Algerian Democratic and Popular Republic
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

20 August 1955 University of Skikda
Faculty of Sciences
Department of Mathematics
Ref:………………

جامعة 20 أوت 1955 –سكيكدة
كلية العـلوم
قسم الرياضيات
المرجع:.........

Thesis

A view to obtaining the diploma of

Master of 2° cycle (LMD) in Mathematics
Option: ***Numerical analysis of PDEs***

Periodic Solutions of a First Order Neutral Differential Equation with Iterative Terms Arising in Life Sciences

Presented by:

Malak KHEMIS

Publicly discussed:

In front of the Jury:

1.	**Abdellah LALLOUCHE**	**M.C.B**	**20 August 1955 University of Skikda**	**President**
2.	**Ahlème BOUAKKAZ**	**M.C.A**	**20 August 1955 University of Skikda**	**Supervisor**
3.	**Yassine BEDRANI**	**M.C.B**	**20 August 1955 University of Skikda**	**Examiner**

University year : 2023/2024

CONTENTS

Abstract in English

Periodic solutions of a first order neutral differential equation with iterative terms arising in the life sciences

The main purpose of the current work is to investigate the existence, uniqueness and continuous dependence on parameters of positive periodic solutions for a first order neutral differential equation with iterative terms.

For achieving our goal, we convert the considered problem into an integral equation with a *Green*'s kernel and then for establishing the desired results, we apply the *Krasnoselskii* and *Banach* fixed point theorems with the help of some useful properties of the obtained *Green*'s kernel.

Keywords. Fixed point theory, *Green*'s function, iterative differential equation, neutral differential equation, periodic solution.

Abstract in French

Solutions périodiques d'une équation différentielle de type neutre avec des termes itératifs du premier ordre survenant dans les sciences de la vie

L'objectif principal de ce travail est d'étudier l'existence, l'unicité et la dépendance continue par rapport aux paramètres des solutions périodiques positives pour une équation différentielle de type neutre du premier ordre avec des termes itératifs.

Pour atteindre notre objectif, nous transformons le probleme considéré en une équation intégrale équivalente dont le noyau est une fonction de *Green*, puis pour établir les résultats souhaités, nous appliquons les théorèmes de point fixe de *Krasnoselskii* et de *Banach* à l'aide de certaines propriétés utiles du noyau de *Green* obtenu.

Mots-clés. Théorie du point fixe, fonction de *Green*, equation différentielle itérative, equation différentielle de type neutre, solution périodique.

الحلول الدورية الموجبة لمعادلة تفاضلية محايدة بحدود تكرارية

من الدرجة الأولى الناشئة في علوم الحياة

الغرض الرئيسي من العمل الحالي هو دراسة الوجود والوحدانية والاعتماد المستمر على المعلمات للحلول الدورية الموجبة لمعادلة تفاضلية محايدة من الدرجة الأولى ذات حدود تكرارية لتحقيق هدفنا , قمنا بتحويل المسالة المدروسة إلى معادلة تكاملية نواتها دالة غرين ولاثبات النتائج المرجوة قمنا بتطبيق نظريتي النقطة الثابتة لكراسنوسيلسكي وبناخ بمساعدة بعض الخواص المفيدة لنواة غرين المتحصل عليها .

كلمات مفتاحية : نظرية النقطة الثابتة , دالة غرين, معادلة تفاضلية تكرارية, معادلة تفاضلية محايدة, حل دوري.

Dedication

*- To my father **Abde Elhafid**, mother **Fedjria** and dearest grandmother **Yamina**, you have always sacrificed for me and held out your hand in difficult times.*

*- To my dearest sister **Marwa**, for the support and encouragement she has shown throughout the duration of this thesis.*

- To all my family, for their love and inspiration.

*- To my friends **Wiam** and **Meriem.***

Acknowledgements

In the name of Allah, the Beneficent, the Merciful. First of all, I thank ALLAH (SWT), the Lord Almighty, for giving me the health, strength and ability to carry out this work.

I would like to express my deep gratitude to my supervisor, Dr. **Ahlème BOUAKKAZ** for her guidance, valuable advice and support during the preparation of this work.

My sincere thanks also go to Dr. **Rabah KHEMIS** for his impactful contributions, encouragement and unwavering support throughout the duration of this work.

To the esteemed members of my Master thesis committee, your expertise and constructive critiques have significantly enriched this work. I am very grateful for the rigorous discussions that have undoubtedly elevated the quality of my Master thesis.

I extend my appreciation to all professors at the Department of Mathematics of 20 August 1955, University of Skikda.

To all my family and friends. Your love and understanding sustained me through the challenges, and for that, I am profoundly grateful.

Acronyms

Abbreviation	**Meaning**
ADE	Advanced differential equation
DDE	Delay differential equation
EPO	Erythropoietin
HSC	Hematopoietic stem cells
IDE	Iterative differential equation
NDE	Neutral differential equation
ODE	Ordinary differential equation
RBCs	Red blood cells
SDDE	State-dependent delay differential equation

List of symbols

Sets and numbers

$\mathbb{R}$: the set of real numbers (1-dimensional real Euclidean space).

$\mathbb{R}^*$: the set of all non-zero real numbers

$\mathbb{R}_+^* = (0, +\infty)$: the set of all non-zero positive real numbers

$\mathbb{R}^n$: n-dimensional real Euclidean space

$[a, b]$: the interval of numbers between a and b, including a and b

(a, b) : an open interval

$[a, +\infty)$: left-closed and right–unbounded interval

$\mathcal{C}([a, b]) := \mathcal{C}([a, b], \mathbb{R})$ désigne l'ensemble des fonctions continues sur $[a, b]$ à valeurs dans $\mathbb{R}$

$\mathcal{C}(\mathbb{R}^{n+1}, (0, +\infty))$ is the space of continuous functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

$\mathcal{C}([a, b], (0, +\infty))$ is the space of continuous functions from $[a, b]$ into $(0, +\infty)$

$\mathcal{C}^1(\mathbb{R}^{n+1}, (0, +\infty))$: space of continuously differentiable functions from $\mathbb{R}^{n+1}$ into $(0, +\infty)$

τ : a delay

T : a period

P_T : the *Banach* space of all continuous and $T-$periodic functions

$P_T(.) = \{x \in P_T,\ m \leq x(t) \leq M,\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}\}$

Functions

$|\cdot|$: absolute value

$\|.\|_{\mathbb{X}}$: a norm on $\mathbb{X}$

$\|f\|_{\infty}$: the uniform norm defined by $\sup|f(x)|$

$x^{[n]}(t)$: the composition of the function $x(t)$ with itself n times or the n^{th} iterate of the function $x(t)$

$\sum_{i=1}^{n}$: the summation from index $i = 1$ to $i = n$

$\lim_{x \longrightarrow x_0}$: limit as x approaches x_0

$\approx$: approximately equal to

$x'(t) := x^{(1)}(t) := \dfrac{dx(t)}{dt}$: the first derivative of the function $x(t)$ with respect to t

$x''(t) := x^{(2)}(t) := \dfrac{d^2x(t)}{dt^2}$: the second derivative of the function $x(t)$ with respect to t

sup : the supremum

max : the maximum

min : the minimum

$\exp x$: the exponential function of x

$G(t, \theta)$: the *Green's* function

Other notations will be clarified upon their initial occurrence.

General introduction

First order iterative-delay functional differential equations appear widely in the modelling of several phenomena such as the transmission of infectious diseases in epidemiology, the two-body problem in classical electrodynamics, population growth in population dynamics and many other remarkable natural phenomena. Equations of this kind which relate unknown functions, their derivatives and their iterates can be considered as a particular type of the class of time and state-dependent delay differential equations (SDDEs) of the following form:

$$x'(t) = f\left(t, x(t), x(t-\tau), x(t-\tau_1(t,x(t))), ..., x(t-\tau_n(t,x(t)))\right),$$

or of the class of advanced differential equations with state-dependent deviating arguments of the following form:

$$x'(t) = f\left(t, x(t), x(t-\tau), x(t+\tau_1(t,x(t))), ..., x(t+\tau_n(t,x(t)))\right),$$

where $\tau_i(t,x(t)$, $i=\overline{1,n}$ are positive functions. Currently, the existence, uniqueness, and stability of positive periodic solutions for first order iterative-delay functional differential equations have been investigated by many researchers where numerous interesting studies have been developed and valuable results have been obtained in this direction in which they have exchanged the difficulties encountered as well as the importance of their findings. Unfortunately, until now, few investigators have investigated neutral differential equations with iterative terms (see [**?**, 64]) since their distinctive characteristics and the presence of the iterative terms impede application of usual methods.

Problem statement

Studying iterative delay differential equations is not always a piece of cake due to many considerations involved when dealing with them; for examples the iterative terms may create some difficulties and hamper the application of several usual method, their theory which is not yet well developed cannot aid effectively in dealing with them, the relationship between the real phenomenon and the mathematical model may not be appropriate, or even the added criteria that help in investigating them may not be compatible with the model.

The red thread of the current thesis is the use a hybrid approach as a melting pot of the fixed point theory and the *Green*'s functions method to deal with a first order neutral differential equation with iterative terms. To be more precise, it aims at answering the following specific questions:

(i) The first question that arises is how to find a functional space and a subset of it that can be able on the one hand to contain the n order composition $x^{[n]}(t)$ of the harvesting function H, and on the other hand they can facilitate the application the used fixed point theorems.

(ii) Does the problem have at least one solution?

(iii) Does the problem have a unique solution?

(iv) Does the unique solution, if it exists, depends on the model parameters?

Objectives

The key objective of this work lies in establishing a set of sufficient criteria that open the door to an easy application of the hybrid technique that combines the fixed point theory with the *Green*'s functions method and hence to a fruitful discussion of the existence, uniqueness and stability of positive periodic solutions for a first-order neutral iterative differential equation.

Methodology

We mainly study a first order neutral functional differential equation with iterative terms which is generally difficult to be investigated. As we said before, this work overcomes the hurdle by using a hybrid technique based on the fixed point theory and the *Green*'s functions method. The methodology used herein is summarized in the following steps: firstly, we construct a *Banach* space and a subset of it as a bedrock of the used technique since they should be able to control the iterative terms, to fulfil at least the minimum requirements of the studied phenomenon, and to pave the way to the application of the fixed point theorems. Secondly, we convert the problem at hand into an integral equation with a *Green*'s function kernel. Thirdly, we define an integral operator that can be written as a sum of two operators, where one of them is assumed to be continuous and compact, while the other operator is a contraction. Fourthly, we apply the *Krasnoselskii* fixed point theorem with the aid of some properties of the obtained *Green*'s function to guarantee the existence of at least one positive periodic solution for the proposed equation. Finally, we add an additional criterion under which the *Banach* fixed point theorem ensures the stability and the existence of a unique positive periodic solution.

Layout of the thesis

This thesis It is divided into three chapters and concludes with a conclusion, and a bibliography. It begins with a general introduction that gives a succinct synopsis of the topic and the studied problem, the research goals and methodology, and the outlines of the thesis. The remainder of the current thesis is planned as follows:

The first chapter renders some basic definitions and concepts as well as certain tools from functional analysis that will be necessary for the understanding of this manuscript and also useful for reaching the desired target. In particular, we introduce the notion of the *Green*'s function, the *Ascoli-Arzela* theorem and the fixed point theorems used to establish the main results.

The second chapter provides the requisite mathematical background that helps in taking a deep dive into understanding the topic. It includes a brief overview on delay differential equations, their types, some methods for solving them and some mathematical models such as the Nicholson's equation and a hematopoiesis model.

The third chapter is devoted to the application of the *Krasnoselskii* and *Banach* fixed point theorems combined with the *Green*'s functions method to show the existence, uniqueness and continuous dependence on parameters of positive periodic solutions for the following first-order neutral differential equation with iterative terms:

$$\begin{aligned} \frac{d}{dt}\left(x(t)-\sum_{i=1}^{2}c_i(t)\,x(t-\tau_i)\right) &= -a(t)\,x(t)+\sum_{i=1}^{2}f_i\left(t,x(t),x(t-\tau_i)\right) \\ &\quad -H(t,x(t),x^{[2]}(t),...,x^{[n]}(t)),x^{[n]}(t). \end{aligned}$$

Here $x^{[0]}(t)=t$, $x^{[1]}(t)=x(t)$, $x^{[2]}(t)=x(x(t))$, ..., $x^{[n]}(t)=x^{[n-1]}(x(t))$ denote the iterates of the function $x(t)$ and $f\in C\left([0,T]\times\mathbb{R}^2,(0,\infty)\right)$ and $H\in C\left([0,T]\times\mathbb{R}^n,(0,\infty)\right)$ are periodic functions.

The thesis ends with a general conclusion that recaps the outline of the technique used to obtain the main outcomes.

CHAPTER 1

Preliminary notions

Contents

This introductory chapter briefly covers the key concepts and fundamental tools that we will need in subsequent chapters.

In this chapter, we introduce some preliminary notions and definitions, fundamental results, and powerful functional analysis tools that will be utilized throughout the remaining chapters. We present some concepts and results on continuous and compact operators, the *Banach*, *Schauder* and *Krasnoselskii* fixed point theorems that will play a crucial role in deriving our desired results. Additionally, we introduce the concept of the *Green's* function.

1.1 Continuous, Lipschitz continuous and compact operators in a normed vector space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two normed vector spaces over the same field $\mathbb{F}$.

Definition 1.1 [24] An operator $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be continuous at a point $M_0 \in \mathbb{X}$ if

$$\lim_{M \longrightarrow M_0} \mathcal{A}x = \mathcal{A}x_0.$$

The continuity at $M_0 \in \mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \exists \eta > 0, \ \forall M \in \mathbb{X}, \ \left(\|x - x_0\|_{\mathbb{X}} < \eta\right) \Longrightarrow \left(\|\mathcal{A}x - \mathcal{A}x_0\|_{\mathbb{Y}} < \varepsilon\right).$$

If $\mathcal{A}$ is continuous at every point of $\mathbb{X}$, then $\mathcal{A}$ is said to be continuous on $\mathbb{X}$. The continuity on $\mathbb{X}$ could be characterized as follows:

$$\forall \varepsilon > 0, \ \forall M \in \mathbb{X}, \exists \eta > 0, \ \forall N \in \mathbb{X}, \ \left(\|x - y\|_{\mathbb{X}} < \eta\right) \Longrightarrow \left(\|\mathcal{A}x - \mathcal{A}y\|_{\mathbb{Y}} < \varepsilon\right).$$

Definition 1.2 [73] A map $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be *Lipschitz continuous* if there is a positive constant c such that

$$\forall M, \ N \in \mathbb{X} : \|\mathcal{A}x - \mathcal{A}y\|_{\mathbb{Y}} \leq c \|x - y\|_{\mathbb{X}}.$$

If $c \in [0, 1[$, $\mathcal{A}$ is called a contraction mapping.

Remark 1.1 If $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{Y}$ then

$$\mathcal{A} \text{ is a contraction} \Longrightarrow \mathcal{A} \text{ is } \textit{Lipschitz} \text{ continuous} \Longrightarrow \mathcal{A} \text{ is continuous on } \mathbb{X}.$$

Theorem 1.1 *[70] A continuous function on a closed bounded interval is bounded and attains its bounds.*

Remark 1.2 The above theorem is essential in obtaining the proof of many theorems and lemmas in the remaining chapters of this thesis.

Definition 1.3 A map $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{Y}$ is called compact if and only if $\mathcal{A}$ maps bounded sets into relatively compact sets, i.e.,

$$[\mathcal{A} \text{ compact}] \Longleftrightarrow \left[\forall \Omega \subset \mathbb{X}, \ \ (\Omega \text{ bounded}) \Longrightarrow \left(\overline{\mathcal{A}(\Omega)} \text{ compact}\right)\right].$$

Equivalently, $\mathcal{A}$ is compact if and only if for every bounded sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, the sequence $(\mathcal{A}M_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}$.

1.2 Completely continuous operator in a Banach space

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two *Banach* spaces over the same field $\mathbb{F}$ and let Ω a subset of $\mathbb{X}$.

Definition 1.4 A continuous map $\mathcal{A} : \Omega \subset \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be completely continuous if and only if $\mathcal{A}$ maps bounded sets of Ω into relatively compact sets, i.e.,

$$[\mathcal{A} \text{ is completely continuous}] \Longleftrightarrow \left[\forall F \subset \Omega, \ \ (F \text{ bounded}) \Longrightarrow \left(\overline{\mathcal{A}(F)} \text{ compact}\right)\right].$$

1.3 Arzelà-Ascoli theorem

The Arzela-Ascoli Theorem is a very important technical result, used in many aspects of mathematics such as functional analysis, ordinary differential equations and complex analysis. It is an efficient tool that allows us, for example, to check of compactness for subsets in spaces of continuous functions or even to prove the compactness of operators.

