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THE EXISTENCE RESULT OF RENORMALIZED SOLUTION FOR NONLINEAR PARABOLIC SYSTEM WITH VARIABLE EXPONENT AND L^1 DATA

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ABSTRACT. In this paper, we prove the existence result of a renormalized solution to a class of nonlinear parabolic systems, which has a variable exponent Laplacian term and a Leary lions operator with data belong to L^1 .

1. Introduction

Let Ω is bounded open domain of \mathbb{R}^N , ($N \geq 2$) with lipschiz boundary $\partial\Omega$, T is a positive number our aim is to study the existence of renormalized solution for a class of nonlinear parabolic systems with variable exponent and L^1 data. More precisely, we study the asymptotic behavior of the problem

$$\left\{ \begin{array}{ll} (b_1(u))_t - \operatorname{div}\mathcal{A}(x, t, \nabla u) + \gamma(u) = f_1(x, t, u, v) & \text{in } Q = \Omega \times]0, T[, \\ (b_2(v))_t - \Delta v = f_2(x, t, u, v) & \text{in } Q = \Omega \times]0, T[, \\ u = v = 0 & \text{on } \Sigma = \partial\Omega \times]0, T[, \\ b_1(u)(t=0) = b_1(u_0) & \text{in } \Omega, \\ b_2(v)(t=0) = b_2(v_0) & \text{in } \Omega, \end{array} \right. \quad (1.1)$$

where $\operatorname{div}\mathcal{A}(x, t, \nabla u) = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is a Leary lions operator (see assumptions (3.1)-(3.3)) with $p : \bar{\Omega} \rightarrow [1, +\infty)$ be a continuous real-valued function and let $p^- = \min_{x \in \bar{\Omega}} p(x)$ and $p^+ = \max_{x \in \bar{\Omega}} p(x)$

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with $1 < p^- \leq p^+ < N$. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ with $\gamma(s) = \lambda |s|^{p(x)-2} s$ is a continuous increasing function for $\lambda > 0$ and $\gamma(0) = 0$ such that γ is assumed to belong to $L^1(Q)$. The function $f_i : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ for $i = \overline{1, 2}$ be a Carathéodory function (see assumptions (3.5)-(3.7)).

Finally the function $b : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing C^1 -function lipchizienne with $b_i(0) = 0$ (see (3.4)), the data f_i and $(b_1(u_0), b_2(v_0))$ is in $(L^1)^2$, for $i = \overline{1, 2}$.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc (see [9], [19]) .

Problems of this type have been studied by serval a authors. In 2007 H. Redwane, studied the existence of solutions for a class of nonlinear parabolic systems see [18], in 2013 Youssef. B and all studied the existence of a renormalized solution for the nonlinear parabolic systems with unbounded nonlinearities see [2] agin in 2016 B . El Hamdaoui and all in [11] studied the renormalized solutions for nonlinear parabolic systems in the Lebesgue Sobolev Space with variable exponent and L^1 data. In 2016 [17] authors proved the existence and uniqueness of renormalized solution of a reaction diffusion systems which has a variable exponent Laplacian term and could be applied to image denoising for the case of parabolic equations. In 2010 T. M. Bendahmane, P. Wittbold and A.Zimmermann [7] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponent and L^1 data. C. Zhang and S. Zhou studied the renormalized and entropy solution for nonlinear parabolic equation with variable exponent and L^1 data. Moreover, they obtain the equivalence of renormalized solution and entropy solution(see [23]).

In the present paper we prove the existence of renormalized solution for nonlinear parabolic systems with variable exponent and L^1 data of (1.1). The notion of renormalized solution was introduced by Diperna and Lions [10] in their study of the Boltzmann equation, and this result can be seen as a generalization of the results obtained by F. Souilah and all in [12].

The paper is organized as follows: Section 2, to recall some basic notations and properties of variable exponent Lebesgue Sobolev space. Section 3, is devoted to specify the assumptions on, $\mathcal{A}(x, t, \xi)$, γ , b_1 , b_2 , f_1 , f_2 , $b_1(u_0)$ and $b_2(v_0)$ needed in the present study. Section 4, to give the definition of a renormalized solution of (1.1), and we establish (Theorem (4.1)) the existence of such a solution.

2. The Mathematical Preliminaries on Variable Exponent Sobolev Spaces

In this section, we first recall some results on generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ where Ω is an open subset of \mathbb{R}^N . We refer to [13] for further properties of Lebesgue Sobolev spaces with variable exponents. Let $p : \overline{\Omega} \rightarrow [1, +\infty)$ be a continuous real-valued function and let $p^- = \min_{x \in \overline{\Omega}} p(x)$, $p^+ = \max_{x \in \overline{\Omega}} p(x)$ with $1 < p(\cdot) < N$.

2.1. Generalized Lebesgue-Sobolev Spaces. First, denote the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u \text{ measurable function in } \Omega : \rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx \right\}$$

which is equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0, \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}. \tag{2.1}$$

The space $L^{p(x)}(\Omega)$ is also called a generalized Lebesgue space.

The space $(L^{p(\cdot)}(\Omega); \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for $x \in \Omega$.

The following inequality will be used later:

$$\min \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\} \leq \int_{\Omega} |u(x)|^{p(x)} dx \leq \max \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\}. \tag{2.2}$$

Finally, the Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p^+} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \tag{2.3}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Next, define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}, \tag{2.4}$$

which is Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \tag{2.5}$$

The space $(W^{1,p(\cdot)}(\Omega); \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. To have the following result:

Proposition 2.1. *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, the following properties hold true.*

- (i) $\|u\|_{p(\cdot)} > 1 \implies \|u\|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$,
- (ii) $\|u\|_{p(\cdot)} < 1 \implies \|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,
- (iii) $\|u\|_{p(\cdot)} < 1$ (respectively $= 1; > 1$) $\iff \rho_{p(\cdot)}(u) < 1$ (respectively $= 1; > 1$),
- (iv) $\|u_n\|_{p(\cdot)} \rightarrow 0$ (respectively $\rightarrow +\infty$) $\iff \rho_{p(\cdot)}(u_n) < 1$ (respectively $\rightarrow +\infty$),

$$(v) \rho_{p(\cdot)} \left(\frac{u}{\|u\|_{p(\cdot)}} \right) = 1.$$

For a measurable function $u : \Omega \rightarrow \mathbb{R}$, we introduce the following notation

$$\rho_{1,p(\cdot)} = \int_{\Omega} |u|^{p(x)} dx + \int_{\Omega} |\nabla u|^{p(x)} dx.$$

Proposition 2.2. *If $u \in W^{1,p(\cdot)}(\Omega)$ and $p^+ < +\infty$, the following properties hold true.*

- (i) $\|u\|_{1,p(\cdot)} > 1 \implies \|u\|_{1,p(\cdot)}^{p^+} < \rho_{1,p(\cdot)}(u) < \|u\|_{1,p(\cdot)}^{p^-}$,
- (ii) $\|u\|_{1,p(\cdot)} < 1 \implies \|u\|_{1,p(\cdot)}^{p^-} < \rho_{1,p(\cdot)}(u) < \|u\|_{1,p(\cdot)}^{p^+}$,
- (iii) $\|u\|_{1,p(\cdot)} < 1$ (respectively $= 1; > 1$) $\iff \rho_{1,p(\cdot)}(u) < 1$ (respectively $= 1; > 1$).

Extending a variable exponent $p : \bar{\Omega} \rightarrow [1, +\infty)$ to $\bar{Q} = [0, T] \times \bar{\Omega}$ by setting $p(x, t) = p(x)$ for all $(x, t) \in \bar{Q}$.

We may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q) = \left\{ u : Q \rightarrow \mathbb{R} \text{ measurable such that } \int_Q |u(x, t)|^{p(x)} d(x, t) < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)} = \inf \left\{ \mu > 0, \int_Q \left| \frac{u(x, t)}{\mu} \right|^{p(x)} d(x, t) \leq 1 \right\},$$

which share the same properties as $L^{p(\cdot)}(\Omega)$.

3. The Assumptions on The Data

This paper, we assume that the following assumptions hold true:

Let Ω be a bounded open set of \mathbb{R}^N ($N \geq 2$), $T > 0$ is given and we set $Q = \Omega \times]0, T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a Carathéodory function such that for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$

$$\mathcal{A}(x, t, \xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \tag{3.1}$$

$$|\mathcal{A}(x, t, \xi)| \leq \beta \left[L(x, t) + |\xi|^{p(x)-1} \right], \tag{3.2}$$

$$(\mathcal{A}(x, t, \xi) - \mathcal{A}(x, t, \eta)) \cdot (\xi - \eta) > 0, \tag{3.3}$$

where $1 < p^- \leq p^+ < +\infty$, α, β are positives constants and L is a nonnegative function in $L^{p'(\cdot)}(Q)$, $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$.

