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Mémoire

Présenté en vue de l'obtention du diplôme de Master en Mathématiques

Existence de solutions positives périodiques pour un système non linéaire d'équations de Lotka-Volterra avec des retards distribués et impulsions

Option : AFA

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CONTENTS

Abstract in English

Fixed point theory has a long history of being used in nonlinear differential equations, in order to prove the existence, uniqueness, or other qualitative properties of solutions. However, using the fixed point theorem for stability and periodicity of solutions have a more recent appearance. In this dissertation, we focus on studying the quantitative and qualitative properties of some nonlinear delay differential equations. We start by giving some fixed point theorems. After we introduce and provide results for delay differential equations and necessary relevant definitions and notions about impulsive differential equations. Finally, we derive sufficient conditions ensuring at least one positive periodic solution of the system, more precisely by using the Krasnoselskii fixed-point theorem in the cone of Banach space. We address the existence of a positive periodic solution for n- species Lotka-Volterra competition system with variable delays and impulses. An example is also given to illustrate the claims established.

Keywords: Fixed points theorem; Krasnoselskii's theorem; Periodicity; Positivity; Delay equations; Lotka-Volterra system; Integral equation; Variable delays, impulses.

Abstract in French

La théorie du point fixe est utilisée depuis longtemps dans les équations différentielles non linéaires, afin de prouver l'existence, l'unicité ou d'autres propriétés qualitatives. Cependant, l'utilisation du théorème du point fixe pour la stabilité et la périodicité des solutions a une apparition plus récente. Dans ce mémoire, nous nous concentrons sur l'étude des propriétés quantitatives et qualitatives de certains types d'équations différentielles de retard non linéaires. Nous commençons par donner quelques théorèmes de point fixe. Ensuite, nous donnons des notions préliminaires sur les équations différentielles à retard et les équations impulsives qui sont nécessaires par la suite. A la fin, en utilisant le théorème du point fixe de la théorie du cône nous étudions l'existence d'une solution périodique positive pour le système de compétition Lotka-Volterra a n-espèces avec de multiples arguments déviants. Finalement, un exemple est également donné pour illustrer l'efficacité des résultats établies.

Mots clés: Points fixe; Theorème de Krasnoselskii; Solutions périodiques positives; Equations à retard; Système de Lotka-Volterra; Equation intégrale; Retards,impulsive

نظرية النقطة الثابتة تستعمل مند سنوات لدراسة المعادلات التفاضلية غير الخطية، لإثبات الوجود و الوحدانية و خواص أخرى نوعية وكمية. لكن استعمالها لإثبات الاستقرار والاستقرار المقارب والحلول الدورية الموجبة للمعادلات ذات تأخر تعتبر حديثة الظهور. في مذكرتنا ركزنا مع الخواص النوعية و الكمية لبعض أنواع المعادلات التفاضلية غير الخطية ذات تأخر. بدأنا ببعض نظريات النقطة الثابتة، بعدها أساسيات حول المعادلات التفاضلية المتأخرة و المعادلات المندفعة اللازمة في عملنا و أخيرا و باستعمال نظرية النقطة الثابتة لكراسنوسكي المعرفة على مخروط من فضاء بناخ درسنا وجود حلول دورية موجبة لجملة معادلات غير خطية نبضية للوطكا فولطرا ذات تأخرات متغيرة بدلالة الزمن وتأخرات توزيعية .في النهاية قدمنا مثال توضيحي يؤكد نجاعة النتائج المدروسة .

الكلمات المفتاحية: نظرية النقطة الثابتة، نظرية كراسنوسكي، حلول دورية موجبة، معادلة متأخرة، نظام لوطكا فولطرا، معا دلة نبضية,معادلة تكامل، تأخر .

Dedication

I dedicate this humble work

*My dear father H*ocine *my dear mother Fatiha*

My Allah protect them

My brother Nejm Edin

My sisters Chaima, Safa and Marwa

My aunts Mahdia and Samia especially

The bird, Abed Eldjalil

My beautiful friends Imene,Doua,Sabrina,Dounia ,

Lamia,Ghada,Fairouz ,Halima and Asma

The two families Lamamra and Bounama

The companions of my journey who shared

The beautiful moments with me my Allah bless them

Everyone who had an impact on my life.

Lamamra Rania

Dedication

I dedicate this humble work

To my family, especially for

My dear parents, my mother

And father for their patience, love,

Support,encouragement and interest.

To my sisters, my brothers, and

My friends. Without forgetting

All the teachers.Thank you

For everything

Mokhbi kaltoum

We dedicate this work to all our loved ones.

Thank you our parents for believing in our abilities.

Thank you our teachers for all the support you have provided at the end.

Thank you our families, all our friends, for everything

Keltoum and Rania

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Acronyms

Abbreviation	**Meaning**
ODEs	Ordinary Differential Equations
FDEs	Functional Differential Equations
DDEs	Delay Differential Equations
NDDEs	Neutral Delay Differential Equations

List of Symbol

Here we state some conventions regarding mathematical notation that we will use in this disserttion.

$\mathbb{R}^n$	Euclidean space of n- dimensions;
$\mathbb{R}^+ = [0, \infty)$	set of positive real numbers;
(X, d)	this is called metric space;
(a, b)	open interval from a to b;
$\mathbb{N}$	set of natural numbers;
$\dfrac{d}{dt}$	first derivative with respect to t;
$[a, b]$	closed interval from a to b;
$\mathbb{C}\left([a, b], \mathbb{R}\right)$	space of continuous functions mapping from the interval $[a, b]$ to $\mathbb{R}$;
ω	period;

$\mathbb{C}([-\tau, 0]; \mathbb{R}^n)$	the space of all continuous $\mathbb{R}^n -$ valued functions φ defined on $[-\tau, 0]$; with a norm $\lVert\varphi\rVert = \sup_{-\tau\leq\theta\leq 0} \lvert\varphi(\theta)\rvert$;
$f : A \to B$	the mapping f from A to B;
$\sigma(C)$	the $\sigma -$ algebra generated by C.
$d(x, y)$	this is called the distance between x and y;

$\|\cdot\|_\infty$	uniform norm;
$\mathbb{C}_\omega$	set of periodic functions;
$\lvert x\rvert$	the Euclidean norm of a vector x;
$\sum_{i=0}^{n}$	summation from index $i = 0$ to $i = n$.

Other notations will be explained when they first appear.

Introduction

The fixed point theory is at the heart of the nonlinear analysis then and it provides the necessary tools to have theorems existence in many different non-linear problems. It uses its tools of analysis and topology. Advancements in fixed point theory enrich many scientific fields such as biology, chemistry, computer science, economics, and game theory. Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory, (see [2], [61], [67]).

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known as the contraction mapping theorem or the contraction mapping principle). It was first started by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space and provides a constructive method to find those fixed points.

Concerning the topological branch, results are obtained using the topological properties of set X. The main result is Schauder's fixed point theorem which was stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem.

In 1955, Krasnoselskii studied a paper by Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated a hybrid theorem known under its name. The reader is referred to the classical textbook on fixed points [62]. This is a captivating result and it has a number of interesting applications. The proof of this result combines the Banach contraction principle and Schauder fixed point theorem and thus it is a blend of the two branches.

Delay differential equations are a class of differential equations where the derivatives at the current time depend on the solution at previous times. Strictly speaking, delay differential equations are a specific example of functional differential equations, in which the functional part of the differential equations is the evaluation of a functional on the past of the process. Mathematical modeling involving delay differential equations is widely used for analysis and predictions in various areas of the life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks, (see for example, [4, 12, 14]), and the references cited therein. The delays can represent gestation times, incubation periods, or transport delays. In many cases, time delays can be substantial such as gestation, forestation, deforestation, and maturation, or can represent little lags such as acceleration and deceleration in physical processes. Due to their importance, numerous applications have been devoted to the existence of periodic solutions of several different types of delay differential equations. For some specific work concerning the existence of periodic solutions to periodic population models which were carried out using the fixed point theory, the reader is referred to see ([41] , [42] , [26], [29], [46]), amongst others, and the references therein.

The theory of impulsive differential equations is an important area of scientific activity. Many evolution processes are characterized by the fact that at certain moments of time they may experience abrupt changes. These short-term perturbations act instantaneously, that is in the form of impulses. For example, many biological phenomena involving thresholds, optimal control models in economics, and frequency modulated systems exhibit impulsive effects. In this way, impulsive differential equations appear as a natural description of observed evolution phenomena of several real-world problems. There are various good monographs on impulsive differential equations, e.g. [2] , [21]. Significant progress has been made in the theory of impulsive differential equations in recent years. In particular, the fundamental theory and some qualitative investigations of solutions to impulsive differential equations have marked a rapid development (see for example, [4] and the sources related).

To this end, in this dissertation, we have been interested in the use of fixed point theory to problem of periodicity and positivity for delay differential and neutral differential equations. We have studied it and have presented interesting results. In this work, we present a collection of results to some problems of neutral differential equations and systems Lotka of delay differential equations using fixed point theory.

This dissertation consists of four chapters

Chapter 1 : The first chapter is devoted to pointing out the tools which are needed in the following chapters. The aim of this chapter is to introduce the basic concepts, notations, and elementary results which will be used throughout this work. We recall some classical results from functional analysis such as the Ascoli-Arzela theorem. The main part of the chapter is dedicated to the presentation of fixed point theorems, the Schauder's and Banach fixed point theorem, and that of Krasnoselskii. Moreover, we analyze some examples to illustrate how to apply these theorems to some specific differential equations.

Chaptre 2 : This chapter is devoted to DDEs analysis, we present some basic preliminaries and we discuss the existence and uniqueness theorem for the solution and properties of them. The authors provide the appropriate mathematical tools which will be needed to understand the concepts that will be developed in this dissertation for the study of the periodicity of delay differential equations (DDEs). A significant and interesting model of delay equations emanating from biology is given at the beginning of this chapter.

Chaptre 3 : We summarize in the third chapter a few concepts and results for impulsive differential equations that will be needed in our arguments.

Chaptre 4 : The goal of this chapter is to present a very recent work published in [9]. We investigate the existence of positive periodic solutions for an n-species Lotka-Volterra system with distributed delays and nonlinear impulses. In the process, we convert the given system into an equivalent integral equation. Then we construct appropriate mappings and use Krasnoselskii's fixed point theorem in a cone of a Banach space to show the existence of a positive periodic solution of the system. Easily verifiable sufficient conditions are established. We discuss our problem in two situations: when impulse functions are subquadratic and when they are sublinear. The technique to deal with the impulsive term is different from earlier approaches. In particular, the results improve some previous ones in the literature. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

$$x_i'(t) = x_i\left(t\right)\left[r_i\left(t\right) - \sum_{j=1}^{n} a_{ij}(t) h_j\left(x_j(t)\right) - \sum_{j=1}^{n} b_{ij}(t) f_j\left(x_j(t-\tau_{ij}\left(t\right))\right)\right.$$
$$\left. - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,\rho) g_j\left(x_j\left(\rho\right)\right) d\rho\right],$$

$t \neq t_k,\ k \in \mathbb{N}^*,$

$$x_i\left(t_k^+\right) - x_i\left(t_k^-\right) = I_{ik}\left(t_k, x_i\left(t_k\right)\right),\ t = t_k,\ k \in \mathbb{N}^*,$$

where $x_i \in \mathbb{R}^+$, $D_{ij} \in C\left(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+\right)$, $h_j, f_j, g_j : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous, and $r_i, a_{ij}, b_{ij}, c_{ij} \in C\left(\mathbb{R}, \mathbb{R}^+\right)$, for $i, j = 1, 2, ..., n$.

CHAPTER 1

Fixed point Theorems

Contents

In this chapter, we will mention (and complete) some important theorems in the theory of the fixed point, as well as some tools of the functional analysis necessary for the rest of this dissertation. In particular, the Ascoli-Arzela theorem is an element it is very useful in this work to prove compactness in function spaces challenged on compact sets and not necessarily compact sets. The theorems of Banach, Schauder, and Krasnoselskii will be presented in this chapter. These elements of the analysis have been presented. Taken from a few books chosen as ((see [15], [61], [68]).

1.1 Notation and priliminaries

This section contains an elementary set of definitions, theorems, and examples which were motivated by the examples in the last section and were formulated to aid us in deciding which fixed point theorem to use and which stability properties to prove.

1.1.1 Normed and Banach space

Definition 1.1 *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ a function. Then d is called.a metric on X if the following properties hold.*

1) $d(x, y) = 0$ *if and only if* $x = y$ *for some* $x, y \in X$;

2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called metric space.

Definition 1.2 The metric space (X, d) is complete if every Cauchy sequence in (X, d) has a limit in that space. Asequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $d(x_n, x_m) < \varepsilon$.

Definition 1.3 *A* vector space $(\mathcal{V}, +, \cdot)$ is a normed space if for each $x, y \in \mathcal{V}$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(i) $\|x\| = 0$ *if and only if* $x = 0$,

(ii) $\|\alpha x\| = |\alpha| \|x\|$ *for each* $\alpha \in \mathbb{R}$, *and*

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Remark 1.1 A normed space is a vector space and it is a metric space with $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 1.4 A Banach space is a complete normed space.

We list two examples, (see Burton, [15]).

Example 1.1 (a) The space $\mathbb{C}([a, b], \mathbb{R}^n)$ consisting of all continuous function $f : [a, b] \to \mathbb{R}^n$ is a vector space over the reals.

b) If $\|f\| = \max_{a \leq t \leq b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then $(\mathbb{C}, \|.\|)$ is a Banach space.

Example 1.2 Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:

i) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$;

ii) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $p \in (1, \infty)$;

iii) $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition 1.5 *A* sequence $\{x_n\}$ in a normed space X is said to be Cauchy if $\lim_{m,n \to \infty} \|x_m - x_n\| = 0$, *i.e.*, for $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq n_0$.

Definition 1.6 *A* normed space $(X, \|.\|)$ is said to be complete if it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.

Definition 1.7 *A* complete normed space is called a Banach space.

Example 1.3 The linear space $\mathbb{C}([a,b])$ of continuous functions on the closed and bounded interval $[a,b]$ is a Banach space with the uniform convergence norm $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$.

Definition 1.8 Every finite- dimensional normed space is a Banach space.

Definition 1.9 *A* closed subspace of a Banach space is a Banach space.

Definition 1.10 A subset $\mathcal{M}$ of X is said to be totally bounded if for each $\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, ..., x_n$ in X such that $\mathcal{M} \subset \bigcup\limits_{i=1}^{n} B_\varepsilon(x_i)$.

Proposition 1.1 *Let X be a metric space. Then the following are equivalent*

i) X is a compact.

ii) Every sequence in X has a convergent subsequence.

iii) X is complete and totally bounded.

Proposition 1.2 *Let X be a subset of a complete metric space X. Then we have the following:*

a) $\mathcal{M}$ is compact if and only if $\mathcal{M}$ is closed and totally bounded.

b) $\overline{\mathcal{M}}$ is compact if and only if $\mathcal{M}$ is totally bounded.

A subset $\mathcal{M}$ of a topological space is said to be relatively compact if its closure is compact, i.e., $\overline{\mathcal{M}}$ is compact. In particular, we have an interesting result.

Proposition 1.3 *Let $\mathcal{M}$ be a closed subset of a complete metric space. Then $\mathcal{M}$ is compact if and only if it is relatively compact.*

Definition 1.11 ([15]) Let $\{f_n\}$ be a sequence of real valued functions with $f_n : [a,b] \to \mathbb{R}$.

a) $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a, b]$.

b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply

$$|f_n(t_1) - f_n(t_2)| < \epsilon$$

for all n.

The following results gives the main method of proving compactness in the spaces in which we are interested.

Definition 1.12 (Ascoli-Arzela, [15]) If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$,then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.

Remark 1.2 Let k_1 and k_2 be two strictly positive real numbers. The subset F of continuous real functions on $[a, b]$, differentiable on $]a, b[$ which satisfy

$$|f(t)| \leq k_1 \text{ et } \sup |f'(c)| \leq k_2,$$

for all $t \in [a, b]$,is relatively compact in $C([a, b], \mathbb{R})$.

Indeed, for all $f \in F$, the finite increment theorem proves that for all t_0, $t \in [a, b]$ there exists $c \in]t_0, t[$ such that

$$|f(t) - f(t_0)| = |f'(c)| |t - t_0|,$$

so $|f(t) - f(t_0)| \leq k_2 |t - t_0|$, let's fixe $t_0 \in [a, b]$. Let $\varepsilon > 0$ and $\eta = \frac{\varepsilon}{k_2}$, then

$$\forall t \in [a, b], |t - t_0| \leq \eta \Rightarrow |f(t) - f(t_0)| \leq \varepsilon.$$

F is said to be "uniformly bounded" if and only if

$$\forall x \in [a, b], \{f(x), f \in F\} \text{ is bounded,}$$

which exactly is the equicontinuity of F at t_0 .As we can take for t_0 any point of $[a; b]$, we deduce that F is equicontinuous. We have $|f(t)| \leq k_1$ for all $f \in F$ wich implies that $\|f(t)\|_\infty \leq k_1$

$$\forall f \in F, B'(0, k_1),$$

i,e

$$F \subset B'(0, k_1),$$

hence the boundedness of F.