Let $\mathbb{Z}$ be a compact subset of a normed vector space over $\mathbb{F}$ and let $\mathcal{C}(\mathbb{Z})$ (can also be written as $\mathcal{C}(\mathbb{Z}, \mathbb{R})$) be the normed vector space of real valued continuous functions on $\mathbb{Z}$ endowed with the uniform norm

$$\|f\|_{\infty} = \sup_{x\in\mathbb{Z}} |f(x)| .$$

Let $\mathcal{F}$ be a collection of real-valued continuous functions defined on $\mathbb{Z}$ i.e., $\mathcal{F}$ is a collection of functions in $\mathcal{C}(\mathbb{Z})$.

Definition 1.5 [16] The collection $\mathcal{F}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\eta > 0$ so that for all $f \in \mathcal{F}$ and $x, y \in \mathbb{Z}$ with $\|x-y\|_{\mathbb{Z}} < \eta$ we have $|f(x) - f(y)| < \varepsilon$, i.e.,

$$\forall \varepsilon > 0,\ \forall x \in \mathbb{Z},\ \exists \eta > 0,\ \forall y \in \mathbb{Z},\ [\|x-y\|_{\mathbb{Z}} < \eta] \Longrightarrow [\forall f \in \mathcal{F},\ \ |f(x) - f(y)| < \varepsilon].$$

Definition 1.6 [16] The collection $\mathcal{F}$ is said to be uniformly bounded if there is an $C \geq 0$ so that $\|f\|_{\infty} = \sup_{x\in\mathbb{Z}} |f(x)| \leq C$ for all $f \in \mathcal{F}$, i.e.,

$$\exists C \geq 0 : \|f\|_{\infty} = \sup_{x\in\mathbb{Z}} |f(x)| \leq C,\ \forall f \in \mathcal{F}.$$

Theorem 1.2 *[16] If $\mathcal{F}$ is a collection of uniformly bounded and equicontinuous functions in $\mathcal{C}(\mathbb{Z})$, then $\mathcal{F}$ is relatively compact in $\mathcal{C}(\mathbb{Z})$.*

1.4 Periodic functions

Periodic functions whose graph exhibits translational symmetry, are often used to model oscillations, waves, and other phenomena that exhibit periodicity.

Let G be a function defined on a set I, and let T be a non-zero real constant.

Definition 1.7 The function G is said to be T-periodic function (also called cyclic function, periodic waveform, or simply periodic wave), if

$$G(t+T) = G(t),$$

for all $t \in I$.

If there exists a least positive constant T that fulfil the above property, it is called the fundamental period (also primitive period, basic period, or prime period.)

Corollary 1.1 *The derivative of a T-periodic function is also a T-periodic function.*

Remark 1.3 The antiderivative of a T-periodic function is not necessarily a T-periodic function.

for $t \in \mathbb{R}$.

1.5 The Banach space of continuous periodic functions

Theorem 1.3 *For $T > 0$, the set*

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t)\},$$

of all continuous and $w-$periodic functions endowed with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

is a Banach space.

Remark 1.4 For $m \geq 0$, $M, L > 0$, let

$$P_T(.) = \{x \in P_T,\ m \leq x(t) \leq M,\ \ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \forall t, t_1, t_2 \in \mathbb{R}\},$$

1) $P_T(.)$ is a closed convex and bounded subset of P_T.
2) It follows from conditions $m \leq x(t) \leq M$ and $|x(t_2) - x(t_1)| \leq L|t_2 - t_1|$ in the definition of $P_T(.)$ that this subset is equicontinuous and uniformly bounded. Hence, as an outcome of the *Arzelà-Ascoli* theorem we conclude that the closed subset $P_T(.)$ is compact (see *Mezghiche* et al. [64]).

1.6 Fixed point theorems

A fixed point (sometimes shortened to fixpoint) is an invariant point that does not change under a given transformation. It can provide a way to establish the existence of solutions to prominent families of equations.

Definition 1.8 [73] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space over $\mathbb{F}$. A fixed point of a mapping $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is an $x \in \mathbb{X}$ which is mapped to itself by $\mathcal{A}$, that is

$$\mathcal{A}(x) = x.$$

1.6.1 Banach's fixed point theorem

Banach's fixed point theorem (also known as the contraction mapping theorem or contractive mapping theorem), is a powerful mathematical tool that is usually used to guarantee the existence and uniqueness of solutions.

Theorem 1.4 *[73] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a Banach space and let $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ be a contraction on $\mathbb{X}$. Then $\mathcal{A}$ has a unique fixed point $x \in \mathbb{X}$ such that*

$$\mathcal{A}(x) = x.$$

Theorem 1.5 *[35] If Ω is a closed subset of a Banach space $\mathbb{X}$ and $\mathcal{A} : \Omega \to \Omega$ is a contraction, then $\mathcal{A}$ has a unique fixed point in Ω.*

1.6.2 Schauder's fixed point theorem

The *Schauder* fixed point theorem is applicable to a compact mapping in a *Banach* space.

Theorem 1.6 *[73] Let Ω be a non-empty bounded closed convex subset of a Banach space $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and let $\mathcal{A} : \Omega \longrightarrow \Omega$ be a compact mapping. Then $\mathcal{A}$ has a fixed point in Ω.*

An alternative version of the *Schauder* fixed point theorem can be expressed as follows:

Theorem 1.7 *[73] Let Ω be a non-empty compact convex subset of a Banach space $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and let $\mathcal{A} : \Omega \longrightarrow \Omega$ be a continuous mapping. Then $\mathcal{A}$ has a fixed point in Ω.*

1.6.3 Krasnoselskii's fixed point theorem

The *Krasnoselskii* fixed point theorem is applicable to a mapping that can be expressed as a sum of two maps, a contraction and a continuous and compact mapping in a *Banach* space.

Theorem 1.8 *[73] Let Ω be a non-empty closed convex bounded subset of a Banach space $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and let $\mathcal{A}_1, \mathcal{A}_2 : \Omega \to \mathbb{X}$ be two mappings such that*

i) $\mathcal{A}_1 x + \mathcal{A}_2 y \in \Omega$ *for all* $x, y \in \Omega$,

ii) $\mathcal{A}_1$ *is continuous and compact,*

iii) $\mathcal{A}_2$ *is a contraction mapping,*

Then $\mathcal{A}_1 + \mathcal{A}_2$ *has a fixed point in* Ω.

Remark 1.5 Note that if $\mathcal{A}_1 = 0$, the theorem becomes the *Banach* fixed point theorem and if $\mathcal{A}_2 = 0$ then the theorem is not other than the *Schauder* fixed point theorem.

1.7 Green's functions for boundary value problems

The method of the *Green's* functions is a valuable device used to solve differential equations with sources whether containing ordinary derivatives or partial derivatives especially those that are often difficult or impossible to solve by other approaches. Generally speaking, a *Green*'s function is an integral kernel that helps us in solving non-homogeneous linear differential equations.
We consider two-point n^{th}-order linear boundary value problem of the form

$$\begin{cases} L_n x(t) = g(t), \ t \in I = [a, b], \\ U_i(x) = \gamma_i, \ i = 1, ..., m. \end{cases} \tag{1.1}$$

where

$$L_n x(t) = c_0(t) x^{(n)}(t) + c_1(t) x^{(n-1)}(t) + ... + c_{n-1}(t) x^{(1)}(t) + c_n(t) x(t),$$

$$U_i(x) = \sum_{j=0}^{n-1} \left(\alpha_j^i x^{(j)}(a) + \beta_j^i x^{(j)}(b) \right), \ i = 1, ..., m, \ m \leq n,$$

α_j^i, β_j^i and γ_i are real constants for all $i = 1, ..., m$ and $j = 0, ..., n-1$, g and c_k are continuous real functions for all $k = 0, ..., n$, and $c_0(t) \neq 0$ for all $t \in I$.

Definition 1.9 [17] We call homogeneous problem associated with (1.1), the problem (1.2) defined by

$$\begin{cases} L_n x(t) = 0, \, t \in I = [a, b], \\ U_i(x) = 0, \, i = 1, ..., m. \end{cases} \tag{1.2}$$

Remark 1.6 [17] The homogeneous problem (1.2) is said to be k-compatible, $0 \leq k \leq n$, if its set of solutions has dimension equals to k.

Theorem 1.9 *The problem*

$$\begin{cases} L_n x(t) = g(t), \, t \in I = [a, b], \\ U_i(x) = \gamma_i, \, i = 1, ..., n. \end{cases} \tag{1.3}$$

where the number of boundary conditions equals the order of the linear differential equation, has a unique solution if and only if the homogeneous problem associated with (1.3) has only the trivial solution.

Theorem 1.10 *[61]*

i) Assume $c_0(t) \neq 0, \forall t \in [a, b]$.

If the homogeneous problem associated with (1.1) has only the trivial solution, then the Green's function for this boundary value problem exists and is unique.

ii) If the above conditions are assumed to be satisfied, then the solution of the nonhomogeneous boundary value problem is unique and given by

$$x(t) = \int_a^b G(t, s) \ g(s) \, ds, \tag{1.4}$$

where the kernel $G(t, s)$ *is a Green's function.*

Remark 1.7 There are several methods to finding particular solutions of nonhomogeneous equations such as the method of variation of parameters, integrating factor method, and the *Green's* function method, to name a few. According to the superposition principle the general solution can be found as a sum of the general solution to the homogeneous equation and a particular solution of the nonhomogeneous equation. So, if the homogeneous problem (1.2) has only the trivial solution and the n^{th}-order boundary value problem (1.1) is linear. Then, the unique solution obeying nonhomogeneous boundary conditions is simply the obtained particular solution.

Definition 1.10 [17] We say that G is a Green's function for problem (1.2) if it satisfies the following properties:

(G$_1$) G is defined on the square $I \times I$.

(G$_2$) For $k = 0, 1, ..., n-2$, the partial derivatives $\dfrac{\partial^k G}{\partial t^k}$ exist and they are continuous on $I \times I$.

(G$_3$) $\dfrac{\partial^{n-1} G}{\partial t^{n-1}}$ and $\dfrac{\partial^n G}{\partial t^n}$ exist and are continuous on the triangles $a \leq s < t \leq b$ and $a \leq t < s \leq b$.

(G$_4$) For each $t \in (a, b)$ there exist the lateral limits

$$\frac{\partial^{n-1} G}{\partial t^{n-1}} \left(t, t^+\right) \text{ and } \frac{\partial^{n-1} G}{\partial t^{n-1}} \left(t, t^-\right),$$

(i.e., the limits when $(t, s) \to (t, t)$ with $s > t$ or with $s < t$) and, moreover

$$\frac{\partial^{n-1} G}{\partial t^{n-1}} \left(t, t^+\right) - \frac{\partial^{n-1} G}{\partial t^{n-1}} \left(t, t^-\right) = -\frac{1}{c_0 (t)}.$$

(G$_5$) For each $s \in (a, b)$, the function $t \to G(t, s)$ is a solution of the differential equation $L_n x = 0$ on $t \in [a, s)$ and $t \in (s, b]$. That is,

$$c_0 (t) \frac{\partial^n G}{\partial t^n} (t, s) + c_1 (t) \frac{\partial^{n-1} G}{\partial t^{n-1}} (t, s) + ... + c_{n-1} (t) \frac{\partial G}{\partial t} (t, s) + c_n (t) G (t, s) = 0,$$

on both intervals.

(G$_6$) For each $s \in (a, b)$, the function $t \to G(t, s)$ satisfies the boundary conditions $U_i (G (., s)) = 0$, $i = 1, ..., m$:

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j G}{\partial t^j} (a, s) + \beta_j^i \frac{\partial^j G}{\partial t^j} (b, s) \right) = 0.$$

Definition 1.11 The problem is said to be regular if in addition to the previous conditions, we have

1) c_0 is nonzero (except perhaps at a finite number of isolated points).

2) The homogeneous problem (1.2) has only the trivial solution.

Remark 1.8 Since G depends on the main part of the differential equation, but not on the source term g, once G is found we can immediately solve the more general problem for any arbitrary source term.

Example 1.1 We aim to find the *Green's function for the following* two-point second order boundary value problem*:*

$$\begin{cases} x^{(2)}(t) = g(t), \; t \in I = (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{1.5}$$

First, we must check that the rank of

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

is 2. Indeed, the first and the third columns ensure that the rank equals 2.

Solutions of the homogeneous differential equation

$$\frac{d^2 x(t)}{\partial t^2} = 0,$$

are of the form

$$x(t) = A_1 t + A_2,$$

where A_1 and A_2 are constants. The boundary conditions of the problem (1.5) give $A_1 = 0$ and $A_2 = 0$, which ensures the existence of a unique *Green's function. Consequently*

1) On $(0, s)$,

$$G(t,s) = A_1(s)\, t + A_2(s),$$

and on $(s, 1)$,

$$G(t,s) = B_1(s)\, t + B_2(s).$$

2) We must have $G(0,s) = 0$, then $A_2(s) = 0$ and we must also have $G(1,s) = 0$ i.e.,

$$B_1(s) + B_2(s) = 0.$$

3) Since G is continuous, then

$$G\left(s^{-},s\right)=\lim_{t\to s^{-}}G\left(t,s\right)=\lim_{t\to s^{+}}G\left(t,s\right)=G\left(s^{+},s\right),$$

this implies that

$$A_1\left(s\right)s+A_2\left(s\right)=B_1\left(s\right)s+B_2\left(s\right).$$

4)

$$\frac{\partial G}{\partial t}\left(s^{+},s\right)=B_1\left(s\right),\ \frac{\partial G}{\partial t}\left(s^{-},s\right)=A_1\left(s\right).$$

Therefore

$$B_1\left(s\right)-A_1\left(s\right)=1.$$

$A_1\left(s\right),\ A_2\left(s\right),\ B_1\left(s\right)$ and $B_2\left(s\right)$ are solution of

$$\begin{cases} A_2\left(s\right)=0 \\ B_1\left(s\right)+B_2\left(s\right)=0 \\ s\left[A_1\left(s\right)-B_1\left(s\right)\right]+A_2\left(s\right)-B_2\left(s\right)=0 \\ A_1\left(s\right)-B_1\left(s\right)+1=0, \end{cases}$$

which gives

$$\begin{cases} A_1\left(s\right)=s-1 \\ A_2\left(s\right)=0 \\ B_1\left(s\right)=s \\ B_2\left(s\right)=-s. \end{cases}$$

So, the *Green's* function is given by

$$G\left(t,s\right)=\begin{cases} t\left(s-1\right) & \text{if } 0\leqslant t\leqslant s \\ s\left(t-1\right) & \text{if } s\leqslant t\leqslant 1. \end{cases}$$

Finally, the solution of the problem (1.5) is

$$\begin{aligned} x\left(t\right) &= \int_0^1 G\left(t,s\right)g\left(s\right)ds \\ &= \int_0^t t\left(s-1\right)g\left(s\right)ds+\int_t^1 s\left(t-1\right)g\left(s\right)ds. \end{aligned} \tag{1.6}$$

CHAPTER 2

Delay differential equations

Contents

This chapter is dedicated to provide a brief background on delay differential equations.

2.1 Historical background

Delay differential equations (DDEs) which depend on previous states or have memory effects in their formulations, now occupy an increasingly preponderant place in the literature. They have long played a crucial role in understanding, modelling, and predicting the future of several phenomena encountered in life sciences, physical sciences, economic sciences, etc.

The interpretation of the delay that can be discrete, time varying, state dependent, or even distributed, differs from one model to another depending on the studied phenomenon. In biology, the delay can represent the duration of the cell cycle, the duration of the transformation of one type of cells into another or the time necessary for the maturation of cells in the dynamics of cell populations, the gestation period, the developmental time, the juvenile phase, the life cycle duration or the period of maturation in the dynamics of human, animal or plant populations or even the incubation period of a contagious disease in epidemiology, etc.

Functional differential equations, whether advanced (ADEs) or retarded (DDEs) ones, have an ancient history dating back to the early of the 18th century. The first attempts have been made by *J. Bernoulli*, *L. Euler*, *J.L. Lagrange*, *P. Laplace*, *S. Poisson*, but unfortunately, equations of this kind have been ignored at that time. This can be due to many factors, including limitation of the theory and mathematical tools that can help deal with these equations and also the fact that the introduction of delays may further complicate the dynamics or may affect the stability of the system.

They had their moment of glory in the 20th century which has been marked by an explosion of scientific researches on this topic at that time. The flame of the passion for investigating delayed real phenomena has grown during the International Congress of Mathematicians in 1908 when *Picard* has revealed the importance of taking into account the effect of delays in modelling physical phenomena.

