_0^{t}\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s))) - (y(t)-h(y(s))) \\
&\quad -\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds \\
&\quad +(\Phi^{-1}(\omega)-I)\int_0^{t}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) \\
&\quad +G(s,y(s),y(s-\tau_2(s)))]ds\Big] \\
&= Q(t,y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds \\
&\quad +\Phi(t)(\Phi^{-1}(\omega)-I)^{-1}\Big[\int_t^{\omega}\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s))) \\
&\quad -(y(t)-h(y(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds \\
&\quad +\int_t^{\omega}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s,y(s),y(s-\tau_2(s)))]ds \\
&\quad +\int_0^{t}\Phi^{-1}(\omega)\Phi^{-1}(s)A(s)[Q(s,y(s-\tau_2(s))) - (y(t)-h(y(s))) \\
&\quad -\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds \\
&\quad +\int_0^{t}\Phi^{-1}(\omega)\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s,y(s),y(s-\tau_2(s)))]ds\Big].
\end{aligned}
$$

By letting $s = v-\omega$ and $U(\omega) = (\Phi^{-1}(\omega)-I)^{-1}$, the above expression yields

$$
\begin{aligned}
y(t) &= Q(t,y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds + \Phi(t)U(\omega)\int_t^{\omega}\Phi^{-1}(s)A(s) \\
&\times\ [Q(s,y(s-\tau_2(s))) - (y(t)-h(y(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds \\
&+\ \Phi(t)U(\omega)\int_t^{\omega}\Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s,y(s),y(s-\tau_2(s)))]ds \\
&+\ \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(\omega)\Phi^{-1}(v-\omega)A(v-\omega)[Q(v-\omega,y(v-\omega-\tau_2(v-\omega))) \\
&-\ (y(v-\omega)-h(y(v-\omega))) - \int_{v-\omega-\tau_1(v-\omega)}^{v-\omega} A(u)h(y(u))du]dv \\
&+\ \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(\omega)\Phi^{-1}(v-\omega)[F(v-\omega)h(y(v-\omega-\tau_1(v-\omega))) \\
&+\ G(v-\omega,y(v-\omega),y(v-\omega-\tau_2(v-\omega)))]dv.
\end{aligned}
\tag{3.12}
$$

Consequently, since (3.3) and (3.4) hold, equation (3.12) becomes

$$\begin{aligned}
&\boldsymbol{y(t)}\\
&= \boldsymbol{Q(t, y(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(y(s))ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_t^{\omega} \Phi^{-1}(s)A(s)[Q(s, y(s-\tau_2(s))) - (y(t) - h(y(s)))}\\
&\quad - \boldsymbol{\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_t^{\omega} \Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s, y(s), y(s-\tau_2(s)))]ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_t^{t+\omega} \Phi^{-1}(s)A(s)[Q(s, y(s-\tau_2(s))) - (y(t) - h(y(s)))}\\
&\quad - \boldsymbol{\int_{s-\tau_1(s)}^{s} A(u)h(y(u))du]ds}\\
&\quad + \boldsymbol{\Phi(t)U(\omega)\int_t^{t+\omega} \Phi^{-1}(s)[F(s)h(y(s-\tau_1(s))) + G(s, y(s), y(s-\tau_2(s)))]ds.}
\end{aligned} \tag{3.13}$$

By combining the two integrals of (3.13), we can easily obtain (3.8). By finding the derivative of (3.8), we can obtain (3.1). The converse implication is easily obtained and the proof is complete. □

3.2 Existence and Uniquness of periodic solution

In this section, we prove the existence of periodic solutions using Krasnoselskii fixed point theorem, and the existence and uniqueness of periodic sulotion by contraction mapping principle. To apply Theorem 1.3.4, we need to define a Banach space $\boldsymbol{E}$, a closed bounded convex subset $\boldsymbol{M}$ of $\boldsymbol{E}$ and construct two mappings: one completely continuous and the other a contraction. So, we let $(\boldsymbol{E}, \|.\|) = (\boldsymbol{P}_\omega, \|.\|)$ and

$$\boldsymbol{M} = \{\varphi \in \boldsymbol{P}_\omega : \|\varphi\| \leq \boldsymbol{L}\} \tag{3.14}$$

where $\boldsymbol{L}$ is a positive constant.

