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Memory
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Homogenzation of the Hyperbolic Problem in perforated domains with small holes

Option : AFA

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Dedication:

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I dedicate this work from my deep heart:

To my dear mother and father for the million things they provided

me, and for the unlimited support they always gave me.

To my dear brother Mouhamed el amine and beloved sister Imene for their love and encouragement.

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Abstract

In this memory we study a class of hyperbolic problems in periodically perforated domains with a homogeneous Newmann condition on the boundary of the holes. We focus on the homogenezation of these equations. The proof is based on th periodic unfolding method in perforated domains.

Key words : hyperbolic problem, periodic unfolding, perforated domain, homognisation.

Résumé

Nous considérons dans cette mémoire une class de Problème hyperbolique dans un domaine perforé avec des petits trous périodiquement et de condition de Newmann hmogène. Nous nous concentrons sur l'homogenisation de ces équation. Lépreuve est basée sur la méthod d'eclatement periodique dans les domaines perforés.

Mots clés : probléme hyperbolique, éclatement périodique, domaine perforé, homogénéisation.

ملخص

في هذا العمل في هذا العمل قمنا بدراسة قسم من المسائل القطع الزائدي في المجالات المسامية مع وضع شروط نيومان المتجانسة على حدود الثقوب نهتم بدراسة تجانس هذه المعادلات و ذلك بالاعتماد على طريقة الانفجار الدوري في المجالات المسامية.

كلمات مفتاحية: مشكلة قطع زائدية، انتشار دوري، مجال مثقب، تجانس.

CONTENTS

INTRODUCTION

An effective mathematical approach to solving complex problems is the theory of homogenization. Developed in the late 1960s and early 1970s, this theory has become a cornerstone in the study of partial differential equations (PDEs) and their numerical approximations. Homogenization theory provides a framework for approximating problems with highly oscillatory coefficients by using problems with simpler, constant coefficients.

The periodic unfolding method was initially introduced by Cioranescu, Damlamian, and Griso [7] for fixed domains (see [6] for further details). It was subsequently extended to perforated domains by Cioranescu, Donato, and Zaki [10]. This method was later adapted for two-component domains separated by a periodic interface by Donato, Nguyen, and Tardieu [17]. Recently, Cioranescu, Damlamian, Donato, Griso, and Zaki [5] provided a comprehensive overview of the periodic unfolding method for perforated domains, addressing both cases where the unit hole is compact within the open unit cell and when this compactness cannot be achieved.

concerning the time-dependent periodic unfolding method for fixed domains, our reference is Gaveau [19], which provides a list of elementary results without proofs. However, to address time-dependent problems in perforated domains, we adapt the results from [3] to accommodate time-dependent functions. Specifically, we focus on treating perforated domains where the unit hole is a compact subset of the open unit cell.

The aim of this work is to study the hyperbolic equation with homogenious Dirichlet-Newman boundary in the perforated domain $\boldsymbol{\Omega}_{\varepsilon}$ and also in the perforated domain $\boldsymbol{\Omega}_{\varepsilon,\delta}$.

This memory is organized as flows:
In the first chapter: we define first the periodic unfolding for one in perforated domains and some properties.
In the second chapter: we define the time periodic unfolding in perforated domains and some properties . In the third chapter: we define the periodic unfolding for tow parametres in perforated domains and some properties.
In the fourth chapter: we study the homoginization of hyperbolic problems in the perforated domain $\boldsymbol{\Omega}_{\varepsilon}$ and also in the perforated domain $\boldsymbol{\Omega}_{\varepsilon,\delta}$.

CHAPTER

1

THE PERIODIC UNFOLDING METHOD IN PERFORATED DOMAINS

In this chapter we recall the definition and some properties of the periodic unfolding operators in perforated domains $\mathcal{T}_\varepsilon$ for the classical homogenization.

1.1 The periodic unfolding operator $\mathcal{T}_\varepsilon$

In this section, we introduce the case of perforated domains introduced by Cioranescu et all In the following we denote:

Definition 1.1 *:[4] Let* $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1+\infty]$. *We define the function:*

$$\mathcal{T}_\varepsilon(\varphi)(x,y)=\tilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_\varepsilon+\varepsilon y\right) \tag{1.1}$$

for every $x \in \mathbb{R}^{\mathbb{N}}$ *and* $y \in Y$.

Remark 1.1 *: Notice that the oscillations due to perforation are shifted into second variable $\boldsymbol{y}$ which belongs to the fixde domain $\boldsymbol{Y}$, while the first variable $\boldsymbol{x}$ belongs to $\mathbb{R}^{\boldsymbol{N}}$.*
One see immediately the interest of the unfolding operator. Indeed, when trying to pass to the limit in a sequence defined on $\boldsymbol{\Omega_\varepsilon}$ one needs first, while using standard methods, to extend it to a fixed domain. With $\mathcal{T}_\epsilon$, such extensions are more necessary.

The main properties given in[5] for fixed domains can easily be adapted for the perforated ones without any major difficulty in the proofs. These properties are listed in the proposition below. To do so, let us first define the following domain:

$$\tilde{\boldsymbol{\Omega}}_\varepsilon = \boldsymbol{int}\Big(\bigcup_{\boldsymbol{\xi}\in\boldsymbol{\Lambda}_\varepsilon}(\boldsymbol{\xi}+\boldsymbol{Y})\Big)$$

where:

$$\boldsymbol{\Lambda}_\varepsilon\{\boldsymbol{\varepsilon}\in\mathbb{Z}^{\boldsymbol{N}}, \boldsymbol{\varepsilon}(\boldsymbol{\xi}+\bar{\boldsymbol{Y}})\cap\boldsymbol{\Omega}\neq\boldsymbol{0}\}.$$

The set $\tilde{\boldsymbol{\Omega}}_\varepsilon$ is the smallest finite union of $\boldsymbol{\varepsilon Y}$ cells containing $\boldsymbol{\Omega}$

Proposition 1.1 *[4] The unfolding operator $\mathcal{T}_\varepsilon$ has the following properties:*

1. *$\mathcal{T}_\varepsilon$ is a linear operator.*

2. $\mathcal{T}_\varepsilon(\varphi)\Big(\boldsymbol{x},\Big\{\dfrac{\boldsymbol{x}}{\boldsymbol{\varepsilon}}\Big\}_{\boldsymbol{Y}}\Big)=\varphi(\boldsymbol{x})\quad\forall\varphi\in\boldsymbol{L}^{\boldsymbol{p}}(\boldsymbol{\Omega}_\varepsilon)\quad\boldsymbol{and}\quad\boldsymbol{x}\in\mathbb{R}^{\boldsymbol{N}}.$

3. $\mathcal{T}_\varepsilon(\varphi\psi)=\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)\quad\forall\varphi,\psi\in\boldsymbol{L}^{\boldsymbol{p}}(\boldsymbol{\Omega}_\varepsilon).$

4. *Let $\varphi\in\boldsymbol{L}^{\boldsymbol{p}}(\boldsymbol{Y})$ or $\boldsymbol{L}^{\boldsymbol{p}}(\boldsymbol{Y})$ be periodic function set $\boldsymbol{\phi}(\boldsymbol{x})=\varphi\Big(\dfrac{\boldsymbol{x}}{\boldsymbol{\varepsilon}}\Big)$ Then* $\mathcal{T}_\varepsilon(\varphi_\varepsilon)(\boldsymbol{x},\boldsymbol{y})=\varphi(\boldsymbol{y})$*. a.e. in* $\tilde{\boldsymbol{\Omega}}_\varepsilon$.

5. *One has the integration formula*

$$\int_{\boldsymbol{\Omega}}\varphi\boldsymbol{dx}=\frac{\boldsymbol{1}}{|\boldsymbol{Y}|}\int_{\boldsymbol{\Omega}\times\boldsymbol{Y}}\mathcal{T}_\varepsilon\varphi\boldsymbol{dxdy}\quad\forall\varphi\in\boldsymbol{L}^{\boldsymbol{1}}(\boldsymbol{\Omega}_\varepsilon).$$

6. *For every* $\varphi \in L^2(\Omega_\varepsilon), \mathcal{T}_\varepsilon(\varphi)$ *belongs to* $L^2(\mathbb{R}^N \times Y)$. *It also belongs to* $L^2(\tilde{\Omega}_\varepsilon \times Y)$.

7. *For every* $\varphi \in L^2(\Omega_\varepsilon)$

$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2(R^N \times Y)} = \sqrt{|Y|}||\varphi||_{L^2(\Omega_\varepsilon)}$$

.