To the best of our knowledge, *Myshkis*, *Krasovskii*, and *Hale* are the ones of the founding scholars of the theory in this direction. The first book about functional differential equations with finite or infinite delay is the monograph of *Myshkis* [66]. *Krasovskii* investigated functional differential equations and their stability using *Lia-*

punov functionals [49, 50]. Whereas, *Hale* has employed the modern tools of functional analysis to set the cornerstones of the theory of linear and nonlinear functional differential equations and he has applied, for the first time, the theory of semigroups to develop a theory of linear systems with finite delay [35], [36].

Delay differential equations continue to be able to have new diversified applications and theoretical findings being discovered all the time. An important type of delay differential equations that have recently gained considerable momentum, is the class of state-dependent delay differential equations (SDDEs) in which delays depend on the state variable itself. Although the fact that the origins of these equations go back to the early 19th century and despite they appear in the modelling of a growing number of phenomena in various fields such as electrodynamics, biology, epidemiology and medicine, etc., their theory is not yet well developed.

A subclass of differential equations with deviating arguments depending on both the time and the state variable is the class of iterative functional differential equations. They relate unknown functions, their derivatives and iterates where the general form of those of first order is as follows:

$$\frac{d}{dt}x(t) = f\left(x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

with $x^{[0]} = t$, $x^{[1]} = x(t)$, $x^{[2]} = x(x(t))$,..., $x^{[n]} = x\left(x^{[n-1]}(t)\right)$ are the iterates of $x(t)$ with $x^{[n]}(t)$ is the composition of x with itself n times.

Equations of this kind have gained considerable interest due to their wide variety of scientific applications in many fields such as physics, biological sciences, and classical electrodynamics. For instance, it can describe infectious disease transmission models in epidemiology, the motion for the two-body problem of classical electrodynamics, and insect population growth in ecology.

Their study began in the early nineteenth century with the now famous "Babbage equation" and the works of *Schröder* and *Abel* [1, 4, 71]. Unfortunately, although iterative differential equations appear widely in many applications and although they have also fascinated many authors and hence gained much momentum recently, publications that handle such equations are still somewhat rare. Their unpopularity is partly due to the fact that their iterative terms that involve compositions of the unknown function with itself, may create some difficulties both when studying them and when applying the usual methods.

2.2 Classification of delay differential equations

Delay functional equations can be classified as linear or non-linear, autonomous or non-autonomous, periodic or not or according to the types of delays. In this section , we are interested in giving a classification of DDEs according to the types of delays cited in the literature. Two main classes can be distinguished, namely delay differential equations and neutral differential equations which in turn may be sub-divided into different categories according to the types of their delays.

In this first class, namely the class of delay differential equations, we find the following categories:

2.2.1 Constant delay differential equations

In their simplest form, a first order functional differential equation with a constant delay (also called discrete delay) is written as follows:

$$x'(t) = f(t, x(t), x(t-\tau)),$$

where f is given and τ is a positive constant. The presence of the term $x(t-\tau)$ indicates that the state of the system at time t depends on its state at a previous time $t-\tau$. For example, we can find this type of equation in describing the evolution of Lucilia cuprina population that exhibit time delays in their dynamics and which is known as the Nicholson's blowflies model (see [47]).

2.2.2 Variable delay differential equations

The delay in this case is time varying or depending on the state. For example, we find differential equations with time-dependent delays in the transport models [47] and that which depends on the state in the model describing the evolution of a population of fish whose larvae consume a food [3] or the two-body problem in classical electrodynamics [27, 28, 29].

Time-varying delay differential equations

In their simplest form, this kind of equation (also called time-dependent delay differential equations) is written as follows:

$$x'(t) = f(t, x(t), x(t - \tau(t))),$$

where f is given.

State-dependent variable delay differential equations

In their simplest form, this kind of equation is written as follows:

$$x'(t) = f(t, x(t), x(t - \tau(x(t)))),$$

where f is given.

Sometimes this classification is not sufficient in the study, which requires adding additional constraints relating to the delay or its derivative. This leads to defining the following new subcategories of variable delay equations:

Arbitrary variable delay differential equations: The delay and its derivative are not limited.

Differential equations with increased delay: This type of equation requires knowing the maximum value of the delay.

$$0 \leq \tau(t) \leq \tau_{\max}.$$

Bounded Delay differential equations (bi-bounded): This kind of equation which is often less studied than the previous case, contains a delay verifying the following constraint:

$$\tau_{\min} \leq \tau(t) \leq \tau_{\max}.$$

Delay differential equations varying slowly over time: $\tau(t)$ is an almost everywhere derivable function such that

$$\dot{\tau}(t) \leq \lambda < 1,$$

which then indicates a limitation on the speed of variation of the delay and that the latter varies slowly over time, in other words that the delayed information arrives in chronological order.

Moderately time-varying delay equations: $\tau(t)$ is an almost everywhere derivable function such that

$$\dot{\tau}(t) \leq \lambda \text{ with } \lambda \geq 1.$$

Delay equations varying rapidly in time: In this type, there are no constraints on the delay and its derivative.

2.2.3 Distributed delay differential equations

In their simplest form, these equations are written as follows:

$$x'(t) = -\alpha x(t) - \beta \int x(t-a)\, d\eta(a).$$

For example, the bat, when hunting, being blind, it emits sounds, to use the walls of the caves, in order to locate its prey. The echo obtained by the bouncing of these cries represents the delay which depends on the state, which is the predator. We also find this type of equation in the [67] of AIDS or the population dynamics model presented by *Volterra* in 1934 where he used a distributed delay term to examine a cumulative effect on the mortality rate of a species

2.2.4 Differential equations with unknown delay

In this case, no assumption about the delay is considered whether it is constant, variable or distributed

2.2.5 Neutral delay differential equations

These equations denoted in abbreviation NDEs differ from DDEs by the fact that the derivative of the state at the current time depends not only on the values of the past state but also on the derivative of the highest order intervening in the equation of the past time. They appear often in describing many phenomena including the motion of radiating electrons, population growth, and the spread of epidemics. In their simplest form, these equations are written as follows:

$$\frac{d}{dt}\left[Dx\left(t-\tau\left(t\right)\right)\right]=f\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right),$$

where f is given and D is an operator. For example, we find this type of equations in the distributed network model [47].

2.3 Existence and uniqueness results

In this section, we present some classic results on existence and uniqueness of solutions.

2.3.1 Delay differential equations

Given a number $\tau \geq 0$, $\mathcal{C}([a,b],\mathbb{R}^n)$, the *Banach* space of continuous functions defined on $[a,b]$ with values in $\mathbb{R}^n$ is provided with the norm of uniform convergence. If $[a,b]=[-\tau,0]$, we put $C=C([-\tau,0],\mathbb{R}^n)$ and we denote the norm of an element $\Phi\in C$ by

$$\|\varphi\|=\sup_{-\tau\leq t\leq 0}\left|\varphi\left(t\right)\right|.$$

If $t_0\in\mathbb{R}$, $A\geq 0$ and $x\in\mathcal{C}\left(\left[t_0-\tau,t_0+A\right],\mathbb{R}^n\right)$, then for $t\in[t_0,t_0+A]$, we define $x_t\in\mathcal{C}$ by

$$x_t(s)=x(t+s),$$

for all $s\in[-\tau,0]$.

Definition 2.1 [35] If $\mathfrak{D}$ is a subset of $\mathbb{R}\times\mathcal{C}$, $f:\mathfrak{D}\longrightarrow\mathbb{R}^n$ is a given function and here represents the derivative on the right, the equation

$$\dot{x}(t):=f(t,x_t), \tag{2.1}$$

where

$$x_t(s):=x(t+s),\ \ s\in[-\tau,0], \tag{2.2}$$

is a delay functional differential equation on $\mathfrak{D}$ denoted and the number τ is called the delay.

It is clear that the case $\tau = 0$ corresponds to the case of ordinary differential equations. It is obvious that an initial condition appropriate to the time $t = t_0$ requires the determination of the function x over the entire interval $[t_0 - \tau, t_0]$.

$$x(t) = \psi(t), \ t \in [t_0 - \tau, t_0], \tag{2.3}$$

where $\psi : [t_0 - \tau, t_0] \longrightarrow \mathbb{R}^n$ is a given function assumed to be continuous called the initial condition of the delay equation (2.1) Thus, the equation (2.1) can be written in the form

$$\begin{cases} x'(t) := f(t, x_t), & t \geq t_0 \\ x(t) = \psi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{2.4}$$

where is ψ a given continuous function on the interval $[t_0 - \tau, t_0]$.

Definition 2.2 [35] Given $\psi \in C$ and $t_0 \in \mathbb{R}$, a solution of the equation (2.1) is a function denoted $x(t)$ such that $x(t) = \psi(t)$ if $t \in [t_0 - \tau; t_0]$ and satisfying (2.1) if $t \in [t_0, t_0 + A]$ with $A > 0$. Such a function $x(t)$ is called the solution of (2.1) through $(t_0, \psi)$and it is often denoted by

$$x(t) = x\left(t_0, \psi, f\right). \tag{2.5}$$

Lemma 2.1 *[35]Let $\psi \in \mathcal{C}$, $t_0 \in \mathbb{R}$ and $f\left(t, \psi\right)$ a continuous function. Finding solutions of the equation (2.1) though (t_0, ψ) is equivalent to solving*

$$\begin{cases} x_{t_0} = \psi, \\ x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u) du, \ t \geq t_0. \end{cases}$$

Theorem 2.1 *(Existence) [35] For equation (2.1), suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$ is a continuous mapping on Ω. if $(t_0, \psi) \in \Omega$, then there exists a solution of the equation (2.1) passing through (t_0, ψ).*

Definition 2.3 [35] The function $f(t, \varphi)$ is said to be a Lipschitz mapping with respect to φ on a compact K of $\mathbb{R} \times \mathcal{C}$ if there is a constant $k > 0$ such that for all $(t, \psi_i) \in K$, $i = 1, 2$, one has

$$|f(t,\psi_1) - f(t,\psi_2)| \leq k \ |\psi_1 - \psi_2| \,. \tag{2.6}$$

Theorem 2.2 *[35] Suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$, $f : \Omega \longrightarrow \mathbb{R}^n$ is continuous and $f(t,\psi)$ is a Lipschitz mapping with respect to ψ on any compact subset Ω. If $(t_0,\psi) \in \Omega$, then there is a unique solution of the equation (2.1) passing through (t_0,ψ).*

2.3.2 Neutral differential equation

Definition 2.4 [36] Suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ of elements (t,ψ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at a point β of Ω. if D, its first and second derivatives are continuous in the sense of Fréchet with respect to ψ and D_ψ, is atomic in β of Ω.

Definition 2.5 [36] Suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$, $D : \Omega \to \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ are two given continuous functions with D is atomic in zero .The relation

$$\frac{d}{dt}D(t,x_t) = f(t,x_t), \tag{2.7}$$

is said to be a differential equation of neutral type.

Theorem 2.3 *[36] (Existence) If Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $(t_0,\psi) \in \Omega$, then there is a solution of the equation (2.7) passing through (t_0,ψ).*

Theorem 2.4 *[36] (Existence and uniqueness) if Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $f(t,\psi)$ is a Lipschitz mapping with respect to ψ on any compact subset of Ω, so for all $(t_0,\psi) \in \Omega$, there is a unique solution for the equation (2.7) passing through (t_0,ψ).*

2.4 Some methods for solving DDEs

2.4.1 Step method

The step method (also called step-by-step method, or method of successive integrations) makes it possible to solve DDE and NDE digitally and at the same time makes it possible to establish the existence and uniqueness of the solution. It was presented in 1965, by *R. Bellman* for constant delays. Others such as *El'sgol'ts* and Norkin (1973) have shown that it also remains valid for variable delays, provided that the delay never cancels. To fix the ideas, consider the following delay linear functional differential equation with variable coefficients:

$$\begin{cases} x'(t) = a_1(t)\,x(t) + a_2(t)\,x(t-\tau) & \text{for all } t \in [0,\tau] \\ x(t) = \varphi(t) & \text{for all } t \in [-\tau,0]. \end{cases} \tag{2.8}$$

If a_1 and a_2 are two real constants, then the equation is said to be of type "*Frisch-Holme*" [32]. Now, let's solve this equation by the step method. The principle of this method is to look for solutions on intervals of the type $[k\tau, (k+1)\tau]$ where $k \in \mathbb{N}$ by pursuing the following steps:

First step: In the interval $[-\tau, 0]$, the function $x(t)$ is the given function $\varphi(t)$, so the equation is solved in the interval $[-\tau, 0]$ and we denote by $x_0(t)$ this solution. It should be pointed out here that if $t \in [0,\tau]$, so $t-\tau$ will reside in $[-\tau, 0]$.

Second step: In the interval $[0,\tau]$, if $t \in [0,\tau]$, so $t-\tau$ will reside in $[-\tau,0]$, so $x(t-\tau) = x_0(t-\tau)$ in the interval $[0,\tau]$ and the system (2.8) becomes

$$\begin{cases} x'(t) = a_1(t)\,x(t) + a_2(t)\,x_0(t-\tau) & \text{pour tout } t \in [0,\tau] \\ x(0) = \varphi(0), \end{cases} \tag{2.9}$$

which is an initial-valued problem for an ordinary differential equation (ODE) where $x_0(t-\tau) = \varphi(t-\tau)$ is known. Thus, we solve this ODE in $[0,\tau]$ using the initial condition $x(0) = \varphi(0)$ and we denote by $x_1(t)$ this solution in $[0,\tau]$.

Third step: In the interval $[\tau, 2\tau]$, the system becomes

$$\begin{cases} x'(t) = a_1(t)\,x(t) + a_2(t)\,x_1(t-\tau) & \text{pour tout } t \in [\tau,2\tau] \\ x(\tau) = x_1(\tau). \end{cases} \tag{2.10}$$

This ODE with the initial condition $x_1(\tau)$ is in turn can be solved to find the solution $x_2(t) \in [\tau, 2\tau]$ and so on.

2.4.2 Runge Kutta's method

The *Runge Kutta* method can be adapted to solve delay differential equations. *Runge Kutta* q method is a one-step method with

$$\begin{aligned} \phi(t_i, h, x_i) &= \sum_{j=1}^{q} w_j k_j \\ k_j &= k_j(t_i) = f(t_i + c_j h, x_i + h \sum_{i=1}^{q} a_{ji} k_j) \;\; 1 \leq j \leq q, \end{aligned} \tag{2.11}$$

where w_j, c_j, a_{ji} are constants. It is

explicit	if $a_{ji} = 0, \forall i \geq j$
semi-explicit	if $a_{ji} = 0, \forall ij > j$
implicit	in the other cases.

Then we have the following relationship, between the rank q and the order p of the method

$$\begin{aligned} q &\leq 4 \quad q = p \\ q &> 4 \quad p \leq q. \end{aligned}$$

For example

$q = 1$ we have *Euler*'s method,

$q = 2$ we have the improved Euler method for example,

$q = 4$ we have the "classic" method of *Runge Kutta*,

$q = 5$ we have the *Runge Kutta* Merson method for example.

The method that we have chosen is the "classic" method of *Runge Kutta* of range 4 and order 4:

$$x_{i+1} = x_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2.12}$$

with

$$\begin{aligned} k_1 &= f(t_i, x_i) \\ k_2 &= f(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1) \\ k_3 &= f(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2) \\ k_4 &= f(t_i + h, x_i + hk_3). \end{aligned} \tag{2.13}$$

2.5 Some pioneering models in life sciences

2.5.1 Malthusian model

Let $x(t)$ denotes the population size at time t; let b and d denote the birth rate and death rate, respectively, then

$$\frac{dx(t)}{dt} = bx(t) - dx(t) = (b-d)\,x(t) = rx(t),$$

where r is the intrinsic growth rate of the population. The solution of this equation with an initial population $x(0) = x_0$ is given by

$$x(t) = x_0\, e^{rt},$$

which represents the traditional exponential growth if $r > 0$ or decay of the population if $r < 0$.

2.5.2 The logistic equation

Such a population growth, due to *Malthus* (1798), may be valid for a short period, but it cannot go on forever. Taking into account the fact that resources are limited, *Verhulst* (1836) proposed the logistic equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right),$$

to resolve the Malthusian dilemma of unbounded growth where $K > 0$ is the carrying capacity of the population.

2.5.3 The delayed logistic equation

In practice, the process of reproduction is not instantaneous and suffers from certain delays. This motivated *Hutchinson* (1948) to propose the familiar logistic equation describing the growth of a single population (also known as *Hutchinson*'s equation or *Wright*'s equation):

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t-\tau)}{K}\right],$$

where r and K have the same meaning as in the logistic equation and τ is a positive constant.