Let $b_i : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing C^1 -function lipchizienne with $b_i(0) = 0$ and for any ρ, τ are positives constants and for $i = \overline{1, 2}$ such that

$$\rho \leq b'_i(s) \leq \tau, \quad \forall s \in \mathbb{R}, \tag{3.4}$$

$f_i : Q \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that for any $k > 0$, there exists $\sigma_k > 0$, $c_k \in L^1(Q)$ such that

$$|f_1(x, t, s_1, s_2)| \leq c_k(x, t) + \sigma_k |s_2|^2, \tag{3.5}$$

for almost every $(x, t) \in (Q)$, for every s_1 such that $|s_1| \leq k$, and for every $s_2 \in \mathbb{R}$.

For any $k > 0$, there exists $\zeta_k > 0$ and $G_k \in L^{p'(\cdot)}(Q)$ such that

$$|f_2(x, t, s_1, s_2)| \leq G_k(x, t) + \zeta_k |s_1|^{p(x)-1}, \tag{3.6}$$

for almost every $(x, t) \in (Q)$, for every s_2 such that $|s_2| \leq k$, and for every $s_1 \in \mathbb{R}$.

$$f_1(x, t, s_1, s_2)s_1 \geq 0 \text{ and } f_2(x, t, s_1, s_2)s_2 \geq 0, \tag{3.7}$$

$$(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2. \tag{3.8}$$

4. The Main Results

In this section, we study the existence of renormalized solutions to problem (1.1).

Definition 4.1. Let $2 - \frac{1}{N+1} < p^- \leq p^+ < N$ and $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$. A measurable functions $(u, v) \in (C(]0, T[; L^1(\Omega)))^2$ is a renormalized solution of the problem (1.1) if ,

$$T_k(u) \in L^{p^-} (]0, T[; W_0^{1,p(\cdot)}(\Omega)), T_k(v) \in L^2(]0, T[; H_0^1(\Omega)) \text{ for any } k > 0 , \tag{4.1}$$

$$\gamma(u) \in L^1(Q) \text{ and } f_i(x, t, u, v) \in (L^1(Q))^2, \quad \forall i = \overline{1, 2},$$

$$b_1(u) \in L^\infty (]0, T[; L^1(\Omega)) \cap L^{q^-} (]0, T[; W_0^{1,q(\cdot)}(\Omega)) \tag{4.2}$$

$$\text{and } b_2(v) \in L^\infty (]0, T[; L^1(\Omega)) \cap L^2(]0, T[; H_0^1(\Omega)),$$

for all continuous functions $q(x)$ on $\overline{\Omega}$ satisfying $q(x) \in [1, p(x) - \frac{N}{N+1})$ for all $x \in \overline{\Omega}$,

$$\lim_{n \rightarrow \infty} \int_{\{n \leq |u| \leq n+1\}} \mathcal{A}(x, t, \nabla u) \nabla u dx dt + \lim_{n \rightarrow \infty} \int_{\{n \leq |v| \leq n+1\}} |\nabla v|^2 dx dt = 0, \tag{4.3}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support on \mathbb{R} , to have,

$$(B_S^1(u))_t - \text{div}(\mathcal{A}(x, t, \nabla u)S'(u)) + S''(u)\mathcal{A}(x, t, \nabla u)\nabla u + \gamma(u)S'(u) \tag{4.4}$$

$$= f_1(x, t, u, v)S'(u) \text{ in } \mathcal{D}'(Q),$$

$$(B_S^2(v))_t - \text{div}(\nabla v S'(v)) + S''(v)\nabla v = f_2(x, t, u, v)S'(v) \text{ in } \mathcal{D}'(Q), \tag{4.5}$$

$$B_S^1(u)(t = 0) = S(b_1(u_0)) \text{ in } \Omega, \tag{4.6}$$

$$B_S^2(v)(t = 0) = S(b_2(v_0)) \text{ in } \Omega, \tag{4.7}$$

where $B_S^i(z) = \int_0^z b'_i(r)S'(r)dr$, for $i = \overline{1,2}$.

The following remarks are concerned with a few comments on definition (4.1).

Remark 4.1. Note that, all terms in (4.4) are well defined. Indeed, let $k > 0$ such that $\text{supp}(S') \subset [K, K]$, we have $B_S^i(u)$ belongs to $L^\infty(Q)$ for all $i = \overline{1,2}$ because

$$|B_S^1(u)| \leq \int_0^u |b'_1(r)S'(r)|dr \leq \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$|B_S^2(v)| \leq \int_0^v |b'_2(r)S'(r)|dr \leq \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$S(u) = S(T_k(u)) \in L^{p^-}([0, T[; W_0^{1,p(\cdot)}(\Omega)), S(v) = S(T_k(v)) \in L^2([0, T[; H_0^1(\Omega))$$

and $\frac{\partial B_S^i(u)}{\partial t} \in (\mathcal{D}'(Q))^2$ for $i = \overline{1,2}$. The term $S'(u)\mathcal{A}(x, t, \nabla T_k(u))$ identifies with $S'(T_k(u))\mathcal{A}(x, t, \nabla(T_k(u)))$ a.e. in Q , where $u = T_k(u)$ in $\{|u| \leq k\}$, assumptions (3.2) imply that

$$\begin{aligned} & |S'(T_k(u))\mathcal{A}(x, t, \nabla T_k(u))| \\ & \leq \beta \|S'\|_{L^\infty(\mathbb{R})} \left[L(x, t) + |\nabla(T_k(u))|^{p(x)-1} \right] \text{ a.e in } Q. \end{aligned} \tag{4.8}$$

Using (3.2) and (4.1), it follows that $S'(u)\mathcal{A}(x, t, \nabla u) \in (L^{p'(\cdot)}(Q))^N$. The term $S''(u)\mathcal{A}(x, t, \nabla u)\nabla(u)$ identifies with $S''(u)\mathcal{A}(t, x, \nabla(T_k(u)))\nabla T_k(u)$ and in view of (3.2), (4.1) and (4.8), to obtain $S''(u)\mathcal{A}(x, t, \nabla u)\nabla(u) \in L^1(Q)$ and $S'(u)\gamma(u) \in L^1(Q)$. Finally $f_1(x, t, u, v) S'(u) = f_1(x, t, T_k(u), v)S'(u)$ a.e in Q . Since $|T_k(u)| \leq k$ and $S'(u) \in L^\infty(Q)$, $c_k(x, t) \in L^1(Q)$, to obtain from (3.5) that $f_1(x, t, T_k(u), v)S'(u) \in L^1(Q)$, and $f_2(x, t, u, v) S'(v) = f_2(x, t, u, T_k(v))S'(v)$ a.e in Q . Since $|T_k(v)| \leq k$ and $S'(v) \in L^\infty(Q)$, $G_k(x, t) \in L^{p'(\cdot)}(Q)$, to obtain from (3.6) that $f_2(x, t, u, T_k(v))S'(v) \in L^1(Q)$. Also $\frac{\partial B_S^1(u)}{\partial t} \in L^{(p^-)' }([0, T[; W^{-1,p'(\cdot)}(\Omega)) + L^1(Q)$ and $B_S^1(u) \in L^{p^-}([0, T[; W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$, and $\frac{\partial B_S^2(v)}{\partial t} \in L^2([0, T[; H^{-1}(\Omega)) + L^1(Q)$ and $B_S^2(v) \in L^2([0, T[; H_0^1(\Omega)) \cap L^\infty(Q)$, which implies that $(B_S^1(u), B_S^2(v)) \in (C([0, T[; L^1(\Omega)))^2$.

4.1. The Existence Theorem.

Theorem 4.1. *Let $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$, assume that (3.1)-(3.8) hold true, then there exists at least one renormalized solution $(u, v) \in (C(]0, T[, L^1(\Omega)))^2$ of Problem (1.1) (in the sens of Definition (4.1)).*

Proof. of Theorem (4.1) The above theorem is to be proved in five steps.