8. $\nabla \mathcal{T}_\varepsilon(\varphi)(x, y) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x \varphi)(x, y)$*for every* $(x, y) \in \Omega \times Y$.

9. *If* $\varphi \in H^1(\tilde{\Omega}_\varepsilon)$, *then* $\mathcal{T}_\varepsilon(\varphi)$ *is in* $L^2(\mathbb{R}^N, H^1(Y))$.

10. *One has the estimate*

$$||\nabla \mathcal{T}_\varepsilon(\varphi)||_{L^2(\Omega \times Y)} = \varepsilon\sqrt{|Y|}||\nabla_x \varphi||_{(L^2(\Omega_\varepsilon))^N}.$$

proof:

1. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $\varphi, \psi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned} \mathcal{T}_\varepsilon\left(\alpha\varphi + \beta\psi\right) &= \varphi\left(\alpha(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y))\right) + \psi\left(\beta(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y)\right) \\ &= \alpha\varphi + \beta\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right) \\ &= \alpha\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y) + \beta\psi(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \\ &= \alpha\mathcal{T}_\varepsilon(\varphi) + \beta\mathcal{T}_\varepsilon(\psi) \end{aligned}$$

$\mathcal{T}_\varepsilon$ *is a linear operator.*

2. $\mathcal{T}_\varepsilon(\varphi)\left(x,\left\{\frac{x}{\varepsilon}\right\}\right)=\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon\left\{\frac{x}{\varepsilon}\right\}\right)=\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+x-\varepsilon\left[\frac{x}{\varepsilon}\right]\right)=\varphi(x)$

3. *Let* $\varphi,\psi\in L^2(\Omega_\varepsilon)$, *we have by definition* 1.1

$$\begin{aligned}\mathcal{T}_\varepsilon(\varphi\psi)&=\varphi\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y\right)\\&=\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y\right)\psi\left(\varepsilon[\frac{x}{\varepsilon}]+\varepsilon y\right)\\&=\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)\end{aligned}$$

4. *by definition for all* $\varphi_\varepsilon=\varphi\left(\frac{x}{\varepsilon}\right)$, $\varphi\in L^2(Y)$, *we have :*

$$\mathcal{T}_\varepsilon\left(\varphi_\varepsilon\right)(x,y)=\varphi\left(\varepsilon\left[\left\{\frac{x}{\varepsilon}\right\}\varepsilon\right]+\varepsilon\left\{\frac{x}{\varepsilon}\right\}\varepsilon\right)$$

$$=\varphi\left\{\frac{x}{\varepsilon}\right\}=\varphi(y)$$

5. *According the definition* 1.1 *we have :*

$$\begin{aligned}\frac{1}{|\bar{Y}|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)&=\frac{1}{|Y|}\int_{\widetilde{\Omega}_\varepsilon\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy\\&=\frac{1}{|Y|}\sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy\end{aligned}$$

on each set $(\varepsilon\xi+\varepsilon Y)\times Y$ *with* $\xi\in\Xi_\varepsilon$ *,the function* $\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy=\varphi(\varepsilon\xi+\varepsilon y)$ *is constant in x, it is consequence of* 1.1. *So, for each integral in the sum of the right member, we have :*

$$\begin{aligned}\int_{(\varepsilon\xi+\varepsilon Y)\times Y}\mathcal{T}_\varepsilon(\varphi)(x,y)dxdy&=|\varepsilon\xi+\varepsilon Y|\int_Y\varphi(\varepsilon\xi+\varepsilon y)dy\\&=\varepsilon^N|Y|\int_Y\varphi(\varepsilon\xi+\varepsilon y)dy=|Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x)dx\end{aligned}\tag{1.2}$$

we suppose $x=\varepsilon\xi+\varepsilon y$, *this implies* $dx=\varepsilon^N dy\Longrightarrow dy=\frac{1}{\varepsilon^N}$ *by summing in* Ξ, *the right member be* $\int_{\widetilde{\Omega}_\varepsilon}\varphi(x)dx.$ *so we get*

$$\sum_{\xi\in\Xi_\varepsilon}\varepsilon^N|Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x)\frac{1}{\varepsilon^N}dx=\int_{\widetilde{\Omega}}\varphi(x)dx$$

Now, we replace this expression in 1.2 we get the result.

6. *For all* $\varphi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned}\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(R^N\times Y)} &= \left(\int |\mathcal{T}_\varepsilon(\varphi)|^2 dxdy\right)^{\frac{1}{2}} = \left(1\int_{\Omega\times Y} |\varphi|^2 dxdy\right)^{\frac{1}{2}}\\ &= \left(\int 1dy\right)^{\frac{1}{2}}\left(\int |\varphi|^2 dx\right)^{\frac{1}{2}}\\ &= \sqrt{|Y|}\|\varphi\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

7. *For all* $\varphi \in L^2(\Omega_\varepsilon)$

$$\begin{aligned}\nabla_y\mathcal{T}_\varepsilon(\varphi)(x,y) &= \nabla_y\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \varepsilon\nabla_x\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_y + \varepsilon y\right)\\ &= \varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)(x,y).\end{aligned}$$

8. *We apply proposition 1.1, we obtain*

$$\begin{aligned}\|\nabla_y\mathcal{T}_\varepsilon(\varphi)\|_{(L^2(\Omega\times Y))^N} &= \|\varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)\|_{L^2(\Omega\times Y))^N}\\ &= \varepsilon\sqrt{|Y|}\|\nabla_x\varphi\|_{(L^2(\Omega_\varepsilon))^N}.\end{aligned}$$

CHAPTER

2

THE TIME-DEPENDENT UNFOLDING OPERATOR IN PERFORATED DOMAINS

In this chapter, we adapt the unfolding operator in[5] to time-dependent functions We present the unfolding operator $\boldsymbol{\mathcal{T}_\varepsilon^*}$ which maps functions defined on the oscillating domain $\boldsymbol{\Omega_\varepsilon^* \times (0,T)}$ into functions defined on the fixed domaine $\boldsymbol{\Omega \times Y^* \times (0,T)}$. This avoids the use of any extension operator. We also give some properties by extending the previous ones (see[5] for more details and [6] for the classical unfolding theory).

For any $\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{n}}$, we use $[\boldsymbol{z}]_{\boldsymbol{y}}$ to denote the, unique integer combinatio $\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} \boldsymbol{k_j b_j}$ of the period such that $\boldsymbol{z} - [\boldsymbol{z}]_{\boldsymbol{Y}} \in \boldsymbol{Y}$ Set:

$$\boldsymbol{\{z\}_Y = z - [z]_Y \in Y \quad a.e. \quad For \quad z \in \mathbb{R}^n.} \tag{2.1}$$

Then for each $x \in \mathbb{R}^n$, we have:

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}\right) \quad a.e. \quad For \quad x \in \mathbb{R}^n.$$

Let us first recall the unfolding operator $\mathcal{T}_\varepsilon$ for the fixed domain $\Omega \times (0, T)$ introduced in[14], where the properties of $\mathcal{T}_\varepsilon$, are shown without proofs.

Definition 2.1 *: For $p \in [1, +\infty)$ and $q \in [1, +\infty]$, let ϕ be in $Ł^q(0, T; L^p(\Omega))$. The unfolding operator $\mathcal{T}_\varepsilon$:$L^q(0, T; L^p(\Omega)) \mapsto L^q(0, T; L^p(\Omega \times Y))$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi)(x, y, t) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y, t\right) & a.e. \quad For (x, y, t) \in \hat{\Omega}_\varepsilon \times Y \times (0, T), \\ 0 & a.e. \quad For (x, y, t) \in \Lambda_\varepsilon \times Y \times (0, T). \end{cases} \tag{2.2}$$

In a similar way, we extend the unfolding operator for the perforated domain Ω^*_ε to the foillowing unfolding operator $\mathcal{T}^*_\varepsilon$ for the perforated domain $\Omega^*_\varepsilon \times (0, T)$.