Define the mapping $\boldsymbol{P} : \boldsymbol{P}_\omega \to \boldsymbol{P}_\omega$ by

$$\begin{aligned}
&(P\varphi)(t) \\
&= Q(t,\varphi(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds + \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(s)A(s) \\
&\quad \times [Q(s,\varphi(s-\tau_2(s))) - (\varphi(s)-h(\varphi(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))du]ds \\
&\quad + \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(s)[F(s)h(\varphi(s-\tau_1(s))) \\
&\quad + G(s,\varphi(s),\varphi(s-\tau_2(s)))]ds.
\end{aligned} \tag{3.15}$$

Therefore, we express the above equation as

$$(P\varphi)(t) = (R\varphi)(t) + (B\varphi)(t), \tag{3.16}$$

where $R, B : P_\omega \to P_\omega$ are given by

$$\begin{aligned}
(R\varphi)(t) &= Q(t,\varphi(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds \\
&+ \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(s)A(s)[Q(s,\varphi(s-\tau_2(s))) \\
&- (\varphi(s)-h(\varphi(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))du]ds.
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
(B\varphi)(t) &= \Phi(t)U(\omega)\int_t^{t+\omega}\Phi^{-1}(s)[F(s)h(\varphi(s-\tau_1(s))) \\
&+ G(s,\varphi(s),\varphi(s-\tau_2(s)))]ds.
\end{aligned} \tag{3.18}$$

Lemma 3.2.1. *[33] Let R be defined by* (3.17)*, and assume that* (3.3)-(3.7) *hold. Then R is continuous and RM is contained in a compact set.*

Proof. First of all, by (3.5)-(3.7), we obtain

$$\begin{aligned}
|Q(t,x)| &\le |Q(t,x) - Q(t,0) + Q(t,0)| \\
&\le k_1\|x\| + |Q(t,0)|,
\end{aligned} \tag{3.19}$$

$$\begin{aligned} |G(t,x,y)| &\le |G(t,x,y)-G(t,0,0)+G(t,0,0)| \\ &\le k_2\|x\|+k_3\|y\|+|G(t,0,0)|, \end{aligned} \tag{3.20}$$

$$|h(x)| \le k_4\|x\|+|h(0)|. \tag{3.21}$$

Let R defined by (3.17), we show that are is continuous in the supremum norm. Let $\varphi_1, \varphi_2 \in M$, for $\epsilon > 0$, choose $\eta = \epsilon/\Delta$, where

$$\Delta = k_1+\alpha|A|k_4+c\omega|A|k_1+c\omega|A|+c\omega|A|k_4+c\omega|A|(\alpha|A|k_4).$$

For $\|\varphi_1-\varphi_2\| < \eta$, we obtain

$$\begin{aligned} &|(R\varphi_1)(t)-(R\varphi_2)(t)| \\ &\le |Q(t,\varphi_1(t-\tau_2(t)))-Q(t,\varphi_2(t-\tau_2(t)))|+\int_{t-\tau_1(t)}^{t}|A||h(\varphi_1(s))-h(\varphi_2(s))|ds \\ &\quad+\int_t^{t+\omega}|[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||Q(s,\varphi_1(s-\tau_2(s)))-Q(s,\varphi_2(s-\tau_2(s)))|ds \\ &\quad+\int_t^{t+\omega}|[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||\varphi_1(s)-\varphi_2(s)|ds \\ &\quad+\int_t^{t+\omega}|[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||h(\varphi_1(s))-h(\varphi_2(s))|ds \\ &\quad+\int_t^{t+\omega}|[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A|\int_{s-\tau_1(s)}^{s}|A||h(\varphi_1(u))-h(\varphi_2(u))|duds \\ &\le k_1\|\varphi_1-\varphi_2\|+\alpha|A|k_4\|\varphi_1-\varphi_2\|+c\omega|A|k_1\|\varphi_1-\varphi_2\|+c\omega|A|\|\varphi_1-\varphi_2\| \\ &\quad+c\omega|A|k_4\|\varphi_1-\varphi_2\|+c\omega|A|(\alpha|A|k_4)\|\varphi_1-\varphi_2\| \\ &= [k_1+\alpha|A|k_4+c\omega|A|k_1+c\omega|A|+c\omega|A|k_4+c\omega|A|(\alpha|A|k_4)]\|\varphi_1-\varphi_2\| < \epsilon, \end{aligned}$$

which shows the continuity of R.

Now, we show that RM is contained in a compact set. Firstly, we prove that for all $\varphi \in M$, $R\varphi$ is uniformly bounded.