- **Step 1: Approximate problem and a priori estimates.** Let us define the following approximation of b and f for $\varepsilon > 0$ fixed and for $i = \overline{1, 2}$

$$b_\varepsilon^i(r) = T_{\frac{1}{\varepsilon}}(b_i(r)) \text{ a.e in } \Omega \text{ for } \varepsilon > 0, \quad \forall r \in \mathbb{R}, \tag{4.9}$$

$$b_\varepsilon^i(u_0^\varepsilon) \text{ are a sequence of } (C_c^\infty(\Omega))^2 \text{ functions such that} \tag{4.10}$$

$$(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon)) \rightarrow (b_1(u_0), b_2(v_0)) \text{ in } (L^1(\Omega))^2 \text{ as } \varepsilon \text{ tends to } 0.$$

$$f_1^\varepsilon(x, t, r_1, r_2) = f_1(x, t, T_{\frac{1}{\varepsilon}}(r_1), r_2), \tag{4.11}$$

$$f_2^\varepsilon(x, t, r_1, r_2) = f_2(x, t, r_1, T_{\frac{1}{\varepsilon}}(r_2)),$$

in view of (3.5), (3.6) and (3.7), there exist $G_k^\varepsilon \in L^{p'(\cdot)}(Q)$, $c_k^\varepsilon \in L^1(Q)$ and $\sigma_k^\varepsilon, \zeta_k^\varepsilon > 0$ such that

$$|f_1^\varepsilon(x, t, s_1, s_2)| \leq c_k^\varepsilon(x, t) + \sigma_k^\varepsilon |s_2|^2, \tag{4.12}$$

$$|f_2^\varepsilon(x, t, s_1, s_2)| \leq G_k^\varepsilon(x, t) + \zeta_k^\varepsilon |s_1|^{p(x)-1}, \tag{4.13}$$

for almost every $(x, t) \in (Q)$, $s_1, s_2 \in \mathbb{R}$,

$$f_1^\varepsilon(x, t, s_1, s_2)s_1 \geq 0 \text{ and } f_2^\varepsilon(x, t, s_1, s_2)s_2 \geq 0. \tag{4.14}$$

Let us now consider the approximate problem:

$$(b_\varepsilon^1(u^\varepsilon))_t - \operatorname{div} \mathcal{A}(x, t, \nabla u^\varepsilon) + \gamma(u^\varepsilon) = f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \text{ in } Q, \tag{4.15}$$

$$(b_\varepsilon^2(v^\varepsilon))_t - \Delta v^\varepsilon = f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \text{ in } Q, \tag{4.16}$$

$$u^\varepsilon = v^\varepsilon = 0 \text{ on }]0, T[\times \partial\Omega, \tag{4.17}$$

$$b_\varepsilon^1(u^\varepsilon)(t = 0) = b_\varepsilon^1(u_0^\varepsilon) \text{ in } \Omega, \tag{4.18}$$

$$b_\varepsilon^2(v^\varepsilon)(t = 0) = b_\varepsilon^2(v_0^\varepsilon) \text{ in } \Omega. \tag{4.19}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}([0, T[; W_0^{1,p(\cdot)}(\Omega))$ and $v^\varepsilon \in L^2([0, T[; H_0^1(\Omega))$ of (4.15)-(4.18) is an easy task (see [15]).

we choose $T_k(u^\varepsilon)\chi_{(0,t)}$ as a test function in (4.15), to get

$$\begin{aligned} \int_{\Omega} B_k^{1,\varepsilon}(u^\varepsilon)(t)dx &+ \int_0^t \int_{\Omega} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon) + \int_0^t \int_{\Omega} \gamma(u^\varepsilon) T_k(u^\varepsilon) dx ds \\ &= \int_0^t \int_{\Omega} f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) T_k(u^\varepsilon) dx ds + \int_{\Omega} B_k^{1,\varepsilon}(u_0^\varepsilon) dx, \end{aligned} \tag{4.20}$$

for almost every t in $(0, T)$, and where

$$B_k^{i,\varepsilon}(r) = \int_0^r T_k(s) \frac{\partial b_\varepsilon^i(s)}{\partial s} ds, \forall i = \overline{1, 2}.$$

Under the definition of $B_k^{i,\varepsilon}(r)$ the inequality

$$0 \leq \int_{\Omega} B_k^{1,\varepsilon}(u_0^\varepsilon)(t) dx \leq k \int_{\Omega} |b_\varepsilon^1(u_0^\varepsilon)| dx, \quad k > 0.$$

Using (3.1), $f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) T_k(u^\varepsilon) \geq 0$, and we have $\gamma(u^\varepsilon) = \lambda |u^\varepsilon|^{p(x)-1} u^\varepsilon \geq 0$ because $1 < p^- \leq p(x) \leq +\infty$ and the definition of $B_k^\varepsilon(r)$ in (4.20), to obtain

$$\int_{\Omega} B_k^\varepsilon(u^{1,\varepsilon})(t) dx + \alpha \int_{E_k} |\nabla u^\varepsilon|^{p(x)} dx ds \leq k \|b_\varepsilon^1(u_0^\varepsilon)\|_{L^1(Q)}, \tag{4.21}$$

where $E_k = \{(x, t) \in Q : |u^\varepsilon| \leq k\}$, using $\overline{B}_k^\varepsilon(u^\varepsilon)(t) \geq 0$ and inequality (2.2) in (4.21), to get

$$\begin{aligned} \alpha \int_0^T \min \left\{ \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p^-}, \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p^+} \right\} &\leq \alpha \int_{\{(x,t) \in Q: |u^\varepsilon| \leq k\}} |\nabla u^\varepsilon|^{p(x)} dx dt \\ &\leq C, \end{aligned} \tag{4.22}$$

then is $T_k(u^\varepsilon)$ is bounded in $L^{p^-}([0, T[; W_0^{1,p(x)}(\Omega))$.

Similarly, we choose $T_k(v^\varepsilon)\chi_{(0,t)}$ as a test function in (4.16), to get

$$\int_{\Omega} B_k^{2,\varepsilon}(v^\varepsilon)(t) dx + \alpha \int_{F_k} |\nabla v^\varepsilon|^2 dx ds \leq k \|b_\varepsilon^2(v_0^\varepsilon)\|_{L^1(Q)}, \tag{4.23}$$

where $F_k = \{(x, t) \in Q : |v^\varepsilon| \leq k\}$, then is $T_k(v^\varepsilon)$ is bounded in $L^2([0, T[; H_0^1(\Omega))$. Adding (4.21) and (4.23), one gets

$$\int_{\Omega} B_k^{1,\varepsilon}(u^\varepsilon)(t) dx + \int_{\Omega} B_k^{2,\varepsilon}(v^\varepsilon)(t) dx \leq k \|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\|_{L^1(Q) \times L^1(Q)}. \tag{4.24}$$

Also, to obtain

$$k \int_{\{(t,x) \in Q: |u^\varepsilon| > k\}} |\gamma(u^\varepsilon)| dx dt \leq k \|b_\varepsilon(u_0^\varepsilon)\|_{L^1(Q)}. \tag{4.25}$$

Hence

$$\begin{aligned}
 k \int_{\{(x,t) \in Q: |u^\varepsilon| > k\}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt &+ k \int_{\{(x,t) \in Q: |v^\varepsilon| > k\}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq k \|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\|_{L^1(Q) \times L^1(Q)}. \tag{4.26}
 \end{aligned}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(b_\varepsilon^1(u^\varepsilon))$ as test function in (4.15). Reasoning as above, by $\nabla T_{k,1}(s) = \nabla s \chi_{\{k \leq |s| \leq k+1\}}$ and the young's inequality, to obtain

$$\begin{aligned}
 \alpha \int_{\{k \leq |b_\varepsilon^1(u^\varepsilon)| \leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon) |\nabla(u^\varepsilon)|^{p(x)} \, dxdt &\leq k \int_{\{|b_\varepsilon^1(u_0^\varepsilon)| > k\}} |b_\varepsilon^1(u_0^\varepsilon)| \, dx \\
 &+ Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)| > k\}} |\gamma(u^\varepsilon)| \, dxdt \\
 &+ Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)| > k\}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq C_1,
 \end{aligned}$$

inequality (2.2) implies that

$$\begin{aligned}
 &\int_0^T \alpha \chi_{\{k \leq |b_\varepsilon^1(u^\varepsilon)| \leq k+1\}} \min \left\{ \|\nabla(b_\varepsilon^1(u^\varepsilon))\|_{L^{p(x)}(\Omega)}^{p^-}, \|\nabla(b_\varepsilon^1(u^\varepsilon))\|_{L^{p(x)}(\Omega)}^{p^+} \right\} \\
 &\leq \alpha \int_{\{k \leq |b_\varepsilon^1(u^\varepsilon)| \leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon) |\nabla(u^\varepsilon)|^{p(x)} \, dxdt \leq C_1. \tag{4.27}
 \end{aligned}$$

Similarly, we choose $T_k(b_\varepsilon^2(v^\varepsilon))$ as test function in (4.16), to have

$$\begin{aligned}
 \int_{\{|b_\varepsilon^2(v^\varepsilon)| \leq k\}} b'_{2,\varepsilon}(v^\varepsilon) |\nabla(v^\varepsilon)|^2 \, dxdt &\leq k \int_{\{|b_\varepsilon^2(v_0^\varepsilon)| > k\}} |b_\varepsilon^2(v_0^\varepsilon)| \, dx \\
 &+ Ck \int_{\{|b_\varepsilon^2(v^\varepsilon)| > k\}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \leq C_2,
 \end{aligned}$$

we know that properties of $B_k^{i,\varepsilon}(u^\varepsilon)$, $(B_k^{i,\varepsilon}(r^\varepsilon) \geq 0, B_k^{i,\varepsilon}(r^\varepsilon)) \geq \rho(|r| - 1)$, for all $i = \overline{1, 2}$, to obtain

$$\begin{aligned}
 \int_\Omega |B_k^{1,\varepsilon}(u^\varepsilon)(t)| \, dx &+ \int_\Omega |B_k^{2,\varepsilon}(v^\varepsilon)(t)| \, dx \leq k \int_\Omega |b_\varepsilon^1(u^\varepsilon)(t)| \, dx + k \int_\Omega |b_\varepsilon^2(v^\varepsilon)(t)| \, dx \\
 &\leq \rho \left(2meas(\Omega) + k \|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\|_{L^1(Q) \times L^1(Q)} \right). \tag{4.28}
 \end{aligned}$$