Finally, as F is bounded and equicontinuous, then the Ascoli-Arzela's theorem ensures that F is relatively compact. But here we manipulate function spaces defined on infinite t-intervals. So, for compactness, we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([18], Theorem 1.2.2 p. 20) and is as follows.

Theorem 1.1 *([15]) Let $\mathbb{R}^+ = [0, \infty)$ and let $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k(t)\}$ is an equicontinuous sequence of $\mathbb{R}^d$-valued functions on $\mathbb{R}^+$ with $|\phi_k(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^d$.*

Proof. It is clear that set of functions $\{\phi_k(t)\}$ is uniformly bounded on $\mathbb{R}^+$. Thus, considering intervals $[0, n]$, n a positive integer, and using a diagonalization process there is a subsequence, say $\{\phi_k(t)\}$ again, converging uniformly on any compact subset of $\mathbb{R}^+$ to some continuous function $\phi(t)$ with $|\phi(t)| \leq q(t)$ for $t \in \mathbb{R}^+$. Because $\phi(t) \to 0$ as $t \to \infty$, it will now be possible to show that $\|\phi_k - \phi\| \to 0$ as $k \to \infty$, where $\|.\|$ denotes the supremum metric on $\mathbb{R}^+$. From the definition of $q(t)$, for any $\varepsilon > 0$ there is a $T > 0$ with $q(t) < \varepsilon/2$ if $t \geq T$, which yields

$$|\phi_k(t) - \phi(t)| \leq 2q(t) < \varepsilon \text{ if } k \in \mathbb{N} \text{ and } t \geq T, \qquad (**)$$

where $\mathbb{N}$ denote the set of positive integers. On the other hand, since $\{\phi_k(t)\}$ converges to $\phi(t)$ uniformly on $[0, T]$ as $k \to \infty$, for the ε there is a $\kappa \in \mathbb{N}$ with

$$|\phi_k(t) - \phi(t)| < \varepsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq T,$$

which together with (**), implies that $\|\phi_k - \phi\| < \varepsilon$ if $k \geq \kappa$. This shows that

$$\|\phi_k - \phi\| \to 0 \text{ as } k \to \infty.$$

Let us give an example for compact set: (see Burton, [15]). ■

1.2 Fixed point theorems

Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory.

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known as the contraction mapping theorem or the contraction mapping principle). It was first started by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space and provides a constructive method to find those fixed points. Concerning the topological branch, results are obtained using the topological properties of set X. The main result is Schauder's fixed point theorem which was stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem. Although historically the two branches of the fixed point theory had separate development. In 1958, Krasnoselskii established that the sum of two operators $A+B$ has a fixed point in a nonempty closed convex subset M of a Banach space X, where one of them is a contraction and another one is compact (see below).

1.2.1 Banach fixed point

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem, the famous contraction principle, which is one of the most important results of the analysis. It is the most widely applied fixed point result in different areas of mathematics and applications. It requires the structure of a complete metric space with the contractive conditions on the map which is easy to test in many situations. It has been generalized in many different directions. Moreover, the proof of the Banach contraction principle gives a sequence of approximate solutions and useful information as regards the rate of convergence toward the fixed point. This is very important since a fundamental principle both in mathematics and computer science is iteration. Particularly, fixed-point iterations and monotone iterative techniques are the core methods when solving a large class of abstract and applied mathematical problems and play an important role in many algorithms.

Let describe this theorem.

Definition 1.13 Let f be a mapping in the set M. we call fixed point of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.

Theorem 1.2 *(Contraction Mapping Principle, Smart* [71] *). Let* $(\mathcal{S}, \rho)$ *be a complete metric space and let* $\mathcal{P} : \mathcal{S} \to \mathcal{S}$*. If there is a constant* $0 \leq \alpha < 1$

such that for each pair $\phi_1, \phi_2 \in \mathcal{S}$, we have

$$\rho(\mathcal{P}\phi_1, \mathcal{P}\phi_2) \leq \alpha\rho(\phi_1, \phi_2),$$

then there is one and only one point $\phi \in \mathcal{S}$ with $\mathcal{P}\phi = \phi$.

Example 1.4 Let $f(t,x)$ be a continuous real-valued function defined for t in the interval $[0,T]$, and x in $\mathbb{R}$. The Cauchy initial value problem is the problem of finding a continuously differentiable function x on $[0,T]$ satisfying the differential equation

$$\begin{cases} x'(t) = f(t, x(t)), \; t \in [0,T] \\ x(0) = \zeta. \end{cases} \tag{1.1}$$

Proof. Consider the space $C([0,T])$ of continuous real-valued functions with standard supremum norm and f is $L-$Lipschitzian with respect to x. Integrating both sides of *(1.1)* we obtain

$$x(t) = \zeta + \int_0^t f(s, x, (s))ds.$$

We denote the function defined by the right side of the above by $\mathcal{P}x$. Precisely,

$$(\mathcal{P}x)(t) = \zeta + \int_0^t f(s, x(s))ds.$$

Thus $\mathcal{P} : \mathbb{C}([0,T] \to \mathbb{C}([0,T])$, and a solution to *(1.1)* corresponds to a fixed point x of $\mathcal{P}$. Observe that for any $x, y \in [0;T]$,

$$\begin{aligned} |(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| &= \left| \int_0^t f(s,x(s))ds - \int_0^t f(s,y(s))ds \right| \\ &\leq \int_0^t |f(s,x(s)) - f(s,y(s))| \, ds \\ &\leq L \int_0^t |x(s) - y(s)| \, ds \\ &= Lt \, \|x - y\| \, . \end{aligned}$$

It follows that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq LT \|x - y\| .$$

If $LT < 1$, then the result is immediate via the Banach Contraction Principle.

■

1.2.2 Schauder's fixed point theorem

Definition 1.14 Let (X, d) be a generalized metric space.Asubset C of X is called compact if every open cover of C has a finite subcover.A subset C of X is sequentially compact if every sequence in C contains a convergent subsequence with limit in C.

Definition 1.15 A set C of topological space is said relatively compact if its closure is compact, i.e., $\overline{C}$ is compact. The set C is sequentially relatively compact if every sequence in C contains a convergent subsequence (the limit need not be an element of C), i.e., $\overline{C}$ is sequentially compact.

Definition 1.16 Let X ,Y be two generalized metrics spaces, $K \subset X$ and $f : K \to Y$ be a an open operator. Then f is called:

(i) compact, if for any bounded subset $A \subset K$ we have $f(A)$ is relatively compact or $\overline{f(A)}$ is compact;

(ii) Complete continuous, if f is continuous and compact;

(iii) with relatively compact range, if f is continuous and $f(K)$ is relatively compact or $\overline{f(K)}$ is compact.

Theorem 1.3 *(Schauder's first point theorem, Burton* [15] *) Let $\mathcal{M}$ be a nonempty compact convex subset of a Banach space and let $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ be continuous. Then $\mathcal{P}$ has a fixed point in $\mathcal{M}$.*

Theorem 1.4 *(Schauder's second fixed point theorem, Burton* [15] *) Let $\mathcal{M}$ be a nonempty convex subset of a normed space and let $\mathcal{P} : \mathcal{M} \rightarrow \mathcal{K}$ where $\mathcal{K}$ is a compact subset of $\mathcal{M}$. Then $\mathcal{P}$ has a fixed point in $\mathcal{K}$.*

1.2.3 Krasnoselskii fixed point

The fixed point theorem of Krasnoselskii combined the two main fixed point theorems, the Banach contraction mapping principle and Schauder fixed point theorem into the following result. Firstly, we recall the theorem of Schauder. Schauder's theorems require compactness instead of completeness and which yields a, possibly nonunique, fixed point. More precisely, Schauder's fixed point theorem shows us that a continuous map on a compact convex subset of a Banach space has a fixed point. The next two results are found in Smart [61] and Burton [15].

Krasnoselskii's theorem may be combined with Banach and Schauder's fixed point theorems. In a certain sense, we can interpret this as follows: if a compact operator has the fixed point property, under a small perturbation, then this property can be inherited. The theorem is useful in establishing the existence results for perturbed operator equations. It also has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of periodic solutions. Thus the existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. (see [61] Smart, 1980;p.31).

Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result:

Theorem 1.5 ***(Krasnoselskii,*** *see Burton* [13] *). Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathcal{M}$ into X such that the following conditions hold*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$, $\forall x, y \in \mathcal{M}$;

(ii) *$\mathcal{A}$ is continuous and $\mathcal{A}\mathcal{M}$ is contained in a compact set;*

(iii) *$\mathcal{B}$ is a contraction with $\alpha < 1$,*

then there is a $z \in \mathcal{M}$, with $z = \mathcal{A}z + \mathcal{B}z$.

Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-\mathcal{B})x - (I-\mathcal{B})y\| &= \|(x-y) - (\mathcal{B}x - \mathcal{B}y)\| \\ &\leq \|x-y\| + \|\mathcal{B}x - \mathcal{B}y\| \\ &\leq \|x-y\| + \alpha\|x-y\| \\ &= (1+\alpha)\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} \|(I-\mathcal{B})x - (I-\mathcal{B})y\| &= \|(x-y) - (\mathcal{B}x - \mathcal{B}y)\| \\ &\geq \|x-y\| - \|\mathcal{B}x - \mathcal{B}y\| \\ &\geq \|x-y\| - \alpha\|x-y\| \\ &= (1-\alpha)\|x-y\|. \end{aligned}$$

In short

$$(1-\alpha)\|x-y\| \leq \|(I-\mathcal{B})x - (I-\mathcal{B})y\| \leq (1+\alpha)\|x-y\|.$$

This inequality shows that $(I-\mathcal{B}) : \mathcal{M} \to (I-\mathcal{B})\mathcal{M}$ is continuous and one to one. Thus, $(I-\mathcal{B})^{-1}$ exist and is continuous. Let us pose $U := (I-\mathcal{B})^{-1}\mathcal{A}$ It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathcal{M} \text{ such that } (I-\mathcal{B})^{-1}\mathcal{A}z{=}z.$$

This is equivalent to $z{=}\ \mathcal{A}z{+}\mathcal{B}z$. ■

Remark 1.3 If $\mathcal{A}=0$, the theorem can be summed up in Banach's theorem and if $\mathcal{B}=0$ then the theorem is none other than Schauder's theorem.

In addition, an example is eventually analyzed to illustrate the effectiveness of the proved results of Krasnoselskii theorem.

Example 1.5 For better understanding this observation, now, we analyze an example to illustrate the application of Krasnoselskii fixed point theorem for proving the existence of $\omega-$periodic solutions of the following differential equation,

$$x'(t) = -a(t)x(t) - g(t,x), \tag{1.2}$$

where $a(t) = a(t+\omega)$, and the function $g(t,x)$ is periodic in t of period ω.

Proof. We can transform this equation in another form while writing, formally

$$x'(t)e^{-\int_0^t a(s)ds} = -a(t)e^{-\int_0^t a(s)ds}x(t) - g(t,x)e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)e^{-\int_0^t a(s)ds} + a(t)e^{-\int_0^t a(s)ds}x(t) = -g(t,x)e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)e^{-\int_0^t a(s)ds}\right)' = -g(t,x)e^{-\int_0^t a(s)ds},$$

then integrating from $t-\omega$ to t, we obtain

$$\int_{t-\omega}^{t}\left(x(u)e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-\omega}^{t} g(u,x)e^{-\int_0^u a(s)ds}du,$$

which gives

$$x(t)e^{-\int_0^t a(s)ds} - x(\omega - t)e^{-\int_0^{\omega-t} a(s)ds} = -\int_{t-\omega}^{t} g(u,x)e^{-\int_0^u a(s)ds}du,$$

or

$$x(t) = x(\omega - t)e^{-\int_{\omega-t}^{t} a(s)ds} - \int_{t-\omega}^{t} g(u,x)e^{-\int_t^u a(s)ds}du. \tag{1.3}$$

If we suppose that $e^{-\int_{\omega-t}^{t} a(s)ds} := \alpha$ and if $(X, \|.\|)$ is a Banach space of functions $\varphi : \mathbb{R} \to X$ continuous and $\omega-$ periodic, then the equation *(1.3)* can be written as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (\mathcal{H}\varphi)(t),$$

where $\mathcal{B}$ is contraction provides that the constant $\alpha < 1$ and $\mathcal{A}$ is compact mapping. ■

This example shows the birth of the mapping $\mathcal{H}\varphi := \mathcal{B}\varphi + \mathcal{A}\varphi$ which is identified with a sum of a contraction and a compact mapping.

Finally, let us indicate a very useful Theorem 2.1 when trying to prove that the existence and positivity of a solution to some problem.

Definition 1.17 *(Krasnoselskii,* [45] *) Let X be a Banach space and let Ω be a closed, nonempty subset of X. Ω is a cone if*

i) $\alpha u + \beta v \in \Omega$ for all $u, v \in \Omega$ and all $\alpha, \beta \geq 0$;

ii) $u, -u \in \Omega$ imply $u = 0$.

The proof of Krasnoselskii's fixed point theorem stated below can be found in [54] .

Theorem 1.6 *(Bellman and K. L. Cooke* [7] *) Let X be a Banach space, and let $\Omega \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{P} : \Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \Omega,$$

be a completely continuous operator such that either

i) $\|\mathcal{P}u\| \leq \|u\|$ *for* $u \in \Omega \cap \partial\Omega_1$ *and* $\|\mathcal{P}u\| \geq \|u\|$ *for* $u \in \Omega \cap \partial\Omega_2$*; or*

ii) $\|\mathcal{P}u\| \geq \|u\|$ *for* $u \in \Omega \cap \partial\Omega_1$ *and* $\|\mathcal{P}u\| \leq \|u\|$ *for* $u \in \Omega \cap \partial\Omega_2$.

Then $\mathcal{P}$ *has a fixed point in* $\Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

CHAPTER 2

Retarded functional differential equations with applications

Contents

This chapter provides background material necessary for the rest of the dissertation. Some preliminaries and basic definitions are given for delay differential equations. Strictly speaking, a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process. Like ordinary differential equations, delay differential equations have several features which make their analysis more complicated. The survey of the theory related to delay differential equations can be found e.g. in books, [4], [15], [7], [25], [26] – [37], [46] – [43].

2.1 Basic concepts of delay differential equations

Motivation

The questions have been asked by many researchers "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. To clarify more, we give a biological system in which the present rate of change of some unknown function depends upon past values of the same function.

Real example of delay differential equation

To have a better understanding and reading of this section, we will focus on a simple real example. The goal is to help the reader to understand the most relevant aspects of delay differential equations. The following is an example presented in [4]. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t = \alpha\rho$. Then the dynamics of N can be described by the linear delay differential equation

$$\frac{d}{dt}N(t) = -\mu N(t) - rN(t-h), \tag{2.1}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Equation (2.1) describes the changes in N.

With deeper study and understanding of population dynamics, people started to consider introducing state-dependent delay into population models, as was pointed out in Arino et al. [4].

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population.

Mathematical point of view:

To determine a solution past time t_0, we need to prescribe the value of $N(t_0 - h)$. Suppose we have the initial value $N(t_0 - h)$. Once we advance, say to $N(\varepsilon)$, with $t_0 < \varepsilon < t_0 + h$ small, notice that to calculate the derivative at $t = \varepsilon$ so that we can advance the next step, we need to know

$$\frac{d}{dt}N(\varepsilon) = -\mu N(\varepsilon) - rN(\varepsilon - h),$$

where $\varepsilon - h \in (t_0 - h, t_0)$. In this manner, we realize that we need to know the values of $N(.)$ on the whole interval $[t_0 - h, t_0]$. If we do not specify these values, we obtain an unsatisfactory notion of uniqueness, as the following example

$$x'(t) = -\frac{\pi}{2}x(t-1), x(0) = \frac{1}{\sqrt{2}}.$$

Here

$$x_1(t) = \sin\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right] \text{ and } x_2(t) = \cos\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right],$$

are both solutions to the above equation at $t_0 = 0$. But if we specify the initial behavior on the interval $[-1, 0]$, we obtain that only one solution exists to each delay differential equations, by the existence-uniqueness result in Theorem 2.1 that we give below.

Clearly, to begin with, an initial value problem requires more information than an analogous problem for a system without delays. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - h, t_0]$. Each such initial function determines a unique solution to the delay differential equation. If we require that initial functions be continuous, then the space of solutions has the same dimensionality as $C([t_0 - h, t_0], \mathbb{R})$.

In the next section, with the previous discussion as a guide, let us now define the DDEs problem for a given initial function.

2.1.1 A general initial value problem

Suppose $\tau > 0$ is a given real number $\tau > 0$, denote $C([a, b], \mathbb{R}^n)$, the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We will denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ as $|x|$ from now on in order to avoid confusion with another norm we shall use. If $[a, b] = [-\tau, 0]$, we let C=$C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by

$$\|\varphi\|_\tau := \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|.$$

Let $\sigma \in \mathbb{R}, A > 0$ and $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$, be defined by

$$x_t = x(t+\theta) \text{ for } -\tau \leq \theta \leq 0.$$

Definition 2.1 ([37]) *If Ω is a subset of $\mathbb{R} \times C$, Let $f : \mathbb{R} \times C \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation*

$$\begin{cases} x'(t) = f(t, x_t), t \geq \sigma, \\ \quad\quad \textit{and } x_\sigma = \varphi, \end{cases} \tag{2.2}$$

is a retarded functional differential equation and we will denote this equation by DDEs.