2.6 Some delay models

2.6.1 Nicholson's blowflies model

In 1950 the famous Australian biologist *Alexander J. Nicholson* conducted a long series of experiments aimed at learning more about the groups of blowflies responsible for 90% of myiasis that threatens the farms of many countries such as Australia, New Zealand and South Africa. This dipteran fly having a round to oval body in length varies from 4.5 to 10 millimeters with reddish eyes and a greenish or green body bluish with coppery highlights is part of the family of "Calliphoridae" and it is known under the name "Australian sheep blowflies" or in Latin "Lucilia cuprina" or "Phaenicia cuprina". In addition, it has two pairs of wings , the first pair are membranous wings and the second pair is modified and the hind wings are modified Known as "dumbbells" that are used to stabilize the flight.

The development cycle of this fly includes four stages of development: egg, larva, pupa and adult. Pregnant female Lucilia attracted by smelly and damp woolen cuts or folds lie on the sheep. An average of 250 eggs on the skin of the animal which will burst and melt in carnivorous larvae after the incubation period does not exceed 24 hours: these the worms feed on wound secretions and the underlying tissues of the sheep during three larval stages lasting $4-5$ days. After the larval stages, fully developed larvae leave fall and sink into the ground to turn into pupa that develop to new young flies. Motivated by the experimental data obtained by *Nicholson*, *Gurney, blyth* and

Nisbet proposed in 1980, which describes the evolution of population dynamics over time.

$$\frac{dN(t)}{dt} = \beta N(t-\tau)\exp\left(\frac{-N(t-\tau)}{k}\right) - \delta N(t), \tag{2.14}$$

$N(t)$ stands for the size of the population at that time t.
$\dfrac{dN(t)}{dt}$ represents the rate of evolution of the number of the population.
β is the maximum per capita daily egg production,
$\dfrac{1}{k}$ is the size at which the population reproduces at its maximum rate,
δ is per capita daily adult death rate,
τ is the duration of the maturation phase.

Developmental Cycle and Growth of Lucilia Cuprina

2.6.2 Lasota-Wazewska and Mackey-Glass models

Throughout more than 60 years of the mathematical modelling of problems arising in hematology, a quite large amount of hematopoiesis models have been investigated by many authors. To our knowledge, the sixties of the past century can be regarded as a watershed in the history of the modelling of blood cell kinetics but the first timid attempts were focused on dealing with quite complex models.

To our knowledge, first models of hematopoiesis dynamics have been proposed by *Bell* and *E. C. Anderson* who were among the first ones to study the dynamics of a cell population structured in age and size in 1967. After a year, *Rubinow* put forward a model of maturity structured population dynamics. In 1978 and inspired by the works of *Lajtha* (1959) and *Bruns Tonnack* (1970), *Mackey* proposed two mathematical models of HSC dynamics. Then models of structured populations have been studied by various researchers. *Swick* in 1980, *Marcati* in 1981, *Gyllenberg* and *Webb* and *Bélair* in 1997, *Diekmann*, *Heijmans*, *Thieme*, *Heijmans*, *Webb* and *Grabosch*...

Starting from 1990, these models saw a significant increase in interest translated by an exponential increase in both quantity and quality of publications on this topic.

Let us now cite what we consider as remarkable contributions to this field because these works have provided a benchmark of excellence in the modelling of physiological control systems including respiratory dynamics as well as hematopoiesis in human and animal whether in health and disease. The end of the following decade witnessed two turning points, the first work carried out by *Wazewska* and *Lasota* in 1976 and the other done by *Mackey* and *Glass* in the following year.

In the end of the seventies of the past century and in one of the earliest papers in this topic which was and still is one of the most important milestones in the history of mathematical modelling of erythropoiesis, *Wazewska-Czyzewska* and *Lasota* introduced, the following delayed differential equation with one constant delay:

$$x'(t) = -ax(t) + b\exp(-\gamma x(t-\tau)),$$

where they were interested in the problem of the existence of periodic solutions to this erythropoiesis model which was aimed at modelling and getting better understanding of the survival of red blood cells in an animal. In medical terms $x(t)$ stands for the density of mature red blood cells in the blood circulation at time t, $a > 0$ is the death rate of red blood cells, the positive constants γ and b are related to the production of red-blood cells per unit time and the time delay required to produce a mature red blood cell for release in circulating bloodstreams is denoted by the positive constant τ.

In 1977, the Canadian physiologists *Leon Glass* and *Michael Mackey* [60] proposed two other pioneering works that have provided some surprising insight into the formation of hematopoietic cells and hence have broken a new ground in mathematical modelling of blood cell production as well as the regulation and control mechanisms of hematologic diseases, are the following famous delay differential equations with a constant delay, which are known as the *Mackey-Glass* equations:

$$x'(t) = -\alpha x(t) + \frac{b}{1 + x(t-\tau)},$$

and

$$x'(t) = -ax(t) + \frac{bx(t-\tau)}{1 + x(t-\tau)},$$

for modelling and getting a better understanding of the erythropoiesis and leukopoiesis. In medical terms, $x(t)$ (cells/kg) represents the density of mature circulating red or white blood cells in the blood circulation at time t, $\alpha x(t)$ (cells/day) is the death term, $\alpha > 0$ (days^{-1}) is called the death rate of blood cells in the circulation, $\dfrac{b}{1+x(t-\tau)}$ and $\dfrac{bx(t-\tau)}{1+x(t-\tau)}$ (cells/kg-day) which depend on the cell density at an earlier time, stands for the blood cells reproduction, $b > 0$ (units cells/kg-day) describes the maximal blood cells production rate that the body can approach when the density of blood cells in the circulation falls below normal and $\tau > 0$ (days) denotes the maturation delay.

2.6.3 Neutral erythropoiesis model with iterative terms

Consider the following neutral *Mackey-Glass* equation with iterative production and harvesting terms:

$$\frac{d}{dt}\left[x(t) - \lambda x(t-\tau(t))\right] = -a(t)x(t) + \sum_{i=2}^{n} \frac{\eta(t)}{1+x^{[i]}(t)} - k\left(t, x(t), ..., x^{[n]}(t)\right), \quad (2.15)$$

where $x(t)$ is the nth iterate $x^{[n]}(t)$ stands for the composition of $x(t)$ with itself n times, $\tau, a, \eta \in C(\mathbb{R}, (0,\infty))$ are common w-periodic functions, $\lambda \in (0,1)$, $\tau(t)$ denotes a transit time needed for the liberation of erythrocytes into the bloodstream and $k \in C(\mathbb{R}^{n+1}, (0,\infty))$ is the harvesting function. which is assumed globally *Lipschitz* in $x_1, x_2, ..., x_n$.

This equation which is now known as *Mackey-Glass* model describes a highly complex process, called erythropoiesis, by which red blood cells (named also erythrocytes, erythroid cells, RBCs) are produced in the marrow of certain bones for releasing them in the bloodstream. Here, $x(t)$ (cells/kg) is the density of circulating mature human erythrocytes, $a(t)x(t)$ (cells/kg-day) denotes the death term, $a(t)$ (days^{-1}) stands for the mortality rate of RBCs, $\sum_{i=2}^{n} \dfrac{\eta(t)}{1+x^{[i]}(t)}$ (cells/kg-day) is the erythroid cells reproduction where $\eta(t)$ (units cells/kg-day) describes the erythrocytes production rate.

In this model, the mortality and the maximal production rates are time varying parameters and H is the blood cell harvesting such as wet cupping, blood sampling or blood donation which plays a crucial role in the blood cell population dynamics and the management of biological renewable resources. Furthermore the iterates $x^{[i]}(t)$ in this equation result from $(n-1)$ delays of the form $\tau_i(t, x(t))$ that describe the mean time durations between the division of multipotent hematopoietic stem cells (HSCs) in the bone marrow and the formation and maturation of erythrocytes. Indeed, these delays should depend on both the time and the current density of mature RBCs $x(t)$ and this is essentially due to the fact that some growth factors and hormones such as the renal erythropoietin (EPO), thyroid and pituitary hormones and sex steroids control the division of the HSCs and stimulate RBC maturation. In other words, when the number of mature erythrocytes is large, the aforementioned hormones with the aid of other growth factors suppress the division of the HSCs and repress the RBC maturation, and in the converse case, they will promote and stimulate them. So, this equation which is a first order iterative differential equation originates from a neutral differential equation with two types of delays, the first one is a time varying lag and the other ones depend on both the state and the time variables.

CHAPTER 3

Existence, uniqueness, and stability results of a neutral delay differential equation

Contents

This chapter is devoted to prove the existence, uniqueness, and continuous dependence on parameters of positive periodic solutions for a first order nonlinear neutral differential equation with iterative terms.

3.1 Introduction

Delay differential equations of neutral type which involve derivatives with respect to time and also with respect to a delayed time, have a rich history and have found numerous applications as a potent tool for comprehending complex retarded systems in various areas of life sciences such as population dynamics, epidemiology, immunology, physiology, neural networks and hematology. Therefore, there is currently significant interest in obtaining sufficient conditions for the existence of periodic solutions for first order nonlinear neutral differential equations. However, neutral differential equation with iterative terms that manifest themselves in many fields, are generally difficult to be investigated. We cite some recent works on this topic.

In [56], the existence of positive periodic solutions for the following neutral differential equation:

$$[x(t) - cx(t - \tau(t))]' = -Q(t)x(t) + f(t, x(t - \tau(t))),$$

is investigated. Here $|c| < 1$, $Q, \tau \in \mathcal{C}(\mathbb{R}, (0, \infty))$ and $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are periodic functions.

Candan [18] established the existence of positive periodic solutions for the following neutral differential equation:

$$[x(t) - P(t)x(t - \tau)]' = -Q(t)x(t) + f(t, x(t - \tau)),$$

where $\tau > 0$, $Q \in \mathcal{C}(\mathbb{R}, (0, \infty))$, $P \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are periodic functions.

Mezghiche et al. [64] discussed the existence, uniqueness and stability of positive periodic solutions for the following neutral differential equation:

$$\begin{aligned} \frac{d}{dt}[x(t) - cx(t - \tau(t))] &= -a(t)x(t) + f(t, x(t - \tau(t))) \\ &\quad -H\left(t, x(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right). \end{aligned}$$

Khemis et al. [38] studied the existence, uniqueness and stability of positive periodic solutions for the following neutral erythropoiesis model with iterative production and harvesting terms:

$$\begin{aligned}\frac{d}{dt}\left[x(t)-\lambda x(t-\tau_1(t))\right] &= -a(t)x(t)+\sum_{i=2}^{n}\frac{p(t)}{1+x^{[i]}(t)}\\ &\quad -H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).\end{aligned}$$

In this chapter, we investigate the following first order nonlinear neutral delay differential equation with iterative terms and two delays:

$$\begin{aligned}&\frac{d}{dt}\left(x(t)-\sum_{i=1}^{2}c_i(t)x(t-\tau_i)\right)\\ &\qquad = -a(t)x(t)+\sum_{i=1}^{2}f_i(t,x(t),x(t-\tau_i))-H(...),\end{aligned} \tag{3.1}$$

where $H(...)=H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)),x^{[n]}(t)\right)$ such as $x^{[0]}(t)=t$, $x^{[1]}(t)=x(t)$, $x^{[2]}(t)=x(x(t)),...,x^{[n]}(t)=x^{[n-1]}(x(t))$ denote the iterates of the function $x(t)$. Equation (3.1) can model many biological and ecological equations such as: Neutral *Mackey-Glass* models with harvesting, Neutral *Wazewska-Lasota* model with harvesting, Neutral *Nicholson*'s blowflies model with harvesting and Neutral houseflies model with harvesting. $x(t)$ can represent the concentration, the density of individuals, the number or the size of the population at each time t, $a(t)$ can stand for the mortality rate,
$\sum_{i=1}^{2}f_i(t,x(t),x(t-\tau_i))$ can represent the recruitment or the production term, and $H(...)$ can describe the harvesting term which can describe capturing, hunting, trapping, fishing, or gathering etc.)

- $x\in\mathbb{R}$, $\tau_i>0$, $H(...)=H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)$, $x^{[n]}(t)$ is the $n-$th iterate of $x(t)$ which stands for x composed with itself n times.

- $c_i\in C([0,T],(0,1))$, $a\in(\mathbb{R},(0,\infty))$, and $f\in C([0,T]\times\mathbb{R}^2,(0,\infty))$ and $H\in C([0,T]\times\mathbb{R}^n,(0,\infty))$ are periodic functions i.e.,

$$c_i(t+T)=c_i(t),\ a(t+T)=a(t),$$

$$f_i(t+T,x(t+T),x(t-\tau_i))=f_i(t,x(t),x(t-\tau_i)),$$

and

$$H(t+T,x(t+T),x^{[2]}(t+T),...,x^{[n]}(t+T))=H(t,x(t),x^{[2]}(t),...,x^{[n]}(t)).$$

- $0 \leq c_i(t) \leq c_i$, $0 < c_i < \frac{1}{2}$, $i = 1, 2$, $c = \max\{c_1, c_2\}$.

Furthermore, we assume that the functions f_i and $H(...)$ satisfy the *Lipshitz* condition in their respective arguments;

$$\begin{aligned}
&|f_1(t,U) - f_1(t,V)| \leq k_1 |u_1 - v_1| + k_2 |u_2 - v_2|, \\
&U + u_1, u_2 \text{ and } V = v_1, v_2, \\
&\left| f_2\left(t, \bar{U}\right) - f_2\left(t, \bar{V}\right) \right| \leq k_3 |u_3 - v_3| + k_4 |u_4 - v_4|, \\
&\bar{U} = u_3, u_4 \text{ and } \bar{V} = v_3, v_4, \ ,
\end{aligned} \tag{3.2}$$

and

$$|H(t,X) - H(t,Y)| \leq \sum_{i=1}^{n} l_i |x_i - y_i|, \tag{3.3}$$

where k_i and l_i are positive constants, and $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$.

3.2 Preliminaries

For $T, M, L > 0$ and $m \geq 0$, let

$$P_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\},$$

be the *Banach* space of continuous and periodic functions endowed with the norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|,$$

and let the subset of P_T

$$P_T(.) = \{x \in P_T, \ m \leq x(t) \leq M, \ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \forall t, \ t_1, t_2 \in \mathbb{R}\}.$$

According to Remark 1.4, $P_T(.)$ is closed, convex, bounded, and compact.

For convenience, we introduce the notations below.

$$\begin{aligned}
a_1 &= \max_{t \in [0,T]} a(t), \ f_1 = \max_{t \in [0,T]} \{|f_1(t,0), f_2(t,0)|\}, \\
H_1 &= \max_{t \in [0,T]} \left|\bar{H}(.)\right|, \ \bar{H} = H(t,0,0,...,0),
\end{aligned}$$

and

$$\eta_1 = \frac{\exp(-A)}{\exp(A)-1},\ \eta_2 = \frac{\exp(A)}{\exp(A)-1},\ \Lambda = \sum_{i=1}^{n} l_i \sum_{j=0}^{i-1} L^j,$$

where

$$A = \int_0^T a(u)du.$$

Furthermore, we assume that the following conditions are satisfied:

Let δ_1, δ_2, c, k, m, M, L, Λ, and $L \in \mathbb{R}$, which are positive and

$$f(t, u_i, v_i) \geq \delta_i > 0,\ i = 1, 2, \forall t \in \mathbb{R},\ \forall u_1, u_2, v_1, v_2 \in (0, \infty), \tag{3.4}$$

$$2[cM + \eta_2 T(kM + f_1)] \leq M, \tag{3.5}$$

$$2\eta_1 T\delta_0 - \eta_2 T(M\Lambda + H_1) - 2cT\eta_2 a_1 M + 2cm \geq m, \tag{3.6}$$

where

$$\delta_0 = \min\{\delta_1, \delta_2\},$$

and

$$2\{\eta_2(2 + a_1 T)[\frac{1}{2}H_1 + f_1 + M(k + \frac{1}{2}\Lambda + ca_1)] + Lc\} \leq L. \tag{3.7}$$

We first give the following lemma, which will be needed later.