Definition 2.2 *: For $p \in [1, +\infty)$ and $q \in [1, +\infty]$ let ϕ be $\in Ł^q(0; T; L^p(\Omega))$ the unfolding operator $\mathcal{T}^*_\varepsilon$:$L^q(0, T; L^p(\Omega)) \to L^q(0, T; L^p\Omega \times Y)$ is defined as follows:*

$$\mathcal{T}^*_\varepsilon(\phi)(x, y, t) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right) & a.e. \quad for (x, y, t) \in \hat{\Omega}_\varepsilon \times Y^* \times (0, T), \\ 0 & a.e. \quad for (x, y, t) \in \Lambda_\varepsilon \times Y^* \times (0, T). \end{cases}$$

From this definition, the following properties are immediate:

$$\begin{aligned} &(i)\ \mathcal{T}^*_\varepsilon(vw) = \mathcal{T}^*_\varepsilon(v)\mathcal{T}^*_\varepsilon(w), \forall w, v \in L^q(0, T; L^p(\Omega^*)), \\ &(ii)\ \mathcal{T}^*_\varepsilon(\psi\varphi) = \varphi\mathcal{T}^*_\varepsilon(v)\mathcal{T}^*_\varepsilon(\psi), \forall \psi \in L^q(\Omega^*_\varepsilon) \quad and \quad \varphi \in L^q(0, T), \\ &(iii)\ \nabla_y(\mathcal{T}^*_\varepsilon(\phi)) = \varepsilon\mathcal{T}^*_\varepsilon(\nabla\phi), \forall \phi \in L^q(0, T; W^{1,P}(\Omega^*_\varepsilon)). \end{aligned} \tag{2.3}$$

Remark 2.1 *: Concerning $\mathcal{T}_\varepsilon$, and $\mathcal{T}^*_\varepsilon$, we have the following:*

$$\begin{aligned} &(i)\ \mathcal{T}^*_\varepsilon(\omega|^*_{\Omega\varepsilon\times(0,T)}) = \mathcal{T}_\varepsilon(\omega)|_{\Omega\times Y^*\times(0,T)}, \\ &(ii)\ \mathcal{T}^*_\varepsilon(\psi) = \mathcal{T}_\varepsilon(\tilde{\psi})|_{\Omega\times Y^*\times(0,T)}, \end{aligned}$$

where w and, ψ are defined on $\Omega \times (0, T)$ and $\Omega^*_\varepsilon \times (0, T)$, respectiuely.

In Definitions 2.1 and 2.2, if ϕ *is independent of* t, *then* $\mathcal{T}_\varepsilon$ *and* $\mathcal{T}_\varepsilon^*$ *are the classical unfoliding operators defined in [7] and [2] ,respectiuely.*

For simplicity, we always write $\mathcal{T}_\varepsilon^*(\phi)$ *instead of* $\mathcal{T}_\varepsilon^*(\phi|_{\Omega_\varepsilon^*\times(0,T)})$ *for any function* ϕ *defined in* $\Omega \times (0,T)$ *.*

Next we list some properties of the unfolding operator $\mathcal{T}_\varepsilon^*$ *used in this paper. The proofs are essentially the same as those in [5] and [6].*

Proposition 2.1 *: For* $p \in [1,+\infty)$ *and* $q \in [1,+\infty]$ *the operator* $\mathcal{T}_\varepsilon^*$ *linear and continuos* $L^q(0,T,L^p(\Omega_\varepsilon^*))$ ***to*** $L^q(0,T,L^p(\Omega \times Y^*)$ *let* $\phi \in L^q(0,T,L^1(\Omega_\varepsilon^*))$ *and* $w, v \in L^q(0,T,L^q(\Omega_\varepsilon^*))$
For a.e. $t \in (0,T)$ *we have :*

- *(i)* $\dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y} \mathcal{T}^*(\phi)(x,y,t)dxdy = \int_{\hat{\Omega}_\varepsilon^*} \phi(x,t)dx = \int_{\Omega_\varepsilon^*} \phi(x,t)dx - \int_{\Lambda_\varepsilon^*} \phi(x,t)dx,$

- *(ii)* $||\mathcal{T}_\varepsilon^*(\omega)||_{L^p} = |Y|^{\frac{1}{p}}||\omega||_{L^p(\hat{\Omega}_\varepsilon^*)} \leqslant |Y|^{\frac{1}{p}}||\omega||_{L^P(\Omega_\varepsilon^*)}.$

Proposition 2.2 *: For* $q \in [1,+\infty]$, *let* ϕ_ε *be in* $L^q(0,T;L^1(\Omega_\varepsilon^*))$ *and satisfy*

$$\int_o^T \int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dxdt \to 0,$$

then

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon dxdt - \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)dxdydt \to 0.$$

As usual,this convergence is denoted by

$$\int_0^T \int_{\Omega_\varepsilon^*} \phi_\varepsilon dxdt \backsimeq \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)dxdydt.$$

Moreover,we have the following convergences

Proposition 2.3 *:*

- *(i) For* $p, q \in (1, +\infty]$ *let* $\phi_\varepsilon \in L^q(0, T; L^p(\Omega^*_\varepsilon))$ *and* $\psi \in L^{q'}(0, T; L^{p'}(\Omega^*_\varepsilon))$ $(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1)$ *such that*

$$||\phi_\varepsilon||_{L^q(0,T;L^P(\Omega^*_\varepsilon))} \leqslant C \quad and \quad ||\psi||_{L^{q'}(0,T;L^{P'}(\Omega^*_\varepsilon))} \leqslant C,$$

then

$$\int_0^T \int_{\Omega^*_\varepsilon} \phi_\varepsilon \psi dxdt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*(\phi_\varepsilon) \mathcal{T}^*(\psi) dxdydt.$$

- *(ii) For* $p, q \in (1, +\infty]$*let* $\phi_\varepsilon \in L^P(0, T; L^p(\Omega^*_\varepsilon))$ *and* $\psi_\varepsilon \in L^q(0, T; L^{p0}(\Omega^*_\varepsilon))$ $(\frac{1}{p} + \frac{1}{p0} < 1, \frac{1}{q} + \frac{1}{q'} = 1)$ *such that*

$$||\phi_\varepsilon||_{L^q(0,T,L^P(\Omega^*_\varepsilon))} \leqslant C \quad and \quad ||\psi_\varepsilon||_{L^{q'}(0,T,L^P0(\Omega^*_\varepsilon))} \leqslant C,$$

then

$$\int_0^T \int_{\Omega^*_\varepsilon} \phi_\varepsilon \psi_\varepsilon dxdt \simeq \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(\phi_\varepsilon) \mathcal{T}^*_\varepsilon(\psi_\varepsilon) dxdydt.$$

Proposition 2.4 *(some convergence properties)*

- *(i) For* $p, q \in [1, \infty), \quad let \quad \omega \in \quad L^q(0, T; L^p(\Omega))$. *Then:*

$$\mathcal{T}^*_\varepsilon(\omega) \to \omega \quad strongly \quad in \quad L^q(0, T; L^p(\Omega \times Y^*)).$$

For

- *(ii)* $p, q \in [1, \infty), let \quad \{\omega_\varepsilon\} be \quad sequence \quad in \quad L^q(0, T; L^p(\Omega))$ *such that:*

$$\omega_\varepsilon \to \omega \quad strongly \quad in \quad L^q(0, T; L^p(\Omega)).$$

then

$$\mathcal{T}^*_\varepsilon(\omega_\varepsilon) \to \omega \quad strongly \quad in \quad L^q(0, T; L^p(\Omega \times Y^*)).$$

- *(iii) For* $p, \in [1, \infty) \quad and \quad q \in \quad (1, \infty], let \quad \{\omega_\varepsilon\} \quad be \quad sequence \quad a \quad in$ $L^q(0, T; L^p(\Omega^*_\varepsilon))$ *such that:*

$$||\omega_\varepsilon||_{L^q(0,T;L^p(\Omega^*_\varepsilon))} \leqslant C$$

if

$$\mathcal{T}^*_\varepsilon(\omega_\varepsilon) \to \hat{\omega} \quad weakly \quad in \quad L^q(0,T;L^P(\Omega\times Y^*)),$$

then we have

$$\hat{\omega}_\varepsilon \to \theta\mathcal{M}.(\hat{\omega}) \quad weakly \quad in \quad L^q(0,T;L^p(\Omega)).$$

For $q = \infty$*, the weak convergence above are replaced by the weak* convergence, respectively.*

proof: *using proposition 2.1 and remark 2.1 we have*

$$\begin{aligned}||\mathcal{T}^*_\varepsilon(\omega)-\omega||_{L^q(0,T;L^p(\Omega\times Y^*))} &= ||\mathcal{T}^*_\varepsilon(\omega-\phi\varphi)+\mathcal{T}^*_\varepsilon(\phi\varphi)-\phi\varphi+\phi\varphi-\omega||_{L^q(0,T;L^p(\Omega\times Y^*))}\\ &\leqslant 2|Y|^{\frac{1}{p}}||\phi\varphi-\omega||_{L^q(0,T;;L^p(\Omega))}+||(\mathcal{T}^*_\varepsilon(\phi)-\phi)\varphi||_{L^q(0,TL^p(\Omega\times Y^*))}\end{aligned}$$

For any $\phi\in\mathcal{D}(\Omega)$ *and* $\varphi\in\mathcal{D}(0,T)$. *Since*

$$\mathcal{T}^*_\varepsilon(\phi)\to\phi \quad strongly \quad in \quad L^p(\Omega\times Y^*),$$

then

$$||(\mathcal{T}^*_\varepsilon(\phi)-\phi)\varphi||_{L^q(0.T;L^p(\Omega\times Y^*))}\to 0.$$

Consequently,we have

$$\lim_{\varepsilon\to 0} sup||\mathcal{T}^*_\varepsilon(\omega)-\omega||_{L^q(0,T;L^p(\Omega\times^*Y))}\leqslant 2|Y|^{\frac{1}{p}}||\phi\varphi-\omega||_{L^q(0,T;L^p(\Omega))},$$

which implies statement (i) due to the density of $\mathcal{D}(0,T)\otimes\mathcal{D}(\Omega)$ *in* $L^q(0,T;L^p(\Omega))$.