Definition 2.2 ([37]) *A function x is said to be a solution of (2.2) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, and x satisfies (2.2) for $t \in [\sigma, \sigma + A]$. In such a case we say that x is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (2.2) with initial value at σ or simply a solution of (2.2) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (2.2) is a very general type of equation and includes ordinary differential equations ($\tau = 0$). Although the structure of these equations is similar to ordinary differential equations, the crucial difference is that a delay differential equation (or a system of equations) is an infinite dimensional problem and the corresponding phase space is a functional space usually the space of continuous functions is considered.

Remark 2.1 The quantitie $\tau \geq 0$, is called the delay. The delay may be constant, function $\tau(t)$ of t (time-dependent delay), or function $\tau(t, x(t))$ (state-dependent delay).

Definition 2.3 *Equation (2.2) is called:*

i) *linear if* $f(t,\varphi) = L(t,\varphi)$*, where* L *is linear in* φ*.*

ii) nonhomogeneous if $f(t,\varphi) = L(t,\varphi) + h(t)$*, where* $h(t) \neq 0$*, it is called homogeneous if* $h = 0$*.*

iii) autonomous if $f(t,\varphi) = g(\varphi)$*, where* g *does not depend on* t*.*

Equation (2.2) is a very general type of equation and includes differential-difference equations. To be more explicit we give some classes of equations that can be expressed by (2.2), we have equations with a fixed delay (the simplest possible case) such as

$$x'(t) = f(t, x(t), x(t-\tau)),$$

or nonlinear nonautonomous differential equations with multiple time varying delays on the same state x

$$x'(t) = f(t, x(t), x(t-\tau_1(t)), ..., x(t-\tau_p(t))),$$

with $0 \leq \tau_i(t) \leq \tau$ for all $i = 1, ..., p$. We also have integrodifferential equations with a distributed delay

$$x'(t) = \int_{-\tau}^{0} g(t, x(t+\theta))\, d\theta,$$

where we see how in the integration process we need to know the values of x in $[t-\tau, t]$ for each t where the vector field is defined.

2.1.2 Existence and uniqueness theory

The existence and uniqueness theory for delay equations can be derived from the more general theory of functional differential equations. Since we intend to consider only equations of the form (2.2) we will not make use of the full generality available. Nevertheless, the more general theory leads to a presentation that is simpler and also benefits from an analogy with similar results in the theory of ordinary differential equations.

We now state the basic theory of DDEs.

Lemma 2.1 *([37]) Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (2.2) through (σ, φ) is equivalent to solving the integral equation:*

$$x(t) = \varphi(\sigma) + \int_{\sigma}^{t} f(s, x_s)\, ds \text{ for } t \geq \sigma, \text{ and } x_\sigma = \varphi.$$

Lemma 2.2 *([37]) If $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma - \tau, \sigma + A]$.*

Proof. Since x is continuous on $[\sigma - \tau, \sigma + A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon, \forall \theta \in [-\tau, 0]$. ■

The existence and uniqueness of the solutions of DDEs are given by the following Theorems.

Theorem 2.1 *(Local existence, [37]) Suppose $\overline{\Omega}$ is an open subset in $\mathbb{R} \times C$ and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous. For any $(\sigma, \varphi) \in \overline{\Omega}$, there exists a solution of equation (2.2) through (σ, φ).*

Definition 2.4 **(** Lipschitzian, [37] **).** *We say $f(t, \varphi)$ is Lipschitz in φ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \varphi_i) \in K, i = 1, 2,$*

$$|f(t, \varphi_1) - f(t, \varphi_2)| < k\, |\varphi_1 - \varphi_2|\,.$$

Theorem 2.2 *(Existence and uniqueness, [45]) Suppose $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$, $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each* *compact set in* $\overline{\Omega}$*. If* $(t_0, \varphi) \in \overline{\Omega}$ *, then there is a unique solution of Eq. (2.2) through* (t_0, φ)*.*

Proposition 2.1 *. If* f *is at most affine i.e.* $f(t, \varphi) \leq a + b\,|\varphi|\,,$ *with* $a, b > 0$*, then there exists a global solution of the equation (2.2) i.e.* $\forall \varphi$*, the solution* $x(\sigma, \varphi)$ *is defined on* $[A, \infty[$*.*

In the following we also require continuous dependence of solutions on initial conditions, for which the following theorem gives a result analogous to that for ordinary differential equations.

Theorem 2.3 *. Suppose* x *is a solution through* (t_0, φ) *of the equation (2.2) and that it is unique on* $[t_0-\tau, \beta]$*. If* $\{(t_n, \varphi_n)\} \subset \mathbb{R}\times C$ *is a sequence such that* $(t_n, \varphi_n) \to (t_0, \varphi)$ *as* $n \to \infty$*, then for all sufficiently large* n *every solution* x_n *through* φ_n *exists on* $[t_n - \tau, \beta]$*, and* $x_n \to x$ *uniformly on* $[t_0 - \tau, \beta]$*.*

2.2 Neutral delay differential equations

Now are ready to give the definition of an other class of delay differential equations so-called the Neutral delay diffferential equations (NDDEs).

Definition 2.5 [37] *Suppose that* $\mathbb{R} \times C$ *is open with elements* (t, φ)*. A function* $D : \overline{\Omega} \to \mathbb{R}^n$ *is said to be atomic at* β *on* $\overline{\Omega}$ *if* D *is continuous together with its first and second Fréchet derivatives with respect to* φ*; and* D_φ*, the derivative with respect to* φ*, is atomic at* β *on* $\overline{\Omega}$*.*

Definition 2.6 [37] *Suppose that* $\overline{\Omega} \subseteq \mathbb{R} \times C$ *is open,* $f : \overline{\Omega} \to \mathbb{R}^n$*,* $D : \overline{\Omega} \to \mathbb{R}^n$ *are given continuous functions with* D *atomic at zero. The equation*

$$\frac{dD}{dt}\left(t, x_t\right) = f\left(t, x_t\right), \tag{2.3}$$

is called the neutral delay differential equation NDDE (D, f)*.*

If the delayed argument occurs in the highest order derivative of the state we call it neutral functional differential equation.

The following equations are some examples of neutral differential equations

Example 2.1 [37] *If* $\tau > 0$, B *is an* $n \times n$ *constant matrix,* $D(\varphi) = \varphi(0) - B\varphi(-\tau)$, *and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - Bx(t-\tau)] = f(t, x_t).$$

Example 2.2 [37] *If* $\tau > 0$, x *is a scalar,* $D(\varphi) = \varphi(0) - \sin(-\tau)$, *and* $f : \overline{\Omega} \to \mathbb{R}^n$ *is continuous, then the pair* (D, f) *defines an NDDE,*

$$\frac{d}{dt}[x(t) - \sin x(t-\tau)] = f(t, x_t). \tag{2.4}$$

Remark 2.2 *Note that when* x *is continuous differentiable, (2.4) is equivalent to*

$$x^{'}(t) - (\cos x(t-\tau))\, x^{'}(t-\tau) = f(t, x_t).$$

Definition 2.7 [37] *A function* x *is said to be a solution of (2.3) on* $[\sigma - \tau, \sigma + A]$ *if there are* $\sigma \in \mathbb{R}$ *and* $A > 0$ *such that*

$$x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n), \;\; (t, x_t) \in \overline{\Omega}, \;\; t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ *is continuously differentiable and satisfies equation (2.3) on* $[\sigma, \sigma + A]$. *For a given* $t_0 \in \mathbb{R}, \varphi \in C$, *and* $(\sigma, \varphi) \in \overline{\Omega}$, *we say* $x(t, \sigma, \varphi)$ *is a solution of equation (2.3) with initial value* φ *at* σ *or simply a solution through* (σ, φ) *if there is an* $A > 0$ *such that* $x(t, \sigma, \varphi)$ *is a solution of equation (2.3) on* $[\sigma - \tau, \sigma + A]$ *and* $x_\sigma(\sigma, \varphi) = \varphi$; *we say* $x(t, \sigma, \varphi)$ *is a solution of (2.3) on* $[\sigma - \tau, \infty)$, *if for every* $A > 0$, $x(t, \sigma, \varphi)$ *is a solution of equation (2.3) on* $[\sigma - \tau, \sigma + A]$ *and* $x_\sigma(\sigma, \varphi) = \varphi$.

Theorem 2.4 *(Existence,* [45] *) If* $\overline{\Omega}$ *is an open set in* $\mathbb{R}\times C$ *and* $(t_0,\varphi)\in\overline{\Omega}$*, then there exists a solution of the NDDE* (L,f) *through* (t_0,φ).

Theorem 2.5 *(Existence and Uniqueness,* [48] *) If* $\overline{\Omega}$ *is an open set in* $\mathbb{R}\times C$ *and* $f:\overline{\Omega}\to\mathbb{R}^n$ *as Lipschitz in on compact sets of* $\overline{\Omega}$*, then, for any* $(t_0,\varphi)\in\overline{\Omega}$*, there exists a unique solution of the NDDE* (D,f) *through* (σ,φ).

2.3 Method of steps

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solve the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solution (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance. The method of steps was first proposed by Bellman and Cooke [7]. This approach, furnishes a method of finding explicit solutions. The desired solution is found on successive intervals by solving ordinary differential equations without delays in each interval. As an illustration to this approach, consider the DDE:

$$\begin{cases} x'(t)=f(t,x(t),x(t-\tau)), t\geq t_0 \\ x(t)=\varphi_0(t), \; t_0-\tau\leq t\leq t_0. \end{cases} \tag{2.5}$$

For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0-\tau,t_0]$, therefore one can obtain the solution in the next step interval $[t_0,t_0+\tau]$ by solving the following ordinary differential equation:

$$x'(t)=f(t,x(t),\varphi_0(t-\tau))=g_0(t,x(t), \;\text{for } t_0\leq t\leq t_0+\tau.$$

Under suitable hypotheses on g_0, existence and uniqueness of a solution of this equation (hence a solution of (2.5)) on $[t_0 - \tau, t_0]$ can be established. Denoting this solution by $\varphi_1(t)$ and restricting equation (2.5) to the interval $[t_0 + \tau, t_0 + 2\tau]$, we find the ordinary differential equations

$$x'(t) = f(t, x(t), \varphi_1(t - \tau)) = g_1(t, x(t) \text{ for } t_0 + \tau \leq t \leq t_0 + 2\tau,$$

with the initial condition $x(t_0+\tau) = \varphi_1(t_0+\tau)$, for which we can again establish existence and uniqueness of a solution φ_2. Thus we have now extended the solution x to the interval $[t_0 + \tau, t_0 + 2\tau]$, and we now have a formula for $x(t)$ when $t \in [t_0 - \tau, t_0 + 2\tau]$.

In general, by assuming that $\varphi_{k-1}(t), \forall(k = 1, 2, ...)$ is defined on the interval $[t_0 + (k-2)\tau, t_0 + (k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$x'(t) = f(t, x(t), \varphi_{k-1}(t - \tau)), \text{ for } t_0 + (k-1)\tau \leq t \leq t_0 + k\tau,$$

with the initial condition $x(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$. We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[t_0 - \tau, \infty)$.

Remark 2.3 *The method of steps can be extended to differential equations with other types of delays, such as multiple delays, variable delay and even state dependent delay or for neutral systems. The difficulty is to locate the primary discontinuities.*

2.3.1 Primary discontinuity of delay differential equation

Definition 2.8 [26] *If the solution of a DDE and its derivatives of order μ are continuous at some point in the time interval, but the derivative of order*

$\mu+1$ is not, then such a point is called a primary discontinuity of the given problem.

Theorem 2.6 [26] *The points $\xi_\mu := \mu\tau$ the primary discontinuities of problem (2.5). More precisely, $x^{(\mu)}$ is continuous at ξ_μ but $x^{(\mu+1)}$ is, in general, not, even if the functions φ and f have continuous derivatives of all orders.*

Proof. See [26]. Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t = 0$, the first derivative $x'(t)$ has a primary discontinuity, since the integrable equation

$$x'(t) = f(t, x(t), \varphi(t-\tau)), t \in [0, \tau]\,,$$

may satisfy the condition $x(0) = \varphi(0)$, but it is unlikely to satisfy the additional condition $x'(0^+) = \varphi'(0^-)$. Only for special choices of the initial function $\varphi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\varphi'(0^-) = f(0, \varphi(0), \varphi(-\tau))$. ■

Example 2.3 *We illustrale this method by using the special cases of equation (2.5), the following is an example presented in ([37]), canada 1998, Let*

$$\begin{aligned} x'(t) &= ax(t-\tau), t \in [0, +\infty) \\ x(t) &= 1, t \in [-\tau, 0]\,, \end{aligned}$$

where a is positive constant. Using the method of steps, it is easy to see that the solution $x(t)$ is a piecewise polynomial. On each subinterval $[i\tau, (i+1)\tau]$, $x(t)$ is an $(i+1)$-th. order polynomial, i.e.,

$$x(t) = \sum_{j=1}^{i+1} \frac{a^j}{j!}(t-(j-1)\tau)^j, i \in \mathbb{N}.$$

It is also clear that integer multiples of τ are primary discontinuities for this particular problem.

As a generalization of (2.5), we consider

$$x'(t) = f(t, x(t), x(t - \tau(t)), x'(t - \tau(t))), t \in [0, \overline{a}],$$

where $t - \tau(t)$ is a strictly increasing function and

$$0 < \tau(t) \leq t,\ \overline{a} = \inf_{t \geq 0} \{t - \tau(t)\}.$$

Remark 2.4 *The method of steps can be extended to delay differential equations with additive noise term (see chapitre 3).*

CHAPTER 3

Impulsives differentials equations

Contents

Impulsive differential equations appear as a natural description of observed evolution phenomena in several real world problems. There are various good monographs on impulsive differential equations [16, 17, 50, 63]. Many processes studied in applied sciences are represented by differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states. For instance, mechanical systems with impact, population dynamics, mathematical economy, chemical technology, electric technology, chemistry, engineering, control theory, medicine, etc.

The theory of impulsive differential systems is an important branch of the differential equations field. Over the last decades, there has been a significant development of this theory, but in spite of its importance, this development has been quite slow due to the special features of impulsive differential equations in general, such as pulse phenomena, Model for Marek's Disease (see, for instance, [60]). An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active ([16]).

3.1 Modeling a problem of impulsive differential equation

3.1.1 Example (Administering drug):

Suppose there is a patient who requires a minimum amount of drug in their body for treatment to be efective. If the patient is initially given a concentration of D.see [60]

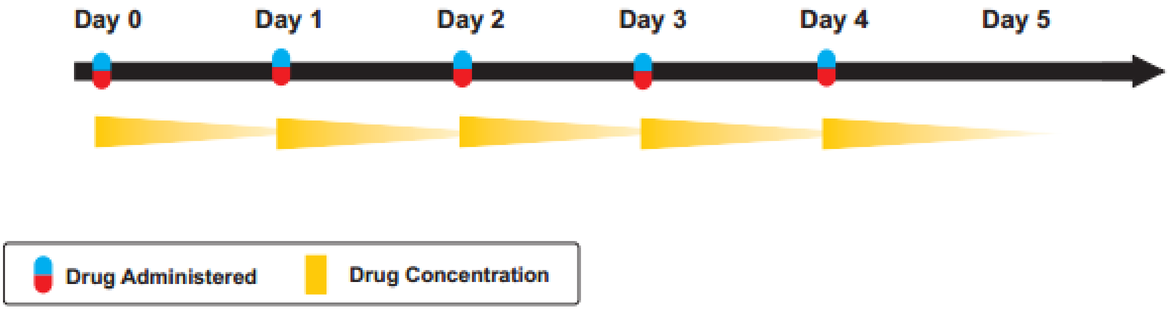

Figure 3.1: Schematic for the daily administering of a drug. the faded yellow signi es a loss in concentration within the body.

Of the drug, we can model the decaying concentration within the patient's body over time with a differential equation (also see Figure (3.1)). If we assume that the concentration of drug decays exponentially at rate k, then the differential equation 3.1 models the rate of change of drug concentration within the body over time.

$$\frac{dx(t)}{dt} = -kx(t), \tag{3.1}$$

where t is time in hours. The solution to the differential equation 3.1 is:

$$x(t) = De^{-kt}. \tag{3.2}$$

The solution curve $x(t)$ models the concentration of drug within the body over time (see Figure (3.2)) and can be used to determine how much of the single dose, D, is left in the patient's body at a given time. Of course the model as it stands is not capable of investigating more than a single dosage of drug. We could of course incorporate the intake of drug directly into the differential equation 3.1, but if we were to do this, there would be the underlying assumption that the drug was being administered continuously.