For $\varphi \in M$, we have

$$\begin{aligned} |(R\varphi)(t)| &\le |Q(t,\varphi(t-\tau_2(t)))|+\int_{t-\tau_1(t)}^{t}|A||h(\varphi(s))|ds \\ &\quad+\int_t^{t+\omega}|[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A|[|Q(s,\varphi(s-\tau_2(s)))| \\ &\quad+|\varphi(s)|+|h(\varphi(s))|+\int_{s-\tau_1(s)}^{s}|A||h(\varphi(u))|du]ds \end{aligned}$$

$$
\begin{aligned}
&\leq k_1 L + \beta + \alpha|A|(k_4 L + \eta) + c\omega|A|[k_1 L + \beta + L + k_4 L + \eta \\
&+\alpha|A|(k_4 L + \eta)] = E,
\end{aligned}
\tag{3.22}
$$

such that E is a constant, and

$$
\alpha = \sup_{t\in[0,\omega]} |\tau_1(t)|, \quad \beta = \sup_{t\in[0,\omega]} |Q(t,0)|, \quad \gamma = \sup_{t\in[0,\omega]} |G(t,0,0)|,
$$

$$
\eta = |h(0)|, \quad c = \sup_{t\in[0,\omega]} \Big(\sup_{s\in[t,t+\omega]} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}| \Big).
$$

Therefore, for all $\varphi \in M$, $R\varphi$ is uniformly bounded.

Second, we show that RM is equicontinuous. Hence, let $\varphi \in M$, without loss of generality, we can pick $z < t$ such that $t - z < \omega$. Then we have

$$
\begin{aligned}
&|(R\varphi)(t) - (R\varphi)(z)| \\
&\leq |Q(t,\varphi(t-\tau_2(t))) - Q(z,\varphi(z-\tau_2(z)))| + \Big| \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds \\
&\quad - \int_{z-\tau_1(z)}^{z} A(s)h(\varphi(s))ds \Big| \\
&\quad + \Big| \int_{t}^{t+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1} A(s) Q(s,\varphi(s-\tau_2(s)))ds \\
&\quad - \int_{z}^{z+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1} A(s) Q(s,\varphi(s-\tau_2(s)))ds \Big| \\
&\quad + \Big| \int_{t}^{t+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1} A(s)\varphi(s)ds \\
&\quad - \int_{z}^{z+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1} A(s)\varphi(s)ds \Big| \\
&\quad + \Big| \int_{t}^{t+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1} A(s)h(\varphi(s))ds \\
&\quad - \int_{z}^{z+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1} A(s)h(\varphi(s))ds \Big| \\
&\quad + \Big| \int_{t}^{t+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1} A(s) \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))duds \\
&\quad - \int_{z}^{z+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1} A(s) \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))duds \Big|.
\end{aligned}
$$

Since (3.19)-(3.21) hold, we have

$$
\begin{aligned}
&\Big| \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds - \int_{z-\tau_1(z)}^{z} A(s)h(\varphi(s))ds \Big| \\
&\leq (k_4 L + \eta)\Big(\int_{z}^{t} |A|ds + \int_{z-\tau_1(z)}^{t-\tau_1(t)} |A|ds \Big),
\end{aligned}
$$