(ii) From proposition 2.1 we have:

$$||(\mathcal{T}^*_\varepsilon(\omega)-\omega)||_{L^q(0.T;L^p(\Omega\times Y^*))}\leqslant|Y|^{\frac{1}{p}}||\omega_\varepsilon-\omega)||_{L^q(0.T;L^p(\Omega))}.$$

Hence statement(ii) followes from statement(i)

(iii) Let $\psi\in L^{q'}(0,T;L^{p'}(\Omega))(\frac{1}{p}+\frac{1}{p'}=1,\frac{1}{q}+\frac{1}{q'}=1)$ *from proposition 2.3(i)*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\tilde{\omega}\psi dxdt=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_\varepsilon}\omega_\varepsilon\psi dxdt=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int^*_{\Omega\times Y}\mathcal{T}^*_\varepsilon(\omega_\varepsilon)\mathcal{T}^*_\varepsilon(\psi)dxdydt.$$

Since statement (i) gives

$$\mathcal{T}^*_\varepsilon(\psi) \to \psi \quad \textit{strongly} \quad \textit{in} \quad L^{q'}(o,T;(L^{p'}\Omega \times Y^*)),$$

then we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \tilde{\omega}\psi dxdt = \int_0^T\int_\Omega \left\{\frac{1}{|Y|}\int_{Y^*}\hat{\omega}dy\right\}\psi dxdt = \theta\int_0^T\int_\Omega \mathcal{M}_{Y^*}(\hat{\omega})\psi dxdydt,$$

which implies the desired result.

Now we complete this subsection with a convergence result related to the space $L^q(0,T,W^{1,p}(\Omega^*_\varepsilon))$ *this proof can be directly obtained from that of Theorem 2.1 in [5].*

Proposition 2.5 *[16, 18]Let* $p \in]1,+\infty[$ *and* $\{\varphi_\varepsilon\}$ *be a sequence in the space* $L^\infty(0,T;W^{1,p}_0)$ *such that*

$$||\nabla\varphi||_{L^\infty(0,T;L^p(\Omega))} \leqslant C.$$

Then there exist $\varphi \in L^\infty(0,T;W^{1,p}_0(\Omega))$ *and* $\hat{\varphi} \in L^\infty(0,T;W^{1,p}_{per}(\Omega))$ *such that up to a subsequence,*

- *(i)* $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi \quad \textit{weakly}^* \quad \textit{in} \quad L^\infty(0,T;L^p(\Omega;W^{1,p}(Y)))$
- *(ii)*$\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon) \to \nabla_x\varphi + \nabla_y\hat{\varphi} \quad \textit{weakly}^* \quad \textit{in} \quad L^\infty(0,T;L^p(\Omega \times Y))$

Theorem 2.1 *For* $p \in (1,+\infty)$*, let* $\{u_\varepsilon\}$ *be a sequence in* $L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R}^N))$ *such that*

$$\|w_\varepsilon\|_{L^p(0,T;L^p(\Omega))} \leq C \quad \textit{and} \quad \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C, \tag{2.17}$$

Then there exists $w \in L^p(0,T;W^{1,p}_0(\Omega))$ *with* $\dfrac{\partial w}{\partial t} = 0$ *and* $\hat{w} \in L^\infty(0,T;L^p(\Omega;W^{1,p}_{per}))$ *with* $\mathcal{M}_{Y^*}(\hat{w}) \equiv 0$ *such that, up to a subsequence,*

(i) $w_\varepsilon \rightharpoonup w$ *weakly* in* $L^p(0,T;L^p(\Omega;W^{1,p}(\Omega)))$*,*

(ii) $\mathcal{T}^*_\varepsilon(\nabla w) \rightharpoonup \nabla w + \nabla\hat{w}$ *weakly* in* $L^p(0,T;L^p(\Omega \times \mathbb{R}^N))$*,*

(iii) $\mathcal{T}^*_\varepsilon\left(\dfrac{\partial w_\varepsilon}{\partial t}\right) \rightharpoonup \dfrac{\partial w}{\partial t}$ *weakly* in* $L^p(0,T;L^p(\Omega \times \mathbb{R}^N))$*,*

(iv) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w$ *strongly in* $L^q(0,T;L^q(\Omega;W^{-1,p}(\Omega)))$,

(v) $||w_\varepsilon - w||_{L^p(0,T;W_0^{1,p}(\Omega))} \to 0$.

where q *any nimber in* $(1,+\infty)$.

We end this section by recalling the definition of the mean value operator $\mathcal{M}_Y$ *and that of the local average operator* $\mathcal{M}_Y^\varepsilon$ *and give some of their properties that will be useful in the sequel.*

Definition 2.3 *: let* $p \in [1,+\infty]$ ***and*** $q \in [1,+\infty]$. *The mean value operator* $\mathcal{M}_Y$:$L^q(0,T;L^p(\Omega \times Y)) \mapsto L^q(0,T;L^p(\Omega))$, *is defined by:*

$$\mathcal{M}_Y(\varphi)(x,t) = \frac{1}{|Y|}\int_Y u(x,y,t)dy,$$

For every $u \in L^p(0,T;L^p(\Omega \times Y))$

Definition 2.4 *: Let* $p \in [1,+\infty]$ ***and*** $q \in [1,+\infty]$, *the local average operator* $\mathcal{M}_Y^\varepsilon$:$L^q(0,T;L^p(\Omega \times Y)) \to L^q(0,T;L^p(\Omega))$ *is defined by:*

$$\mathcal{M}_Y^\varepsilon(u)(x,t) = \frac{1}{Y}\int_Y \mathcal{T}_\varepsilon(\varphi)(x,y,t)dy,$$

For every $u \in L^q(0,T;L^p(\Omega))$.

Remark 2.2 *: In connection, some of the properties of* $\mathcal{T}_\varepsilon$ *(in the case of dependence on time) can be derived directly for those of the unfolding operator for fixed domains from [8] with the time t as a mere parameter.*
As a consequence, we have the following result.

Proposition 2.6 *Let* $p \in [1,\infty[$ ***and*** $q \in [1,\infty]$
for $\varphi \in L^q(0,T;L^p(\Omega))$, *one has*

$$\mathcal{T}_\varepsilon(\mathcal{M}_Y^\varepsilon(\varphi))(x,y,t) = \mathcal{M}_Y(\mathcal{T}_\varepsilon(\varphi))(x,t) = \mathcal{M}_Y^\varepsilon(\varphi)(x,t) \quad \textit{in} \quad \Omega \times [0,t].$$

Let $\{\omega_\varepsilon\}$ *be a sequence in* $L^q(0,T,L^P(\Omega))$ *such that*

$$\omega_\varepsilon \to \omega \quad \textbf{\textit{strongly}} \quad \textbf{\textit{in}} \quad L^q(0,T,L^p(\Omega)).$$

Then

$$\mathcal{M}_Y^\varepsilon(\omega_\varepsilon) \to \mathcal{M}_Y(\omega) = \omega \quad strongly \quad in \quad L^q(0,T,L^p(\Omega)).$$

For any $\varphi \in L^q(0,T;L^p(\Omega))$

$$||\mathcal{M}_Y(\varphi)||_{L^q(0,T;L^p(\Omega))} \leqslant |Y|^{\frac{1-p}{p}}||\varphi||_{L^q(0,T;L^p(\Omega))}.$$

CHAPTER

3

UNFOLDING OPERATOR IN DOMAINS DEPENDING ON TWO PARAMETERS

In this chapter we recall the definition and some of its properties of the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending on two mall parameters ε and δ, as introduced in [9].