From the estimation (4.22), (4.23), (4.27), (4.28) and the properties of $B_k^{i,\varepsilon}$ and $b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon)$, we deduce that

$$b_\varepsilon^1(u^\varepsilon) \text{ and } b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^\infty(]0, T[; L^1(\Omega)), \tag{4.29}$$

$$u^\varepsilon \text{ and } v^\varepsilon \text{ is bounded in } L^\infty(]0, T[; L^1(\Omega)), \tag{4.30}$$

and

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{p^-} (]0, T[; W_0^{1,p(x)}(\Omega)), \tag{4.31}$$

and

$$b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^2 (]0, T[; H_0^1(\Omega)), \tag{4.32}$$

by (4.27), (4.28) and Lemma 2.1 in [7] by and if

$$2 - \frac{1}{N+1} < p(\cdot) < N,$$

to obtain

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{q^-} (]0, T[; W_0^{1,q(x)}(\Omega)), \tag{4.33}$$

for all continuous variable exponents $q \in C(\overline{\Omega})$ satisfying

$$1 \leq q(x) < \frac{N(p(x) - 1) + p(x)}{N + 1},$$

for all $x \in \Omega$.

And

$$T_k(u^\varepsilon) \text{ is bounded in } L^{p^-} (]0, T[; W_0^{1,p(\cdot)}(\Omega)), \tag{4.34}$$

and

$$T_k(v^\varepsilon) \text{ is bounded in } L^2 (]0, T[; H_0^1(\Omega)). \tag{4.35}$$

By (4.25) and (4.26), we may conclude that

$$\gamma(u^\varepsilon) \text{ is bounded in } L^1 (]0, T[; L^1(\Omega)), \tag{4.36}$$

and

$$f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \text{ and } f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \text{ is bounded in } L^1 (]0, T[; L^1(\Omega)), \tag{4.37}$$

independently of ε .

Proceeding as in [3], [4] that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact ($\text{supp } S' \subset [-k, k]$),

$$S(u^\varepsilon) \text{ is bounded in } L^{p^-} (]0, T[; W_0^{1,p(\cdot)}(\Omega)), \tag{4.38}$$

and

$$S(v^\varepsilon) \text{ is bounded in } L^2 (]0, T[; H_0^1(\Omega)), \tag{4.39}$$

and

$$(S(u^\varepsilon))_t \text{ is bounded in } L^1(Q) + L^{(p^-)'} (]0, T[; W^{-1,p'(\cdot)}(\Omega)), \tag{4.40}$$

and

$$(S(v^\varepsilon))_t \text{ is bounded in } L^1(Q) + L^2 (]0, T[; H^{-1}(\Omega)). \tag{4.41}$$

In fact, as a consequence of (4.34), by Stampacchia’s Theorem, we obtain (4.38). To show that (4.40) holds true, we multiply the equation (4.15) by $S'(u^\varepsilon)$ and the equation (4.16) by $S'(v^\varepsilon)$, to obtain

$$\begin{aligned} (B_S^1(u^\varepsilon))_t &= \operatorname{div}(S'(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon)) - \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla (S'(u^\varepsilon)) \\ &\quad - \gamma(u^\varepsilon) S'(u^\varepsilon) + f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{4.42}$$

And

$$\begin{aligned} (B_S^2(v^\varepsilon))_t &= \operatorname{div}(S'(v^\varepsilon) \nabla v^\varepsilon) - \nabla (S'(v^\varepsilon)) \\ &\quad + f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{4.43}$$

Since $\operatorname{supp}(S')$ and $\operatorname{supp}(S'')$ are both included in $[-k; k]$; u^ε may be replaced by $T_k(u^\varepsilon)$ in $\{|u^\varepsilon| \leq k\}$. To have

$$\begin{aligned} &|S'(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon)| \\ &\leq \beta \|S'\|_{L^\infty} \left[L(x, t) + |\nabla T_k(u^\varepsilon)|^{p(x)-1} \right], \end{aligned} \tag{4.44}$$

as a consequence, each term in the right hand side of (4.42) is bounded either in $L^{(p-)' } (]0, T[; W^{-1, p'(\cdot)}(\Omega))$ or in $L^1(Q)$, and obtain (4.40).

Now we look for an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(s) = T_{n+1}(s) - T_n(s) = \begin{cases} 0 & \text{if } |s| \leq n, \\ (|s| - n) \operatorname{sign}(s) & \text{if } n \leq |s| \leq n + 1, \\ \operatorname{sign}(s) & \text{if } |s| \geq n. \end{cases}$$

Remark that $\|\theta_n\|_{L^\infty} \leq 1$ for any $n \geq 1$ and that $\theta_n(s) \rightarrow 0$, for any s when n tends to infinity.

Using the admissible test function $\theta_n(u^\varepsilon)$ in (4.15) leads to

$$\begin{aligned} \int_\Omega \widetilde{\theta}_n(u^\varepsilon)(t) \, dx &+ \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla (\theta_n(u^\varepsilon)) \, dxdt + \int_Q \gamma(u^\varepsilon) \theta_n(u^\varepsilon) \, dxdt \\ &= \int_Q f^\varepsilon(x, t, u^\varepsilon) \theta_n(u^\varepsilon) \, dxdt + \int_\Omega \widetilde{\theta}_n(u_0^\varepsilon) \, dx, \end{aligned} \tag{4.45}$$

where $\widetilde{\theta}_n(r)(t) = \int_0^r \theta_n(s) \frac{\partial b_\varepsilon^i(s)}{\partial s} \, ds$, for all $i = \overline{1, 2}$,

for almost any t in $]0, T[$ and where $\widetilde{\theta}_n(r) = \int_0^r \theta_n(s) \, ds \geq 0$. Hence, dropping a nonnegative term

$$\int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \, dxdt \tag{4.46}$$

$$\begin{aligned} &\leq \int_Q \gamma(u^\varepsilon) \theta_n(u^\varepsilon) dxdt + \int_Q f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \theta_n(u^\varepsilon) dxdt + \int_\Omega \widetilde{\theta}_n(u_0^\varepsilon) dx \\ &\leq \int_{\{|u^\varepsilon| \geq n\}} |\gamma(u^\varepsilon)| dxdt + \int_{\{|u^\varepsilon| \geq n\}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt + \int_{\{|b_\varepsilon^1(u_0^\varepsilon)| \geq n\}} |b_\varepsilon^1(u_0^\varepsilon)| dx. \end{aligned}$$

Similarly, we take test function $\theta_n(v^\varepsilon)$ in (4.16) leads to

$$\int_{\{n \leq |v^\varepsilon| \leq n+1\}} |\nabla v^\varepsilon|^2 dxdt \tag{4.47}$$

$$\begin{aligned} &\leq \int_Q f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \theta_n(v^\varepsilon) dxdt + \int_\Omega \widetilde{\theta}_n(v_0^\varepsilon) dx \leq \int_{\{|v^\varepsilon| \geq n\}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &+ \int_{\{|b_\varepsilon^2(v_0^\varepsilon)| \geq n\}} |b_\varepsilon^2(v_0^\varepsilon)| dx. \end{aligned}$$

Next, we study the convergence of $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ in $C([0, T[; L^1(\Omega))$.

Lemma 4.1. *Both $(u^{\varepsilon_n})_{n \in \mathbb{N}}$ and $(v^{\varepsilon_n})_{n \in \mathbb{N}}$ are Cauchy sequences in $C([0, T[; L^1(\Omega))$.*

Proof. Let ε_n and ε_m two positive integers. It follows from (4.15) and (4.16) that

$$\begin{aligned} &\int_\Omega \frac{\partial b_{\varepsilon_n}^1(u^{\varepsilon_n} - u^{\varepsilon_m})}{\partial t} \varphi dx + \int_0^t \int_\Omega (\mathcal{A}(x, t, \nabla u^{\varepsilon_n}) - \mathcal{A}(x, t, \nabla u^{\varepsilon_m})) \nabla \varphi dxdt \\ &+ \int_0^t \int_\Omega \lambda \left[|u^{\varepsilon_n}|^{p(x)-2} u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2} u^{\varepsilon_m} \right] \phi dxds \\ &= \int_0^t \int_\Omega [f_1^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_1^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] \varphi dxds, \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} &\int_\Omega \frac{\partial b_{\varepsilon_n}^2(v^{\varepsilon_n} - v^{\varepsilon_m})}{\partial t} \phi dx + \int_0^t \int_\Omega (\nabla v^{\varepsilon_n} - \nabla v^{\varepsilon_m}) \nabla \phi dxdt \\ &= \int_0^t \int_\Omega [f_2^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_2^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] \phi dxds, \end{aligned} \tag{4.49}$$