Many drugs are not consumed continuously, but instead

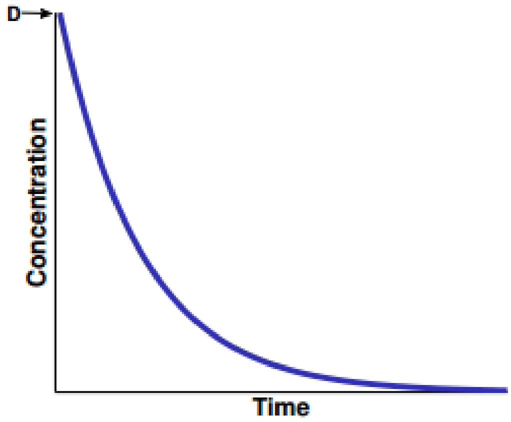

Figure 3.2: Solution curve for the differential equation representing exponential decay in drug concentration.

They are ingested at consecutive moments in time. We need to show incorporate the consecutive administering of drug into the model. This will be the "impulsive" component of the model Suppose a dose of the same concentration, D, is given at the moments of time:

$$0 < t_1 < t_2 < t_3 < t_4 < \cdots < t_n,$$

where t_n is the time in which the nth dose, and nal dose is given. The immediate change in concentration of drug within the body upon taking the dose can be modelled as:

$$x\left(t_i^+\right) = x\left(t_i^-\right) + D \text{ when } t = t_i, \quad i = 1, 2, \ldots$$

Notice that we have introduced a new notation, superscripts $(+)$ and $(-)$. The impulsive condition, taking the drug, occurs instantaneously, but is described by a mapping. Therefore the superscript $(-)$ signies the concentration of drug directly.

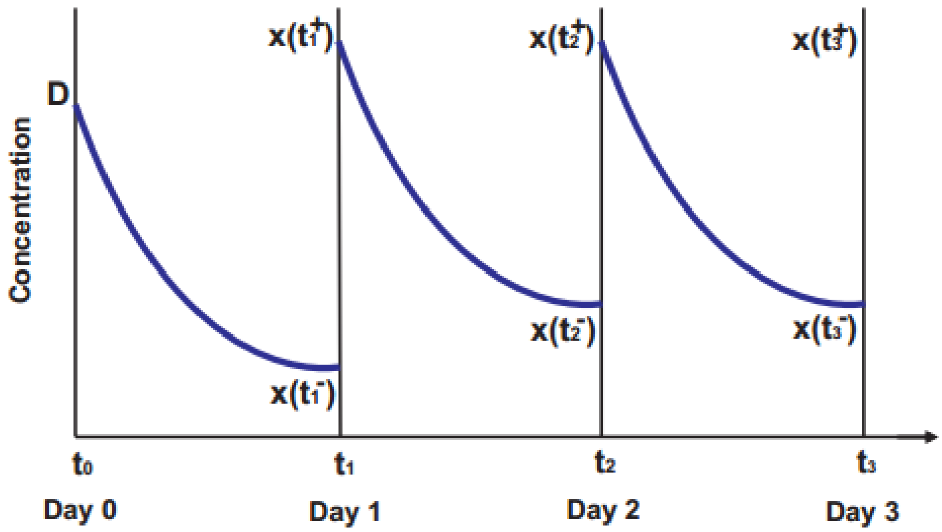

Figure 3.3: Piecewise continuous solution curve for impulsive differential equation describing periodic drug in take.

Before the change in state while the superscript (+) represents the concentration of drug directly after the change in state. By combining both the differential equation and the impulsive condition, we arrive at the simple impulsive differential equation:

$$\begin{cases} \frac{dx(t)}{dt} = -kx(t), \text{ for } t \neq t_i, \\ x\left(t_i^+\right) = x\left(t_i^-\right) + D, \text{ for } t \neq t_i \quad i = 1, 2, \ldots, \\ x\left(0\right) = D, \end{cases} \tag{3.3}$$

Figure (3.3) is an example of a numerical solution to model 3.3 and in Table (3.1) we have calculated the concentration of drug at the moments directly before and after impulses (i.e. directly before and after the drug is ingested), under the assumption that the intervals, $t_{i+1} - t_i = T$ are all the same. Note that T could represent 24 hours but we will leave it as T to be more general.

Time	concentration before impulse $x(t^-)$	concentration after impulse $x(t^+)$
Day 0		D
Day 1	De^{-Tk}	$De^{-Tk}+D$
Day 2	$De^{-2Tk}+De^{-Tk}$	$De^{-2Tk}+De^{-Tk}+D$
Day 3	$De^{-3Tk}+De^{-2Tk}+De^{-Tk}$	$De^{-3Tk}+De^{-2Tk}+De^{-Tk}+D$
.	.	.
.	.	.
.	.	.

Table 3.1: Concentration of drug within the body at the moments directly before and after impulse every day and under the assumption that drug is taken every T hours exactly.

The concentration of drug before the impulse is determined by the differential equation. The concentration after the impulse is determined by the impulsive condition and is exactly one dosage, D, greater. Take note of the expressions above, which could be simplifed.They are left in this form to illustrate what remains of any particular dose consumed previously. For example, what remains of the initial dose D on day three is De^{-3Tk}. As you can see from the pattern in Table (3.1) in general the concentration of drug at the start of the mth day, prior to administering is:

$$x\left(t_m^-\right)=D\sum_{j=1}^{m}e^{-jTk}, \tag{3.4}$$

Proof. We will prove 3.4 by induction. Consider the drug concentration at the moment before impulse on day 1 ($m=1$). According to 3.4 we have:

$$x\left(t_1^-\right)=D\sum_{j=1}^{1}e^{-jTk}=De^{-Tk},$$

which is exactly what we have in Table (3.1), so 3.4 is true for $m = 1$. Now, let $h \in \mathbb{Z}^+$ and suppose 3.4 is true for $m = h$. According to the impulsive differential equation model (3.3), the drug concentration at time t_{h+1}^- (i.e. $m = h + 1$) is:

$$x\left(t_{h+1}^-\right) = (D \sum_{j=1}^{h} e^{-jTk} + D)e^{Tk},$$

i.e. an additional dose of D was taken and the entire new concentration decayed according to the differential equation. We can simplify the above:

$$\begin{aligned} x\left(t_{h+1}^-\right) &= \left(D \sum_{j=1}^{h} e^{-jTk} + D\right) e^{-Tk} \\ &= D \sum_{j=1}^{h} e^{-jTk} e^{Tk} + De^{-Tk} \\ &= D \sum_{j=1}^{h} e^{-Tk(j+1)} + De^{-Tk} \\ &= D \sum_{j=2}^{h+1} e^{-Tkj} + De^{-Tk} \\ &= D \sum_{j=1}^{h+1} e^{-jTk}, \end{aligned}$$

thus 3.4 holds for $m = h + 1$ and this concluding the proof.The solution, 3.4, gives us the minimum concentration of drug on any particular day and can be used to investigate best practices which minimize waste and maintain an e ective dosage within the patient. Of course this simple example does not require an impulsive differential equation to answer these questions, but it is my hope that it has familiarized the reader with the model structure as we move forward. ∎

3.2 Classes of Impulsive Differential Equations

There are three classes of impulsive differential equations see [60]:

Class 1: Equations with fxed moments of the impulse effect

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \quad t \neq t_k \\ \Delta x &= I_k(x), \quad t = t_k. \end{aligned} \tag{3.5}$$

The impulse is fixed before hand by defining the sequence $t_k : t_k < t_{k+1}, (k \in \Bbbk \subset \mathbb{Z})$. For $t \in (t_k, t_{k+1}]$ the solution $x(t)$ of equation 3.5 satisfies the equation $\frac{dx}{dt} = f(t,x)$, and for $t = t_k$ satisfies the relation

$$x\left(t_k^+\right) = x\left(t_k^-\right) + I_k\left(x\left(t_k^-\right)\right).$$

Class 2: Equations with state-dependent moments of the impulse effect

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) \quad t \neq t_k(x) \\ \Delta x &= I_k(x) \quad t = t_k(x), \end{aligned} \tag{3.6}$$

where $t_k : \Omega \to \mathbb{R}$ and $t_k < t_{k+1} (k \in \Bbbk \subset \mathbb{Z}, x \in \Omega)$. The impulse occurs when the mapping point (t,x) meets some hypersurface σ_k of the equation $t = t_k(x)$.

Class 3: Autonomous impulsive equations

$$\begin{aligned} \frac{dx}{dt} &= f(x), \quad x \notin \sigma \\ \Delta x &= I_k(x), \quad x \in \sigma, \end{aligned} \tag{3.7}$$

where σ is an $(n-1)-$ dimensional manifold contained in the phase space $\Omega \subset \mathbb{R}^n$.The impulse occurs when the solution $x(t)$ meets the manifold $.\sigma$.

3.2.1 Space of piecewise continuous functions

First at all, we give the definition of a piecewise continuous function followed by two illustrative examples.

Definition 3.1 $(t_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of real numbers in $I =]t_0, +\infty[$, such as

$$|t_k| \to \infty \text{ when } k \to \infty, k \in \mathbb{N}.$$

We say that the function $g : I \to \mathbb{R}$ belongs to the space $PC(I, \mathbb{R})$ if:

– *g is continuous on each interval $]t_k, t_{k+1}[$.*

– *g is continuous on each interval $]t_k, t_{k+1}[$.*

– *g has a right limit $g(t_k^+) = lim_{t\to t_k^+} g(t)$ and $g(t_k^-) = lim_{t_k^-} g(t)$ exists such that $g(t_k^-) = g(t_k)$.*

Similarly, g is said to be in the space $PC^1(I, \mathbb{R})$ if g and g' belong to $PC(I, \mathbb{R})$. Since the solution of an impulsive differential equation are piecewise continuous, then the appropriate functional space for such solutions is the piecewise continuous function space defined by:

$$PC(I, \mathbb{R}) = \left\{ \begin{array}{l} x : I \to \mathbb{R} \text{ is continuous everywhere exept for } t = t_k \text{ where} \\ x\left(t_k^-\right) \text{ et } x\left(t_k^-\right) \text{ ,} k = 1, 2, \ldots, m \text{ exist and } x\left(t_k^-\right) = x\left(t_k\right) \end{array} \right\},$$

for example, let the function g defined by:

$$\begin{cases} g(t) = t + 1, \text{ si } t \neq n, \\ g(n^+) = g(n) + 2. \end{cases}$$

3.3 Impulsive differential equations with fixed moments

Definition 3.2 (Description of impulsive systems) A differential equation Impulsive represents a combination of a continuous process described by an ordinary differential equation and instantaneous jumps in state called impulses. In this dissertation, we are interested for impulsive equations with fixed impulses, of the form:

$$\begin{cases} x'(t) = f(x(t)), t \neq t_k, \\ x\left(t_k^+\right) = I(x(t_k)), k \in \mathbb{N}^* \\ x\left(t_0^+\right) = x(t_0) \end{cases}, \tag{3.8}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, with is a domain of $\mathbb{R}^n$, $I : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. Any solution $x(t)$ of the system 3.8 verifies:

– $\lim\limits_{t \to 0^+} x(t) = x_0 = x(0^+)$.
– when $t \in]t_0, +\infty[$, $t \neq t_i$ so $x'(t) = f(x(t))$.
– $x(t_i^-) = x(t_i)$ and $x(t_i^+) = I(x(t_i)), i \in \mathbb{N}$.

Let $I \subset \mathbb{R}$ be an interval, and denote by $PC(I, \mathbb{R}^n)$ the set of operators $x : I \to \mathbb{R}$ which are continuous for $t \in I$, $t \neq t_k$ and have discontinuities of the first kind at the points $t_k \in I, (k \in \mathbb{N}^*)$ but are continuous from the left at these points.

We summarize in the following definition that will be needed in the last chapter.

Definition 3.3 (see [33] [50]) *A function $x : \mathbb{R} \to (0, +\infty)$ is said to be a solution of 3.5, if the following conditions are satisfied:*

1) $x(t)$ is absolutely continuous on each (t_k, t_{k+1});

2) for each $k \in \mathbb{N}^*$, $x\left(t_k^+\right)$ and $x\left(t_k^-\right)$ exist and $x\left(t_k^-\right) = x(t_k)$;

3) $x(t)$ satisfies the first equation of 3.5 for almost everywhere in $\mathbb{R}$ and $x(t_k)$ satisfies the second equation of 3.5 at impulsive point t_k, $k \in \mathbb{N}^*$.

3.3.1 Explicit solutions of impulsive differential equations

We present here two examples to illustrate how to solve this type of equation explicitly, (see [66]).

Example 3.1 We consider the following linear impulsive differential equation

$$\begin{cases} x'(t) = \alpha x(t), \text{ si } t \in]t_n, t_{n+1}], n \in \mathbb{N} \\ x(t_n^+) = x(t_n) + \beta, \text{ avec } \alpha, \beta \in \mathbb{R}, \\ x(0^+) = x_0. \end{cases} \tag{3.9}$$

To find the solution of the system 3.9, it is necessary to solve the equation on each subinterval $]t_n, t_{n+1}]$:

– if $t \in]0, t_1]$, so,

$$x(t) = x_0 exp(\alpha t).$$

– if $t \in]t_1, t_2]$, we know that,

$$x(t_1^+) = x_0 exp(\alpha t_1) + \beta,$$

so,

$$\begin{aligned} x(t) &= x(t_1^+) exp(t - t_1) \\ &= (x_0 exp(\alpha t_1) + \beta) exp(\alpha(t - t_1)). \end{aligned}$$

– if $t \in]t_2, t_3]$, from the above, we have,

$$x(t_2^+) = x_0 exp(\alpha t_2) + \beta \exp(t_2 - t_1) + \beta,$$

so,

$$\begin{aligned} x(t) &= x(t_2^+) exp(t - t_2) \\ &= x_0 exp(\alpha t) + \beta exp(t - t_1) + \beta exp(\alpha(t - t_2)). \end{aligned}$$

Similarly, on $]t_n, t_{n+1}]$, we obtain,

$$x(t) = x_0 exp(\alpha t) + \beta exp(\alpha(t - t_1)) + \beta exp(\alpha(t - t_2)) + ... + \beta exp(\alpha(t - t_n)).$$

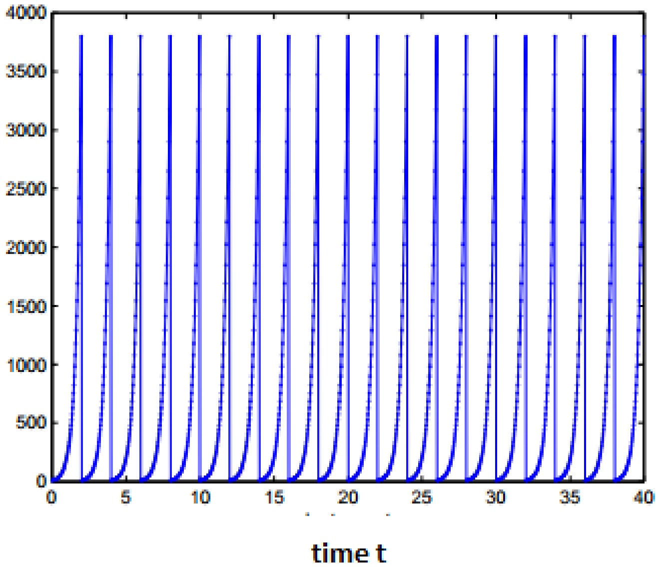

Figure of the exact solution of the system (3.9)

Example 3.2 see [66]

$$\begin{cases} x'(t) = \alpha, \text{ if } t \in]t_n, t_{n+1}], n \in \mathbb{N}, \\ x(t_n^+) = \beta x(t_n), \text{ where } \alpha, \beta \in \mathbb{R}, \\ x(0^+) = x_0. \end{cases} \tag{3.10}$$

By solving the system 3.10 on each sub-interval $]t_n, t_{n+1}]$, we find:

– If $t \in]0, t_1]$, so,

$$x(t) = \alpha t + x_0.$$

– If $t \in]t_1, t_2]$, we have,

$$x(t_1^+) = \beta(\alpha t_1 + x_0).$$

so,

$$\begin{aligned} x(t) &= \alpha(t - t_1) + x(t_1^+) \\ &= \alpha(t - t_1) + \beta(\alpha t_1 + x_0). \end{aligned}$$

– If $t \in]t_2, t_3]$, we have,

$$x(t_2^+) = \alpha\beta(t_2 - t_1) + \beta^2(\alpha t_1 + x_0).$$

Then, we find,

$$x(t) = \alpha(t - t_2) + \alpha\beta(t_2 - t_1) + \beta^2(\alpha t_1 + x_0).$$

By repeating the process, we get

$$x(t) = \alpha(t-t_n)+\alpha\beta(t_n-t_{n-1})+\alpha\beta^2(t_{n-1}-t_{n-2})+...+\alpha\beta^{n-1}(t_2-t_1)+\beta^n(\alpha t_1+x_0),$$

when $t \in]t_n, t_{n+1}]$.