and

$$\begin{aligned}
&\Big|\int_t^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\\
&\quad-\int_z^{z+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\quad=\Big|\int_t^{\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\\
&\quad+\int_{z+\omega}^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\\
&\quad+\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)\\
&\quad\times A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\le|\Phi(\omega)-I|\Big|\int_\omega^t[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\quad+\Big|\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)\\
&\quad\times A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\le c|\Phi(\omega)-I|(k_1L+\beta)\int_z^t|A|ds+(k_1L+\beta)\\
&\quad\times\int_0^\omega|\Phi(t)-\Phi(z)|\Big|U(\omega)\Phi(\omega)\Phi^{-1}(s)\Big||A|ds,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_t^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad-\int_z^{z+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}A(s)\varphi(s)ds\Big|\\
&=\Big|\int_t^{z}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad+\int_{z+\omega}^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad+\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)\varphi(s)ds\Big|\\
&\le|\Phi(\omega)-I|\Big|\int_z^t[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\varphi(s)ds\Big|\\
&\quad+\Big|\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)\varphi(s)ds\Big|\\
&\le c|\Phi(\omega)-I|L\int_z^t|A|ds+L\int_0^\omega|\Phi(t)-\Phi(z)|\Big|U(\omega)\Phi(\omega)\Phi^{-1}(s)\Big||A|ds,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_t^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)h(\varphi(s))ds \\
&\quad-\int_z^{z+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}A(s)h(\varphi(s))ds\Big| \\
&=\Big|\int_t^{z}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)h(\varphi(s))ds \\
&\quad+\int_{z+\omega}^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)h(\varphi(s))ds \\
&\quad+\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)h(\varphi(s))ds\Big| \\
&\le|\Phi(\omega)-I|\Big|\int_z^{t}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)h(\varphi(s))ds\Big| \\
&\quad+\Big|\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)h(\varphi(s))ds\Big| \\
&\le c|\Phi(\omega)-I|(k_4L+\eta)\int_z^t|A|ds+(k_4L+\eta) \\
&\quad\times\int_0^{\omega}|\Phi(t)-\Phi(z)|\Big|U(\omega)\Phi(\omega)\Phi^{-1}(s)\Big||A|ds,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_t^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^{s}A(u)h(\varphi(u))duds \\
&\quad-\int_z^{z+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}A(s)\int_{s-\tau_1(s)}^{s}A(u)h(\varphi(u))duds\Big| \\
&=\Big|\int_t^{z}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^{s}A(u)h(\varphi(u))duds \\
&\quad+\int_{z+\omega}^{t+\omega}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^{s}A(u)h(\varphi(u))duds \\
&\quad+\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)\int_{s-\tau_1(s)}^{s} \\
&\quad\times A(u)h(\varphi(u))duds\Big| \\
&\le|\Phi(\omega)-I|\Big|\int_z^{t}[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^{s}A(u)h(\varphi(u))duds\Big| \\
&\quad+\Big|\int_z^{z+\omega}\Big([\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}-[\Phi(s)U^{-1}(\omega)\Phi^{-1}(z)]^{-1}\Big)A(s)\int_{s-\tau_1(s)}^{s} \\
&\quad\times A(u)h(\varphi(u))duds\Big| \\
&\le|\Phi(\omega)-I|\alpha|A|(k_4L+\eta)\int_z^t|A|ds+\alpha|A|(k_4L+\eta) \\
&\quad\times\int_0^{\omega}|\Phi(t)-\Phi(z)|\Big|U(\omega)\Phi(\omega)\Phi^{-1}(s)\Big||A|ds,
\end{aligned}$$

which implies

$$
\begin{aligned}
&|(R\varphi)(t) - (R\varphi)(z)| \\
&\leq |Q(t, \varphi(t - \tau_2(t))) - Q(z, \varphi(z - \tau_2(z)))| + (k_4 L + \eta) \\
&\quad \times \left(\int_z^t |A| ds + \int_{z-\tau_1(z)}^{t-\tau_1(t)} |A| ds \right) \\
&\quad + c|\Phi(\omega) - I|(k_1 L + \beta) \int_z^t |A| ds + (k_1 L + \beta) \\
&\quad \times \int_0^\omega |\Phi(t) - \Phi(z)| \left| U(\omega)\Phi(\omega)\Phi^{-1}(s) \right| |A| ds \\
&\quad + c|\Phi(\omega) - I| L \int_z^t |A| ds + L \int_0^\omega |\Phi(t) - \Phi(z)| \left| U(\omega)\Phi(\omega)\Phi^{-1}(s) \right| |A| ds \\
&\quad + c|\Phi(\omega) - I|(k_4 L + \eta) \int_z^t |A| ds + (k_4 L + \eta) \\
&\quad \times \int_0^\omega |\Phi(t) - \Phi(z)| \left| U(\omega)\Phi(\omega)\Phi^{-1}(s) \right| |A| ds \\
&\quad + |\Phi(\omega) - I| \alpha |A| (k_4 L + \eta) \int_z^t |A| ds + \alpha |A| (k_4 L + \eta) \\
&\quad \times \int_0^\omega |\Phi(t) - \Phi(z)| \left| U(\omega)\Phi(\omega)\Phi^{-1}(s) \right| |A| ds.
\end{aligned}
$$

By the dominated convergence theorem, we have $|(R\varphi)(t) - (R\varphi)(z)| \to 0$ as $t - z \to 0$ for all $\varphi \in M$. Then, RM is equicontinuous.
Hence, by Ascoli-Arzela's theorem, RM is contained in a compact set. □

Lemma 3.2.2. *[33] Suppose* (3.3)-(3.7) *hold and*

$$c\omega(k_4|F| + k_2 + k_3) < 1. \tag{3.23}$$

If B is defined by (3.18)*, and F by* (3.9)*, then B is a contraction.*

Proof. Let B is defined by (3.18), and $\varphi, \psi \in M$. By (3.5)-(3.7), we have

$$
\begin{aligned}
&|(B\varphi)(t) - (B\psi)(t)| \\
&= \Bigg| \int_t^{t+\omega} [\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1} \Big[F(s)(h(\varphi(s - \tau_1(s))) - h(\psi(s - \tau_1(s)))) \\
&\quad + (G(s, \varphi(s), \varphi(s - \tau_2(s))) - G(s, \psi(s)\psi(s - \tau_2(s)))) \Big] ds \Bigg| \\
&\leq c\omega(k_4|F| + k_2 + k_3)\|\varphi - \psi\|.
\end{aligned}
$$