where $\varphi \in L^\infty(]0, T[; W^{1,p(\cdot)}(\Omega))$ and $\phi \in L^2(]0, T[; H_0^1(\Omega))$. To do this fix $\tau \in [0, T]$. Taking $\varphi = \frac{1}{k}T_k(u^{\varepsilon_n} - u^{\varepsilon_m})1_{\{]0, \tau[\}}$ in (4.48) and $\phi = \frac{1}{k}T_k(v^{\varepsilon_n} - v^{\varepsilon_m})1_{\{]0, \tau[\}}$ in (4.49), one gets

$$\begin{aligned} & \frac{1}{k} \int_{\Omega} B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau) - u^{\varepsilon_m}(\tau))dx - \frac{1}{k} \int_{\Omega} B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(0) - u^{\varepsilon_m}(0))dx \\ & + \int_0^\tau \int_{\Omega} \frac{1}{k} (\mathcal{A}(x, t, \nabla u^{\varepsilon_n}) - \mathcal{A}(x, t, \nabla u^{\varepsilon_m})) \nabla T_k(u^{\varepsilon_n} - u^{\varepsilon_m}) dxdt \\ & + \int_0^\tau \int_{\Omega} \frac{\lambda}{k} [|u^{\varepsilon_n}|^{p(x)-2} u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2} u^{\varepsilon_m}] T_k(u^{\varepsilon_n} - u^{\varepsilon_m}) dxds \\ & = \int_0^t \int_{\Omega} \frac{1}{k} [f_1^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] T_k(u^{\varepsilon_n} - u^{\varepsilon_m}) dxds, \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} & \frac{1}{k} \int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau))dx - \frac{1}{k} \int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(0) - v^{\varepsilon_m}(0))dx \\ & + \frac{1}{k} \int_0^t \int_{\Omega} \nabla(v^{\varepsilon_n} - v^{\varepsilon_m}) \nabla T_k(v^{\varepsilon_n} - v^{\varepsilon_m}) dxdt \\ & = \int_0^t \int_{\Omega} \frac{1}{k} [f_2^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] T_k(v^{\varepsilon_n} - v^{\varepsilon_m}) dxds, \end{aligned} \tag{4.51}$$

where

$$B_k^{i,\varepsilon_n}(r) = \int_0^r T_k(s) \frac{\partial b_{\varepsilon_n}^i(s)}{\partial s} ds. \forall i = \overline{1, 2},$$

adding (4.50) and (4.51), we get

$$\begin{aligned} & \frac{1}{k} \int_{\Omega} B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau) - u^{\varepsilon_m}(\tau))dx + \frac{1}{k} \int_{\Omega} B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau))dx \\ & \leq \int_0^\tau \int_{\Omega} \lambda [|u^{\varepsilon_n}|^{p(x)-2} u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2} u^{\varepsilon_m}] dxdt + \\ & \int_0^\tau \int_{\Omega} [f_1^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dxdt + \\ & \int_0^\tau \int_{\Omega} [f_2^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dxdt + \\ & \int_{\Omega} |b_{\varepsilon_n}^1(u_0^{\varepsilon_n} - u_0^{\varepsilon_m})| dx + \int_{\Omega} |b_{\varepsilon_n}^2(v_0^{\varepsilon_n} - v_0^{\varepsilon_m})| dx, \end{aligned}$$

since $B_k^{i,\varepsilon_n}(r) \geq \rho \int_0^r T_k(s) ds \geq \rho(|s| - 1) \cdot \mathcal{N}i = \overline{1, 2}$

$$\begin{aligned} & \int_{\Omega} |u^{\varepsilon_n}(\tau) - u^{\varepsilon_m}(\tau)| dx + \int_{\Omega} |v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau)| dx \\ & \leq 2k \text{ meas}(\Omega) + \int_0^{\tau} \int_{\Omega} k\lambda \left[|u^{\varepsilon_n}|^{p(x)-2} u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2} u^{\varepsilon_m} \right] dx dt \\ & + k \int_0^{\tau} \int_{\Omega} [f_1^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_1^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dx dt \\ & + k \int_0^{\tau} \int_{\Omega} [f_2^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_2^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dx dt \\ & + k \int_{\Omega} |b_{\varepsilon_n}^1(u_0^{\varepsilon_n} - u_0^{\varepsilon_m})| dx + k \int_{\Omega} |b_{\varepsilon_n}^2(v_0^{\varepsilon_n} - v_0^{\varepsilon_m})| dx, \end{aligned}$$

letting $\varepsilon_n, \varepsilon_m \rightarrow \infty$ and then $k \rightarrow 0$, to obtain

$$\begin{aligned} & \sup_{\tau \in [0, T]} \int_{\Omega} |u^{\varepsilon_n}(\tau) - u^{\varepsilon_m}(\tau)| dx + \sup_{\tau \in [0, T]} \int_{\Omega} |v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau)| dx \\ & \leq \int_0^{\tau} \int_{\Omega} k\lambda \left[|u^{\varepsilon_n}|^{p(x)-2} u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2} u^{\varepsilon_m} \right] dx dt \\ & + k \int_0^{\tau} \int_{\Omega} [f_1^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_1^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dx dt \\ & + k \int_0^{\tau} \int_{\Omega} [f_2^{\varepsilon_n}(x, t, u^{\varepsilon_n}, v^{\varepsilon_n}) - f_2^{\varepsilon_m}(x, t, u^{\varepsilon_m}, v^{\varepsilon_m})] dx dt \\ & + k \int_{\Omega} |b_{\varepsilon_n}^1(u_0^{\varepsilon_n} - u_0^{\varepsilon_m})| dx + k \int_{\Omega} |b_{\varepsilon_n}^2(v_0^{\varepsilon_n} - v_0^{\varepsilon_m})| dx. \end{aligned}$$

□

- **Step 2: The limit of the solution of the approximated problem.** Arguing again as in [3], [4], [5] estimates (4.38), (4.40), (4.39) and (4.41) imply that, for a subsequence still indexed by ε ,

$$(u^\varepsilon, v^\varepsilon) \text{ converge almost every where to } (u, v), \tag{4.52}$$

using (4.15), (4.34), (4.35) and (4.44), to get

$$T_k(u^\varepsilon) \text{ converge weakly to } T_k(u) \text{ in } L^{p^-} \left(]0, T[; W_0^{1,p(\cdot)}(\Omega) \right), \tag{4.53}$$

and

$$T_k(v^\varepsilon) \text{ converge weakly to } T_k(v) \text{ in } L^2 \left(]0, T[; H_0^1(\Omega) \right), \tag{4.54}$$

$$\chi_{\{|u^\varepsilon| \leq k\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(\cdot)}(Q) \right)^N, \tag{4.55}$$

as ε tends to 0 for any $k > 0$ and any $n \geq 1$ and where for any $k > 0$, η_k belongs to $\left(L^{p'(\cdot)}(Q) \right)^N$. Since $\gamma(u^\varepsilon)$ is a continuous increasing function, from the monotone convergence theorem and (4.25) and by (4.52), to obtain that

$$\gamma(u^\varepsilon) \text{ converge weakly to } \gamma(u) \text{ in } L^1(Q). \tag{4.56}$$

We now establish that $(b_1(u), b_2(v))$ belongs to $(L^\infty(]0, T[; L^1(\Omega)))^2$. Indeed using (4.20) and $|B_k^{i,\varepsilon}(s)| \geq \rho(|s| - 1), \forall i = \overline{1, 2}$, leads to

$$\begin{aligned} \int_{\Omega} |b_\varepsilon^1(u^\varepsilon)|(t) dx + \int_{\Omega} |b_\varepsilon^2(v^\varepsilon)|(t) dx &\leq \rho(2\text{meas}(\Omega)) \\ &+ \|(f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon), f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon))\|_{(L^1(Q))^2} \\ &+ k \|\gamma(u^\varepsilon)\|_{L^1(Q)} \\ &+ k \|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\|_{(L^1(\Omega))^2}. \end{aligned}$$

By lemma (4.1) and (4.46), (4.47), we conclude that there exist two subsequences of u^{ε_n} and v^{ε_n} , still denoted by themselves for convenience, such that u^{ε_n} converges to a function u in $C(]0, T[; L^1(\Omega))$, v^{ε_n} converges to a function v in $C(]0, T[; L^1(\Omega))$. Using (4.25) and (4.10), (4.26), we have $(b_1(u), b_2(v))$ belongs to $(L^\infty(]0, T[; L^1(\Omega)))^2$. We are now in a position to exploit (4.46) and (4.47). Since $(u^\varepsilon, v^\varepsilon)$ is bounded in $(L^\infty(]0, T[; L^1(\Omega)))^2$, to get

$$\lim_{n \rightarrow +\infty} \left(\sup_\varepsilon \text{meas} \{|u^\varepsilon| \geq n\} \right) = 0. \tag{4.57}$$

and

$$\lim_{n \rightarrow +\infty} \left(\sup_\varepsilon \text{meas} \{|v^\varepsilon| \geq n\} \right) = 0. \tag{4.58}$$