Definition 3.1 *[9] Let $p \in [1, +\infty[$. For $\varphi \in L^p(\Omega)$ the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ is the function $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ defined by*

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi) & if \quad (x,z) \in \hat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y \\ 0 & \qquad otherwise, \end{cases}$$

where is $\mathcal{T}_\varepsilon$ the operator for fixed domains as introduced in [5]. To go further, let us introduce what is called a perforated domain with small holes denoted here $\Omega^*_{\varepsilon,\delta}$. Let $B \subset\subset Y$ and *denote* $Y_\delta^* = \dfrac{Y}{\delta \bar{B}}$. *Then* $\Omega_{\varepsilon,\delta}^*$ *is defined as*

$$\Omega_{\varepsilon,\delta}^* = \{x \in \Omega \quad \text{such that} \quad \{\frac{x}{\varepsilon}\}_Y \in Y_\delta^*\}$$

, where $\delta \to 0$. *This definition means that* $\Omega_{\varepsilon,\delta}^*$, *is a domain* ε*-periodically perforated by holes* $\varepsilon\delta B$,

Remark 3.1 *As shown in [9], it turns out that the operator* $\mathcal{T}_{\varepsilon,\delta}$ *is well-adapted for domains with small holes when dealing with functions which vanish on the boundary of* $\Omega_{\varepsilon,\delta}^*$. *It is precisely the case we treat in this work. We will deal with functions belonging in particular, to* $H_0^1(\Omega_{\varepsilon,\delta}^*)$. *The extensions of these functions by zero to the whole of* Ω, *belong to* $H_0^1(\Omega)$. *Consequently in the sequel, we will not distinguish the elements of* $H_0^1(\Omega_{\varepsilon,\delta}^*)$ *and their extensions from* $H_0^1(\Omega)$.

Proposition 3.1 *[9]*

1. *For any* $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.

2. *For any* $u \in L^1(\Omega)$,

$$\delta^N \int_{\Omega \times \mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)| \, dx \, dz \leq \int_\Omega |u| \, dx.$$

3. *For any* $u \in L^2(\Omega)$,

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|_{L^2(\Omega \times \mathbb{R}^N)}^2 \leq \frac{1}{\delta^N} \|u\|_{L^2(\Omega)}^2.$$

4. *For any* $u \in L^1(\Omega)$,

$$\left| \int_\Omega u \, dx - \delta^N \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u) \, dx \, dz \right| \leq \int_{\Lambda_\varepsilon} |u| \, dx.$$

5. *Let* $u \in H^1(\Omega)$. *Then*

$$\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta} \nabla_z (\mathcal{T}_{\varepsilon,\delta}(u)), \quad \text{in } \Omega \times \frac{1}{\delta} Y.$$

6. *Suppose* $N \geq 3$ *and let* $\omega \subset \mathbb{R}^N$ *be open and bounded. The following estimates hold:*

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon,\delta}(u-\mathcal{M}_{\varepsilon Y}(u))\|^2_{L^2(\Omega;L^2_*(\mathbb{R}^N))} \leq \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\omega)} \leq \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{2/N}\|\nabla u\|^2_{L^2(\Omega)} + 2|\omega|\|u\|^2_{L^2(\Omega)},$$

where C *is the Sobolev-Poincaré-Wirtinger constant for* $H^1(Y)$*.*

7. *Suppose* $N \geq 3$ *and let* $\{w_{\epsilon,\delta}\}$ *be a sequence in* $H^1(\Omega)$ *which is uniformly bounded as both* ϵ *and* δ *approach 0. Then there exists* W *in* $L^2(\Omega; L^2_*(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Omega \times \mathbb{R}^N)$ *such that, up to a subsequence,*

$$\frac{\delta^{N/2-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - \mathcal{M}_{\varepsilon Y}(w_{\varepsilon,\delta}) \rightharpoonup W \quad \textit{weakly in } L^2(\Omega; L^2_*(\mathbb{R}^N)),$$

and

$$\frac{\delta^{N/2-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})) \rightharpoonup \nabla_z W \quad \textit{weakly in } L^2(\Omega \times \mathbb{R}^N).$$

Furthermore, if

$$\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{N/2-1}}{\varepsilon} < +\infty,$$

then one can choose the subsequence above and some $U \in L^2(\Omega; L^2_{loc}(\mathbb{R}^N))$ *such that*

$$\frac{\delta^{N/2-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \textit{weakly in } L^2(\Omega; L^2_{loc}(\mathbb{R}^N)).$$

Definition 3.2 *A sequence* $\{v_{\varepsilon,\delta}\}$ *in* $L^1(\Omega)$ *satisfies the unfolding criterion for integrals(u.c.i) if*

$$\int_\Omega v_{\varepsilon,\delta} - \delta^N \int_{\Omega\times R^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})dxdz \to 0,$$

for every sequence $(\varepsilon,\delta) \to (0^+,0^+)$ *this property is denoted*

$$\int_\Omega v_{\varepsilon,\delta}dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})dxdz.$$

Proposition 3.2 *([9]u.c.i) If v_ε is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |u_\varepsilon| dx \to 0,$$

then it satisfies u.c.i..

corollary 3.1 *[9] Let$\{u_\varepsilon\}$ be bounded in$L^2(\Omega)$ and $\{v_\varepsilon\}$ be bounded in $L^p(\Omega)$ with $p \geqslant 2$. Then $\{u_\varepsilon, v_\varepsilon\}$ satisfies u.c.i*

Remark 3.2 *As observed in [9], for any $\psi \in \mathcal{D}(\Omega)$, one has*

$$||\mathcal{T}_{\varepsilon,\delta(\psi)-\psi}||_{L^\infty(\Omega\times\frac{1}{\delta}Y)} \to 0.$$

3.1 Time-dependent unfolding operator in domains with two parameters

In this section, we extend the operator $\mathcal{T}_{\varepsilon,\delta}$, defined in the previous section to time-dependent functions by adapting what is done in [16]. We start by defining the unfolding operator for time-dependent functions in the domain $\Omega^*_{\varepsilon,\delta}\times]0,T[$, depending on ε and δ
In what follows, we have$(\varepsilon,\delta) \to (0,0)$ through any sequence and subsequence

Definition 3.3 *Let $p \in [1,+\infty[$ and $q \in L^q(0,T;L^p(\Omega))$ The unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^q(0,T;L^p(\Omega)) \to L^q(0,T;L^p(\Omega))$ is definded as*

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x,\delta z,t) \quad if \quad (x,z,t) \in \hat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y\times]0,T[, \\ \qquad 0 \qquad\qquad otherwise. \end{cases}$$

that is,

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon\delta z, t) \quad if \quad (x,z,t) \in \hat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y\times]0,T[, \\ \qquad 0 \qquad\qquad otherwise. \end{cases}$$

As mentioned above, for $\delta = 1$ we are in presence of the unfolding operator for fixed domains introduced in [5].

Remark 3.3 *From now on, if a function does not depend on t, by $\mathcal{T}_{\varepsilon,\delta}$ we simply mean the operator introduced in Definition 3.1.*

Being defined by means of the operator $\mathcal{T}_{\varepsilon}$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$, inherits most of the general properties of it. In particular, the following proposition is straightforward:

Proposition 3.3 *Let $p \in [1,+\infty[$ and $q \in [1,+\infty[$.*

- $\mathcal{T}_{\varepsilon,\delta}$ *is linear and continuous from* $L^q(0,T;L^p(\Omega))$ *to* $L^q(0,T;L^p(\Omega \times \mathbb{R}^N))$.
- $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(\omega)$ *for every* $v, w \in L^q(0,T;L^p(\Omega))$.
- $\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi) = \varepsilon\delta\mathcal{T}_{\varepsilon,\delta}(\nabla\varphi)$ *in* $\Omega \times \frac{1}{\delta}Y \times]o,T[$ *for all*$\varphi \in L^q(0,T;H^1_0(\Omega))$.

Theorem 3.1 *Let $p \in [1,+\infty[$ and $q \in [1,+\infty[$.*

- *Let* $\varphi \in L^q(0,T,L^p(\Omega))$

$$\frac{\delta^N}{|Y|}\int_{\Omega\times R^N}\mathcal{T}_{\varepsilon,\delta}(x,z,t)dxdz = \int_{\hat{\Omega}_\varepsilon}\varphi(x,t)dx$$
$$= \int_{\Omega}\varphi(x,t)dx - \int_{\Lambda_\varepsilon}\varphi(x,t)dx.$$

For a.e. $t \in]0,T[$.