Lemma 3.1 *$x \in P_T(.) \cap C^1(\mathbb{R}, \mathbb{R})$ and x is a solution to the NDE (3.1) if and only if $x \in P_T(.)$ satisfies the following integral equation:*

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t,s) \left\{ \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) - H(\ldots) \right. \\ &\quad \left. - \sum_{i=1}^{2} c_i(s) a(s) x(s-\tau_i) ds \right\} + \sum_{i=1}^{2} c_i(t) x(t-\tau_i), \end{aligned}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u) du\right)}{\exp\left(\int_t^{t+T} a(u) du\right) - 1}.$$

Proof. Suppose that $x \in P_T(.) \cap \mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfies the equation (3.1). Then, by multiplying both sides of equation (3.1) by $\exp\left(\int_0^t a(u)\,du\right)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(x(t)-\sum_{i=1}^{2} c_i(t)\,x(t-\tau_i)\right)\exp\left(\int_0^t a(u)\,du\right)\\
&=\left(-a(t)\,x(t)+\sum_{i=1}^{2} f_i(t,x(t),x(t-\tau_i))-H(\ldots)\right)\exp\left(\int_0^t a(u)\,du\right).
\end{aligned}$$

The integration from t to $t+T$ leads to

$$\begin{aligned}
&\int_t^{t+T}(x'(s)+a(s)\,x(s))\exp\left(\int_0^s a(u)\,du\right)ds\\
&-\int_t^{t+T}\frac{d}{ds}\left(\sum_{i=1}^{2} c_i(s)\,x(s-\tau_i)\right)\exp\left(\int_0^s a(u)\,du\right)ds\\
&=\int_t^{t+T}\left(\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i))-H(\ldots)\right)\exp\left(\int_0^s a(u)\,du\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^{2} c_i(t)\,x(t-\tau_i)\right)\exp\left(\int_0^t a(u)\,du\right)\\
&\qquad=\frac{d}{dt}\left(\sum_{i=1}^{2} c_i(t)\,x(t-\tau_i)\exp\int_0^t a(u)\,du\right)\\
&\qquad-a(t)\sum_{i=1}^{2} c_i(t)\,x(t-\tau_i)\exp\left(\int_0^t a(u)\,du\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_t^{t+T}(x'(s)+a(s)\,x(s))\exp\left(\int_0^s a(u)\,du\right)ds\\
&-\int_t^{t+T}\frac{d}{ds}\left(\sum_{i=1}^{2} c_i(s)\,x(s-\tau_i)\exp\int_0^s a(u)\,du\right)ds\\
&=\int_t^{t+T}\left(\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i))-H(\ldots)-a(s)\sum_{i=1}^{2} c_i(s)\,x(s-\tau_i)\right)\\
&\times\exp\left(\int_0^s a(u)\,du\right)ds.
\end{aligned}$$

Hence

$$\int_t^{t+T}\frac{d}{ds}\left(\left(x(s)-\sum_{i=1}^{2}c_i(s)\,x(s-\tau_i)\right)\exp\int_0^s a(u)\,du\right)ds$$
$$=\int_t^{t+T}\left(\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\ldots)-a(s)\sum_{i=1}^{2}c_i(s)\,x(s-\tau_i)\right)$$
$$\times\exp\left(\int_0^s a(u)\,du\right)ds.$$

The property $x(t)=x(t+T)$ implies that

$$\left(x(t)-\sum_{i=1}^{2}c_i(t)\,x(t-\tau_i)\right)\left(\exp\left(\int_t^{t+T}a(u)\,du\right)-\exp\left(\int_0^t a(u)\,du\right)\right)$$
$$=\exp\left(\int_0^t a(u)\,du\right)\left(\exp\left(\int_t^{t+T}a(u)\,du\right)-1\right)\left(x(t)-\sum_{i=1}^{2}c_i(t)\,x(t-\tau_i)\right)$$
$$=\int_t^{t+T}\left(\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\ldots)-a(s)\sum_{i=1}^{2}c_i(s)\,x(s-\tau_i)\right)$$
$$\times\exp\left(\int_0^s a(u)\,du\right)ds.$$

Thus

$$x(t)=\int_t^{t+T}\left(\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\ldots)\right.$$
$$\left.-a(s)\sum_{i=1}^{2}c_i(s)\,x(s-\tau_i)\right)\frac{\exp\left(\int_0^s a(u)\,du\right)\exp\left(-\int_0^t a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right)-1}ds$$
$$+\sum_{i=1}^{2}c_i(t)\,x(t-\tau_i).$$

Therefore

$$\begin{aligned}x(t) &= \int_t^{t+T}G(t,s)\left(\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\ldots)\right.\\ &\qquad\left.-\sum_{i=1}^{2}c_i(s)\,a(s)\,x(s-\tau_i)\,ds\right)+\sum_{i=1}^{2}c_i(t)\,x(t-\tau_i).\end{aligned}$$

Conversely, we assume that $x\in P_T(.)$ satisfies the integral equation,so by differentiating the integral equation with respect to t we get

$$\begin{aligned}\frac{dx}{dt} &= G(t,t+T)\left[\sum_{i=1}^{2} f_i\left(t+T, x(t+T), x(t+T-\tau_i)\right)\right.\\
&\quad -H\left(t+T, x(t+T), ..., x^{[n]}(t+T)\right)\\
&\quad \left.-\sum_{i=1}^{2} c_i(t+T)\, a(t+T)\, x(t+T-\tau_i)\right]\\
&\quad -G(t,t)\left[\sum_{i=1}^{2} f_i\left(t, x(t), x(t-\tau_i)\right) - H\left(t, x(t), ..., x^{[n]}(t)\right)\right.\\
&\quad \left.-\sum_{i=1}^{2} c_i(t)\, a(t)\, x(t-\tau_i)\right]\\
&\quad +\int_t^{t+T} \frac{d}{dt} G(t,s)\left[\sum_{i=1}^{2} f_i\left(s, x(s), x(s-\tau_i)\right) - H\left(s, x(s), ..., x^{[n]}(s)\right)\right.\\
&\quad \left.-\sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i)\right] ds\\
&\quad +\sum_{i=1}^{2} c_i'(t)\, x(t-\tau_i) + \sum_{i=1}^{2} c_i(t)\, x'(t-\tau_i).\end{aligned}$$

But

$$\begin{aligned}
& G(t,t+T)\left[\sum_{i=1}^{2} f_i\left(t+T,x(t+T),x(t+T-\tau_i)\right)\right. \\
& -H\left(t+T,x(t+T),...,x^{[n]}(t+T)\right) \\
& \left.-\sum_{i=1}^{2} c_i(t+T)\,a(t+T)\,x(t+T-\tau_i)\right] \\
& -G(t,t)\left[\sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)-H\left(t,x(t),...,x^{[n]}(t)\right)\right. \\
& \left.-\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i)\right] \\
= & \frac{e^{\int_t^{t+T} a(u)du}}{e^{\int_0^T a(u)du}-1}\left[\sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)-H\left(t,x(t),...,x^{[n]}(t)\right)\right. \\
& \left.-\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i)\right] \\
& -\frac{e^{\int_t^{t} a(u)du}}{e^{\int_0^T a(u)du}-1}\left[\sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)-H\left(t,x(t),...,x^{[n]}(t)\right)\right. \\
& \left.-\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i)\right] \\
= & \left[\frac{e^{\int_0^T a(u)du}}{e^{\int_0^T a(u)du}-1}-\frac{1}{e^{\int_0^T a(u)du}-1}\right]\left[\sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)\right. \\
& \left.-H\left(t,x(t),...,x^{[n]}(t)\right)-\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i)\right] \\
= & \left[\frac{e^{\int_0^T a(u)du}-1}{e^{\int_0^T a(u)du}-1}\right]\left[\sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)-H\left(t,x(t),...,x^{[n]}(t)\right)\right. \\
& \left.-\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i)\right] \\
= & \sum_{i=1}^{2} f_i\left(t,x(t),x(t-\tau_i)\right)-H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right) \\
& -\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i).
\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{dt}G(t,s) &= \frac{d}{dt}\left(\frac{e^{\int_t^s a(u)du}}{e^{\int_0^T a(u)du}-1}\right) = \frac{1}{e^{\int_0^T a(u)du}-1}\times\left(-a(t)\,e^{\int_t^s a(u)du}\right) \\ &= -a(t)\frac{e^{\int_t^s a(u)du}}{e^{\int_0^T a(u)du}-1} = -a(t)\,G(t,s).\end{aligned}$$

So,

$$\begin{aligned}x'(t) &= \sum_{i=1}^{2} f_i(t,x(t),x(t-\tau_i)) - H\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right) \\ &\quad -\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i) \\ &\quad -a(t)\int_t^{t+T} G(t,s)\left[\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H\left(s,x(s),...,x^{[n]}(s)\right)\right. \\ &\quad \left.-\sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\right] ds \\ &\quad +\sum_{i=1}^{2} c_i'(t)\,x(t-\tau_i) + \sum_{i=1}^{2} c_i(t)\,x'(t-\tau_i).\end{aligned}$$

From the integral equation, we have

$$\begin{aligned}&\int_t^{t+T} G(t,s)\left[\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H\left(s,x(s),...,x^{[n]}(s)\right)\right. \\ &\quad \left.-\sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\right] ds \\ &= x(t) - \sum_{i=1}^{2} c_i(t)\,x(t-\tau_i).\end{aligned}$$

Then

$$\begin{aligned}x'(t) &= \sum_{i=1}^{2} f_i(t,x(t),x(t-\tau_i)) - H\left(t,x(t),...,x^{[n]}(t)\right) \\ &\quad -\sum_{i=1}^{2} c_i(t)\,a(t)\,x(t-\tau_i) \\ &\quad -a(t)\,x(t) + a(t)\sum_{i=1}^{2} c_i(t)\,x(t-\tau_i) + \\ &\quad \sum_{i=1}^{2} c_i'(t)\,x(t-\tau_i) + \sum_{i=1}^{2} c_i(t)\,x'(t-\tau_i).\end{aligned}$$

Which implies that

$$\begin{aligned} & x'(t) - \sum_{i=1}^{2} c_i'(t)\, x(t-\tau_i) - \sum_{i=1}^{2} c_i(t)\, x'(t-\tau_i) \\ = & \sum_{i=1}^{2} f_i(t, x(t), x(t-\tau_i)) - H\left(t, x(t), ..., x^{[n]}(t)\right) \\ & - \sum_{i=1}^{2} c_i(t)\, a(t)\, x(t-\tau_i) \\ & - a(t)\, x(t) + \sum_{i=1}^{2} c_i(t)\, a(t)\, x(t-\tau_i). \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\left[x(t) - \sum_{i=1}^{2} c_i(t)\, x(t-\tau_i)\right] = & -a(t)\, x(t) + \sum_{i=1}^{2} f_i(t, x(t), x(t-\tau_i)) \\ & - H\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right). \end{aligned}$$

This completes the proof. ■

Lemma 3.2 *[84] If $x, y \in P_T(.)$, then*

$$\left\| x^{[v]} - y^{[v]} \right\| \leq \sum_{k=0}^{v-1} L^k \|x - y\|,$$

where $v \in \mathbb{N}^+$.

Remark 3.1 Proof. We will prove this estimate by induction. So, the proof can be given in two steps:

The basis step: For $v = 1$, we have

$$\|x - y\| \leq \|x - y\|.$$

then, the estimate holds for $v = 1$.

The inductive step: Now, we suppose that the estimate holds for a given $v = s$ and we want to prove that it also holds for $v = s + 1$. Assume that

$$\left\| x^{[s]} - y^{[s]} \right\| \leq \sum_{k=0}^{v-1} L^k \|x - y\|,$$

then

$$\begin{aligned}\left|x^{[s+1]}(t)-y^{[s+1]}(t)\right| &\leq \left|x\left(x^{[s]}(t)\right)-x\left(y^{[s]}(t)\right)\right|+\left|x\left(y^{[s]}(t)\right)-y\left(y^{[s]}(t)\right)\right| \\ &\leq L\left|x^{[s]}(t)-y^{[s]}(t)\right|+\left|x\left(y^{[s]}(t)\right)-y\left(y^{[s]}(t)\right)\right|,\end{aligned}$$

so

$$\begin{aligned}\left\|x^{[s+1]}-y^{[s+1]}\right\| &\leq L\left\|x^{[s]}-y^{[s]}\right\|+\|x-y\| \\ &\leq L\sum_{k=0}^{s-1}L^{k}\|x-y\|+\|x-y\| \\ &\leq \left(\sum_{k=0}^{s-1}L^{k+1}+1\right)\|x-y\| \\ &\leq \sum_{k=0}^{s}L^{k}\|x-y\|.\end{aligned}$$

By induction we infer that

$$\left\|x^{[s]}-y^{[s]}\right\| \leq \sum_{k=0}^{s-1}L^{k}\|x-y\|, \quad \forall s\in\mathbb{N},$$

which finishes the proof. ■

Lemma 3.3 *We have*

(1) $0<\eta_1\leq G(t,s)\leq\eta_2$,

(2) $G(t,s)$ *is* T *periodic in* $t,s\in\mathbb{R}$,

(3)

$$\int_{t_1}^{t_1+1}\left|G(t_2,s)-G(t_1,s)\right|ds\leq Ta_1\eta_2\left|t_2-t_1\right|,$$

where $t,s,t_1,$*and* t_2 *belong to* $\mathbb{R}$,

(4) *By virtue of Remark 3.2 and estimates (3.2) and (3.3), we find that*

$$|f_i(s,x(s),x(s-\tau))|\leq kM+f_1,(i=1,2),$$

and

$$|H(...)|\leq M\Lambda+H_1,$$

for all $x\in P_T(.)$.

Proof.

(**1**) On the one hand, we have

$$\int_t^s a(u)\,du \le \int_0^T a(u)\,du.$$

Then

$$\exp\left(\int_t^s a(u)\,du\right) \le \exp\left(\int_0^T a(u)\,du\right),$$

and

$$-\int_0^T a(u)\,du \le \int_t^s a(u)\,du.$$

Then

$$\exp\left(-\int_0^T a(u)\,du\right) \le \exp\left(\int_t^s a(u)\,du\right).$$

Since

$$\exp\left(\int_0^T a(u)\,du\right) - 1 > 0.$$

Therefore

$$\begin{aligned}\frac{\exp\left(\int_0^T a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right)-1} &= \eta_1 \le G(t,s) = \frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right)-1} \\ &\le \frac{\exp\left(\int_0^T a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right)-1} = \eta_2.\end{aligned}$$

(**2**) Now we will prove that $G(t+T,s+T)=G(t,s)$, we have

$$G(t+T,s+T) = \frac{\exp(\int_{t+T}^{s+T} a(u)du)}{\exp(\int_0^T a(u)du)-1}.$$

$$v = u - T,$$

then

$$\begin{aligned} du &= dv, a(u) = a(v+T) \\ u &= t+T \Longrightarrow v = (t+T)-T = t, \\ u &= s+T \Longrightarrow v = (s+T)+T = s. \end{aligned}$$

So

$$G(t+T,s+T) = \frac{\exp(\int_{t+T}^{s+T} a(u)du)}{\exp(\int_0^T a(u)du)-1} = \frac{\exp(\int_t^s a(v+T)dv)}{\exp(\int_0^T a(v+T)dv)-1}.$$

Since, α is T-periodic, then $a(v+T)=a(v)$. Hence

$$G(t+T,s+T)=\frac{\exp(\int_{t+T}^{s+T}a(u)du)}{\exp(\int_0^T a(u)du)-1}=\frac{\exp(\int_{t+T}^{s+T}a(v)dv)}{\exp(\int_0^T a(v)dv)-1}=G(t,s).$$

(**3**) Let $t_1,t_2\in[0,T]$ (with $t_1<t_2$), then

$$\begin{aligned}&\int_{t_1}^{t_1+T}\left|\exp\left(\int_{t_2}^{s}a(u)\,du\right)-\exp\left(\int_{t_1}^{s}a(u)\,du\right)\right|ds\\ =&\int_{t_1}^{t_1+T}\exp\left(\int_{t_2}^{s}a(u)\,du\right)\left|1-\exp\left(\int_{t_1}^{t_2}a(u)\,du\right)\right|ds.\end{aligned}$$

By applying the mean value theorem to the function $f(t)=\exp\left(\int_t^{t_2}a(u)\,du\right)$ over the interval $[t_1,t_2]$, we obtain

$$1-\exp\left(\int_{t_1}^{t_2}a(u)\,du\right)=a(\delta)\exp\left(\int_{\delta}^{t_2}a(u)\,du\right)(t_2-t_1).$$

Hence,

$$\begin{aligned}&\int_{t_1}^{t_1+T}\left|\exp\left(\int_{t_2}^{s}a(u)\,du\right)-\exp\left(\int_{t_1}^{s}a(u)\,du\right)\right|ds\\ =&\int_{t_1}^{t_1+T}\exp\left(\int_{t_2}^{s}a(u)\,du\right)\left|a(\delta)\exp\left(\int_{\delta}^{t_2}a(u)\,du\right)(t_2-t_1)\right|ds\\ =&\int_{t_1}^{t_1+T}|a(\delta)|\exp\left(\int_{\delta}^{s}a(u)\,du\right)|t_2-t_1|\,ds\\ \leq&\ a_1|t_2-t_1|\int_{t_1}^{t_1+T}\exp\left(\int_0^T a(u)\,du\right)ds\\ \leq&\ a_1T|t_2-t_1|\exp\left(\int_0^T a(u)\,du\right).\end{aligned}$$

That is,

$$\begin{aligned}&\int_{t_1}^{t_1+T}\left|\frac{\exp\left(\int_{t_2}^{s}a(u)\,du\right)-\exp\left(\int_{t_1}^{s}a(u)\,du\right)}{\exp\left(\int_0^T a(u)du\right)-1}\right|ds\\ =&\int_{t_1}^{t_1+T}|G(t_2,s)-G(t_1,s)|\,ds\\ \leq&\ a_1T\frac{\exp\left(\int_0^T a(u)\,du\right)}{\exp\left(\int_0^T a(u)du\right)-1}|t_2-t_1|\\ =&\ a_1T\eta_2|t_2-t_1|.\end{aligned}$$

In addition, it follows from 3.2 that

$$\begin{aligned}
|f_i(s,x(s),x(s-\tau_i))| &= |f_1(s,x(s),x(s-\tau_i)) - f_1(s,0,0) + f_1(s,0,0)| \\
&\leq |f_1(s,x(s),x(s-\tau_i)) - f_1(s,0,0)| + |f_1(s,0,0)| \\
&\leq k_1 |x(s) - 0| + k_2 |x(s-\tau_i) - 0| + |f_1(s,0,0)| \\
&\leq k_1 M + k_2 M + f_1 \\
&\leq kM + f_1,
\end{aligned}$$

where $k = \max\{k_1, k_2\}$.