The proof is complete. □

Theorem 3.2.1. *[33] Let the hypothesis of Lemmas 3.2.1 and 3.2.2 hold, $\boldsymbol{M}$ defined by* (3.14).
If there exist a constant $\boldsymbol{L > 0}$ such that

$$\begin{gathered}\boldsymbol{k_1 L + \beta + \alpha|A|(k_4 L + \eta) + c\omega|A|[k_1 L + \beta + L + (k_4 L + \eta) + \alpha|A|(k_4 L + \eta)]} \\ \boldsymbol{+ c\omega[|F|(k_4|L + \eta) + (k_2 + k_3)L + \gamma] \leq L,}\end{gathered}$$

then (3.1) *has a $\boldsymbol{\omega}$-periodic solution in $\boldsymbol{M}$.*

Proof. By Lemma 3.2.1, $\boldsymbol{R}$ is continuous and $\boldsymbol{RM}$ is contained in a compact set, by Lemma 3.2.2, $\boldsymbol{B}$ is a contraction. Next, we will show that if $\boldsymbol{\varphi, \psi} \in \boldsymbol{M}$, we have $\boldsymbol{R\varphi + B\psi_i nM}$. Let $\boldsymbol{\varphi, \psi} \in \boldsymbol{M}$, we have

$$\begin{aligned}\boldsymbol{\|R\varphi + B\psi\|} &\leq \boldsymbol{k_1 L + \beta + \alpha|A|(k_4 L + \eta) + c\omega|A|[k_1 L + \beta + L + (k_4 L + \eta)} \\ &\quad \boldsymbol{+\alpha|A|(k_4 L + \eta)] + c\omega[|F|(k_4 L + \eta) + (k_2 + k_3)L + \gamma] \leq L.}\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's fixed point theorem are satisfied. Thus, there exists a fixed point $\boldsymbol{w} \in \boldsymbol{M}$ such that $\boldsymbol{w = Rw + Bw}$.
By Lemma 3.1.1, this fixed point is a solution of (3.1). Hence (3.1) has a $\boldsymbol{\omega}$-periodic solution in $\boldsymbol{M}$. □

Theorem 3.2.2. *[33] Suppose* (3.5)-(3.7) *hold. If*

$$\begin{gathered}\boldsymbol{k_1 + \alpha|A|k_4 + c\omega|A|k_1 + c\omega|A| + c\omega|A|k_4 + c\omega|A|(\alpha|A|k_4)]} \\ \boldsymbol{+ c\omega|F|k_4 + c\omega(k_2 + k_3) < 1,}\end{gathered} \tag{3.24}$$

then (3.1) *has a unique $\boldsymbol{\omega}$-periodic solution in $\boldsymbol{P_\omega}$.*

Proof. Let the mapping $\boldsymbol{P}$ given by (3.15). For all $\varphi_1, \varphi_2 \in \boldsymbol{P}_\omega$, we have

$$
\begin{aligned}
&|(P\varphi_1)(t) - (P\varphi 2)(t)| \\
&\le |Q(t, \varphi_1(t - \tau_2(t))) - Q(t, \varphi_2(t - \tau_2(t)))| + \int_{t-\tau_1(t)}^{t} |A||h(\varphi_1(s)) - h(\varphi_2(s))|ds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||Q(s, \varphi_1(s - \tau_2(s))) - Q(s, \varphi_2(s - \tau_2(s)))|ds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||\varphi_1(s) - \varphi_2(s)|ds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A||h(\varphi_1(s)) - h(\varphi_2(s))|ds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||A| \int_{s-\tau_1(s)}^{s} |A||h(\varphi_1(u)) - h(\varphi_2(u))|duds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||F||h(\varphi_1(s - \tau_1(s))) - h(\varphi_2(s - \tau_1(s)))|ds \\
&\quad + \int_t^{t+\omega} |[\Phi(s)U^{-1}(\omega)\Phi^{-1}(t)]^{-1}||G(s, \varphi_1(s), \varphi_1(s - \tau_2(s))) \\
&\quad - G(s, \varphi_2(s)\varphi_2(s - \tau_2(s)))|ds \\
&\le k_1\|\varphi_1 - \varphi_2\| + \alpha|A|k_4\|\varphi_1 - \varphi_2\| + c\omega|A|k_1\|\varphi_1 - \varphi_2\| + c\omega|A|\|\varphi_1 - \varphi_2\| \\
&\quad + c\omega|A|k_4\|\varphi_1 - \varphi_2\| + c\omega|A|(\alpha|A|k_4)\|\varphi_1 - \varphi_2\| + c\omega|F|k_4\|\varphi_1 - \varphi_2\| \\
&\quad + c\omega(k_2 + k_3)\|\varphi_1 - \varphi_2\| \\
&= [k_1 + \alpha|A|k_4 + c\omega|A|k_1 + c\omega|A| + c\omega|A|k_4 + c\omega|A|(\alpha|A|k_4) \\
&\quad + c\omega|F|k_4 + c\omega(k_2 + k_3)]\|\varphi_1 - \varphi_2\|.
\end{aligned}
$$