The equi-integrability of the sequence $f_i^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)$ in $(L^1(Q))^2$. We shall now prove that $f_i^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)$ converges to $f_i(x, t, u, v)$ strongly in $(L^1(Q))^2$, for all $i = \overline{1, 2}$ by using Vitali's theorem. Since $f_i^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \rightarrow f_i(x, t, u, v)$ a.e in Q it suffices to prove that $f_i^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)$ are equi-integrable in Q . Let $\delta_1 > 0$ and \mathbf{A} be a measurable subset belonging to $\Omega \times]0, T[$, we define the following sets

$$G_{\delta_1} = \{(x, t) \in Q : |u_n| \leq \delta_1\}, \tag{4.59}$$

$$F_{\delta_1} = \{(x, t) \in Q : |u_n| > \delta_1\}. \tag{4.60}$$

Using the generalized Hölder's inequality and Poincaré inequality, to have

$$\int_{\mathbf{A}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dx dt = \int_{\mathbf{A} \cap G_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dx dt + \int_{\mathbf{A} \cap F_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dx dt,$$

therefore

$$\begin{aligned}
 \int_{\mathbf{A}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt &\leq \int_{\mathbf{A} \cap G_{\delta_1}} \left(c_{k,\varepsilon}(x, t) + \sigma_{k,\varepsilon} |v^\varepsilon|^2 \right) \, dxdt \\
 &+ \int_{\mathbf{A} \cap F_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq \int_{\mathbf{A}} c_{k,\varepsilon}(x, t) \, dxdt + \sigma_{k,\varepsilon} \int_Q |\nabla T_{\delta_1}(v^\varepsilon)|^2 \, dxdt \\
 &+ \int_{\mathbf{A} \cap F_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq \int_{\mathbf{A}} c_{k,\varepsilon}(x, t) \, dxdt + \sigma_{k,\varepsilon} (\text{meas}(\mathbf{Q}) + 1)^{\frac{1}{2}} \\
 &\quad \left(\int_{Q_T} |\nabla T_{\delta_1}(v^\varepsilon)|^2 \, dxdt \right)^{\frac{1}{2}} + \int_{\mathbf{A} \cap F_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq K_1 + C_2 \left(\frac{k}{\alpha} \|b_\varepsilon^2(v_0^\varepsilon)\|_{L^1(\Omega)} \right)^{\frac{1}{2}} \\
 &+ \int_{\mathbf{A} \cap F_{\delta_1}} \frac{1}{|u^\varepsilon|} |u^\varepsilon f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq K_2 + \int_{\mathbf{A} \cap F_{\delta_1}} \frac{1}{\delta_1} |u^\varepsilon f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| \, dxdt \\
 &\leq K_2 + \frac{1}{\delta_1} \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \left(\int_{\mathbf{A} \cap F_{\delta_1}} |u^\varepsilon|^{p(x)} \, dxdt \right)^{\frac{1}{p^-}} \\
 &\quad \left(\int_{\mathbf{A} \cap F_{\delta_1}} |f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)|^{p'(x)(p(x)-1)} \, dxdt \right)^{\frac{1}{p'^-}} \\
 &\rightarrow 0 \text{ when } \text{meas}(\mathbf{A}) \rightarrow 0.
 \end{aligned}$$

Which shows that $f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)$ is equi-integrable. By using Vitali's theorem, to get

$$f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \rightarrow f_1(x, t, u, v) \text{ strongly in } L^1(Q). \tag{4.61}$$

Now we prove that

$$f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \rightarrow f_2(x, t, u, v) \text{ strongly in } L^1(Q). \tag{4.62}$$

Let $\delta_2 > 0$ and \mathbf{A} be a measurable subset belonging to $\Omega \times]0, T[$, we define the following sets

$$G_{\delta_2} = \{(x, t) \in Q : |v_n| \leq \delta_2\}, \tag{4.63}$$

$$F_{\delta_2} = \{(x, t) \in Q : |v_n| > \delta_2\}. \tag{4.64}$$

Using the generalized Hölder’s inequality and Poincaré inequality, to get

$$\int_{\mathbf{A}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt = \int_{\mathbf{A} \cap G_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt + \int_{\mathbf{A} \cap F_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt,$$

therefore

$$\begin{aligned} \int_{\mathbf{A}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt &\leq \int_{\mathbf{A} \cap G_{\delta_2}} \left(G_k^\varepsilon(x, t) + \xi_k^\varepsilon |u^\varepsilon|^{p(x)-1} \right) dxdt \\ &+ \int_{\mathbf{A} \cap F_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &\leq \int_{\mathbf{A}} G_k^\varepsilon(x, t) dxdt + \xi_k^\varepsilon \int_Q |\nabla T_{\delta_2}(u^\varepsilon)|^{p(x)-1} dxdt \\ &+ \int_{\mathbf{A} \cap F_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &\leq \int_{\mathbf{A}} G_k^\varepsilon(x, t) dxdt + \xi_k^\varepsilon \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) (\text{meas}(\mathbf{Q}) + 1)^{\frac{1}{p^-}} \\ &\quad \left(\int_{Q_T} |\nabla T_{\delta_2}(u^\varepsilon)|^{(p(x)-1)p'(x)} dxdt \right)^{\frac{1}{p'^-}} \\ &+ \int_{\mathbf{A} \cap F_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &\leq K_3 + C_4 \left(\frac{k}{\alpha} \|b_\varepsilon^1(u_0^\varepsilon)\|_{L^1(\Omega)} \right)^{\frac{1}{2}} \\ &+ \int_{\mathbf{A} \cap F_{\delta_2}} \frac{1}{|v^\varepsilon|} |v^\varepsilon f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &\leq K_4 + \int_{\mathbf{A} \cap F_{\delta_2}} \frac{1}{\delta_2} |v^\varepsilon f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)| dxdt \\ &\leq K_4 + \frac{1}{\delta_2} \left(\int_{\mathbf{A} \cap F_{\delta_2}} |v^\varepsilon|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{\mathbf{A} \cap F_{\delta_2}} |f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)|^2 dxdt \right)^{\frac{1}{2}} \\ &\rightarrow 0 \text{ when } \text{meas}(\mathbf{A}) \rightarrow 0. \end{aligned}$$

Which shows that $f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)$ is equi-integrable. By using Vitali’s theorem, to get

$$f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) \rightarrow f_2(x, t, u, v) \text{ strongly in } L^1(Q). \tag{4.65}$$

Using (4.56), (4.61) and the equi-integrability of the sequence $|b_\varepsilon^1(u_\varepsilon^0)|$ in $L^1(\Omega)$ and $|b_\varepsilon^2(v_\varepsilon^0)|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n \rightarrow +\infty} \left(\sup_\varepsilon \left(\int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon dxdt + \int_{\{n \leq |v^\varepsilon| \leq n+1\}} |\nabla v^\varepsilon|^2 dxdt \right) \right) = 0. \tag{4.66}$$

• **Step 4: Strong convergence.** The specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence in $L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \leq k, \forall \mu > 0$, and $v_0^\mu \rightarrow T_k(u_0)$ a.e in Ω with $\frac{1}{\mu} \|v_0^\mu\|_{L^{p(\cdot)}(\Omega)} \rightarrow 0$ as $\mu \rightarrow +\infty$.

For fixed $k \geq 0$ and $\mu > 0$, let us consider the unique solution $T_k(u)_\mu \in L^\infty(\Omega) \cap L^{p^-}([0, T]; W_0^{1,p(\cdot)}(\Omega))$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t} + \mu (T_k(u)_\mu - T_k(u)) = 0 \text{ in } \mathcal{D}'(Q), \tag{4.67}$$

$$T_k(u)_\mu(t = 0) = v_0^\mu. \tag{4.68}$$

The behavior of $T_k(u)_\mu$ as $\mu \rightarrow +\infty$ is investigated in [9] and we just recall here that (4.67)-(4.68) imply that

$$T_k(u)_\mu \rightarrow T_k(u) \text{ strongly in } L^{p^-}([0, T]; W_0^{1,p(\cdot)}(\Omega)) \text{ a.e in } Q, \text{ as } \mu \rightarrow +\infty, \tag{4.69}$$

with $\|T_k(u)_\mu\|_{L^\infty(\Omega)} \leq k$, for any μ , and $\frac{\partial T_k(u)_\mu}{\partial t} \in L^{(p^-)' }([0, T]; W^{-1,p'(\cdot)}(\Omega))$.

The main estimate is the following

Lemma 4.2. *Let S be an increasing $C^\infty(\mathbb{R})$ – function such that $S(r) = r$ for $r \leq k$, and $\text{supp} S'$ is compact. Then*

$$\liminf_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{\partial B_S^1(u^\varepsilon)}{\partial t}, (T_k(u^\varepsilon)_\mu - T_k(u)) \right\rangle dt \geq 0,$$

where here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$, and where $B_S^1(z) = \int_0^z b_1(r) S'(r) dr$.

Proof. See [5], Lemma 1. □

Now we are to prove that the weak limit η_k and we prove the weak L^1 convergence of the "truncated" energy $\mathcal{A}(x, t, \nabla T_k(u^\varepsilon))$ as ε tends to 0. In order to show this result we recall the lemma below.