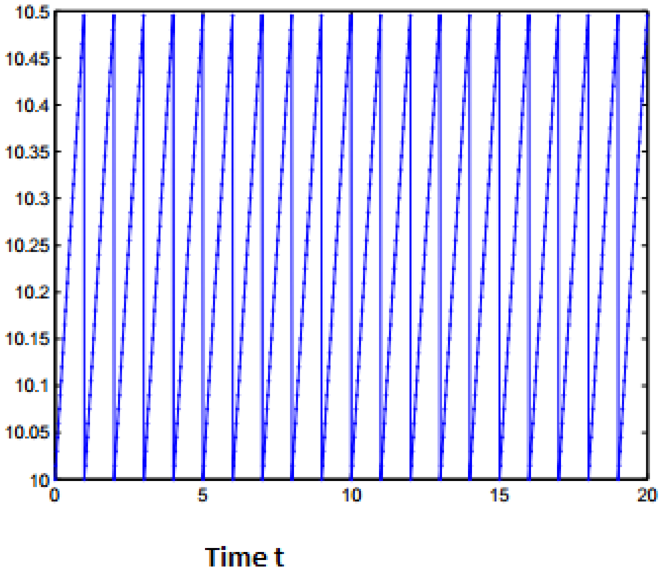

Figure of the exact solution of the system (3.10)

3.3.2 Local existence and uniqueness

In this paragraph, we will give the theorem which ensures the local existence and the uniqueness of the solution of the system 3.8 see [6]

Theorem 3.1 *If the function f is continuous on $\Omega \subset \mathbb{R}^n$, then for any initial condition x_0, there exists an $\alpha > t_0$ such that*

$$x :]t_0 - \alpha, t_0 + \alpha[\to \mathbb{R}^n,$$

is a solution of system 3.8. Moreover, If f is locally Lipschitz in x, then this solution is unique.

CHAPTER 4

Positive periodic solutions for n-species Lotka-Volterra competitive systems with variable delays and impulses

Contents

In this chapter, we expose results published by Benhadri and Caraballo in [9]. When we construct appropriate maps and use Krasnoselskii's fixed point theorem in a cone of a Banach space to show the existence of a positive periodic solution for n-species Lotka-Volterra competitive systems with variable delays and impulses. Easily verifiable sufficient conditions are established. We discuss our problem in two situations: when the impulse functions are subquadratic and when they are sublinear.

4.1 Short biographies of Lotka and Volterra

The equations which model the struggle for existence of two species (prey and predators) bear the name of two scientists: Lotka and Volterra. They lived in different countries, had distinct professional and life trajectories, but they are linked together by their interest and results in mathematical modeling.

Lotka's work in mathematical demography began in 1907 and continued until 1939. In 1920 , where he proved by his model that undamped, permanent oscillations arise in biological systems.

In 1926 he published a paper in the field of bibliometrics, studying the number of scientific publications in specific fields. His ideas eventually contributed to scientometrics – the scientific study of scientific publications.

Lotka published almost a hundred articles on various themes in chemistry, physics, epidemiology or biology, about half of them being devoted to population issues. He also wrote six books. He began this study in1884 and in 1896 he published papers on what is now called integral equations of Volterra type. The theory of functionals as a generalization of the idea of a function of several independent variables was developed by Volterra in a series of papers published since 1887 and was inspired by the problems of the calculus of variations. These papers initiated the modern theory of functional analysis, the name functional being introduced later by Hadamard. In 1892, he became professor of mechanics at the University of Turin and then, in 1900, professor of mathematical physics at the University of Rome

La Sapienza, (see [5]).

Volterra turned his attention to the application of his mathematical ideas to biology, principally reiterating and developing the work of Pierre Francois Verhulst. In the , he studied the ecological problem of a predator population interacting with the prey one. In the following years he published more results, intended to arrive at a mathematical theory of the struggle for existence. A considerable part of the work of Volterra in the latter part of his life was devoted to the applications of mathematics to biology. The subject of these investigations was mainly the study of biological associations of animals of different species living together. In other words he was interested in a mathematical theory of the "survival of the fittest." While there are today other methods of a stochastic nature, the work of Volterra still exerts a dominant influence on several modern and quite recent developments in mathematical biology.

The work of Lotka and Volterra overlapped in the discussion of predatorprey interaction . The problem was discussed by Lotka in 1920 and by Volterra in 1926, their conclusion being the same, that the interaction of the two species would give rise to periodic oscillation in their populations. Volterra acknowledged Lotka's priority, but he mentioned the differences in their papers. They even exchanged some respectful letters. In the case of the predator-prey interaction, the priority of Lotka was firmly established, and the equations with periodic solutions are called Lotka-Volterra equations. Volterra produced more general equations, for more than two species and considering also their interactions in the past. For worked examples of such equations, (see [5]).

Simple population Models

The simplest mathematical model of population growth assumes that the rate of increase of population is proportional to the size of population at any time. Let us denote by $P(t)$ the population at the time t and by k a positive constant, (see [5]). Then

$$\frac{dP}{dt} = kP,$$

which gives by integration

$$P(t) = P_0 exp(kt),$$

where P_0 denotes the population at the time $t = 0$. This law is called the Malthusian growth model and predicts an exponential growth in the population with time. It describes pretty well what happens for certain bacteria or cultures of cells for a short time.

A more realistic model is

$$\frac{dP}{dt} = (B - D)\, P,$$

where $B(t)$ and $D(t)$ denote the birth rate and death rate per individual, respectively. The exponential law corresponds to the case $B(t) = k$ and $D(t) = 0$. Let us assume that the birth rate per individual remain constant, while the death rate per individual is directly proportional to the existing population.

We obtain

$$\frac{dP}{dt} = (B_0 - D_0)\, P,$$

where B_0 and D_0 are positive constants. We can write the equation in a simpler way

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right) P,$$

where $r = B_0$ and $C = \dfrac{B_0}{D_0}$. Its solution is

$$P\,(t) = \frac{CP_0}{P_0 + (C - P_0)\exp{(-rt)}},$$

where $P_0 = P(0)$.

LOTKA-VOLTERRA EQUATIONS

The first and the simplest Lotka–Volterra model (or predator-prey) involves two species. One of them (the predators) feeds on the other species (the prey), which in turn feeds on some third food available around. A standard example is a population of foxes and rabbits in a woodland. The assumptions about the environment and evolution of the predator and prey populations see [5] are:

- The prey population have an unlimited food supply at all times.

- In the absence of predators, the prey population x would grow proportionally to its size, $dx/dt = \alpha x, \alpha > 0$. The coefficient α was named by Volterra the coefficient of auto-increase. (This Malthus-type equation gives by integration the geometrical law of increase $x(t) = x_0 exp(\alpha t)$).

- In the absence of prey, the predator population y would decline proportionally to its size, $dy/dt = -\gamma y, \gamma > 0$. (By integration we get in this case $y(t) = y_0 exp(-\gamma t)$, meaning the final extinction of this population).

- When both predator and prey are present, a decline in the prey population and a growth in the predator population will occur, each at a rate proportional to the frequency of encounters between individuals of the two species ($-\beta xy$ for prey, δxy for predators,$\beta, \delta > 0$).

When the interaction rate is adjoined to the natural rate, the prey equation becomes

$$\frac{dx}{dt} = \alpha x - \beta xy,$$

and may be interpreted as: the change in the prey's numbers is given by its own growth minus the rate at which it is preyed upon. Similarly, the predator

$$\frac{dy}{dt} = -\gamma y + \delta xy,$$

where δxy represents the growth of the predator population. Hence the equation expresses the change in the predator population as growth determined by the food supply, minus natural death.

4.2 Existence of positive periodic solutions

A qualitative analysis such as periodicity, positivity of solutions of delay differential equations has been studied extensively by many authors. Indeed, a famous model for population dynamics is the Lotka-Volterra competition system. Due to the various seasonal effects present in real life situations, it is reasonable and practical to study Lotka-Volterra competition systems with periodic coefficients (see [8],[22],[32],[38],[47],[48],[49], [67], [51], [53]-[65], [73]). The existence of positive periodic solutions of delay differential equations with or without impulsive effects has been an object of active research. Some appropriate references are ,[42], [41] , [8], [49], [67],[53] ,[65] , [61],[73] . In particular, Tang and Zhou [64] investigated the existence of positive periodic solutions of the following system with variable delays,

$$x_i'(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}(t))\right],\ i = 1, 2, ..., n, \tag{4.1}$$

by using Krasnoselskii's fixed point theorem. By the same method, Zhang *et al.*[73] investigated the existence and global attractivity of positive periodic solutions of 3-species Lotka-Volterra predator-prey systems with infinite delays given by

$$\begin{cases} x_1'(t) = x_1(t)\Big(r_1(t) - c_{11}(t)x_1(t) - c_{12}(t)\int_{-\infty}^{t} K_{12}(s-t)\,x_2(s)ds \\ \qquad\qquad + c_{13}(t)\int_{-\infty}^{t} K_{13}(s-t)\,x_3(s)ds\Big), \\ x_2'(t) = x_2(t)\Big(r_2(t) - c_{21}(t)\int_{-\infty}^{t} K_{21}(s-t)\,x_1(s)ds - c_{22}(t)x_2(t) \\ \qquad\qquad + c_{23}(t)\int_{-\infty}^{t} K_{23}(s-t)\,x_3(s)ds\Big), \\ x_3'(t) = x_3(t)\Big(r_3(t) + c_{31}(t)\int_{-\infty}^{t} K_{31}(s-t)\,x_1(s)ds \\ \qquad\qquad + c_{32}(t)\int_{-\infty}^{t} K_{32}(s-t)\,x_2(s)ds - c_{33}(t)x_3(t)\Big). \end{cases} \tag{4.2}$$

Recently, Benhadri *et al.* [8] improved the results of Tang and Zhou [64] to a generalized nonlinear Lotka-Volterra competition with variable delays (and without impulses) of the form:

$$\begin{aligned} x_i'(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\Big], \\ &\quad i = 1,2,...,n. \end{aligned} \tag{4.3}$$

Namely, the authors derived some sufficient conditions for the existence of positive periodic solutions of (4.3).

Luo and Luo [58] used a fixed point theorem of strict-set-contraction to study the existence of positive periodic solutions of the following impulsive neutral Lotka-Volterra system with distributed delays

$$\begin{aligned} x_i'(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)\int_{-\tau_{ij}}^{0} f_{ij}(\xi)\,x_j(t+\xi)d\xi \\ &\quad - \sum_{j=1}^{n} c_{ij}(t)\int_{-\sigma_{ij}}^{0} g_{ij}(\xi)\,x_j'(t+\xi)d\xi\Big], \\ &\quad i = 1,2,...,n,\ t \neq t_k, \\ &\quad x_i(t_k^+) - x_i(t_k^-) = -I_{ik}(x_i(t_k)),\ t = t_k,\ k \in \mathbb{N}^*. \end{aligned} \tag{4.4}$$

The technique and theory to treat the periodicity for the solution of system 4.4 with the negativity of the nonlinear impulse function Δx_i and the term neutral in [58] are completely different from these of this paper. A typical result based on fixed point theory arguments follows a number of arguments adapted to the special structure of the equation under consideration. This leads to many different results in the literature for different classes of equations, for example, with time dependent delays, distributed delays, neutral terms, and certain nonlinearities (see, for example [8],[49], [64], [73]).

In this chapter, mainly motivated by the content in [8], [49] and [64], Benhadri el al. (see[??]) generalized system 4.1 to a model with distributed delays and impulses,

$$x_i'(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)h_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) - \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} D_{ij}(t,\rho)g_j(x_j(\rho))\,d\rho\right], \tag{4.5}$$

$t \neq t_k,\ k \in \mathbb{N}^*,$

$$x_i(t_k^+) - x_i(t_k^-) = I_{ik}(t_k, x_i(t_k)),\ t = t_k,\ k \in \mathbb{N}^*,$$

where $x_i \in \mathbb{R}^+$, $D_{ij} \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$, $h_j, f_j, g_j : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous, and $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$, for $i, j = 1, 2, ..., n$. Moreover, the expression $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(t_{ik}, x_i(t_k))$ denotes the impulse at moment t_k, $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and the left-hand limits of $x_i(t)$ at the impulsive moment t_k respectively, and $I_{ik}(\cdot,\cdot) \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$, $k \in \mathbb{N}^*$. We assume that there exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $I_{i(k+q)}(t_{k+q}, x_i(t_{k+q})) = I_{ik}(t_k, x_i(t_k))$, $k \in \mathbb{N}^*$, where $0 < t_1 < t_2 < ... < t_q < \omega$.

We emphasize that this assumption on the impulse times and functions is crucial for our main results, and therefore our results can only be obtained in this situation, which is also used in previous published literature (see, e.g. Luo and Luo [49]).

Since we are searching for the existence of periodic solutions for system 4.5, it is natural to assume that

$$\begin{aligned} a_{ij}(t+\omega) &= a_{ij}(t), \tau_{ij}(t+\omega) = \tau_{ij}(t), \ D_{ij}(t+\omega, s+\omega) = D_{ij}(t,s), \\ b_{ij}(t+\omega) &= b_{ij}(t), c_{ij}(t+\omega) = c_{ij}(t), \ r_i(t+\omega) = r_i(t), i,j = 1,2,...,n, \end{aligned}$$

for all $t \in \mathbb{R}$, with τ_{ij} being scalar functions, continuous, and $\tau_{ij}(t) \geq \tau_{ij}^* > 0$.

Therefore, the aim of this chapter is to study the approach provided by Krasnoselskii's fixed point theorem in a cone of a Banach space, and to obtain recent contributions in the literature by considering the above general class of equation 4.5 with impulsive effects. For this type of equations, we combine different techniques to prove new results of the existence of positive periodic solutions.

The chapter is organized as follows. In Section 2, we recall some results which are necessary for our analysis. The existence of positive periodic solutions of system 4.5 by using the Krasonoselskii fixed point theorem is proved in Section 3, splitting the analysis into two cases for the impulsive functions: subquadratic and sublinear. Finally, in Section 4, we exhibit an example to illustrate the validity of our result.

4.3 Inversion of equation

Before stating the main result of this paper, we establish the equivalent integral formulation for the solution of equation 4.5.

Lemma 4.1 *A function $x(\cdot)$ is an $\omega-$periodic solution of equation 4.5 if and only if $x(\cdot)$ is an $\omega-$periodic solution of the following equation*

$$\begin{aligned} x_i(t) &= \int_t^{t+\omega} G_i(t,s)\,x_i(s) \times \left[\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) \right. \\ &\quad + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \\ &\quad \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\ &\quad + \sum_{t \le t_k < t+\omega} G_i(t,t_k)\, I_{ik}(t_k, x_i(t_k)), \qquad (4.6) \\ i &= 1,2,...,n, \end{aligned}$$

where

$$G_i(t,s) = \frac{1}{1-e^{-\int_0^\omega r_i(z)dz}} \exp\left(-\int_t^s r_i(z)\,dz\right), \; t \le s \le t+\omega, \qquad (4.7)$$

and

$$e^{-\int_0^\omega r_i(z)dz} \ne 1.$$

Proof. Unlike the procedure carried out in [64], where the authors used the variation of constants formula to rewrite the original equation as an integral equation, we have to proceed in a very different way which is motivated and justified by the appearance of the impulsive terms in our problem. Assume that $x = (x_1, x_2, ..., x_n)^T \in X$, is a periodic solution of equation 4.5. For any $t \in \mathbb{R}$, there exists $k \in \mathbb{N}^*$ such that t_k is the first impulsive point after t. Multiplying both sides of 4.5 by $\exp\left(-\int_0^t r_i(z)\,dz\right)$ and then integrating from t to $u \in [t, t_k]$, $k \in \mathbb{N}^*$, yields

$$\begin{aligned}
&\int_t^u \left[x_i(s)\exp\left(-\int_0^s r_i(z)\,dz\right)\right]' \\
&= \int_t^u \exp\left(-\int_0^s r_i(z)\,dz\right) x_i(s) \\
&\quad \times \left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. - \sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\rho\right] ds,
\end{aligned}$$

or

$$\begin{aligned}
&x_i(u)\exp\left(-\int_0^u r_i(z)\,dz\right) \\
&= x_i(t)\exp\left(-\int_0^t r_i(z)\,dz\right) + \int_t^u \exp\left(-\int_0^s r_i(z)\,dz\right) x_i(s) \\
&\quad \times \left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. - \sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(u,\rho)g_j\left(x_j(\rho)\right)d\rho\right] ds,
\end{aligned}$$

then

$$\begin{aligned}
x_i(u) &= x_i(t)\exp\left(-\int_u^t r_i(z)\,dz\right) + \int_t^u \exp\left(-\int_u^s r_i(z)\,dz\right) x_i(s) \\
&\quad \times \left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. - \sum_{j=1}^{n} c_{ij}(u)\int_{-\infty}^{s} D_{ij}(u,\rho)g_j\left(x_j(\rho)\right)d\rho\right] ds, \\
i &= 1, 2, ..., n,
\end{aligned}$$

hence

$$\begin{aligned}
x_i(t_k) &= x_i(t)\exp\left(-\int_{t_k}^{t} r_i(z)\,dz\right) + \int_{t}^{t_k}\exp\left(-\int_{t_k}^{s} r_i(z)\,dz\right)x_i(s) \\
&\quad \times\left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. - \sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\rho\right] ds, \qquad (4.8) \\
i &= 1,2,...,n.
\end{aligned}$$