The continuity of the operator $\mathcal{T}_{\varepsilon,\delta}$, *as follows:*

$$||\mathcal{T}_{\varepsilon,\delta}(\varphi)||_{L^q(0,T;L^p(\Omega))} \leqslant \left(\frac{|Y|}{\delta^N}\right)^{\frac{1}{p}} ||\varphi||_{L^q(0,T;L^p(\Omega))}.$$

- *Let* $\varphi \in L^q(0,T;H^1(\Omega))$ *and* $N \geqslant 3$. *Then for a.e.* $t \in]0,T[$

$$||\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi))||_{L^p(\Omega\times\frac{1}{\delta}Y))} \leqslant \left(\frac{\varepsilon|Y|}{\delta^{\frac{N}{p}-1}}\right)^{\frac{1}{p}} ||\nabla\varphi||_{L^p(\Omega)}.$$

Proposition 3.4 *Let* $q \in [1, +\infty]$ *and* $\varphi_\varepsilon \in L^q(0, T, L^1(\Omega))$ *satisfying*

$$\int_0^T \int_{\Lambda_\varepsilon} \varphi_\varepsilon dx dt \to 0$$

then

$$\int_0^T \int_\Omega \varphi_\varepsilon dx dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times R^N} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon) dx dz dt.$$

The proof of the following proposition is essentially the same as that of [15,Proposition 2.6].

Proposition 3.5 *Let* q*,* $q \in [1, +\infty]$*. Let* $\{\varphi_\varepsilon\}$*be a sequence in* $L^q(0, T, L^p(\Omega))$ *and* $\{\psi_\varepsilon\}$ *be a sequence in* $L^q(0, t, L^p 0(\Omega))$ *such that*

$$||\varphi_\varepsilon||_{L^q(0,T,L^p(\Omega))} \leqslant C \quad and \quad ||\psi_\varepsilon||_{L^{q'}(0,T,L^{p0}(\Omega))} \leqslant C,$$

where $\frac{1}{p} + \frac{1}{p0}$ *and* $\frac{1}{q} + \frac{1}{p'} = 1$*. Then,*

$$\int_0^T \int_{\hat{\Omega}} \varphi_\varepsilon \psi_\varepsilon dx dt dx dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times \frac{1}{\delta} Y} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon \psi_\varepsilon) dx dz dt.$$

The next two propositions extend to time-dependent functions some properties given in [[9], Theorem 2.11].

Proposition 3.6 *Let* $u \in L^q(0, T; H^1(\Omega))$ *for* $q \in [1, +\infty[$ *one has estimates*

$$||\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}_Y^\varepsilon(u))||_{L^q(0,T;L^p(\Omega;L^{p*}(\mathbb{R}^N)} \leqslant \frac{C\varepsilon |Y|^{\frac{1}{p}}}{\delta^{\frac{N}{p}-1}} ||\nabla||_{L^q(0,T;L^p(\Omega)},$$

and for ω *an open and bounded subset of* $\mathbb{R}^N$

$$||\mathcal{T}_{\varepsilon,\delta}(u)||_{L^q(0,T;L^p(\Omega \times \omega))} \leqslant \frac{2C\varepsilon |Y|^{\frac{1}{p}}}{\delta^{\frac{N}{p}-1}} ||\nabla u||_{L^q(0,T;L^p(\Omega))} + 2|\omega||Y|^{\frac{1-p}{p}} ||u||_{L^q(0,T;L^p(\Omega))},$$

where C *is the Sobolev-Poincare-Wirtinger constant for* $H^1(Y)$*.*

Theorem 3.2 *Let* $p \in [1,+\infty[$, $q \in [1,+\infty]$, $N \geq 3$, $\{w_{\varepsilon,\delta}\}$ *be a sequence in* $L^q(0,T;H^1(\Omega))$ *which is uniformly bounded with respect to* ε *and* δ *as* $(\varepsilon,\delta) \to (0,0)$*. Then up to a subsequence, there exists* W *in* $L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))$ *with* $\nabla_z W$ *in* $L^q(0,T;L^p(\Omega\times\mathbb{R}^N))$ *such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\left(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - \mathcal{M}_\varepsilon Y(w_{\varepsilon,\delta})1_{1/\delta Y}\right) \rightharpoonup W \text{ weakly in } L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))),$$

and

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\epsilon,\delta})1_{1/\delta Y}) \rightharpoonup \nabla_z W \quad \text{weakly} \quad \text{in} \quad L^q(0,T;L^p(\Omega\times\mathbb{R}^N)). \tag{3.1}$$

Furthermore, if

$$k^* = \limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon} < +\infty, \tag{3.2}$$

then one can choose the subsequence above and some U *in* $L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N)))$ *with*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \text{ weakly in } L^q(0,T;L^p(\Omega;L^p_{loc}(\mathbb{R}^N))). \tag{3.3}$$

CHAPTER

4

HOMOGENIZATION OF A CLASS OF HYPERBOLIC PROBLEMS

4.1 Homogenization of hyperbolic problem in the perforated domain Ω_ε

In this section,we use the adapted unfolding method presented in Section 2 to study the asymptotic behavior of a class of hyperbolic problems in perforated domains.
To introduce the coefficient matrix, we define, for $\alpha, \beta, \Omega \in \mathbb{R}$ with $0 < \alpha < B,$ the set $M(\alpha, \beta, \Omega)$ of the nxn matrix-valued functions in $L^\infty(\Omega)$ such that

$$(A(x)\lambda, \lambda) \geqslant \alpha|\lambda|, \qquad |A(x)\lambda| \geqslant \beta|\lambda|$$

For any $\lambda \in \mathbb{R}^n$ and a.e. on Ω.

For any ε, we suppose that

$$\begin{cases} A^\varepsilon \in M(\alpha, \beta, \Omega) \\ A^\varepsilon symmetric \end{cases} \tag{4.1}$$

Consider the following hyperbolic problem with homogeneous Dirichlet-Neumann boundary

$$(P_\varepsilon) = \begin{cases} u''_\varepsilon - div(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & in \quad \Omega^*_\varepsilon \times (0,T), \\ u_\varepsilon = 0 & \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 & on \quad \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u^0_\varepsilon, \qquad u'_\varepsilon(x,0) = u^1_\varepsilon & in \quad \Omega^*_\varepsilon, \end{cases} \tag{4.2}$$

where n_ε is the outward unit normal vector field defined on ∂S_ε

we suppose that

$$V^\varepsilon = \{v \in H^1_0(\Omega), \quad v = 0 \quad in \quad \partial\Omega\}$$

$$\begin{cases} u^0 \in V^\varepsilon, \\ u^1_\varepsilon \in L^2(\Omega^*_\varepsilon), \\ f_\varepsilon \in L^2(0,T,L^2(\Omega^*_\varepsilon)). \end{cases} \tag{4.3}$$

$$\mathcal{W} = \{v_\varepsilon | v_\varepsilon \in L^2(0,T,V^\varepsilon), \quad v' \in L^2(0,T,L^(\Omega^*_\varepsilon))\} \tag{4.4}$$

with the norm defined by

$$||v_\varepsilon||_{\mathcal{W}_\varepsilon} = ||v_\varepsilon||_{L^2(0,T,V^\varepsilon)} + ||v'||_{L^2(0;L^2(\Omega^*_\varepsilon))}.$$

4.1.1 variational formulation :

we multiply the equation of problem (4.2) by a test function v where $v \in V^\varepsilon$ we get:

$$\int_{\Omega^*_\varepsilon} u''_\varepsilon v dx - \int_{\Omega^*_\varepsilon} (div A^\varepsilon \nabla u_\varepsilon) v dx = \int_{\Omega^*_\varepsilon} f_\varepsilon v dx, \tag{4.5}$$

we integrate on Ω^*_ε we get

$$-\int_{\Omega^*_\varepsilon} div(A^\varepsilon \nabla u_\varepsilon) v dx = \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v dx - \int_{\partial S_\varepsilon \cap \partial\Omega} A^\varepsilon \nabla u_\varepsilon v n_\varepsilon d\Gamma,$$

by Green's formula for the second integral we get

$$-\int_{\Omega^*_\varepsilon} div(A^\varepsilon \nabla u_\varepsilon) v dx = \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v dx - \int_{\partial S_\varepsilon} A^\varepsilon \nabla u_\varepsilon v n_\varepsilon d\Gamma + \int_{\partial\Omega} A^\varepsilon \nabla u_\varepsilon v n_\varepsilon d\Gamma, \tag{4.6}$$

then

$$-\int_{\Omega^*_\varepsilon} div(A^\varepsilon \nabla u_\varepsilon) v dx = \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v dx, \tag{4.7}$$

we replace (4.7) in (4.5) we get

$$\int_{\Omega^*_\varepsilon} u'' v dx + \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v dx = \int_{\Omega^*_\varepsilon} f_\varepsilon v dx.$$