Lemma 4.3. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon) dxdt \leq \int_Q \eta_k \nabla T_k(u) dxdt, \tag{4.70}$$

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_Q \left[\mathcal{A} \left(x, t, \nabla u_{\chi_{\{|u^\varepsilon| \leq k\}}}^\varepsilon \right) - \mathcal{A} \left(x, t, \nabla u_{\chi_{\{|u| \leq k\}}} \right) \right] \\ & \qquad \qquad \qquad \times \left[\nabla u_{\chi_{\{|u^\varepsilon| \leq k\}}}^\varepsilon - \nabla u_{\chi_{\{|u| \leq k\}}} \right] dxdt = 0 \end{aligned} \tag{4.71}$$

$\eta_k = \mathcal{A} \left(x, t, \nabla u_{\chi_{\{|u| \leq k\}}} \right)$ a.e in Q , for any $k \geq 0$, as ε tends to 0.

$$\mathcal{A} \left(x, t, \nabla u^\varepsilon \right) \nabla T_k(u^\varepsilon) \rightarrow \mathcal{A} \left(x, t, \nabla u \right) \nabla T_k(u) \text{ weakly in } L^1(Q). \tag{4.72}$$

Proof. Let us introduce a sequence of increasing $C^\infty(\mathbb{R})$ -functions S_n such that, for any $n \geq 1$

$$\begin{cases} S_n(r) = r \text{ if } |r| \leq n, \\ \text{supp}(S'_n) \subset [-(n+1), (n+1)], \\ \|S''_n\|_{L^\infty(\mathbb{R})} \leq 1. \end{cases} \tag{4.73}$$

For fixed $k \geq 0$, we consider the test function $S'_n(u^\varepsilon) \left(T_k(u_\varepsilon) - (T_k(u))_\mu \right)$ in (4.15), we use the definition (4.73) of S'_n and we define $W_\mu^\varepsilon = T_k(u_\varepsilon) - (T_k(u))_\mu$, to get

$$\begin{aligned} & \int_0^T \langle (B_S^1(u^\varepsilon))_t, W_\mu^\varepsilon \rangle dt + \int_Q S'_n(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla W_\mu^\varepsilon dxdt \\ & + \int_Q S''_n(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon W_\mu^\varepsilon dxdt + \int_Q \gamma(u^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt \\ & = \int_Q f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt. \end{aligned} \tag{4.74}$$

Now we pass to the limit in (4.74) as $\varepsilon \rightarrow 0$, $\mu \rightarrow +\infty$, $n \rightarrow +\infty$ for k real number fixed. In order to perform this task, we prove below the following results for any $k \geq 0$:

$$\liminf_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_0^T \langle (B_S^1(u^\varepsilon))_t, W_\mu^\varepsilon \rangle dt \geq 0 \text{ for any } n \geq k, \tag{4.75}$$

$$\lim_{n \rightarrow +\infty} \lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q S''_n(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon W_\mu^\varepsilon dxdt = 0, \tag{4.76}$$

$$\lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q \gamma(u^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{4.77}$$

$$\lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1. \tag{4.78}$$

Proof of (4.75). In view of the definition W_μ^ε , we apply lemma (4.2) with $S = S_n$ for fixed $n \geq k$. As a consequence, (4.75) hold true. □

Proof of (4.76). For any $n \geq 1$ fixed, we have $\text{supp}(S''_n) \subset [-(n+1), -n] \cup [n, n+1]$, $\|W_\mu^\varepsilon\|_{L^\infty(Q)} \leq 2k$ and $\|S''_n\|_{L^\infty(\mathbb{R})} \leq 1$, to get

$$\begin{aligned} & \left| \int_Q S''_n(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon W_\mu^\varepsilon dxdt \right| \\ & \leq 2k \int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon dxdt, \end{aligned} \tag{4.79}$$

for any $n \geq 1$, by (4.66) it possible to establish (4.76) □

Proof of (4.77). For fixed $n \geq 1$ and in view (4.56) . Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon \rightarrow 0} \int_Q \gamma(u^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt = \int_Q \gamma(u) S'_n(u) (T_k(u) - T_k(u)_\mu) dxdt. \tag{4.80}$$

Appealing now to (4.69) and passing to the limit as $\mu \rightarrow +\infty$ in (4.80) allows to conclude that (4.77) holds true. □

Proof of (4.78). By (4.11), (4.61) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$, it is possible to pass to the limit for $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} \int_Q f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon dxdt = \int_Q f_1(x, t, u, v) S'_n(u) (T_k(u) - T_k(u)_\mu) dxdt,$$

using (4.69) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (4.78). □

Now turn back to the proof of Lemma (4.3), due to (4.75)-(4.78), we are in a position to pass to the limit-sup when $\varepsilon \rightarrow 0$, then to the limit-sup when $\mu \rightarrow +\infty$ and then to the limit as $n \rightarrow +\infty$ in (4.74). Using the definition of W_μ^ε , we deduce that for any $k \geq 0$,

$$\lim_{n \rightarrow +\infty} \limsup_{\mu \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) S'_n(u^\varepsilon) \nabla (T_k(u^\varepsilon) - T_k(u)_\mu) dxdt \leq 0.$$

Since $\mathcal{A}(x, t, \nabla u^\varepsilon) S'_n(u^\varepsilon) \nabla T_k(u^\varepsilon) = \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon)$ for $k \leq n$, the above inequality implies that for $k \leq n$,

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon) dxdt \\ & \leq \lim_{n \rightarrow +\infty} \limsup_{\mu \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(t, x, \nabla u^\varepsilon) S'_n(u^\varepsilon) \nabla T_k(u)_\mu dxdt. \end{aligned} \tag{4.81}$$

Due to (4.55), to have

$$\mathcal{A}(x, t, \nabla u^\varepsilon) S'_n(u^\varepsilon) \rightarrow \eta_{n+1} S'_n(u) \text{ weakly in } \left(L^{p'(\cdot)}(Q) \right)^N \text{ as } \varepsilon \rightarrow 0,$$

and the strong convergence of $T_k(u)_\mu$ to $T_k(u)$ in $L^{p^-}([0, T]; W_0^{1,p}(\Omega))$ as $\mu \rightarrow +\infty$, to get

$$\begin{aligned} & \lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) S'_n(u^\varepsilon) \nabla T_k(u)_\mu dxdt \\ &= \int_Q S'_n(u) \eta_{n+1} \nabla T_k(u) dxdt = \int_Q \eta_{n+1} \nabla T_k(u) dxdt, \end{aligned} \tag{4.82}$$

as soon as $k \leq n$, since $S'_n(s) = 1$ for $|s| \leq n$. Now, for $k \leq n$, to have

$$S'_n(u^\varepsilon) \mathcal{A}(x, t, \nabla u^\varepsilon) \chi_{\{|u^\varepsilon| \leq k\}} = \mathcal{A}(x, t, \nabla u^\varepsilon) \chi_{\{|u^\varepsilon| \leq k\}} \text{ a.e in } Q.$$

Letting $\varepsilon \rightarrow 0$, to obtain

$$\eta_{n+1} \chi_{\{|u| \leq k\}} = \eta_k \chi_{\{|u| \leq k\}} \text{ a.e in } Q - \{|u| = k\} \text{ for } k \leq n.$$

Recalling (4.81) and (4.82) allows to conclude that (4.70) holds true. □

Proof of (4.71). Let $k \geq 0$ be fixed. We use the monotone character (3.3) of $\mathcal{A}(x, t, \xi)$ with respect to ξ , to obtain

$$I^\varepsilon = \int_Q (\mathcal{A}(x, t, \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}}) - \mathcal{A}(x, t, \nabla u \chi_{\{|u| \leq k\}})) (\nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} - \nabla u \chi_{\{|u| \leq k\}}) dxdt \geq 0. \tag{4.83}$$

Inequality (4.83) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$\begin{aligned} I_1^\varepsilon &= \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}}) \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} dxdt, \\ I_2^\varepsilon &= - \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}}) \nabla u \chi_{\{|u| \leq k\}} dxdt, \\ I_3^\varepsilon &= - \int_Q \mathcal{A}(x, t, \nabla u \chi_{\{|u| \leq k\}}) (\nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} - \nabla u \chi_{\{|u| \leq k\}}) dxdt. \end{aligned}$$

We pass to the limit-sup as $\varepsilon \rightarrow 0$ in I_1^ε , I_2^ε and I_3^ε . Let us remark that we have $u^\varepsilon = T_k(u^\varepsilon)$ and $\nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} = \nabla T_k(u^\varepsilon)$ a.e in Q , and we can assume that k is such that $\chi_{\{|u^\varepsilon| \leq k\}}$ almost everywhere converges to $\chi_{\{|u| \leq k\}}$ (in fact this is true for almost every k , see Lemma 3.2 in [6]). Using (4.70), to obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_1^\varepsilon &= \lim_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon) dxdt \\ &\leq \int_Q \eta_k \nabla T_k(u) dxdt. \end{aligned} \tag{4.84}$$

In view of (4.53) and (4.55), to have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_2^\varepsilon &= -\lim_{\varepsilon \rightarrow 0} \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}}) (\nabla T_k(u)) \, dxdt \\ &= -\int_Q \eta_k (\nabla T_k(u)) \, dxdt. \end{aligned} \tag{4.85}$$

As a consequence of (4.53), we have for all $k > 0$

$$\lim_{\varepsilon \rightarrow 0} I_3^\varepsilon = -\int_Q \mathcal{A}(x, t, \nabla u \chi_{\{|u| \leq k\}}) (\nabla T_k(u^\varepsilon) - \nabla T_k(u)) \, dxdt = 0. \tag{4.86}$$

Taking the limit-sup as $\varepsilon \rightarrow 0$ in (4.83) and using (4.84), (4.85) and (4.86) show that (4.71) holds true. □

Proof of (4.72). Using (4.71) and the usual Minty argument applies it follows that (4.72) holds true.