(**4**) From (3.3) and Remark 3.2 we get

$$\begin{aligned}
\left|H\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right| &= \left|H\left(s,x(s),...,x^{[n]}(s)\right) - H(s,0,...,0)\right. \\
&\quad \left. + H(s,0,...,0)\right| \\
&\leq \left|H\left(s,x(s),...,x^{[n]}(s)\right) - H(s,0,...,0)\right| \\
&\quad + |H(s,0,...,0)| \\
&\leq \sum_{i=1}^{n} l_i \left|x^{[i]}(s) - 0\right| + H_1 \\
&\leq \sum_{i=1}^{n} l_i \sum_{j=0}^{i-1} L^j \|x\| + H_1 \\
&\leq M\Lambda + H_1.
\end{aligned}$$

■

3.3 Existence results

In this section, we use the *Krasnoselskii* fixed point theorem for the NDE 3.1. For this, according to Lemma 3.1, let $N : P_T(.) \to P_T$ be an operator written as follows:

$$(Nx)(t) = (F_1x)(t) + (F_2x)(t),$$

where $F_1 : P_T(.) \to P_T$ and $F_2 : P_T(.) \to P_T$ are defined by

$$\begin{aligned}
(F_1x)(t) &= \int_t^{t+T} G(t,s) \left(\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H(...) \right. \\
&\quad \left. - \sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i) \right) ds,
\end{aligned}$$

and

$$(F_2x)(t) = \sum_{i=1}^{2} c_i(t)\, x(s - \tau_i).$$

Hence the existence of solutions of the NDE (3.1) is equivalent to the existence of fixed points of N and vice versa.

We first need to prove that F_2 is contraction.

Lemma 3.4 *The operator F_2 is a contraction.*

Proof. For $x, y \in P_T(.)$, we have

$$\begin{aligned} |(F_2x)(t) - (F_2y)(t)| &= \left| \sum_{i=1}^{2} c_i(t)\, x(s-\tau_i) - \sum_{i=1}^{2} c_i(t)\, y(s-\tau_i) \right| \\ &\leq \sum_{i=1}^{2} c_i(t)\, |x(s-\tau_i) - y(s-\tau_i)| \\ &\leq \sum_{i=1}^{2} c_i(t)\, \|x-y\| \\ &= c_1 \|x-y\| + c_2 \|x-y\| \\ &\leq 2c \|x-y\|, \end{aligned}$$

where $c = \max\{c_1, c_2\}$. Thanks to the inequality $0 \leq c_i \leq c < \frac{1}{2}$, F_2 is a contraction. ∎

We will now prove that F_1 is compact and continuous.

Lemma 3.5 *The operator F_1 is continuous and compact.*

Proof. Since $P_T(.)$ is a compact subset of P_T and since any continuous operator maps every compact set into compact one, then to prove that F_1 is a compact operator it is enough to prove that it is continuous. Indeed, for all $x, y \in P_T(.)$, we have

$$\begin{aligned}
&(F_1x)(t) - (F_1y)(t) \\
&= \int_t^{t+T} G(t,s) \left\{ \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) - H(\ldots) - \sum_{i=1}^{2} c_i(s) a(s) x(s-\tau_i) \right\} ds \\
&- \int_t^{t+T} G(t,s) \left\{ \sum_{i=1}^{2} f_i(s, y(s), y(s-\tau_i)) - H(\ldots) - \sum_{i=1}^{2} c_i(s) a(s) y(s-\tau_i) \right\} ds \\
&\leq \int_t^{t+T} |f_1(s, x(s), x(s-\tau_1)) - f_1(s, y(s), y(s-\tau_1))| \, G(t,s) ds \\
&+ \int_t^{t+T} |f_2(s, x(s), x(s-\tau_2)) - f_2(s, y(s), y(s-\tau_2))| \, G(t,s) ds \\
&+ \int_t^{t+T} G(t,s) \, |H(\ldots) - H(\ldots)| \, ds \\
&+ \int_t^{t+T} a(s) \left| \sum_{i=1}^{2} c_i(t) x(s-\tau_i) - \sum_{i=1}^{2} c_i(t) y(s-\tau_i) \right| G(t,s) ds.
\end{aligned}$$

Next, it follows from (2.2), (2.3), and $0 < \eta_1 \leq G(t,s) \leq \eta_2$ that

$$\begin{aligned}
&\|F_1x - F_1y\| \\
&\leq \int_t^{t+T} k_1 |x(s) - y(s)| \, G(t,s) ds + \int_t^{t+T} k_2 |x(s-\tau_1) - y(s-\tau_1)| \, G(t,s) ds \\
&+ \int_t^{t+T} k_3 |x(s) - y(s)| \, G(t,s) ds + \int_t^{t+T} k_4 |x(s-\tau_2) - y(s-\tau_2)| \, G(t,s) ds \\
&+ \int_t^{t+T} \eta_2 \sum_{i=1}^{n} l_i \|x_i - y_i\| \, ds + \int_t^{t+T} \{a_1 c_1 \eta_2 \|x-y\| + a_1 c_1 \eta_2 \|x-y\|\} ds \\
&\leq 4k\eta_2 T \|x-y\| + \eta_2 T A \|x-y\| + 2\eta_2 c a_1 T \|x-y\| \\
&= 2\eta_2 T(2k + ca + \frac{\Lambda}{2}) \|x-y\|,
\end{aligned}$$

where

$$k = max\{k_1, k_2, k_3, k_4\}.$$

This shows that F_1 is *Lipshitz* continuous and hence F_1 is compact. ■

Lemma 3.6 *If (3.4)-(3.7) are fulfilled, then* $F_1x + F_2y \in P_T(.)$, *where* $x, y \in P_T(.)$.

Proof. Let $x, y \in P_T(.)$. Using the previous estimates, we obtain

$$\begin{aligned}(F_1x)(t) - (F_1y)(t) &= \int_t^{t+T} G(t,s)\left\{\sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) - H(...)\right.\\ &\left. -\sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i)\right\} ds + \sum_{i=1}^{2} c_i(t)\, y(t-\tau_i)\\ &\leq \sum_{i=1}^{2} c_i(t)\, y(t-\tau_i) + \int_t^{t+T} G(t,s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i))\, ds\\ &\leq 2cM + 2\eta_2(kM + f_1)T\\ &\leq M.\end{aligned}$$

Similarly, we obtain from (3.4) and (3.6) that

$$\begin{aligned}(F_1x)(t) + (F_1y)(t) &= \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) - H(...)\\ &-\sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i)\} ds + \sum_{i=1}^{2} c_i(t)\, y(t-\tau_i)\}\\ &\geq \int_t^{t+T} 2\eta_1 \delta_0 ds - \int_t^{t+T} \eta_2(M\Lambda + H_1) ds\\ &- \int_t^{t+T} \eta_2 2ca_1 M ds + 2cm\\ &= 2\eta_1\delta_0 ds - \eta_2(M\Lambda + H_1)T - \eta_2 2ca_1 MT + 2cm\\ &\geq m.\end{aligned}$$

Consequently,

$$m \leq (F_1x)(t) + (F_1y)(t) \leq M,$$

for all $x, y \in P_T(.)$.

Let $x, y \in P_T(.)\,,\, t_1, t_2 \in [0, T]$ with $t_1 < t_2$. For all $x, y \in P_T(.)$, we have

$$\begin{aligned}
& |((F_1x) + (F_2y))(t_2) - ((F_1x) + (F_2y))(t_1)| \\
\leq & \left| \int_{t_2}^{t_2+T} G(t_2, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) \right. \\
& \left. - \int_{t_1}^{t_1+T} G(t_1, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i))\, ds \right| \\
& + \left| \int_{t_2}^{t_2+T} G(t_2, s) H(...)ds - \int_{t_1}^{t_1+T} G(t_1, s) H(...)ds \right| \\
& + \left| \int_{t_2}^{t_2+T} G(t_2, s) \sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i)\} ds \right. \\
& \left. - \int_{t_1}^{t_1+T} G(t_2, s) \sum_{i=1}^{2} c_i(s)\, a(s)\, x(s-\tau_i)\} ds \right| \\
& + \left| \sum_{i=1}^{2} c_i(t_2)\, y(t_2 - \tau_i) - \sum_{i=1}^{2} c_i(t_1)\, y(t_1 - \tau_i) \right| \\
& \quad |((F_1x) + (F_2y))(t_2) - ((F_1x) + (F_2y))(t_1)|\,,
\end{aligned}$$

which gives

$$\begin{aligned}
& \left| \int_{t_2}^{t_2+T} G(t_2, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) \right. \\
& \left. - \int_{t_1}^{t_1+T} G(t_1, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i))\, ds \right| \\
& = \left| \int_{t_2}^{t_1} G(t_2, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) \right. \\
& + \int_{t_1}^{t_1+T} G(t_2, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) \\
& + \int_{t_{1+T}}^{t_2+T} G(t_2, s) \sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) \\
& \left. - \int_{t_1}^{t_1+T} G(t_1, s) \{\sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i))\, ds \right| \\
& \leq 2 \left| \int_{t_2}^{t_2+T} \eta_2(kM + f_1)ds \right| + 2 \left| \int_{t_1}^{t_1+T} \eta_2(kM + f_1)ds \right| \\
& + 2 \int_{t_1}^{t_1+T} |G(t_2, s) - G(t_1, s)|\, (kM + f_1)ds \\
& \leq 2(2 + a_1T)\eta_2(kM + f_1)\, |t_2 - t_1|\,.
\end{aligned}$$

Therefore

$$
\begin{aligned}
& \left|\int_{t_2}^{t_2+T} G(t_2,s)\,H(...)ds - \int_{t_1}^{t_1+T} G(t_1,s)\,H(...)ds\right| \\
= & \left|\int_{t_2}^{t_1} G(t_2,s)\,H(...)ds + \int_{t_1}^{t_1+T} G(t_2,s)\,H(...)\right. \\
& \left. + \int_{t_1+T}^{t_2+T} G(t_2,s)\,H(...)ds - \int_{t_1}^{t_1+T} G(t_1,s)\,H(...)ds\right| \\
\leq & \int_{t_2}^{t_1} |\eta_2(M\Lambda + H_1)ds| + \left|\int_{t_1+T}^{t_2+T} \eta_2(M\Lambda + H_1)ds\right| \\
& + \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\,(M\Lambda + H_1)ds \\
= & (2 + a_1 T)\eta_2(M\Lambda + H_1)\,|t_2 - t_1|\,.
\end{aligned}
$$

In view of $0 < \eta_1 \leq G(t,s) \leq \eta_2$, we obtain

$$
\begin{aligned}
& \left|\int_{t_2}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\,ds - \int_{t_1}^{t_1+T} G(t_1,s) \sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\,ds\right| \\
\leq & \left|\int_{t_2}^{t_1} G(t_2,s) \sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\,ds\right| + \left|\int_{t_1+T}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\,ds\right| \\
& + \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)| \sum_{i=1}^{2} c_i(s)\,a(s)\,x(s-\tau_i)\,ds \\
\leq & \left|\int_{t_1}^{t_2} \eta_2 a_1 2cM ds\right| + \left|\int_{t_1+T}^{t_2+T} \eta_2 a_1 2cM ds\right| + 2cMa_1 \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\,ds \\
= & 2(2 + a_1 T)\eta_2 cMa_1\,|t_2 - t_1|\,.
\end{aligned}
$$

Since $x, y \in P_T(.)$, then

$$
\begin{aligned}
& \left|\sum_{i=1}^{2} c_i(t_2)\,y(t_2-\tau_i) - \sum_{i=1}^{2} c_i(t_1)\,y(t_1-\tau_i)\right| \\
& |c_1(t_2)y(t_2-\tau_1) + c_2(t_2)y(t_2-\tau_2) \\
& -c_1(t_1)y(t_1-\tau_1) - c_2(t_1)y(t_1-\tau_2)| \\
& |c_1(t_2)y(t_2-\tau_1) - c_1(t_1)y(t_1-\tau_1)| \\
& |c_2(t_2)y(t_2-\tau_2) - c_2(t_1)y(t_1-\tau_2)| \\
\leq & c\,|y(t_2-\tau_1) - y(t_1-\tau_1)| + c\,|y(t_2-\tau_2) - y(t_1-\tau_2)| \\
= & 2Lc\,|t_2 - t_1|\,.
\end{aligned}
$$

Thus, it follows from (3.7) that

$$\begin{aligned}
& |((F_1x)+(F_2y))(t_2)-((F_1x)+(F_2y))(t_1)| \\
\leq\; & 2(2+a_1T)\eta_2(kM+f_1)\,|t_2-t_1| \\
& +(2+a_1T)\eta_2(M\Lambda+H_1)\,|t_2-t_1| \\
& +2(2+a_1T)\eta_2 cMa_1\,|t_2-t_1| \\
& +2Lc\,|t_2-t_1| \\
\leq\; & L\,|t_2-t_1|\,.
\end{aligned}$$

From 3.5, we obtain the sought result. ■

Theorem 3.1 *Let $\tau \in P_T(.)$. If conditions 3.4-3.7 are satisfied, then the NDE 3.1 includes at least one positive periodic solution in $P_T(.)$*

Proof. Thanks to Lemmas 3.4-3.6, the Krasnoselskii fixed point theorem guarantees the existence of at least one positive periodic solution for the equation (3.1). ■

3.4 Existence, uniqueness and stability results

In this section, we establish the existence, uniqueness, and continuous dependence on parameters of solutions.

Theorem 3.2 *Let $\tau \in P_T(.)$. If conditions (3.4)-(3.7) and estimate*

$$2T\eta_2(2k+\frac{\Lambda}{2}+ca_1)+c<1, \tag{3.8}$$

are fulfilled, then equation (3.1) has a unique positive periodic solution $x \in P_T(.)$.

Proof. Let $x, y \in P_T(.)$. According to Lemmas 3.4-3.5, we infer that $N(P_T(.)) \subset P_T(.)$ and we have

$$\|(Nx)-Ny)\| \leq (2T\eta_2(2k+\frac{\Lambda}{2}+ca_1)+c)\,\|x-y\|\,.$$

■

From estimate (3.8) and the *Banach* fixed point theorem, N is a contraction. So, it has a unique fixed point which is the unique solution of equation (3.1).