The variational formulation of problem (4.2) is the following

$$\begin{cases} \int_{\Omega^*_\varepsilon} u'' v dx, + \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v dx = \int_{\Omega^*_\varepsilon} f_\varepsilon v \; dx, \\ in \quad \mathcal{D}'(0,T) \quad for \quad all \quad v \in V^\varepsilon, \\ u_\varepsilon(x,0) = u^0_\varepsilon, \quad u'_\varepsilon(x,0) = u^1_\varepsilon \quad in \quad \Omega^*_\varepsilon. \end{cases} \tag{4.8}$$

For every fixed ε classical provide that problem (4.8) has a unique solution u_ε such that

$$u_\varepsilon \in \mathcal{C}^0([0,T]; V^\varepsilon) \cap \mathcal{C}^1([0,T]; L^2(\Omega^*_\varepsilon)).$$

In order to study the homoginization of problem (4.2), we suppose that there exists a matix $A = (a_{ij})_{1<i,j<n}$ such that

$$\mathcal{T}(A^\varepsilon) \to A \quad strogly \quad in \quad (L^1(\Omega \times Y^*))^{n\times n}, \tag{4.9}$$

which implies Ae$A[(\alpha, \beta, \Omega \times Y)$ (see also [5] and [6]).

Concerning the initial data, we assume that

$$\begin{cases} ||u_\varepsilon^0||_{V^\varepsilon} \leqslant C, \\ \tilde{u}_\varepsilon^0 \to \theta u^0 \quad weakly \quad in \quad L^2(\Omega), \\ \tilde{u}_\varepsilon^1 \to \theta u^1 \quad weakly \quad in \quad L^2(\Omega), \\ \tilde{f}_\varepsilon \to \theta f \quad weakly \quad in \quad L^2(0,T;L^2(\Omega)), \end{cases} \tag{4.10}$$

where c is a constant independent of eunder these assumptions, classical results showthat problem (4.2) has a unique solution u'' with the following uniform estimate

$$||u||_{L^\infty(0.T;H^1(\Omega))} + ||u'||_{L^\infty(0.T;L^2(\Omega))} \leqslant C, \tag{4.11}$$

where the constant C does not depend on ε.

Now we state the main theorem of this section.

Theorem 4.1 *Let*A^ε *satisfy (4.1) and, (4.9) Suppose that* u_ε *is the solution of problem (4.2) with (4.3) and (4.10).Then there existe* $u \in L^\infty(0,T;H_0^1(\Omega))$ *with* $u' \in L^\infty(0,T;L^2(\Omega))$ *and* $\hat{u} \in L^\infty(0.T;H_p^1er(Y^*))$ *with* $\mathcal{M}_Y^*(\hat{u}) = 0$*, such that*

$$\begin{cases} \mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega,H^1(Y^*)), \\ \mathcal{T}_\varepsilon^*(u_\varepsilon') \rightharpoonup u' \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega \times Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \hat{u} \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega \times Y^*))), \\ \mathcal{T}_\varepsilon^*(u_\varepsilon) \to u \quad stongly \quad in \quad L^q(0,T;L^2(\Omega,H^1(Y^*))), \\ ||u_\varepsilon - u||_{L^q(0,T,L^2(\Omega_\varepsilon^*))} \to 0, \end{cases} \tag{4.12}$$

where q is any number in $(1,+\infty)$. The pair $(u,\hat{u})$ with $\mathcal{M}_{Y^*}(\hat{u}) = 0$ is the unique solution of the following Problem

$$\begin{cases} \theta\displaystyle\int_0^T\int_\Omega u\Psi\varphi'' dx+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla\Phi)dxdydt=\theta\int_o^T\int_\Omega f\Psi\varphi dxdt \\ for \quad any \quad \Psi\in H_0^1(\Omega),\Phi\in L^2(\Omega;H^1_{per}(Y^*)) \quad and \quad \varphi\in\mathcal{D}(0,T), \\ u=0 \quad on \quad \Omega\times(0,T), \\ u(x,0)=u^0, \quad u'(x,0)=u^1 \quad in \quad \Omega. \end{cases} \tag{4.13}$$

We have also

$$\hat{u}=\sum_{j=1}^n\frac{\partial u}{\partial x_j}\mathcal{X}_j \tag{4.14}$$

with $\mathcal{X}_j\in L^\infty(\Omega;H_p^1er(Y^*))(j=1,..n)$ begin the solution of the cell problem

$$\begin{cases} -div_y(A\nabla_y(\mathcal{X}+y_j))=0 \quad in \quad Y^* \\ (A\nabla_y(\mathcal{X}+y_j)).n_1=0 \quad on \quad \partial S \\ \mathcal{M}_Y.(\mathcal{X}_|)(x,.)=0 \quad \mathcal{X}_|(x,.) \quad Y-periodic \end{cases} \tag{4.15}$$

Moreover,

$$\begin{aligned} (i)\tilde{u}_\varepsilon\to\theta u \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega)), \\ (ii)A^\varepsilon\nabla\tilde{u}_\varepsilon\to\theta A^0\nabla u \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega)), \end{aligned} \tag{4.16}$$

Proof of Theorem 4.1. In view of ((4.11)) and Theorem 2.1, we get that there exist $u\in L^\infty(0,T;H_0^1(\Omega))$ and $u'\in L^\infty(0,T;L^2(\Omega))$and $\hat{u}\in L^\infty(0,,TH^1_{per}(Y^*))$ with$\mathcal{M}_Y^*(\hat{u})$ such that, up to a subsequence (still denoted by ε), (4.12) holds. From Proposition (2.4)(iii), we further get that

$$\begin{aligned} &(i)\tilde{u}\rightharpoonup\theta\mathcal{M}_Y^*(u) \qquad weakly^* \quad in \quad L^\infty(o,T;L^2(\Omega)), \\ &(ii)A^\varepsilon\widetilde{\nabla u}_\varepsilon\rightharpoonup\theta\mathcal{M}_{Y^*}[A(\nabla u+\nabla_y\hat{u})] \qquad weakly^* \quad in \quad L^\infty(o,T;L^2(\Omega)). \end{aligned} \tag{4.17}$$

Since u is independent of y, then convergence ((4.17))(i) holds true for the above subsequence.

Let $\Psi, \phi \in \mathcal{D}$ and $\psi \in H^1_{per}(Y^*)$. set

$$v_\varepsilon = \Psi(x) + \varepsilon\varphi(x)\psi^\varepsilon(x), \quad with \quad \psi^\varepsilon(x) = \psi(\frac{x}{\varepsilon}), \tag{4.18}$$

then

$$\begin{aligned}
\nabla_x v_\varepsilon &= \Psi(x) + \varepsilon\varphi(x)\psi^\varepsilon(x))\\
\nabla_x v_\varepsilon &= \nabla_x\left(\Psi(x) + [\varepsilon\phi(x)\psi^\varepsilon(x)]\right)\\
\nabla_x v_\varepsilon &= \nabla_x\Psi(x) + \varepsilon\nabla[\phi(x)\psi\left(\frac{x}{\varepsilon}\right)]\\
\nabla_x v_\varepsilon &= \nabla_x\Psi(x) + \varepsilon\nabla\phi(x)\psi\left(\frac{x}{\varepsilon}\right) + \varepsilon \times \frac{1}{\varepsilon}\phi(x)\nabla_y\psi\left(\frac{x}{\varepsilon}\right)\\
\nabla_x v_\varepsilon &= \nabla_x\Psi(x) + \varepsilon\psi^\varepsilon\nabla_\phi + \phi(\nabla_y\psi)\left(\frac{\cdot}{\varepsilon}\right).
\end{aligned}$$

by proposition 2.4(ii) we have

$$\begin{cases}
\mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi \quad strongly \quad in \quad L^2(\Omega\times Y^*),\\
\mathcal{T}^*_\varepsilon(\phi\psi^\varepsilon) \to \Phi \quad strongly \quad in \quad L^2(\Omega\times Y^*) with \quad \Phi = \phi(x)\psi(x),\\
\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \to \nabla\Psi + \nabla_y\Phi \quad strongly \quad in \quad L^2(\Omega\times Y^*).
\end{cases} \tag{4.19}$$