Lemma 4.4. $\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$.

Proof. Denote $V_\mu^\varepsilon = T_k(v_\varepsilon) - (T_k(v))_\mu$ and choose $S'_n(v^\varepsilon) \left(T_k(v_\varepsilon) - (T_k(v))_\mu \right)$ the test function in (4.16). One can get that

$$\begin{aligned} &\int_0^T \langle (B_S^2(v^\varepsilon))_t, V_\mu^\varepsilon \rangle \, dt + \int_Q S'_n(v^\varepsilon) \nabla v^\varepsilon \nabla V_\mu^\varepsilon \, dxdt \\ &+ \int_Q S''_n(v^\varepsilon) |\nabla v^\varepsilon|^2 V_\mu^\varepsilon \, dxdt = \int_Q f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'_n(v^\varepsilon) V_\mu^\varepsilon \, dxdt. \end{aligned} \tag{4.87}$$

By a similar discussion, one has

$$\liminf_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_0^T \langle (B_S^2(v^\varepsilon))_t, V_\mu^\varepsilon \rangle \, dt \geq 0 \text{ for any } n \geq k, \tag{4.88}$$

$$\lim_{n \rightarrow +\infty} \lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q S''_n(v^\varepsilon) |\nabla v^\varepsilon|^2 V_\mu^\varepsilon \, dxdt = 0, \tag{4.89}$$

and

$$\lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon) S'_n(v^\varepsilon) V_\mu^\varepsilon \, dxdt = 0, \text{ for any } n \geq 1. \tag{4.90}$$

Hence

$$\lim_{n \rightarrow +\infty} \lim_{\mu \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_Q S'_n(v^\varepsilon) \nabla v^\varepsilon \nabla V_\mu^\varepsilon \, dxdt \leq 0. \tag{4.91}$$

□

Similarly, one gets that $\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$. □

• **Step 5:** In this step we prove that (u, v) satisfies (4.3), (4.4)-(4.7) . For any fixed $n \geq 0$ one has

$$\begin{aligned} & \int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon dxdt \\ = & \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_{n+1}(u^\varepsilon) dxdt - \int_Q \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla T_n(u^\varepsilon) dxdt. \end{aligned}$$

According to (4.55) and (4.72) one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 1$ and to obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon dxdt \tag{4.92} \\ = & \int_Q \mathcal{A}(x, t, \nabla u) \nabla T_{n+1}(u) dxdt - \int_Q \mathcal{A}(x, t, \nabla u) \nabla T_n(u) dxdt \\ = & \int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u) \nabla u dxdt. \end{aligned}$$

Letting n tends to $+\infty$ in (4.92), it follows from estimate (4.66), that

$$\limlim_{\varepsilon \rightarrow 0} \int_{\{n \leq |u^\varepsilon| \leq n+1\}} \mathcal{A}(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon dxdt = 0.$$

Similarly, one can prove

$$\limlim_{\varepsilon \rightarrow 0} \int_{\{n \leq |v^\varepsilon| \leq n+1\}} |\nabla v^\varepsilon|^2 dxdt = 0.$$

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\text{supp}(S') \subset [-k, k]$. Pontwise multiplication of that approximate equation (4.15) by $(S'(u^\varepsilon), S'(v^\varepsilon))$ leads to

$$\begin{aligned} & (B_S^1(u^\varepsilon))_t - \text{div}(S'(u^\varepsilon)\mathcal{A}(x, t, \nabla u^\varepsilon)) \tag{4.93} \\ & + S''(u^\varepsilon)\mathcal{A}(x, t, \nabla u^\varepsilon)\nabla(u^\varepsilon) + \gamma(u^\varepsilon)S'(u^\varepsilon) = f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q), \end{aligned}$$

and

$$\begin{aligned} & (B_S^2(v^\varepsilon))_t - \text{div}(S'(v^\varepsilon)\nabla v^\varepsilon) \tag{4.94} \\ & + S''(v^\varepsilon)|\nabla(v^\varepsilon)|^2 = f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q). \end{aligned}$$

In what follows to pass to the limit as ε tends to 0 in each term of (4.93). Since S is bounded, and $(S(u^\varepsilon), S(v^\varepsilon))$ converges to $(S(u), S(v))$ a.e in Q and in $(L^\infty(Q))^2$ *-weak, then

$((B_S^1(u^\varepsilon))_t, (B_S^2(v^\varepsilon))_t)$ converges to $((B_S^1(u))_t, (B_S^1(v))_t)$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\text{supp}(S') \subset [-k, k]$,

$$S'(u^\varepsilon)\mathcal{A}(t, x, \nabla u^\varepsilon) = S'(u^\varepsilon)\mathcal{A}(x, t, \nabla u^\varepsilon)\chi_{\{|u^\varepsilon| \leq k\}} \text{ a.e in } Q.$$

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (4.72) of Lemma(4.3) imply that $S'(u^\varepsilon)\mathcal{A}(x, t, \nabla u^\varepsilon)$ converges to $S'(u)\mathcal{A}(x, t, \nabla u)$ weakly in $(L^{p'(\cdot)}(Q))^N$ as ε tends to 0, because $S'(u) = 0$ for $|u| \geq k$ a.e in Q and $S'(v^\varepsilon)\nabla v^\varepsilon$ converges to $S'(v)\nabla v$ weakly in $L^2(Q)$ as ε tends to 0. The pointwise convergence of u^ε to u , the bounded character of S' , S'' and (4.72) of Lemma (4.3) allow to conclude that

$$S''(u^\varepsilon)\mathcal{A}(x, t, \nabla u^\varepsilon)\nabla T_k(u^\varepsilon) \rightarrow S''(u)\mathcal{A}(x, t, \nabla u)\nabla T_k(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \rightarrow 0$, and lemma (4.1) shows that

$$S''(v^\varepsilon)\nabla^\varepsilon v \nabla T_k(v^\varepsilon) \rightarrow S''(v)\nabla v \nabla T_k(v) \text{ weakly in } L^1(Q).$$

The use of (4.56) to obtain that $\gamma(u^\varepsilon)S'(u^\varepsilon)$ converges to $\gamma(u)S'(u)$ in $L^1(Q)$, and we use (4.11), (4.53) and we obtain that

$$f_1^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)S'(u^\varepsilon) \text{ converges to } f_1(x, t, u, v)S'(u) \text{ in } L^1(Q)$$

and

$$f_2^\varepsilon(x, t, u^\varepsilon, v^\varepsilon)S'(v^\varepsilon) \text{ converges to } f_2(x, t, u, v)S'(v) \text{ in } L^1(Q).$$

As a consequence of the above convergence result, the position to pass to the limit as ε tends to 0 in equation (4.93) and (4.94), we conclude that (u, v) satisfies (4.4) and (4.5).

It remains to show that $S(u)$ satisfies the initial condition (4.6) and $S(v)$ satisfies the initial condition (4.7). To this end, firstly remark that, S being bounded, $(S(u^\varepsilon), S(v^\varepsilon))$ is bounded in $(L^\infty(Q))^2$, $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ is bounded in $L^\infty(Q) \times L^\infty(Q)$. Secondly, (4.93) and (4.94), the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^1(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p^-)' }([0, T[; W^{-1, p'(\cdot)}(\Omega))$ and $\frac{\partial B_S^2(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^2([0, T[; H_0^1(\Omega))$. As a consequence, an Aubin's type lemma ([20], Corollary 4) implies that $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ lies in a compact set of $(C([0, T[; L^1(\Omega)))^2$. It follows that, on the one hand, $B_S^1(u^\varepsilon)(t = 0)$ converges to $B_S^1(u)(t = 0)$ strongly in $L^1(\Omega)$ and $B_S^2(v^\varepsilon)(t = 0)$ converges to $B_S^2(v)(t = 0)$ strongly in $L^1(\Omega)$. Due to (4.10), to conclude that (4.6) and (4.7) holds true. As a conclusion of **Step 3** and **Step 5**, the proof of Theorem (4.1) is complete.

□

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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