Lemma 3.7 *If*

$$G_1(t,s)=\frac{e^{\int_t^s v_1(u)du}}{e^{\int_0^T v_1(u)du}-1},\ G_2(t,s)=\frac{e^{\int_t^s v_2(u)du}}{e^{\int_0^T v_2(u)du}-1},$$

then

$$\int_t^{t+T}|G_2(t,s)-G_1(t,s)|\le\mu\|v_1-v_2\|,\tag{3.9}$$

where

$$\mu=\frac{T^2e^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du}-1\right|}+\frac{T^2e^{T(\|v_1\|+\max\{\|v_1\|,\|v_2\|\})}}{\left|e^{\int_0^T v_2(u)du}-1\right|\left|e^{\int_0^T v_1(u)du}-1\right|}.$$

Proof. We have,

$$\begin{aligned}
&|G_2(t,s)-G_1(t,s)|\\
&=\left|\frac{e^{\int_t^s v_2(u)du}}{e^{\int_0^T v_2(u)du}-1}-\frac{e^{\int_t^s v_1(u)du}}{e^{\int_0^T v_1(u)du}-1}\right|\\
&\le\left|\frac{e^{\int_t^s v_2(u)du}-e^{\int_t^s v_1(u)du}}{e^{\int_0^T v_2(u)du}-1}\right|+\left|e^{\int_t^s v_1(u)du}\right|\left|\frac{1}{e^{\int_0^T v_2(u)du}-1}-\frac{1}{e^{\int_0^T v_1(u)du}-1}\right|\\
&\le\left|\frac{e^{\int_t^s v_2(u)du}-e^{\int_t^s v_1(u)du}}{e^{\int_0^T v_2(u)du}-1}\right|\\
&+\frac{\left|e^{\int_t^s v_1(u)du}\right|}{\left|e^{\int_0^T v_2(u)du}-1\right|\left|e^{\int_0^T v_1(u)du}-1\right|}\left|e^{\int_0^T v_2(u)du}-e^{\int_0^T v_1(u)du}\right|.
\end{aligned}$$

By applying the mean value theorem on the interval $\left[\int_t^s v_1(u)\,du,\int_t^s v_2(u)\,du\right]$, we arrive at

$$\begin{aligned}
\left|e^{\int_t^s v_2(u)du}-e^{\int_t^s v_1(u)du}\right| &= e^{\zeta}\left|\int_t^s v_2(u)\,du-\int_t^s v_1(u)\,du\right|\\
&\le e^{\zeta}\int_0^T|v_2(u)-v_1(u)|\,du\\
&\le T\|v_2-v_1\|\,e^{\max\left\{\int_t^s v_1(u)du,\int_t^s v_2(u)du\right\}}\\
&\le T\|v_2-v_1\|\,e^{\max\left\{\int_0^T v_1(u)du,\int_0^T v_2(u)du\right\}}\\
&\le T\|v_2-v_1\|\,e^{T\max\{\|v_1\|,\|v_2\|\}},
\end{aligned}$$

where $\zeta\in\left]\int_t^s v_1(u)\,du,\int_t^s v_2(u)\,du\right[$. Similarly, we obtain

$$\begin{aligned}
\left|e^{\int_0^T v_2(u)du}-e^{\int_0^T v_1(u)du}\right| &= e^{\eta}\left|\int_0^T v_2(u)\,du-\int_0^T v_1(u)\,du\right|\\
&\le T\|v_2-v_1\|\,e^{T\max\{\|v_1\|,\|v_2\|\}},
\end{aligned}$$

where $\eta \in \left]\int_0^T v_1(u)\,du, \int_0^T v_2(u)\,du\right[$. So,

$$\begin{aligned}
&|G_2(t,s) - G_1(t,s)| \\
&\leq \frac{Te^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|}\|v_2 - v_1\| + T\frac{\left|e^{\int_0^T v_1(u)du}\right| e^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|\left|e^{\int_0^T v_1(u)du} - 1\right|}\|v_2 - v_1\| \\
&\leq \frac{Te^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|}\|v_2 - v_1\| + T\frac{e^{T\|v_1\|} e^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|\left|e^{\int_0^T v_1(u)du} - 1\right|}\|v_2 - v_1\|.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\int_t^{t+T} |G_2(t,s) - G_1(t,s)|\,d\sigma &\leq \frac{T^2 e^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|}\|v_2 - v_1\| \\
&\quad + \frac{T^2 e^{T(\|v_1\| + \max\{\|v_1\|,\|v_2\|\})}}{\left|e^{\int_0^T v_2(u)du} - 1\right|\left|e^{\int_0^T v_1(u)du} - 1\right|}\|v_2 - v_1\| \\
&\leq \mu\|v_2 - v_1\|,
\end{aligned}$$

where

$$\mu = \frac{T^2 e^{T\max\{\|v_1\|,\|v_2\|\}}}{\left|e^{\int_0^T v_2(u)du} - 1\right|} + \frac{T^2 e^{T(\|v_1\| + \max\{\|v_1\|,\|v_2\|\})}}{\left|e^{\int_0^T v_2(u)du} - 1\right|\left|e^{\int_0^T v_1(u)du} - 1\right|}.$$

■

Theorem 3.3 *If the conditions of Theorem 3.2 are fulfilled, then the unique solution of the equation (3.1) depends continuously on the functions a, f and H.*

Proof. Let x be the unique solution ofequation (3.1), so x satisfies the integral equation of Lemma 3.1, i.e.,

$$\begin{aligned}
x(t) &= \int_t^{t+T} G(t,s)\left\{\sum_{i=1}^{2} f_i(s, x(s), x(s-\tau_i)) - H(\ldots)\right. \\
&\quad \left. - \sum_{i=1}^{2} c_i(s)\, v_1(s)\, x(s-\tau_i)\right\} ds + \sum_{i=1}^{2} c_i(t)\, x(t-\tau_i),
\end{aligned}$$

and and let $\widetilde{x}$ be a solution of the perturbed equation with small perturbations in functions a and H which satisfy the requirements of Theorem 3.2. So, $\widetilde{x}$ satisfies the following integral equation:

$$\begin{aligned}
\widetilde{x}(t) &= \int_t^{t+T} \widetilde{G}(t,s)\left\{\sum_{i=1}^{2} f_i(s, \widetilde{x}(s), \widetilde{x}(s-\tau_i)) - \widetilde{H}(\ldots)\right. \\
&\quad \left. - \sum_{i=1}^{2} c_i(s)\, v_2(s)\, \widetilde{x}(s-\tau_i)\right\} ds + \sum_{i=1}^{2} c_i(t)\, \widetilde{x}(t-\tau_i),
\end{aligned}$$

with

$$\widetilde{H}(.) = \widetilde{H}(s, \widetilde{x}(s), \widetilde{x}(s), \widetilde{x}^{[2]}(s), ..., \widetilde{x}^{[n]}(s)),$$

and

$$\widetilde{G}\left(t,s\right) = \frac{\exp\left(\int_t^s \widetilde{a}\left(u\right)du\right)}{\exp\left(\int_t^{t+T} \widetilde{a}\left(u\right)du\right) - 1}$$

Then,

$$\begin{aligned}
&\left|x(t) - \widetilde{x}(t)\right| \\
&\leq \int_t^{t+T} \Bigg| G\left(t,s\right) \sum_{i=1}^{2} f_i\left(s, x\left(s\right), x\left(s-\tau_i\right)\right) \\
&- \widetilde{G}(t,s) \sum_{i=1}^{2} f_i(s, \widetilde{x}\left(s\right), \widetilde{x}\left(s-\tau_i\right)) ds \Bigg| \\
&+ \int_t^{t+T} \left| G\left(t,s\right) H(...) - \widetilde{G}\left(t,s\right) \widetilde{H}(...) \right| ds \\
&+ \int_t^{t+T} \Bigg| G\left(t,s\right) \sum_{i=1}^{2} c_i\left(s\right) v_1\left(s\right) x\left(s-\tau_i\right) \\
&- \widetilde{G}\left(t,s\right) \sum_{i=1}^{2} c_i\left(s\right) v_2\left(s\right) \widetilde{x}\left(s-\tau_i\right) \Bigg| \\
&+ \left| \sum_{i=1}^{2} c_i\left(t\right) x\left(t-\tau_i\right) - \sum_{i=1}^{2} c_i\left(t\right) \widetilde{x}\left(t-\tau_i\right) \right|.
\end{aligned}$$

From 3.3 and 3.9 that

$$\begin{aligned}
&\int_t^{t+T} \left| G\left(t,s\right) \sum_{i=1}^{2} f_i\left(s, x\left(s\right), x\left(s-\tau_i\right)\right) - \widetilde{G}(t,s) \sum_{i=1}^{2} f_i(s, \widetilde{x}\left(s\right), \widetilde{x}\left(s-\tau_i\right)) ds \right| \\
&\leq \int_t^{t+T} \left| G\left(t,s\right) f_1\left(s, x\left(s\right), x\left(s-\tau_i\right)\right) - \widetilde{G}(t,s) f_1(s, \widetilde{x}\left(s\right), \widetilde{x}\left(s-\tau_i\right)) ds \right| \\
&+ \int_t^{t+T} \left| G\left(t,s\right) f_2\left(s, x\left(s\right), x\left(s-\tau_i\right)\right) - \widetilde{G}(t,s) f_2(s, \widetilde{x}\left(s\right), \widetilde{x}\left(s-\tau_i\right)) ds \right| \\
&\leq \int_t^{t+T} \left| G\left(t,s\right) (kM + f_1) - \widetilde{G}(t,s)(kM + f_1) \right| ds \\
&+ \int_t^{t+T} \left| G\left(t,s\right) (kM + f_1) - \widetilde{G}(t,s)(kM + f_1) \right| ds \\
&= 2(kM + f_1) \int_t^{t+T} \left| G\left(t,s\right) - \widetilde{G}(t,s) \right| ds \\
&\leq 2(kM + f_1)\mu \left\| v_1 - v_2 \right\|. \qquad (3.10)
\end{aligned}$$

So

$$\int_t^{t+T} \left| G(t,s) H(...) - \widetilde{G}(t,s) \widetilde{H}(...) \right| \tag{3.11}$$
$$\leq T\eta_2 \left\| H - \widetilde{H} \right\| + \mu(\Lambda M + H) \|v_1 - v_2\| + T\eta_2 \|x - y\| .$$

It follows from (3.9) that

$$\begin{aligned} &\int_t^{t+T} \left| G(t,s) \sum_{i=1}^{2} c_i(s) v_1(s) x(s-\tau_i) - \widetilde{G}(t,s) \sum_{i=1}^{2} c_i(s) v_2(s) \widetilde{x}(s-\tau_i) \right| \\ \leq\ & \int_t^{t+T} c\|v_1\| \eta_2 \|x - \widetilde{x}\| ds + \int_t^{t+T} c\|v_1\| M \left| G(t,s) - \widetilde{G}(t,s) \right| ds \\ &+ \int_t^{t+T} cM\eta_2 \|v_1 - v_2\| ds + \int_t^{t+T} c\|v_1\| \eta_2 \|x - \widetilde{x}\| ds \\ &+ \int_t^{t+T} c\|v_1\| M \left| G(t,s) - \widetilde{G}(t,s) \right| ds + \int_t^{t+T} cM\eta_2 \|v_1 - v_2\| ds \\ \leq\ & 2\eta_2 cT \|v_1\| \|x - \widetilde{x}\| + 2cM(\|v_1\| \mu + T\eta_2) \|v_1 - v_2\| . \end{aligned} \tag{3.12}$$

Using $c_i(t) \leq c_i$, $i = 1, 2$, $c = \max\{c_1, c_2\}$, we get

$$\begin{aligned} &\left| \sum_{i=1}^{2} c_i(t) x(t-\tau_i) - \sum_{i=1}^{2} c_i(t) \widetilde{x}(t-\tau_i) \right| \\ &\leq \left| c \sum_{i=1}^{2} x(t-\tau_i) - \sum_{i=1}^{2} \widetilde{x}(t-\tau_i) \right| \\ &\leq c|x(t-\tau_i) - \widetilde{x}(t-\tau_i)| + c|x(t-\tau_i) - \widetilde{x}(t-\tau_i)| \\ &\leq 2c\|x - \widetilde{x}\| . \end{aligned} \tag{3.13}$$

Using (3.11)-(3.13), we obtain

$$\begin{aligned} \|x - \widetilde{x}\| \ \leq\ & 2(kM + f_1)\mu \|v_1 - v_2\| + T\eta_2 \left\| H - \widetilde{H} \right\| + \mu(\Lambda M + H) \|v_1 - v_2\| \\ & + T\eta_2 \Lambda \|x - y\| + 2c(T\eta_2 \|v_1\| + 1) \|x - \widetilde{x}\| \\ & + 2(c\|v_1\| M\mu + T\eta_2 cM) \|v_1 - v_2\| \\ =\ & [2(kM + f_1)\mu + \mu(\Lambda M + H) + 2c\|v_1\| M\mu + 2T\eta_2 cM] \|v_1 - v_2\| \\ & + [2\eta_2 cT\|v_1\| + 2c] \|x - \widetilde{x}\| + T\eta_2 (\left\| H - \widetilde{H} \right\| + \Lambda \|x - y\|), \\ & \|x - \widetilde{x}\| [1 - 2c(\eta_2 T \|v_1\| + 1)] \\ \leq\ & [2(kM + f_1)\mu + \mu(\Lambda M + H) + 2c\|v_1\| M\mu + 2T\eta_2 cM] \|v_1 - v_2\| \\ & + T\eta_2 (\left\| H - \widetilde{H} \right\| + \Lambda \|x - y\|), \end{aligned}$$

and

$$\begin{aligned}\|x-\widetilde{x}\| &\leq \frac{1}{1-2c(\eta_2\|v_1\|+1)}\{[2(kM+f_1)\mu \\ &+\mu(\Lambda M+H)+2c\|v_1\|M\mu+2T\eta_2cM]\|v_1-v_2\| \\ &+T\eta_2(\left\|H-\widetilde{H}\right\|+\Lambda\|x-y\|),\end{aligned}$$

which completes the proof. ■

Now, we give an example to illustrate the main results.

Example 3.1 Consider the following neutral equation:

$$\begin{aligned}&\frac{d}{dt}\left[x(t)-\left(\frac{1}{1000}\sin\left(\frac{2\pi}{45}t\right)+\frac{1}{1000}\sin^2\left(\frac{2\pi}{45}t\right)\right)x(t-\tau)\right] \\ = &-\left(\frac{1}{90}+\frac{1}{90}\sin^2\left(\frac{2\pi}{45}t\right)\right)x(t) \\ &+\left[\frac{1}{3\pi^4}+\frac{1}{80\pi^4}\sin^2\left(\frac{2\pi}{45}t\right)+\frac{1}{18}\sin^2\left(\frac{2\pi}{45}t\right)x(t)\right. \\ &\left.+\frac{1}{10\pi^4}\sin^2\left(\frac{2\pi}{45}t\right)x(t-\tau)\right] \\ &\left[-\frac{1}{20\pi^2}\sin^2\left(\frac{2\pi}{45}t\right)+\frac{1}{25\pi^8}\left(\frac{2\pi}{45}t\right)x(t)+\frac{1}{35\pi^8}\sin^2\left(\frac{2\pi}{45}t\right)x^{[2]}(t)\right].\end{aligned} \tag{3.14}$$

Let $m=0.04$, $M=1.7, L=2\pi$, $c=0.002$, and $P_T(L,m,L,M)=P_{45}(2\pi,0.04,1.7)$. Then, according to the conditions of

Theorem 3.1 and Theorem 3.2, we have from (3.14) that

$$
\begin{aligned}
&\iota_1 = \frac{1}{25\pi^2}, \iota_2 = \frac{1}{35\pi^2}, H_1 = \frac{1}{20\pi^8}, \Lambda = 0.526339 \times 10^{-4},\\
&\delta_0 = \frac{1}{3\pi^4}, f_1 = \frac{83}{240\pi^4}, k = \frac{1}{10\pi^4}, a_1 = \frac{1}{45}, \eta_1 \approxeq 0.42289, \eta_2 = 1.89526,\\
&2\left[cM + \eta_2 T(kM + f_1)\right] = 2\left[0.002 \times 1.7 + 1.83526 \times 45\left(\frac{1}{10\pi^4} \times 1.7 + \frac{83}{240\pi^4}\right)\right]\\
&= 0.91007755413 \leq 1.7 = M,\\
&2\eta_1 T\delta_0 - \eta_2 T\left(M\Lambda + H_1\right) - 2cT\eta_2 a_1 M + 2cm\\
&= 2 \times 0.002 \times 45 \times \times \frac{1}{3\pi^4} - 1.89526\\
&\times 45\left(1.7 \times 0.526339 \times 10^{-4} + \frac{1}{20\pi^8}\right)\\
&-2 \times 0.002 \times 45 \times 1.89526 \times \frac{1}{45} \times 1.7 + 2 \times 0.002 \times 0.04\\
&= 0.10943299735 \geq 0.04 = m,\\
&2\left\{\eta_2(2 + a_1 T)\left[\frac{1}{2}H_1 + +f + M\left(k + \frac{1}{2}\Lambda + ca_1\right)\right] + Lc\right\}\\
&= \left\{21.89526\left(2 + \frac{1}{45} \times 45\right)\left[\frac{1}{2} \times \frac{1}{20\pi^8} + \frac{83}{240\pi^8}\right.\right.\\
&\left.\left.+1.7\left(\frac{1}{10\pi^4} + \frac{1}{2}0.526339 \times 10^{-4} + 0.002 \times \frac{1}{45}\right) + 2 \times \pi \times 0.002\right]\right\}\\
&= 0.08674914062 \leq 2\pi = L,
\end{aligned}
$$

and

$$
\begin{aligned}
&2T\eta_2(2k + \frac{\Lambda}{2} + ca_1) + c\\
= \;& 2 \times 1.98526 \times 45 \times \left(\frac{2}{45\pi^4} + 0.002\frac{1}{2} + 0.526339 \times \frac{1}{2}\right) + 0.002\\
= \;& 0.1891803597 < 1.
\end{aligned}
$$

This shows that the conditions of Theorems 3.1 and 3.2 are satisfied. Hence, the NDE (3.14) has a unique solution in $P_T(L, m, M) = P_{45}(2\pi, 0.04, 1.7)$ that depends continuously upon on the functions a, f and H.