Since (3.24) holds, with the contraction mapping principle there exists a unique $\boldsymbol{w} \in \boldsymbol{P}_\omega$ with $\boldsymbol{Pw} = \boldsymbol{w}$, which is a solution of (3.1). □

Example 3.2.1. *We consider the two-dimensional system*

$$
\begin{aligned}
\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} &= \begin{pmatrix} p(t) & 1 \\ 0 & q(t) \end{pmatrix} \begin{pmatrix} h(x_1(t - \tau_1(t))) \\ h(x_2(t - \tau_1(t))) \end{pmatrix} + \frac{d}{dt} \begin{pmatrix} 0 \\ V(t, x_1(t - \tau_2(t))) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ W(t, x_1(t - \tau_2(t)), x_2(t - \tau_2(t))) \end{pmatrix}
\end{aligned} \tag{3.25}
$$

where $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *are positive periodic continuous functions with period* $\boldsymbol{\omega}$.
The functions $\boldsymbol{h} : \mathbb{R} \to \mathbb{R}$, $\boldsymbol{V} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{W} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous in their respective arguments,* $\tau_1(.), \tau_2(.)$ *satisfy* (3.3).

Functions $V(.,.)$ *and* $W(.,.,.)$ *are periodic in* t *with period* ω. *They are also globally Lipschitz continuous in* x *and* y *respectively. That is*

$$V(t+T,x) = V(t,x), \tag{3.26}$$

$$W(t+T,x,y) = W(t,x,y), \tag{3.27}$$

and there are positive constants k_1, k_2, k_3 *and* k_4 *such that*

$$|V(t,x)-V(t,y)| \leq k_1\|x-y\|, \tag{3.28}$$

$$|W(t,x,y)-W(t,z,w)| \leq k_2\|x-z\| + k_3\|y-w\|, \tag{3.29}$$

$$|h(x)-h(y)| \leq k_4\|x-y\|. \tag{3.30}$$

Let,

$$p(t) = q(t) = -1,\ h(x) = \frac{1}{3}\sin(x+\frac{\pi}{3}),\ \tau_1(t) = \frac{\pi}{50}\sin^2(\pi t),$$

$$V(t,x) = \frac{1}{5}\sin(2\pi t)\sin(x+\frac{\pi}{6}),$$

$$W(t,x,y) = \frac{1}{8}\cos(2\pi t)\sin(x) + \frac{1}{7}\sin(y+\frac{\pi}{3}),$$

and $\tau_2(.)$ *is a non-negative and continuous function with period* ω.
Consider the Banach space $(E, \|.\|)$

$$E = \{\varphi \in C(\mathbb{R},\mathbb{R}), \varphi(t+1) = \varphi(t), t \in \mathbb{R}\},$$

and the closed bounded convex subset

$$M = \{\varphi \in E : \|\varphi\| \leq \pi\}.$$

Then for $x, y, z, w \in M$, *we have*

$$|V(t,x)-V(t,y)| \leq \frac{1}{5}\|x-y\|,$$

$$|W(t,x,y)-W(t,z,w)| \leq \frac{1}{8}\|x-z\| + \frac{1}{7}\|y-w\|,$$

$$|h(x) - h(y)| \le \frac{1}{3}\|x - y\|,$$

$$\alpha = \sup_{t\in[0,T]} |\tau_1(t)|, \qquad \beta = \sup_{t\in[0,T]} |V(t,0)| = \frac{1}{10},$$

$$\gamma = \sup_{t\in[0,T]} |W(t,0,0)| = \frac{1}{7}, \qquad \eta = |h(0)| = \frac{\sqrt{3}}{6},$$