Similarly, for $u \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
x_i(u) &= x_i(t_k^+)\exp\left(-\int_{u}^{t_k} r_i(z)\,dz\right) + \int_{t_k}^{u}\exp\left(-\int_{u}^{s} r_i(z)\,dz\right)x_i(s) \\
&\quad \times\left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. - \sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\rho\right] ds,
\end{aligned}$$

$$\begin{aligned}
&= x_i(t_k^-)\exp\left(-\int_u^{t_k} r_i(z)\,dz\right) + \int_{t_k}^{u}\exp\left(-\int_u^{s} r_i(z)\,dz\right)x_i(s)\\
&\quad\times\left[-\sum_{j=1}^{n}a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n}b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right.\\
&\quad\left.-\sum_{j=1}^{n}c_{ij}(s)\int_{-\infty}^{s}D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\rho\right]ds\\
&\quad+\exp\left(-\int_u^{t_k} r_i(z)\,dz\right)I_{ik}\left(t_k, x_i(t_k)\right)\\
&= x_i(t_k)\exp\left(-\int_u^{t_k} r_i(z)\,dz\right) + \int_{t_k}^{u}\exp\left(-\int_u^{s} r_i(z)\,dz\right)x_i(s)\\
&\quad\times\left[-\sum_{j=1}^{n}a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n}b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right.\\
&\quad\left.-\sum_{j=1}^{n}c_{ij}(s)\int_{-\infty}^{s}D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\sigma\right]ds\\
&\quad+\exp\left(-\int_u^{t_k} r_i(z)\,dz\right)I_{ik}\left(t_k, x_i(t_k)\right),
\end{aligned}$$

for $i = 1, 2, ..., n$. Substituting 4.8 in the above equality, we obtain

$$\begin{aligned}
x_i(u) &= x_i(t)\exp\left(-\int_u^{t} r_i(z)\,dz\right) + \int_{t}^{u}\exp\left(-\int_u^{s} r_i(z)\,dz\right)x_i(s)\\
&\quad\times\left[-\sum_{j=1}^{n}a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n}b_{ij}(s)f_j\left(x_j(s-\tau_{ij}(s))\right)\right.\\
&\quad\left.-\sum_{j=1}^{n}c_{ij}(s)\int_{-\infty}^{s}D_{ij}(s,\rho)g_j\left(x_j(\rho)\right)d\rho\right]ds\\
&\quad+\exp\left(-\int_u^{t_k} r_i(z)\,dz\right)I_{ik}\left(t_k, x_i(t_k)\right).
\end{aligned}$$

Repeating the above procedure for $u \in [t, t+\omega]$, we obtain

$$\begin{aligned} x_i(u) &= x_i(t)\exp\left(-\int_u^t r_i(z)\,dz\right) + \int_t^u \exp\left(-\int_u^s r_i(z)\,dz\right) x_i(s) \\ &\times \left[-\sum_{j=1}^n a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\ &\left. - \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,\rho) g_j(x_j(\rho))\,d\rho\right] ds \\ &+ \sum_{t\le t_k<t+\omega} \exp\left(-\int_u^{t_k} r_i(z)\,dz\right) I_{ik}(t_k, x_i(t_k)), \end{aligned}$$

for $i = 1, 2, ..., n$. Let $u = t+\omega$ in the above equality. Then

$$\begin{aligned} x_i(t+\omega) &= x_i(t)\exp\left(\int_t^{t+\omega} r_i(z)\,dz\right) + \int_t^{t+\omega} \exp\left(-\int_{t+\omega}^s r_i(z)\,dz\right) x_i(s) \\ &\times \left[-\sum_{j=1}^n a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\ &\left. - \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,\rho) g_j(x_j(\rho))\,d\rho\right] ds \\ &+ \sum_{t\le t_k<t+\omega} \exp\left(\int_{t+\omega}^{t_k} r_i(z)\,dz\right) I_{ik}(t_k, x_i(t_k)), \end{aligned}$$

$i = 1, 2, , ..., n$. Notice that $x_i(t+\omega) = x_i(t)$ and $\exp\left(\int_t^{t+\omega} r_i(z)\,dz\right) = \exp\left(\int_0^{\omega} r_i(z)\,dz\right)$, we obtain

$$\begin{aligned} \left(1-\exp\left(\int_0^{\omega} r_i(z)\,dz\right)\right) x_i(t) &= \int_t^{t+\omega} \exp\left(-\int_{t+\omega}^s r_i(z)\,dz\right) x_i(s) \\ &\times \left[-\sum_{j=1}^n a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\ &\left. - \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,\rho) g_j(x_j(\rho))\,d\rho\right] ds \\ &+ \sum_{t\le t_k<t+\omega} \exp\left(-\int_{t+\omega}^{t_k} r_i(z)\,dz\right) I_{ik}(t_k, x_i(t_k)), \quad (4.9) \\ i &= 1, 2, , ..., n. \end{aligned}$$

It follows from 4.9 that

$$\begin{aligned}
x_i(t) &= \int_t^{t+\omega} \frac{\exp\left(-\int_{t+\omega}^{s} r_i(z)\,dz\right)}{\left(1-\exp\left(\int_0^{\omega} r_i(z)\,dz\right)\right)} x_i(s) \\
&\quad \times \left[-\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) - \sum_{j=1}^{n} b_{ij}(s) f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. -\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,\rho) g_j\left(x_j(\rho)\right) d\rho\right] ds \\
&\quad + \sum_{t\le t_k<t+\omega} \frac{\exp\left(-\int_{t+\omega}^{t_k} r_i(z)\,dz\right)}{\left(1-\exp\left(\int_0^{\omega} r_i(z)\,dz\right)\right)} I_{ik}\left(t_k, x_i(t_k)\right) \\
&= \int_t^{t+\omega} G_i(t,s)\, x_i(s) \\
&\quad \times \left[\sum_{j=1}^{n} a_{ij}(s)h_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j\left(x_j(s-\tau_{ij}(s))\right)\right. \\
&\quad \left. +\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,\rho) g_j\left(x_j(\rho)\right) d\rho\right] ds \\
&\quad + \sum_{t\le t_k<t+\omega} G_i(t,t_k)\, I_{ik}\left(t_k, x_i(t_k)\right)
\end{aligned}$$

for $i = 1, 2, , ..., n$. It is clear that, for $(t, s) \in \mathbb{R}^2$, we have

$$\begin{aligned}
G_i(t+\omega, s+\omega) &= G_i(t,s), G_i(t,t+\omega) - G_i(t,t) = -1, \\
(\partial G_i(t,s)/\partial t) &= r_i(t)\, G_i(t,s), \ \ i = 1, 2, ..., n.
\end{aligned}$$

Next, we prove the converse. Let $x = (x_1(t), x_2(t), ..., x_n(t))^T$ be a periodic solution of 4.6.

Then, if $t \neq t_k, k \in \mathbb{N}^*$, we have

$$\begin{aligned} x_i'(t) &= G_i(t, t+\omega)\, x_i(t+\omega) \\ &\quad \times \left[\sum_{j=1}^{n} a_{ij}(t+\omega) h_j(x_j(t+\omega)) + \sum_{j=1}^{n} b_{ij}(t+\omega) f_j(x_j(t+\omega-\tau_j(t+\omega)) \right. \\ &\qquad \left. + \sum_{j=1}^{n} c_{ij}(t+\omega) \int_{-\infty}^{t+\omega} D_{ij}(t+\omega, \rho) g_j(x_j(\rho)) d\rho \right] \\ &\quad - G_i(t,t)\, x_i(t) \left[\sum_{j=1}^{n} a_{ij}(t) h_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t)) \right. \\ &\quad \left. + \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,\rho) g_j(x_j(\rho)) d\rho \right] + r_i(t) x_i(t) \\ &= x_i(t) \left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t) h_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t)) \right. \\ &\quad \left. - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,\rho) g_j(x_j(\rho)) d\rho \right]. \end{aligned}$$

If $t = t_j, j \in \mathbb{N}^*$, we obtain

$$\begin{aligned} & x_i(t_j^+) - x_i(t_j^-) \\ = & \sum_{t_j^+ \leq t_k < t_j^+ + \omega} G_i(t_j, t_k) I_{ik}(t_k, x_i(t_k)) - \sum_{t_j^- \leq t_k < t_j^- + \omega} G_i(t_j, t_k) I_{ik}(t_k, x_i(t_k)) \\ = & G_i(t_j, t_j+\omega) I_{ij}(t_j+\omega, x_i(t_j+\omega)) - G_i(t_j, t_j) I_{ij}(t_j, x_i(t_j)) \\ = & I_{ij}(t_j, x_i(t_j)), i = 1, 2, , ..., n. \end{aligned}$$

Hence x is a positive $\omega-$periodic solution of 4.5. Thus, the proof of Lemma 4.1 is completed. ■

4.4 Krasnoselskii's fixed point theorem and Existence of positive periodic solutions

As we mentioned previously, one of our main objectives in this paper is to improve the work carried out in [64], and to extend it to investigate a wider class of differential equations with impulsive effects presented in 4.5. In particular, by using Krasnoselskii's fixed point theorem in a cone of a Banach space, we will establish a sufficient condition ensuring the existence of positive $\omega-$periodic solutions of equation 4.5. This section will be splitted into two parts: in the first one, we will focus on the existence of positive periodic solutions when we use subquadratic impulse functions, while in the second part, we will consider the case in which the impulse effects are sublinear (most frequently used in the published literature).

Throughout this paper, we make the following assumptions.

(H_1) There exist nonnegative constants $\overline{T}_j, T_j,\ \overline{F}_j, F_j,\ \overline{R}_j, R_j,\ \overline{E}_{ij}, E_{ij}$ such that for all $x \in \mathbb{R}^+$, and all $t \in \mathbb{R}^+$,

$$\overline{T}_j x \leq h_j\left(x\right) \leq T_j x,\ \ j = 1, 2, ..., n. \tag{4.10}$$

$$\overline{F}_j x \leq f_j\left(x\right) \leq F_j x,\ \ j = 1, 2, ..., n. \tag{4.11}$$

$$\overline{R}_j x \leq g_j\left(x\right) \leq R_j x,\ \ j = 1, 2, ..., n. \tag{4.12}$$

$$\overline{E}_{ij} \leq \int_{-\infty}^{t} D_{ij}(t,s)ds \leq E_{ij},\ \ i, j = 1, 2, ..., n. \tag{4.13}$$

To simplify our description, we introduce the following notations:

$$\widehat{r}_i = \frac{1}{\omega}\int_0^\omega r_i(s)ds > 0, \qquad \widehat{a}_{ij} = \frac{T_j}{\omega}\int_0^\omega a_{ij}(s)ds \geq 0,$$

$$\widehat{b}_{ij} = \frac{F_j}{\omega}\int_0^\omega b_{ij}(s)ds \geq 0, \qquad \widehat{c}_{ij} = \frac{R_j E_{ij}}{\omega}\int_0^\omega c_{ij}(s)ds \geq 0,$$

for $i, j = 1, 2, ..., n$, where T_j, F_j and R_j, E_{ij} are given in 4.10−4.13.

To apply Theorem 4.1, we need to define a Banach space C_ω, a closed subset S of C_ω and construct one mapping. Thus, we let $(C_\omega, \|.\|) = (X, \|.\|)$ where

$$C_\omega = \left\{ x = (x_1, x_2, ..., x_n)^T : x \in PC(\mathbb{R}, \mathbb{R}^n), x(t+\omega) = x(t), t \in \mathbb{R} \right\}, \tag{4.14}$$

with the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|_0, \ \ |x_i|_0 = \max_{t\in[0,\omega]} |x_i(t)|, i = 1, 2, ..., n, \ \forall x \in C_\omega. \tag{4.15}$$

Then, C_ω with the norm $\|.\|$ is a Banach space.

Define a cone K in C_ω by

$$K = \left\{ x(\cdot) = (x_1, x_2, ..., x_n)^T \in C_\omega : x_i(t) \geq \sigma |x_i|_0, i = 1, 2, ..., n, \forall t \in \mathbb{R} \right\},$$

where $\sigma = \min\left\{ e^{-\widehat{r}_i \omega}, \ i = 1, 2, \ldots, n \right\}$.

Use 4.6 to define the operator $\Phi : C_\omega \to C_\omega$ by

$$(\Phi x)(t) := \left[(\Phi_1 x)(t), (\Phi_2 x)(t), ..., (\Phi_n x)(t) \right]^T,$$

where

$$\begin{aligned} (\Phi_i x)(t) \ &= \int_t^{t+\omega} G_i(t,s)\, x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) \right. \\ &+ \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s - \tau_{ij}(s))) \\ &+ \left. \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\ &+ \sum_{t \leq t_k < t+\omega} G_i(t, t_k)\, I_{ik}(t_k, x_i(t_k)). \end{aligned} \tag{4.16}$$

By 4.6, it is easy to check that $x \in C_\omega$ is an $\omega-$periodic solution of equation 4.5 provided x is a fixed point of Φ.

Lemma 4.2 *Assume that (H_1) holds. Then $\Phi : K \to K$ defined by Equation 4.16 is well defined, namely, $\Phi K \subset K$.*

Proof. From 4.16 it is easy to verify that $(\Phi x)(\cdot)$ is continuous in (t_k, t_{k+1}), and $(\Phi x)\left(t_k^+\right)$ and $(\Phi x)\left(t_k^-\right)$ exist, and $(\Phi x)\left(t_k^-\right) = (\Phi x)(t_k)$ for $k \in \mathbb{N}^*$. Moreover, for any $x \in K$,

$$
\begin{aligned}
&(\Phi_i x)(t+\omega)\\
&= \int_{t+\omega}^{t+2\omega} G_i(t+\omega, s)\, x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s))\right.\\
&\quad + \sum_{j=1}^{n} b_{ij}(s) f_j\left(x_j(s-\tau_{ij}(s))\right)\\
&\left.+ \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho\right] ds\\
&\quad + \sum_{t+\omega \le t_k < t+2\omega} G_i(t, t_k)\, I_{ik}(t_k, x_i(t_k))
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{t}^{t+\omega} G_i(t+\omega, s+\omega)\, x_i(s+\omega) \times\\
&\times \left[\sum_{j=1}^{n} a_{ij}(s+\omega) h_j(x_j(s+\omega)) + \sum_{j=1}^{n} b_{ij}(s+\omega) f_j(x_j(s+\omega))\right.\\
&\left.+ \sum_{j=1}^{n} c_{ij}(s+\omega) \int_{-\infty}^{s+\omega} D_{ij}(s+\omega,\rho) g_j(u_j(\rho)) d\rho\right] ds\\
&+ \sum_{t \le t_k < t+\omega} G_i(t, t_k)\, I_{ik}(t_k, x_i(t_k))
\end{aligned}
$$

$$\begin{aligned}
&= \int_t^{t+\omega} G_i(t,s)\, x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\
&\quad \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\
&\quad + \sum_{t \le t_k < t+\omega} G_i(t,t_k)\, I_{ik}(t_k, x_i(t_k)) \\
&= (\Phi_i x)(t), \ \ i = 1,2,...,n.
\end{aligned}$$

That is $(\Phi_i x)(t+\omega) = (\Phi_i x)(t)$, $t \in \mathbb{R}, i = 1,2,...,n$. Thus $\Phi x \in C_\omega$. It is easy to see that for $s \in [t, t+\omega]$, thanks to 4.7, we have

$$A_i := \frac{e^{-\widehat{r}_i \omega}}{1 - e^{-\widehat{r}_i \omega}} \le G_i(t,s) \le \frac{1}{1 - e^{-\widehat{r}_i \omega}} =: B_i, \ \ i = 1,2,...,n. \tag{4.17}$$

From 4.16 and 4.17, we have for $x \in K$

$$\begin{aligned}
|(\Phi_i x)|_0 \ \ \le \ \ & B_i \int_0^{\omega} x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) \right. \\
& + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \\
& \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\
& + B_i \sum_{k=1}^{q} I_{ik}(t_k, x_i(t_k)),
\end{aligned}$$

and

$$\begin{aligned}
(\Phi_i x) &\geq A_i \int_0^{\omega} x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j\left(x_j(s)\right) + \sum_{j=1}^{n} b_{ij}(s) f_j\left(x_j(s-\tau_{ij}(s))\right) \right. \\
&\quad \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j\left(x_j(\rho)\right) d\rho \right] ds \\
&\quad + A_i \sum_{k=1}^{q} I_{ik}\left(t_k, x_i\left(t_k\right)\right) \\
&\geq \frac{A_i}{B_i} \left|(\Phi_i x)\right|_0 \\
&\geq \sigma \left|(\Phi_i x)\right|_0, \; i = 1, 2, ..., n.
\end{aligned}$$

Hence, $\Phi K \subset K$. This completes the proof of Lemma 4.2. ■

Before establishing the main result, we first introduce the following notation:

$$\varphi^M = \max_{t\in[0,\omega]} \left\{ |\varphi(t)| \right\},$$

where φ is a continuous and $\omega-$periodic function.