Conclusion and perspectives

First order iterative delay differential equations can find use in many applied problems whose evolution rate depends on the past history and the composition of the state with itself as many times as needed. For example, medical and life sciences and electrodynamics are typical fields where such phenomena is believed to arise.

The work presented in this thesis focuses on the application of a powerful technique to study the existence, uniqueness, and stability of positive periodic solutions for a first order neutral differential equation with iterative terms that often appear in the modelling of many phenomena in life sciences.

The essence of this technique consists, firstly, in constructing an *Banach* space and a suitable subset of it for fulfilling certain mathematical and biological requirements. Secondly, it is also based on converting the considered equation into an integral one whose kernel is a *Green's* function before constructing an integral operator and applying the *Krasnoselskii* and the *Banach* fixed point theorems with the help of some useful properties of the obtained kernel. The *Krasnoselskii* fixed point theorem guarantees the existence of at least one positive periodic solution whereas the *Banach* fixed point theorem ensures the existence of a unique solution that depends continuously on parameters.

In conclusion, more work can be done on this topic, especially since the theory of delay-iterative differential equations is not very well developed yet. So, this work offer many research perspectives. For instance,

- The technique used here can be used successfully for investigating higher order delay-iterative differential equations or fractional differential equations.

_ It would appear valuable to study the existence of anti-periodic, almost-periodic, pseudo-almost-periodic solutions for delay-iterative differential equations.

- It would also seem crucial to employ numerical methods or software to get approximate solutions or even to provide numerical simulations to show the outcomes.

- It would seem important also to study the existence of almost-periodic, pseudo-almost-periodic, or anti-periodic solutions for delay iterative problems.

BIBLIOGRAPHY

[1] Abel, N.H.: *Oeuvres compétes*. Christiana I I, 36-39, (1881)

[2] Alzabut, J., Khuddush, M., Selvam, A.G.M. et al.: Second Order Iterative Dynamic Boundary Value Problems with Mixed Derivative Operators with Applications. Qual. Theory Dyn. Syst. **22**(1), No. 32 (2023)

[3] Arino, O., Hbid, M. L., Bravo, R.: A mathematic model of growth of population of fish in the larval stage: density-dependence effects. Math. Biosci. **150**(1),1-120 (1998)

[4] Babbage, C.: An essay towards the calculus of functions, Philosophical transactions of the royal society of London, 105,389-432, (1815)

[5] Ben Fredj, H., Chérif, F.: Positive pseudo almost periodic solutions to a class of hematopoiesis model: oscillations and dynamics. J. Appl. Math. Comput. **63**, 479–500 (2020)

[6] Berinde, V.: Existence and approximation of solutions of some first order iterative differential equations, Miskolc Math. **11**(1), 13–26 (2010).

[7] Bohner, M., Streipert, S.: Optimal harvesting policy for the Beverton-Holt model. Math. Biosci. Eng. **13**(4), 673–695 (2016)

[8] Bohner, M., Streipert, S.: Optimal harvesting policy for the Beverton-Holt quantum difference model. Math. Morav. **20**(2), 39–57 (2016)

[9] Bouakkaz, A., Ardjouni, A., Djoudi, A.: Existence of positive periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point theorem. Nonlinear Dyn. Syst. Theory, **17**(3), 230–238 (2017)

[10] Bouakkaz, A., Ardjouni, A., Khemis, R. et al. Periodic solutions of a class of third-order functional differential equations with iterative source terms. Bol. Soc. Mat. Mex. **26**, 443–458 (2020)

[11] Bouakkaz, A.: Bounded solutions to a three-point fourth-order iterative boundary value problem. Rocky Mountain J. Math. **52**(3), 793-803 (2022)

[12] Bouakkaz, A.: Positive periodic solutions for a class of first-order iterative differential equations with an application to a hematopoiesis model. Carpathian J. Math. **38**(2), 347-355 (2022)

[13] Bouakkaz, A., Khemis, R.: Existence, uniqueness and stability of solutions to a delay hematopoiesis model. J. Innov. Appl. Math. Comput. Sci. **2**(2), 23–30 (2022)

[14] Bouakkaz, A., Khemis, R.: Positive periodic solutions for a class of second-order differential equations with state-dependent delays. Turkish J. Math. **44**(4), 1412–1426 (2020)

[15] Bouakkaz, A., Khemis, R.: Positive periodic solutions for revisited Nicholson's blowflies equation with iterative harvesting term. J. Math. Anal. Appl. **494**(2), No. 124663 (2021)

[16] Brown, R. F.: A Topological Introduction to Nonlinear Analysis. Third Edition. Springer International Publishing Birkhauser. (2014)

[17] Cabada, A., Cid, J.A., Villamarin, B.M.: Compulation of Green's functions for boundary value problems with Mathematica. Appl. Math. Comput. **219**(4), 1919-1936 (2012)

[18] Candan, T.: *Existence of positive periodic solutions of first order neutral differential equation with variable coefficients*, Appl. Math. Lett. **52** 142-148 (2016)

[19] Cheraiet, S., Bouakkaz, A., Khemis, R.: Some new findings on bounded solution of a third order iterative boundary-value problem. J. Interdiscip. Math. **25**(4), 1153-1162 (2022)

[20] Cheraiet, S., Bouakkaz, A., Khemis, R.: Bounded positive solutions of an iterative three-point boundary-value problem with integral boundary conditions, J. Appl. Math. Comp. **65**, 597–610 (2021)

[21] Chouaf, S., Bouakkaz, A., Khemis, R.: On bounded solutions of a second-order iterative boundary value problem. Turkish J. Math. **46**(2), 453–464 (2022)

[22] Chouaf, S., Khemis, R., Bouakkaz, A.: Some existence results on positive solutions for an iterative second-order boundary-value problem with integral boundary conditions. Bol. Soc. Parana. Mat. (3). **40,** 1-10 (2022)

[23] Clarke, F.: Functional Analysis. Calculus of Variations and Optimal Control. Graduate Texts in Mathematics. Springer. London. **264**, (2013)

[24] Danchin, R.: Cours de topologie et d'analyse fonctionnelle Master première année. Année 2009-2010

[25] Diagana, T., Zhou, H.: Existence of positive almost periodic solutions to the hematopoiesis model. Appl. Math. Comput. **274**, 644-648 (2016)

[26] Ding, H. S., Liu, Q. L., Nieto, J. J.: Existence of positive almost periodic solutions to a class of hematopoiesis model. Appl. Math. Modeling. **40**(4), 3289-3297 (2016)

[27] Driver, R.D.: A *functional-differential system of neutral type arising in a two-body problem of classical electrodynamics*. Proc. Inter Symposium on Nonli Diff Eqs and Nonlinear Mechanics, Academic Press, New York, 474-484 (1963)

[28] Driver, R.D.: *A two-body problem of classical electrodynamics: the one dimensional case*. Ann. Phys. **21**, 122-142 (1963)

[29] Driver, R.D.: *Existence theory for a delay differential system*, Contrib. Differential Eq. **1**, 317-336 (1963)

[30] Duan, L., Huang, L., Chen, Y.: Global exponential stability of periodic solutions to a delay Lasota–Wazewska model with discontinuous harvesting. Proc. Am. Math. Soc. **144**, 561-573 (2016)

[31] Faria, T., Oliveira, J. J.: Global asymptotic stability for a periodic delay hematopoiesis model with impulses. Appl. Math. Model. **79**, 843-864 (2020)

[32] Frisch, R., Holme, H.: The Characteristic Solutions of Mixed Difference and Differential Equations. Econometrica. (1935)

[33] Gozen, M.: On the existence and uniqueness of positive periodic solutions of neutral differential equations. J. Nonlinear Var. Anal. **7**(3), 367-379 (2023)

[34] Guerfi, A., Ardjouni, A.: Periodic solutions for second order totally nonlinear iterative differential equations. J. Anal. **30**(1), 353-367 (2021)

[35] Hale, J.K., Verduyn Lunel, S.M.: Introduction to functional Differential equations. Springer Science and Business Media, Berlin. **99** (2013)

[36] Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer-Verlag, New York. (1993)

[37] Han, X,. Lei, C.: Existence of positive periodic solutions for first-order nonlinear differential equations with multiple time-varying delays. Open Math. **20**(1), 1380–1393 2022

[38] Khemis, M., Bouakkaz, A.: Existence and uniqueness results for a neutral erythropoiesis model with iterative production and harvesting terms. Bol. Soc. Parana. Mat. (3). **41,** 1-9 (2023)

[39] Khemis, M., Bouakkaz, A., Khemis, R.: Existence, uniqueness and stability results of an iterative survival model of red blood cells with a delayed nonlinear harvesting term. J. Math. Model. **10**(3), 515-528. (2022)

[40] Khemis, M., Bouakkaz, A., Khemis, R.: Positive periodic solutions for a delay model of erythropoiesis with iterative terms. Appl. Anal. **103**(1), 340–352 (2024).

[41] Khemis, M., Bouakkaz, A., Khemis, R.: Positive periodic solutions of a leukopoiesis model with iterative terms. Bol. Soc. Mat. Mex. **30**(1), 1–20 (2024)

[42] Khemis, R., Ardjouni, A., Bouakkaz, A., Djoudi, A.: Periodic solutions of a class of third-order differential equations with two delays depending on time and state. Comment. Math. Univ. Carolin. **60**(3), 379-399 (2019)

[43] Khemis, R., Bouakkaz, A., Chouaf, S.: On the existence of periodic solutions of a second order iterative differential equation. Acta Math. Univ. Comenian. (N.S.) **92** (1), 9-22 (2023)

[44] Khemis, R.: Existence, uniqueness and stability of positive periodic solutions for an iterative Nicholson's blowflies equation, J. Appl. Math. Comput. **69**, 1903–1916 (2023)

[45] Khuddush, M., Prasad, K.R.: Nonlinear two-point iterative functional boundary value problems on time scales. J. Appl. Math. Comput. **68**(6), 4241-4251 (2022)

[46] Kirk, J., Orr, J.S., Hope, C.S.: A Mathematical Analysis of Red Blood Cell and Bone Marrow Stem Cell Control Mechanisms. Br. J. Haematol. **15,** 35-46 (1968)

[47] Kolmanovski, V., Myshkis, A.: Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Acad. Publ. (1999)

[48] Kolmanovskii,V., Myshkis,A.: Applied Theory of Functional Differential Equations. Kluwer Academic Publishers, Dordrecht, (1992)

[49] Krasovskii, N.: Stability of Motion. Stanford Univ. Press, 1963. Translation with additions of the (1959) (Russian edition).

[50] Krasovskii, N.: Theory of A.M. Lyapunov's second method for investigating stability. (1956)

[51] Kulenović, M.R.S., Ladas, G., Sficas, Y.G.: Global attractivity in population dynamics. Comput. Math. Appl. **18,** 925-928 (1989)

[52] Li, J.W., Wang, Z.C.: Existence and Global Attractivity of positive periodic solutions of a survival model of red blood cells. Comput. Math. Appl. **50**, 41-47 (2005)

[53] Liu, G., Yan, J., Zhang, F.: Existence and global attractivity of unique positive periodic solution for a model of hematopoiesis. J. Math. Anal. Appl. **334**(1),157-171 (2007)

[54] Liu, B.: New results on the positive almost periodic solutions for a model of hematopoiesis, Nonlinear. Anal. Real. World. Appl. **17**, 252-264 (2014)

[55] Liu, X., Jia, M.: A class of iterative functional fractional differential equation on infinite interval. Appl. Math. Lett. **136**, No. 108473 (2023)

[56] Luo, Y., Wang, W., Shen, J.: *Existence of positive periodic solutions for two kinds of neutral functional differential equations*, Appl. Math. Lett. **21** 581-587 (2008)

[57] Liu, B., Tunç, C.: Pseudo almost periodic solutions for a class of first order differential iterative equations. App. Math. Lett. **40,** 29–34 (2015)

[58] Liu, B., Tunç, C.: Pseudo almost periodic solutions for CNNs with leakage delays and complex deviating arguments. Neural Comput & Applic. **26**, 429–435 (2015)

[59] Liu, G., Zhao, A., Yan, J.: Existence and global attractivity of unique positive periodic solution for a Lasota-Wazewska model. Nonlinear Anal. **64**, 1737-1746 (2006)

[60] Mackey, M. C., Glass L.: Oscillation and chaos in physiological control system. Science. **197**, 287-289 (1977)

[61] Martin, A.: Equations aux dérivées partielles. Exercices résolus. Dunod. (1991)

[62] Mansouri, B., Ardjouni, A., Djoudi, A.: Periodicity and continuous dependence in iterative differential equations. Rend. Circ. Mat. Palermo. **69**, 561–576 (2020)

[63] Mezghiche, L., Khemis, R.: On periodic solutions of a recruitment model with iterative terms and a nonlinear harvesting. Bol. Soc. Parana. Mat. (3). **41,** 1-9 (2023)

[64] Mezghiche, L., Khemis, R., Bouakkaz, A.: Positive periodic solutions for a neutral differential equation with iterative terms arising in biology and population dynamics. International Journal of Nonlinear Analysis and Applications. **13**(2), 1041–1051 (2022)

[65] Mezghiche, L., Khemis, R., Bouakkaz, A.: Some existence and uniqueness results of a houseflies model with a delay depending on time and state. International Journal of Nonlinear Analysis and Applications. **14**(1), 865–876 (2023)

[66] Myshkis, A.D.*:* Lineare Differential gleichungen mit nacheilenden Argumentom. Deutscher Verlag. Wiss. Berlin, 1955. Traduction du 1951 Russian edition.

[67] Nelson, P.W., Perelson, A. S.: Mathematical analysis of delay differential equation models of HIV-1 infection. Mathematical Biosciences **179**, 73-94 (2002)

[68] Nooney, G.C.: Iron kinetics and erythron development. Biophys. J. **5**, 755-765 (1965)

[69] Rihan, F. A.: Delay Differential Equations and Applications to Biology. Springer, Berlin, Germany, (2021). https://doi.org/10.1007/978-981-16-0626-7

[70] Rynne, B.P., Youngson, M.: Linear Functional Analysis. Springer Undergraduate Mathematics Series. (2008)

[71] Schröder, E.: Über iterate funktionen. Math. Ann. **3** 295-322 (1871)

[72] Sharney, L., Wasserman, L.R., Schwartz, L., Tendler, D.: Multiple pool analysis as applied to erythro-kinetics. Ann. N.Y. Acad. Sci. **108**, 230-249 (1963)

[73] Smart, D. R.: Fixed point theorems. Cambridge Tracts in Mathematics. no. 66. Cambridge University Press. London-New York. (1974)

[74] Song, Q., Hao, X.: Positive solutions for fractional iterative functional differential equation with a convection term. Electron. Res. Arch. **31**(4), 1863–1875 (2023)

[75] Tunç, C., Liu, B.: Global Stability of Pseudo Almost Periodic Solutions for a Nicholson's Blowflies Model with a Harvesting Term. Vietnam J. Math. **44**, 485–494 (2016)

[76] Wang, X., Zhang, H.: A new approach to the existence, nonlinear and uniqueness of positive almost periodic solution for a model of hematopoiesis, Nonlinear. Anal. Real. World. Appl. **11**, 60-66 (2010)

[77] Ważewska-Czyżewska, M., Lasota, A.: Mathematical problems of the dynamics of the red blood cells system. Math. Appl. **4**, 23-40 (1976)

[78] Wu, X. M., Li, J. W., Zhou, H. Q.: A necessary and sufficient condition for the existence of positive periodic solutions of a model of hematopoiesis, Comput. Math. Appl. **54**, 840-849 (2007)

[79] Xu, C., Liao, M., Li, P., Guo, Y., Liu, Z.: Bifurcation properties for fractional order delayed BAM neural networks. Cogn. Comput. **13**, 322–356 (2021)

[80] Xu, C., Liu, Z., Aouiti, C., Li, P., Yao, L., Yan, J.: New exploration on bifurcation for fractional-order quaternion-valued neural networks involving leakage delays. Cogn. Neurodyn. **16**, 1233–1248 (2022)

[81] Yang, X.: Existence and global attractivity of unique positive almost periodic solution for a model of hematopoiesis. Appl. Math. J. Chinese Univ. Ser. B **25(1),** 25–34 (2010)

[82] Yang, D., Zhang, W.: Solutions of equivariance for iterative differential equations, Appl. Math. Lett. **17**(7), 59-765 (2004)

[83] Zhao, H.Y., Fečkan, M.: Periodic solutions for a class of differential equations with delays depending on state. Math. Commun. **23**, 29–42 (2018)

[84] Zhao, H.Y., Liu, J.: Periodic solutions of an iterative functional differential equation with variable coefficients. Math. Methods Appl. Sci. **40**, 286–292 (2017)