$$|F| = |A(t) - (1 - \tau_1'(t))A(1 - \tau_1(t))| < 0.04, \qquad c \le 0.34.$$

Consequently,

$$\begin{aligned}
&k_1 L + \beta + \alpha|A|(k_4 L + \eta) + c\omega|A|[k_1 L + \beta + L + (k_4 L + \eta) + \alpha|A|(k_4 L + \eta)] \\
&\quad + c\omega[|F|(k_4 L + \eta) + (k_2 + k_3)L + \gamma] \\
&\le \frac{\pi}{5} + \frac{1}{10} + \frac{\pi}{50}(\frac{\pi}{3} + \frac{\sqrt{3}}{6}) + 0.34[\frac{\pi}{5} + \frac{1}{10} + \pi + \frac{\pi}{3} + \frac{\sqrt{3}}{6} + \frac{\pi}{50}(\frac{\pi}{3} + \frac{\sqrt{3}}{6})] \\
&\quad + 0.34[0.04(\frac{\pi}{3} + \frac{\sqrt{3}}{6}) + (\frac{1}{7} + \frac{1}{8})\pi + \frac{1}{7}] \le \pi.
\end{aligned}$$

Then (3.25) *has a* $\mathbf{1}$*-periodic solution in* M*. Moreover,*

$$\begin{aligned}
&k_1 + \alpha|A|k_4 + c\omega|A|k_1 + c\omega|A| + c\omega|A|k_4 + c\omega|A|(\alpha|A|k_4) + c\omega|F|k_4 \\
&\quad + c\omega(k_2 + k_3) \\
&\le \frac{1}{5} + \frac{\pi}{50}\frac{1}{3} + 0.34\frac{1}{5} + 0.34 + 0.34\frac{1}{3} + 0.34\frac{\pi}{50}(\frac{\pi}{50}\frac{1}{3}) + 0.34(0.04)\frac{1}{3} + 0.34\frac{15}{56} \\
&\le 1.
\end{aligned}$$

Then (3.25) *has a unique* $\mathbf{1}$*-periodic solution in* M*.*

Conclusion

In conclusion, this work is devoted to study the existence of periodic solutions for a kind of nonlinear differential systems with two variable delays. Our results obtained by applying Krasnoselskii's fixed point theorem to original operator constructed here on a convex, closed-bounded subset of a Banach space using some functional analysis tools. In addition, we also used contraction mapping principle to show the uniqueness of periodic solution of the proposed system. Furthermore, with the help of fundamental matrix solution coupled with Floquet theory, we could transform the differential system into integral system to extract the operator.

The key contribution of this work is to generalize the work [35], replacing the linear term $\boldsymbol{A(t)y(t)}$ by the nonlinear term $\boldsymbol{A(t)h(y(t))}$.

Future research should focus on extending the application of this precise approach to more complex models, such as the equations with damped stochastic perturbations and other variants.

Bibliography

[1] M. Adivar and Y. N. Raffoul, Existence of periodic solutions in totally nonlinear delay dynamic equations, Electronic Journal of Qualitative Theory of Differential Equations 2009(1) (2009), 1–20.

[2] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Springer Dordrecht Heidelberg London New York.

[3] A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale, Acta Univ. Palacki. Olomnc., Fac. rer. nat., Mathematica, 52(1) (2013), 5–19.

[4] A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for a second-order nonlinear neutral differential equation with variable delay, Adv. Nonlinear Anal. 2 (2013), 151-161.

[5] A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, Malaya Journal of Matematik, 2(1) (2013), 60–67.

[6] J. Arino, *Fundamental Theory of Ordinary Differential Equations*, University of Manitoba. MATH 8430 Lecture Notes, Fall 2006.

[7] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York-London, 1963.

[8] L. Brouwer, Uber Abbildungen von Mannigfaltigkeit, Math. Ann., 70 (1912) pp. 97-115.

[9] R.F. Brown, *A Topological Introduction to Nonlinear Analysis.* Third Edition. Springer International Publishing Birkhauser. (2014)

[10] T.A. Burton, Integral equations, implicit functions, and fixed points, Proc. Amer. Math. Soc., 124 (1996), 2383-2390.

[11] T. A. Burton; Stability by fixed point theory for functional differential equations, Dover Publication, New York, 2006.

[12] T. A. Burton and T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, Dynam. Systems Appl. 11, 499-519,(2002).

[13] T. Bühler and D.A. Salamon, *Functional Analysis*, vol.(19)1, American Mathematical Society, 2018.

[14] R. Castelli, J.P. Lessard, J.D. Mireles James, Analytic enclosure of the fundamental matrix solution, Appl Math, vol(60), 617-636 (2015).