4.4.1 The case of subquadratic impulses.

In this section we consider subquadratic impulse functions.

Lemma 4.3 *In addition to hypothesis (H_1), we further assume the following one:*

(H_2) There exist nonnegative functions $\overline{\lambda}_{ik}, \lambda_{ik} \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for all $x \in \mathbb{R}^+$,

$$\overline{\lambda}_{ik}(t) x^2 \leq I_{ik}(t,x) \leq \lambda_{ik}(t) x^2, \; k \in \mathbb{N}^*, \; i = 1, 2, ..., n.$$

Then $\Phi : K \to K$ defined by equation 4.16 is completely continuous.

Proof. Set

$$\Gamma_i\left(t,x\right)(t) = x_i(t)\left[\sum_{j=1}^{n} a_{ij}(t)h_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij}(t)f_j\left(x_j(t-\tau_{ij}\left(t\right))\right) + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} D_{ij}(t,\rho)g_j\left(x_j(\rho)\right)d\rho\right],\ t \in \mathbb{R}. \tag{4.18}$$

We first show that Φ is continuous. Since h_j, f_j, g_j and I are continuous in x, it follows that, for any $L_0 > 0$ and $\varepsilon > 0$, there exists $\mu_1 > 0$ such that for $\|x\| \leq L_0$, $\|y\| \leq L_0$, and $\|x-y\| < \mu_1$ it follows

$$\left|\Gamma_i\left(s,x\right)(s) - \Gamma_i\left(s,y\right)(s)\right| < \frac{\varepsilon}{2nB\omega},\ \ s \in \mathbb{R}^+,\ \ i = 1,2,...,n, \tag{4.19}$$

where $B = \max\limits_{1\leq i\leq n} B_i$. For any $L_0 > 0$ and $\varepsilon > 0$, there exists $\mu_2 > 0$ such that for $\|x\| \leq L_0$, $\|y\| \leq L_0$, and $\|x-y\| < \mu_2$

$$\left|I_{ik}\left(t_k, x_i\left(t_k\right)\right) - I_{ik}\left(t_k, y_i\left(t_k\right)\right)\right| < \frac{\varepsilon}{2qBn},\ \ q \in \mathbb{Z}^+\ i = 1,2,...,n. \tag{4.20}$$

Therefore, if $x, y \in C_\omega$ with $\|x\| \leq L_0$, $\|y\| \leq L_0$, and $\|x-y\| \leq \mu$, where $\mu = \min\left(\mu_1, \mu_2\right)$ then, from 4.16,4.17,4.19 and 4.20,

$$\begin{aligned}
\left|(\Phi_i x) - (\Phi_i y)\right|_0 &\leq B\int_t^{t+\omega}\left|\Gamma_i\left(s,x\right)(s) - \Gamma_i\left(s,y\right)(s)\right| ds \\
&\quad + B\sum_{k=1}^{q}\left|I_{ik}\left(t_k, x_i\left(t_k\right)\right) - I_{ik}\left(t_k, y_i\left(t_k\right)\right)\right| \\
&\leq B\frac{\omega\varepsilon}{2nB\omega} + Bq\frac{\varepsilon}{2qBn} \\
&< \frac{\varepsilon}{n},\ \ i = 1,2,...,n.
\end{aligned}$$

This yields

$$\|\Phi x - \Phi y\| = \sum_{i=1}^{n}\left|(\Phi_i x) - (\Phi_i y)\right|_0 < \varepsilon,$$

which implies that Φ is continuous on K.

We let

$$S = \{x(\cdot) = \left(x_1\left(\cdot\right), x_2\left(\cdot\right), ..., x_n\left(\cdot\right)\right)^T \in C_\omega :\ \ ||x|| \leq L\ \},$$

where L is a non-negative constant. For any $x \in S$, it follows from 4.16 and 4.17 that

$$\begin{aligned}
(\Phi_i x)(t) &= \int_t^{t+\omega} G_i(t,s)\, x_i(s) \left[\sum_{j=1}^n a_{ij}(s) h_j(x_j(s)) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\
&\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\
&\quad + \sum_{t \le t_k < t+\omega} G_i(t,t_k)\, I_{ik}(t_k, x_i(t_k)) \\
&\le B_i L^2 \int_0^{\omega} \left[\sum_{j=1}^n T_j a_{ij}(s) + \sum_{j=1}^n F_j b_{ij}(s) + \sum_{j=1}^n R_j E_{ij} c_{ij}(s) \right] ds \\
&\quad + B_i L^2 \sum_{0 \le t_k < \omega} \lambda_{ik}(t_k) \\
:&= B_i^*, \quad i = 1,2,...,n,
\end{aligned}$$

and, consequently,

$$\|\Phi x\| = \sum_{i=1}^n |(\Phi_i x)|_0 \le \sum_{i=1}^n B_i^*, \quad \forall x \in S.$$

This shows that $\Phi(S)$ is uniformly bounded.

To show that $\Phi(S)$ is equicontinuous, let $x \in S$, we calculate $\frac{d}{dt}(\Phi_i x)(t)$ and show that it is uniformly bounded. Indeed, by taking derivative in (4.16) we have

$$\begin{aligned}
\left|(\Phi_i x)'(t)\right| &\le \left| r_i(t)(\Phi_i x)(t) - x_i(t) \left[\sum_{j=1}^n a_{ij}(t) h_j(x_j(t)) \right.\right. \\
&\quad + \sum_{j=1}^n b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\
&\quad \left.\left. + \sum_{j=1}^n c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,\rho) g_j(x_j(\rho)) d\rho \right] \right| \\
&\le r_i^M B_i^* + L^2 \sum_{j=1}^n \left(T_j a_{ij}^M + F_j b_{ij}^M + E_{ij} R_j c_{ij}^M \right), \\
i &= 1,2,...,n,
\end{aligned}$$

and

$$\| (\Phi x)' \| \leq \sum_{j=1}^{n} \left[r_i^M B_i^* + L^2 \sum_{j=1}^{n} \left(T_j a_{ij}^M + F_j b_{ij}^M + E_{ij} R_j c_{ij}^M \right) \right].$$

Hence, $\Phi S \subset C_\omega$ is a family of uniformly bounded and equi-continuous functions. By the Ascoli-Arzelà Theorem, the operator Φ is compact, and therefore completely continuous. The proof is complete. ■

Throughout the next steps, we denote $\theta = \min(\theta_1, \theta_2, \theta_3, \theta_4)$, where

$$\begin{aligned} \theta_1 &= \min_{j=\overline{1,n}} \left(\frac{\overline{T}_j}{T_j} \right), \theta_2 = \min_{j=\overline{1,n}} \left(\frac{\overline{F}_j}{F_j} \right), \\ \theta_3 &= \min_{j=\overline{1,n}} \left[\min_{i=\overline{1,n}} \left(\frac{\overline{E}_{ij}}{E_{ij}} \right) \frac{\overline{R}_j}{R_j} \right], \theta_4 = \min_{k=\overline{1,q}} \left[\min_{i=\overline{1,n}} \left(\frac{\overline{\lambda}_{ik}(t_k)}{\lambda_{ik}(t_k)} \right) \right]. \end{aligned} \tag{4.21}$$

We can now state and prove our main result in this paper.

Theorem 4.1 *Assume hypothesis (H_1) and the next one:(H_3) The linear system*

$$\sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j + \widehat{\beta}_{ik} x_i = \widehat{r}_i, \;\; k \in \mathbb{N}^*, i = 1, 2, ..., n, \tag{4.22}$$

where

$$\widehat{\beta}_{ik} = \frac{1}{\omega} \sum_{k=1}^{q} \lambda_{ik}(t_k), \; (\lambda_{ik}(t_k) \neq 0),$$

possesses a unique positive solution. Then, system (4.5) possesses at least one positive $\omega-$periodic solution.

Proof. Let

$$x^* = (x_1^*, x_2^*, ..., x_n^*)^T$$

with $x_i^* > 0, i = 1, 2, ..., n$, be a positive solution of (4.22). Set

$$\begin{aligned} m_0 &= \min_{1 \leq i \leq n} \{\widehat{r}_i A_i\}, \\ M_0 &= \max_{1 \leq i \leq n} \{\widehat{r}_i B_i\}. \end{aligned}$$

Then $0 < m_0 < M_0 < +\infty$. Choose a constant $M \geq M_0$ such that $\dfrac{1}{M\omega} < 1$. Let $\alpha_1 = \dfrac{1}{M\omega}$ and

$\Omega_1 = \left\{ x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C_\omega : |x_i|_0 < \alpha_1 x_i^*, i = 1, 2, ..., n \right\}.$

If $x \in K \cap \partial\Omega_1$, then

$$\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \alpha_1 x_i^*, i = 1, 2, ..., n,$$

and

$$\begin{aligned}
(\Phi_i x)(t) \leq\; & B_i \int_0^\omega x_i(s) \left[\sum_{j=1}^n a_{ij}(s) h_j(x_j(s)) \right. \\
& + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s - \tau_{ij}(s))) \\
& \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s, \rho) g_j(x_j(\rho)) d\rho \right] ds \\
& + B_i \sum_{0 \leq t_k < \omega} I_{ik}(t_k, x_i(t_k)) \\
\leq\; & B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n T_j a_{ij}(s) |x_j|_0 \, ds + B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n F_j b_{ij}(s) |x_j|_0 \, ds \\
& + B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n R_j E_{ij} c_{ij}(s) |x_j|_0 \, ds + B_i |x_i|_0 \sum_{0 \leq t_k < \omega} \lambda_i(t_k) |x_i|_0 \\
\leq\; & \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^* + \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
& + \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^* + \alpha_1 B_i \omega |x_i|_0 \widehat{\beta}_{ik} x_i^* \\
=\; & \alpha_1 B_i \omega |x_i|_0 \left[\sum_{j=1}^n \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j^* + \widehat{\beta}_{ik} x_i^* \right] \\
=\; & (B_i \widehat{r}_i) \alpha_1 \omega |x_i|_0 \\
\leq\; & \alpha_1 M_0 \omega |x_i|_0 \\
\leq\; & |x_i|_0, \quad i = 1, 2, ..., n.
\end{aligned}$$

Hence for any $x \in K \cap \partial\Omega_1$

$$\|\Phi x\| = \sum_{i=1}^{n} |(\Phi_i x_i)|_0 \leq \sum_{i=1}^{n} |x_i|_0 = \|x\|.$$

On the other hand, choose $0 < m \leq m_0$ such that $\dfrac{1}{\sigma^2 m\theta\omega} > 1$. Let $\alpha_2 = \dfrac{1}{\sigma^2 m\theta\omega}$ and

$$\Omega_2 = \{x \in C_\omega : |x_i|_0 < \alpha_2 x_i^*, i = 1, 2, ..., n\}.$$

If $x \in K \cap \partial\Omega_2$, then $\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \alpha_2 x_i^*, i = 1, 2, ..., n$, and, consequently

$$\begin{aligned}
(\Phi_i x)(t) \geq\; & A_i \int_0^\omega x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s - \tau_{ij}(s))) \right. \\
& \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \right] ds \\
& + A_i \sum_{0 \leq t_k < \omega} I_{ik}(t_k, x_i(t_k)) \\
\geq\; & \sigma^2 A_i |x_i|_0 \sum_{j=1}^{n} \int_0^\omega a_{ij}(s) T_j \left[\min_{j=\overline{1,n}} \left(\frac{\overline{T}_j}{T_j} \right) \right] |x_j|_0 \, ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^{n} \int_0^\omega F_j \left[\min_{i=\overline{1,n}} \left(\frac{\overline{F}_j}{F_j} \right) \right] b_{ij}(s) |x_j|_0 \, ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^{n} \int_0^\omega E_{ij} R_j \left[\left[\min_{i=\overline{1,n}} \left(\frac{\overline{E}_{ij}}{E_{ij}} \right) \right] \times \left(\frac{\overline{R}_j}{R_j} \right) \right] c_{ij}(s) |x_j|_0 \, ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{k=1}^{q} \lambda_{ik}(t_k) \left[\min_{i=\overline{1,n}} \left(\frac{\overline{\lambda}_{ik}(t_k)}{\lambda_{ik}(t_k)} \right) \right] |x_i(t_k)|_0
\end{aligned}$$

$$\begin{aligned}
\geq\; & \theta_1 \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{a}_{ij} x_j^* + \theta_2 \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{b}_{ij} x_j^* \\
& + \theta_3 \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{c}_{ij} x_j^* + \theta_4 \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \widehat{\beta}_{ik} x_i^*
\end{aligned}$$

$$\begin{aligned}
&\geq \theta \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{a}_{ij} x_j^* + \theta \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{b}_{ij} x_j^* \\
&\quad + \theta \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \sum_{j=1}^{n} \widehat{c}_{ij} x_j^* \\
&\quad + \theta \times \sigma^2 A_i \omega \alpha_2 |x_i|_0 \widehat{\beta}_{ik} x_i^* \\
&= \theta A_i \omega \sigma^2 \alpha_2 |x_i|_0 \left(\sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j^* + \widehat{\beta}_{ik} x_i^* \right) \\
&= (A_i \widehat{r}_i) \theta \omega \sigma^2 \alpha_2 |x_i|_0 \\
&\geq m_0 \theta \omega \sigma^2 \alpha_2 |x_i|_0 \\
&\geq |x_i|_0, i = 1, 2, ..., n,
\end{aligned}$$

and thus

$$\|\Phi x\| = \sum_{i=1}^{n} |(\Phi_i x_i)|_0 \geq \sum_{i=1}^{n} |x_i|_0 = \|x\|, \forall x \in K \cap \partial\Omega_2.$$

Obviously, Ω_1 and Ω_2 are open bounded subsets of C_ω with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Hence, $\Phi : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator and satisfies condition (a) in Theorem 2.1. By Krasnoselskii's Theorem, there exists a fixed point $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), ..., x_n^*(\cdot))^T \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $x^*(\cdot) = (\Phi x^*)(\cdot)$, i.e., x^* is a positive $\omega-$ periodic solution of system 4.5 The proof is completed. ■

4.4.2 The case of sublinear impulse functions

In the case of sublinear impulses we can prove similar results for the system 4.5.

Theorem 4.2 . *Assume that (H_1) holds, and further assume that:*

(H_4) *There exist nonnegative functions $\overline{\zeta}_{ik}, \zeta_{ik} \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for all $x \in \mathbb{R}^+$,*

$$\overline{\zeta}_{ik}(t) x \leq I_{ik}(t, x) \leq \zeta_{ik}(t) x, i = 1, 2, ..., n, k = 1, 2, ...$$

(H_5) *The linear system*

$$\sum_{j=1}^{n}\left(\widehat{a}_{ij}+\widehat{b}_{ij}+\widehat{c}_{ij}\right)x_j=\widehat{r}_i, i=1,2,...,n,\ \ k=1,2,... \tag{4.23}$$

possesses a unique positive solution. Then, system 4.5 possesses at least one positive $\omega-$periodic solution.

Proof. To prove that $\Phi : K \to K$ is completely continuous is similar to the corresponding proof in Lemma 4.3. We only need to prove condition (a) in Theorem 4.1. Let

$$x^* = (x_1^*, x_2^*, ..., x_n^*)^T ,$$

with $x_i^* > 0, i = 1, 2, ..., n,$ be a positive solution of 4.22. We denote $\overline{\theta} =$ $\min\left(\theta_1, \theta_2, \theta_3, \theta_5\right),$ where

$$\theta_5=\min_{k=\overline{1,q}}\left[\min_{i=\overline{1,n}}\left(\frac{\overline{\zeta}_{ik}\left(t_k\right)}{\zeta_{ik}\left(t_k\right)}\right)\right],$$

and $\theta_1, \theta_2, \theta_3$ are given in 4.21. We also set

$$\widehat{\gamma}_{ik}=\frac{1}{\omega}\sum_{k=1}^{q}\zeta_{ik}\left(t_k\right),\ \ (\zeta_{ik}\left(t_k\right)\neq 0),$$

and

$$\begin{aligned}\widetilde{m}_0 &= \min\left\{\min_{1\le i\le n}\left\{\widehat{r}_iA_i\right\},\min_{1\le i\le n}\left\{A_i\widehat{\gamma}_{ik}\right\}\right\},\\ \widetilde{M}_0 &= \max\left\{\max_{1\le i\le n}\left\{\widehat{r}_iB_i\right\},\max_{1\le i\le n}\left\{B_i\widehat{\gamma}_{ik}\right\}\right\}.\end{aligned}$$

Choose a constant $\widetilde{M}\ge\widetilde{M}_0$ such that $0<\dfrac{1-\widetilde{M}\omega}{\widetilde{M}\omega}<1$ where $0<\widetilde{M}\omega<1.$ Let $\eta_1=\dfrac{1-\widetilde{M}\omega}{\widetilde{M}\omega}$ and

$$\widehat{\Omega}_1=\left\{x(t)=\left(x_1\left(t\right),x_2\left(t\right),...,x_n\left(t\right)\right)^T\in C_\omega:\left|x_i\right|_0<\eta_1x_i^*,i=1,2,...,n\right\}.$$

If $x \in K \cap \partial\widehat{\Omega}_1$, then

$$\sigma |x_i|_0 \le x_i(t) \le |x_i|_0 = \eta_1 x_i^*, i = 1, 2, ..., n,$$

and

$$\begin{aligned}
(\Phi_i x)(t) &\le B_i \int_0^\omega \Bigg[x_i(s) \sum_{j=1}^n a_{ij}(s) h_j(x_j(s)) \\
&\quad + x_i(s) \sum_{j=1}^n b_{ij}(s) f_j(x_j(s - \tau_{ij}(s))) \\
&\quad + x_i(s) \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \Bigg] ds \\
&\quad + B_i \sum_{0 \le t_k < \omega} I_{ik}(t_k, x_i(t_k)) \\
&\le B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n T_j a_{ij}(s) |x_j|_0 ds + B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n F_j b_{ij}(s) |x_j|_0 ds \\
&\quad + B_i \int_0^\omega |x_i|_0 \sum_{j=1}^n R_j E_{ij} c_{ij}(s) |x_j|_0 ds + B_i \sum_{0 \le t_k < \omega} \zeta_i(t_k) |x_i|_0 \\
&\le \eta_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^* + \eta_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
&\quad + \eta_1 B_i \omega |x_i|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^* + B_i \omega \widehat{\gamma}_{ik} |x_i|_0 \\
&= \eta_1 B_i \omega |x_i|_0 \left[\sum_{j=1}^n \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j^* \right] + B_i \omega \widehat{\gamma}_{ik} |x_i|_0 \\
&= \omega (B_i \widehat{r}_i) \eta_1 |x_i|_0 + \omega (B_i \widehat{\gamma}_{ik}) |x_i|_0 \\
&\le \eta_1 \widetilde{M}_0 \omega |x_i|_0 + \widetilde{M}_0 \omega |x_i|_0 \\
&\le |x_i|_0, \quad i = 1, 2, ..., n.
\end{aligned}$$

Hence for any $x \in K \cap \partial\widehat{\Omega}_1$

$$\|\Phi x\| = \sum_{i=1}^n |(\Phi_i x)|_0 \le \sum_{i=1}^n |x_i|_0 = \|x\|.$$

On the other hand, choose $0 < \widetilde{m} \leq \widetilde{m}_0$ such that $\dfrac{1-\sigma\widetilde{m}\theta\omega}{\sigma^2\widetilde{m}\theta\omega} > 1$ where $0 < \sigma\widetilde{m}\theta\omega < 1$. Let $\eta_2 = \dfrac{1-\sigma\widetilde{m}\theta\omega}{\sigma^2\widetilde{m}\theta\omega}$ and

$$\widehat{\Omega}_2 = \left\{x \in C_\omega : |x_i|_0 < \eta_2 x_i^*, i = 1, 2, ..., n\right\}.$$

If $x \in K \cap \partial\widehat{\Omega}_2$, then $\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \eta_2 x_i^*, i = 1, 2, ..., n$, and, consequently

$$\begin{aligned}
(\Phi_i x)(t) \geq\ & A_i \int_0^\omega \Bigg[x_i(s)\sum_{j=1}^n a_{ij}(s) h_j(x_j(s)) + x_i(s)\sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \\
& + x_i(s)\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,\rho) g_j(x_j(\rho)) d\rho \Bigg] ds \\
& + \sum_{0\leq t_k<\omega} G_i(t,t_k) I_{ik}(t_k, x_i(t_k)) \\
\geq\ & \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int_0^\omega a_{ij}(s) T_j \left[\min_{j=\overline{1,n}}\left(\frac{\overline{T}_j}{T_j}\right)\right] |x_j|_0 ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int_0^\omega F_j \left[\min_{i=\overline{1,n}}\left(\frac{\overline{F}_j}{F_j}\right)\right] b_{ij}(s) |x_j|_0 ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int_0^\omega E_{ij} R_j \left[\left[\min_{i=\overline{1,n}}\left(\frac{\overline{E}_{ij}}{E_{ij}}\right)\right] \times \left(\frac{\overline{R}_j}{R_j}\right)\right] c_{ij}(s) |x_j|_0 ds \\
& + \sigma A_i \sum_{k=1}^q \zeta_{ik}(t_k) \left[\min_{i=\overline{1,n}}\left(\frac{\overline{\zeta}_{ik}(t_k)}{\zeta_{ik}(t_k)}\right)\right] |x_i(t_k)|_0 \\
\geq\ & \theta_1 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^* + \theta_2 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
& + \theta_3 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^* + \theta_5 \times \sigma A_i \omega |x_i|_0 \widehat{\gamma}_{ik}
\end{aligned}$$

$$
\begin{aligned}
&\geq \overline{\theta} \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^{n} \widehat{a}_{ij} x_j^* + \overline{\theta} \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^{n} \widehat{b}_{ij} x_j^* \\
&\quad + \overline{\theta} \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^{n} \widehat{c}_{ij} x_j^* \\
&\quad + \overline{\theta} \times \sigma A_i \omega |x_i|_0 \widehat{\gamma}_{ik} \\
&= \overline{\theta} A_i \omega \sigma^2 \eta_2 |x_i|_0 \sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j^* \\
&\quad + \overline{\theta} \times \sigma \left(A_i \widehat{\gamma}_{ik} \right) \omega |x_i|_0 \\
&= \left(A_i \widehat{r}_i \right) \overline{\theta} \omega \sigma^2 \eta_2 |x_i|_0 + \left(A_i \widehat{\gamma}_{ik} \right) \omega \overline{\theta} \sigma |x_i|_0 \\
&\geq \widetilde{m}_0 \overline{\theta} \omega \sigma^2 \eta_2 |x_i|_0 + \widetilde{m}_0 \overline{\theta} \omega \sigma |x_i|_0 \\
&\geq |x_i|_0 \,, i = 1, 2, ..., n,
\end{aligned}
$$

and therefore

$$
\|\Phi x\| = \sum_{i=1}^{n} |(\Phi_i x)|_0 \geq \sum_{i=1}^{n} |x_i|_0 = \|x\| \,, \forall x \in K \cap \partial \widehat{\Omega}_2.
$$

Hence, $\Phi : K \cap (\overline{\widehat{\Omega}}_2 \backslash \widehat{\Omega}_1) \to K$ is a completely continuous operator and satisfies condition (a) in Theorem 4.1. Consequently, there exists a fixed point $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), ..., x_n^*(\cdot))^T \in K \cap (\overline{\widehat{\Omega}}_2 \backslash \widehat{\Omega}_1)$ such that $x^*(\cdot) = (\Phi x^*)(\cdot)$. Therefore, system 4.5 has a positive ω-periodic solution. The proof is completed. ■

Remark 4.1 The method applied in this paper can be used to treat a more general nonlinear impulse function. For instance, assuming that $I_{ik}(\cdot, x)$ satisfies

(H_6) There exist nonnegative functions $\overline{\phi}_{ik}, \phi_{ik} \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $F \in C\left(\mathbb{R}_+^n, \mathbb{R}_+^n\right)$ such that for all $x \in \mathbb{R}_+^n$,

$$
\overline{\phi}_{ik}(t) F(\|x\|) \leq I_{ik}(t, x) \leq \phi_{ik}(t) F(\|x\|), i = 1, 2, ..., n, k = 1, 2, ...,
$$

Note that (H_2) and (H_4) are special cases of condition (H_6) which has been used in [29],[30].

Remark 4.2 Notice that when $a_{ij} = 0$ in the second term on the right hand side of 4.5, $I_{ik}(t_k, x(t_k)) = 0$, $f_j(x_j) = x_j$, and $g_j(x_j) = 0$, we can easily derive the corresponding results in [**?**]. Therefore, the results presented in this paper improve and extend the main results in Ref.[64] .

4.5 An example

To illustrate how our main results can be used in practice, we present an example.

Example 4.1 **4.1** Let us consider the following system:

$$\begin{aligned} x_i'(t) &= x_i(t)\left[r_i(t) - \sum_{j=1}^{2} b_{ij}(t) f_j\left(x_j(t - \tau_{ij}(t))\right)\right. \\ &\quad \left. - \sum_{j=1}^{2} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,\rho) g_j\left(x_j(\rho)\right) d\rho\right], \\ t &\neq t_k = \frac{1}{2} k\pi, k \in \mathbb{N}^*, \\ x_i(t_k^+) - x_i(t_k^-) &= I_{ik}\left(t_k, x_i(t_k)\right), \ t = t_k, i = 1, 2. \end{aligned}$$

Corresponding to system 4.5, we let $n = 2, \omega = 2\pi$, $r_1(t) = \frac{1}{2}(1 + \sin 2t)$, $r_2(t) = \frac{1}{3}(1 + \cos 2t)$, $b_{11}(t) = \frac{2(1 + \cos 2t)}{3e^2}$, $b_{12}(t) = 1 + \sin 2t$, $b_{21}(t) = 0$, $b_{22}(t) = \cos(4t)$, $c_{11}(t) = 0$, $c_{12}(t) = (1 + \cos 4t)$, $c_{21}(t) = \frac{3(1 + \sin 2t)}{4}$, $c_{22}(t) = \frac{(1 + \sin 2t)}{2}$, and $D_{12}(t,s) = \frac{1}{2} e^{s-t}(\cos t + 3)$, $D_{21}(t,s) = 2e^{s-t}\left(e^{|\cos t| - 1}\right)$, $D_{11}(t,s) = e^{3(s-t)}\left(e^{3(|\sin 2t| - 1)}\right)$, $D_{22}(t,s) = \frac{1}{3} e^{s-t}(\sin t + 2)$, where $\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22} \in$

$C(\mathbb{R},\mathbb{R}^+)$ are 2π-periodic functions. Also assume that

$$\begin{aligned} f_1(x) &= \frac{x}{2}e^{(\sin x)+1}, \qquad f_2(x) = \frac{x}{4}(|\cos x|+1), \\ g_1(x) &= \frac{x}{3}(|\cos x|+1), \qquad g_2(x) = \frac{x}{2}(|\sin x|+1), \\ I_1(t,x) &= \frac{\pi}{4}x^2(\cos 2t+1)(|\cos x|+1), \qquad I_2(t,x) = \frac{1}{14}x^2(|\sin 2t|+\pi). \end{aligned}$$

Thanks to straightforward computations, we obtain

$$\begin{aligned} \widehat{r}_1 &= \frac{1}{\omega}\int_0^\omega r_1(t)dt = \frac{1}{4\pi}\int_0^{2\pi}(1+\sin 2t)\,dt = \frac{1}{2}, \\ \widehat{r}_2 &= \frac{1}{\omega}\int_0^\omega r_2(t)dt = \frac{1}{6\pi}\int_0^{2\pi}(1+\cos 2t)\,dt = \frac{1}{3}. \end{aligned}$$

A simple calculation shows that $A_i \le G_i(t,s) \le B_i$, for $i=1,2$, where

$$G_i(t,s) = \frac{1}{1-e^{-\widehat{r}_i\omega}}\exp\left(-\int_t^s r_i(z)\,dz\right),\ i=1,2,$$

and

$$\begin{aligned} A_1 &:= \frac{e^{-\widehat{r}_1\omega}}{1-e^{-\widehat{r}_1\omega}} = \frac{e^{-\pi}}{1-e^{-\pi}}, \qquad A_2 := \frac{e^{-\widehat{r}_2\omega}}{1-e^{-\widehat{r}_2\omega}} = \frac{e^{-\frac{2\pi}{3}}}{1-e^{-\frac{2\pi}{3}}}, \\ B_1 &:= \frac{1}{1-e^{-\widehat{r}_1\omega}} = \frac{1}{1-e^{-\pi}}, \qquad B_2 := \frac{1}{1-e^{-\widehat{r}_2\omega}} = \frac{1}{1-e^{-\frac{2\pi}{3}}}. \end{aligned}$$

Since $|\sin x| \le 1$ and $|\cos x| \le 1$ for $x \in \mathbb{R}^+$, we have

$$\begin{aligned} \overline{F}_j x &\le f_j(x) \le F_j x,\ j=1,2, \\ \overline{R}_j x &\le g_j(x) \le R_j x,\ j=1,2, \\ \overline{\lambda}_i(t)\,x^2 &\le I_i(t,x) \le \lambda_i(t)\,x^2,\ i=1,2, \end{aligned}$$

and

$$\overline{E}_{ij} \le \int_{-\infty}^t D_{ij}(t,s)\,ds \le E_{ij},\ i,j=1,2,$$

where $\overline{F}_1 = \frac{1}{2}$, $F_1 = \frac{e^2}{2}$, $\overline{F}_2 = \frac{1}{4}$, $F_2 = \frac{1}{2}$, $\overline{R}_1 = \frac{1}{3}$, $R_1 = \frac{2}{3}$, $\overline{R}_2 = \frac{1}{2}$, $R_2 = 1$, $\overline{\lambda}_1(t) = \frac{\pi}{\cos 2t}$, $\overline{E}_{12} = 1$, $E_{12} = 2$, $\overline{E}_{21} = 2e^{-1}$, $E_{21} = 2$, $\overline{E}_{11} = e^{-3}$, $E_{11} =$

$1,\ \overline{E}_{22}=\frac{2}{3},\ E_{22}=1,\ \overline{\lambda}_1(t)=\frac{\pi}{4}(\cos 2t+1),\ \lambda_1(t)=\frac{\pi}{2}(\cos 2t+1),\ \overline{\lambda}_2(t)=\frac{\pi}{14},\ \lambda_2(t)=\frac{1}{14}(2t+\pi)$.

For the above parameters, it is easy to verify that

$$\begin{aligned}\widehat{b}_{11} &= \frac{F_1}{\omega}\int_0^\omega b_{11}(s)ds=\frac{1}{3}, \widehat{b}_{12}=\frac{F_2}{\omega}\int_0^\omega b_{12}(s)ds=\frac{1}{2},\\ \widehat{b}_{21} &= \frac{F_1}{\omega}\int_0^\omega b_{21}(s)ds=0,\ \widehat{b}_{22}=\frac{F_2}{\omega}\int_0^\omega b_{22}(s)ds=0,\end{aligned}$$

and

$$\begin{aligned}\widehat{c}_{11} &= \frac{R_1E_{11}}{\omega}\int_0^\omega c_{11}(s)ds=0, \widehat{c}_{12}=\frac{R_2E_{12}}{\omega}\int_0^\omega c_{12}(s)ds=2,\\ \widehat{c}_{21} &= \frac{R_1E_{21}}{\omega}\int_0^\omega c_{21}(s)ds=1, \widehat{c}_{22}=\frac{R_2E_{22}}{\omega}\int_0^\omega c_{22}(s)ds=\frac{1}{2}.\end{aligned}$$

Choosing $q=4$, we have $t_{k+q}=t_k+\omega,\ I_{i(k+q)}\left(t_{k+q},x_i\left(t_{k+q}\right)\right)=I_{ik}\left(t_k,x_i\left(t_k\right)\right),$ $i=1,2$, and

$$\begin{aligned}\widehat{\beta}_{1k} &= \frac{1}{\omega}\sum_{0\le t_k<\omega}\lambda_{1k}\left(t_k\right)=\frac{1}{2\pi}\sum_{k=1}^{q}\frac{\pi}{2}\left(\cos 2t_k+1\right)=1\\ \widehat{\beta}_{2k} &= \frac{1}{\omega}\sum_{0\le t_k<\omega}\lambda_{2k}\left(t_k\right)=\frac{1}{28\pi}\sum_{k=1}^{q}\left(2t_k+\pi\right)=\frac{1}{2}.\end{aligned}$$

Moreover, it is easy to verify that the corresponding system of nonlinear equation 4.5,

$$\begin{cases}\sum_{j=1}^{2}\left(\widehat{b}_{1j}+\widehat{c}_{1j}\right)x_j+\widehat{\beta}_{1k}x_1=\widehat{r}_1\\ \sum_{j=1}^{2}\left(\widehat{b}_{2j}+\widehat{c}_{2j}\right)x_j+\widehat{\beta}_{2k}x_2=\widehat{r}_2,\end{cases}$$

has a unique positive solution $x=(x_1,x_2)=\left(\frac{6}{21},\frac{1}{21}\right)$. It is straightforward to show that all conditions of Theorem 4.2 are fulfilled. Hence, we conclude that this system possesses at least one positive $2\pi-$periodic solution.

Conclusion

This dissertation studies the positivity of periodic solution of Lotka - Volterra differential equations with distributed delays and impulses. The approach used in our project is based on the fixed point technique, particularly, using the Krasnoselskii fixed point theorem. This approach relies mainly on necessary arguments: an elementary variation of parameters formula, a non-empty cone subset of a Banach space, and a fixed point application. The benefit of this approach is that the fixed point arguments can yield existence and positive periodic solutions for an equation in one step. The main difficulty of this approach is to define a cone K of a Banach space and a suitable mapping when the mapping is compact on K. The results in this work extend and improve some existing results in the literature in some ways. Recently, Benhadri et al [10] have addressed this technique to investigate the periodicity of another class of delay differential equations with variable delays. However, there are many problems to be solved for fractional differential equations and other variants, persistence, and so on. We leave these for our -future work.

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