[15] C. Chicone, *Ordinary Differential Equations with Applications*, Springer New York, NY, 2006.

[16] K. Cook, D. Krumme, Differential difference equations and nonlinear initial-boundary-value problems for linear hyperbolic partial differential equations, J. Math. Anal. Appl. 24 (1968) 372–387.

[17] D. Daners, *Introduction to Functional Analysis*, University of Sydney, Australia, 2017.

[18] Y. M. Dib, M. R. Maroun, Y. N. Raffoul; Periodicity and stability in neutral nonlinear differential equations with functional delay, Electron. J. Differential Equations. 2005 (2005), no. 142, 1-11.

[19] R. D. Driver, *Ordinary and delay differential equations*, Springer Verlag, New York,1977.

[20] E. Eder, The functional differential equationx x (t) = x (x (t)), Journal of Differential Equations, 54 (1984) 390–400.

[21] K. Gopalsamy, X. He, L. Wen, On a periodic neutral logistic equation. Glasgow Math. J. 33, 281–286 (1991)

[22] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[23] J.K.Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[24] M. N. Islam, Y. N. Raffoul, Periodic solutions of neutral nonlinear system of differential equations with functional delay, *Journal of Mathematical Analysis And Applications.* **15** (2007), 1175–1186.

[25] C.H. Jin, J.W. Luo, Fixed points and stability in neutral differential equations with variable delays. Proc. Am. Math. Soc. 136(3), 909–918 (2008).

[26] W. Ke, On the equation x'(t) = f(x(x(t))); Funkcial. Ekvac, 33 (1990), 405-425.

[27] V. B. Kolmanovskii, A. D. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic, Dordrecht, (1992).

[28] V. B. Kolmanovskii and A. D. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Acad, Publ, 1999.

[29] H. C. Koyuncuoğlu and M. Adıvar, Positive periodic solutions of iterative functional differential equations, *Bull. Int. Math. Virtual Inst.*, 12(2)(2022), 227-235.

[30] M. A. Krasnoselskii, Positive solutions of operator equations, Noordhoof, Gronignen, 1964.

[31] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley and sons, 1989.

[32] Y. Kuang, *Delay differential equations with applications to population dynamics*, Academic Press, Boston, (1993).

[33] Y. LI, G. CHEN, Existence of periodic solutions and stability for a nonlinear system of neutral differential equations, *Electronic Journal of Differential Equations*, Vol. 2024 (2024), No. 21, pp. 1–21.

[34] M. Maroun, Y. N. Raffoul, Periodic solutions in nonlinear neutral difference equations with functional delay. J. Korean Math. Soc. 42, 255–268 (2005)

[35] M. B. Mesmouli, A. Ardjouni, A. Djoudi, Existence and uniqueness of periodic solutions for a system of nonlinear neutral functional differential equations with two functional delays, *Rendiconti del Circolo Matematico di Palermo.* **63** (2014), 409–424.

[36] Y. N. Raffoul, positive periodic solutions of nonlinear functional difference equations, Electr. J. Di§. Eq. 55 (2002), 1-8.

[37] V.P. Rubanik, Oscillations of Quasilinear Systems with Retardation, Nauka, Moscow, 1969.

[38] J. G. Si, X. P. Wang, Smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients, Journal of Mathematical Analysis and Applications, 226 (1998) 377–392.

[39] D.R. Smart, *Fixed Points Theorems*, Cambridge Univ. Press, Cambridge, 1980.

[40] S. Stanek, On global properties of solutions of functional differential equations x'(t) = x(x(t)) + x(t); Dynamic Systems and Applications, 4 (1995), 263-278.

[41] Vito Volterra, *Leçons sur la théorie mathématiques de la lutte pour la vie*, Gauthier-Villars et cie., 1931.

[42] E. Yankson, Periodicity in a system of differential equations with finite delay. Caspian J. M. Sci. 2(2), 137–147 (2013)

[43] E. Yankson, Positive solutions for a system of periodic neutral delay difference equations, African Diaspora Journal of Mathematics , 11 (2) (2011), 90-97.

[44] E. Zeidler, *Nonlinear functional analysis and its applications: Fixed point theorem*, Springer Verlag, New York Berlin Heidelberg, Tokyo 1985.

[45] H.Y. Zhao, M. Fe ckan, Periodic solutions for a class of differential equations with delays depending on state. Math. Commun. 23, 29ñ42 (2018)

[46] H. Y. Zhao and J. Liu, Periodic solutions of an iterative functional differential equation with variable coefficients, Math. Meth. Appl. Sc., 40 (2017) 286–292.