Let $\varphi \in \mathcal{D}(0, T)$. From (4.12) and (4.19) and proposition 4.10 we deduce

$$\int_0^T\int_{\Omega^*_\varepsilon} u''_\varepsilon v_\varepsilon\varphi dxdt \to \theta\int_0^T\int_\Omega u\Psi\varphi'' dxdt. \tag{4.20}$$

$$\int_0^T\int_{\Omega^*_\varepsilon} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dxdt \to \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u + \nabla_y\tilde{u})(\nabla\Psi + \nabla_y\Phi)\varphi dxdydt. \tag{4.21}$$

$$\int_0^T\int_{\Omega^*_\varepsilon} f_\varepsilon v_\varepsilon\varphi dxdt \to \theta\int_0^T\int_\Omega f\Psi\varphi dxdt. \tag{4.22}$$

Now choossing $v_\varepsilon\varphi$ as test function in the variational formulation we obtain

$$\int_0^T\int_{\Omega^*_\varepsilon} u'' v_\varepsilon\varphi dxdt - \int_0^T\int_{\Omega^*_\varepsilon} A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dxdt = \int_0^T\int_{\Omega^*_\varepsilon} f_\varepsilon v_\varepsilon\varphi dx. \tag{4.23}$$

Passing to the limit in this exepression term by term we start by the first term we applay here the operator $\mathcal{T}_\varepsilon$ we obtain

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dxdt &= \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega\times Y} u_\varepsilon'' v_\varepsilon \varphi dxdt \\ &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon v_\varepsilon \varphi'' dxdt \\ &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi'') dxdydt\end{aligned} \tag{4.24}$$

by (4.12) and (4.19) we have

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi'') dxdydt = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega} u\Psi\varphi'' dxdydt, \tag{4.25}$$

now by Fubini's theorem we get

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega} u\Psi\varphi'' dxdt\int_{Y*} dy = \frac{|Y^*|}{|Y|}\int_0^T\int_{\Omega} u\Psi\varphi'' dxdt = \theta\int_0^T\int_{\Omega} u\Psi\varphi'' dxdt,$$

then

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dxdt = \theta\int_0^T\int_{\Omega} u\Psi\varphi'' dxdt, \tag{4.26}$$

then

$$\int_0^T\int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dxdt \to \theta\int_0^T\int_{\Omega} u\Psi\varphi'' dxdt. \tag{4.27}$$

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dxdt &= \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega\times Y^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dxdt \\ &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathcal{T}_\varepsilon(\varphi) dxdt,\end{aligned} \tag{4.28}$$

by (4.12) and (4.19) and also (4.9) we have :

$$\begin{aligned}&\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \to \nabla u + \nabla_y \hat{u} \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega, L^2(\Omega\times Y^*))), \\ &\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \to \nabla\Psi + \nabla\Phi \quad strongly \quad in \quad L^2(0,T;L^2(\Omega, L^2(\Omega\times Y^*))), \\ &\mathcal{T}_\varepsilon^*(A^\varepsilon) \to A \quad srongly \quad in \quad L^1(\Omega\times Y^*),\end{aligned}$$

we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega_\varepsilon^*}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)dxdydt\\ &=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt,\end{aligned}$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dxdt \; =\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt, \tag{4.29}$$

this imply

$$\int_0^T\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dxdt\to\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\hat{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt. \tag{4.30}$$

$$\int_0^T\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dxdt=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega\times Y}f_\varepsilon v_\varepsilon\varphi dxdt=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(f_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)dxdydt, \tag{4.31}$$

by (4.19) we get

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(f_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)dxdydt=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}f\Psi\varphi dxdt, \tag{4.32}$$

now by Fubini's theorem we get

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_\Omega f\Psi\varphi dxdt\int_{Y^*}dy=\lim_{\varepsilon\to 0}\frac{|Y|^*}{|Y|}\int_0^T\int_\Omega f\psi\varphi dxdt=\theta\int_0^T\int_\Omega f\Psi\varphi dxdt,$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dxdt=\theta\int_0^T\int_\Omega f\Psi\varphi dxdt, \tag{4.33}$$

then

$$\int_0^T\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dxdt\to\theta\int_0^T\int_\Omega f\Psi\varphi dxdt, \tag{4.34}$$

Thus, combining (4.26) with (4.29) and (4.33), we get

$$\begin{aligned}&\theta\int_0^T\int_\Omega u_\varepsilon\Psi\varphi'' dxdt+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt\\ &=\theta\int_0^T\int_\Omega f\Psi\varphi dxdt.\end{aligned} \tag{4.35}$$

Now, in order to check the initial condition let v_ε be given by (4.18) and $\varphi\in C^\infty[0,T]$ with $\varphi(0)=1$ and $\varphi(T)=0$. choosing $v_{\varepsilon\varphi}$ as test function in the variational formulation

(4.8).

Using the initial conditions in (4.2) and by integration by parts, we have

$$\int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx - \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx$$

$$-\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx$$

$$-\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx + \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon^*} u_\varepsilon'' v_\varepsilon \varphi dx$$

$$-\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx + \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon} u_\varepsilon' v_\varepsilon \varphi \mid_0^T dx - \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon' v_\varepsilon \varphi' dxdt$$

$$-\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi dx + \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi dx = \int_{\Omega_\varepsilon} u'(0,x) v_\varepsilon dx - \int_0^T \int_{\Omega_\varepsilon} v_\varepsilon u_\varepsilon' \varphi' dxdt$$

Passing to the limit and making use of (4.20)-(4.22), we get

$$-\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} A(\nabla u + \nabla_y \tilde{u})(\nabla\Psi + \nabla_y \Phi)\varphi dxdydt + \theta \int_0^T \int_\Omega f\psi\varphi dxdt$$

$$= -\theta \int_\Omega u^1 \Psi dx - \theta \int_0^T \int_\Omega u'' \Psi \varphi dxdt,$$

by integration by parts the them $\int_0^T \int_\Omega u'' \Psi \varphi dxdt$ we get

$$-\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} A(\nabla u + \nabla_y \tilde{u})(\nabla\Psi + \nabla_y \Phi)\varphi dxdydt + \theta \int_0^T \int_\Omega f\psi\varphi dxdt$$

$$= -\theta \int_\Omega u^1 \Psi dx + \theta \int_\Omega u' \Psi |_0^T dxdt - \theta \int_0^T \int_\Omega \Psi u'' \varphi dxdt$$

$$= -\theta \int_\Omega u^1 \Psi dx + \theta \int_\Omega u'(0,x) \Psi dxdt - \theta \int_0^T \int_\Omega \Psi u'' \varphi dxdt,$$

we have

$$\begin{aligned} &-\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y^*} A(\nabla u + \nabla_y \tilde{u})(\nabla\Psi + \nabla_y \Phi)\varphi dxdydt + \theta \int_0^T \int_\Omega f\psi\varphi dxdt \\ &= -\theta \int_\Omega u^1 \Psi dx + \theta \int_\Omega u'(0,x) \Psi dxdt + \theta \int_0^T \int_\Omega u'' \Psi \varphi dxdt, \end{aligned} \tag{4.36}$$

Combining (4.13) with (4.36)we obtian

$$
\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt+\theta\int_0^T\int_\Omega f\Psi\varphi dxdt\\
&+\theta\int_\Omega u^1\Psi dx-\theta\int_\Omega u'(0,x)\Psi dxdt+\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt-\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt\\
&+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt-\theta\int_0^T\int_\Omega f\Psi\varphi dxdt.
\end{aligned}
$$

Then we obtain

$$u'(x,0)=u^1. \tag{4.37}$$

Choosing $\varphi\in C^\infty([0,T])$ with $\varphi(0)=\varphi(T)=\varphi'(T)=0,\varphi(0)=1$,and taking $v_\varepsilon\varphi$ as test function in the variational fornulation (4.10).

Using the initial conditions in (4.2) and by integration by parts, we have

$$\int_{\Omega_\varepsilon^*}u''v_\varepsilon\varphi dx-\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dx=\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dx \tag{4.38}$$

$$-\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dx+\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dx=\int_{\Omega_\varepsilon^*}u''v_\varepsilon\varphi dx$$

$$-\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dx+\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dx=\int_{\Omega_\varepsilon}uv_\varepsilon\varphi'\ |_0^T\ dx-\int_0^T\int_{\Omega_\varepsilon}u'v_\varepsilon\varphi'dxdt$$

$$-\int_{\Omega_\varepsilon^*}A^\varepsilon\nabla u_\varepsilon\nabla v_\varepsilon\varphi dx+\int_{\Omega_\varepsilon^*}f_\varepsilon v_\varepsilon\varphi dx=-\int_{\Omega_\varepsilon}u_\varepsilon(0)v_\varepsilon dx-\int_0^T\int_{\Omega_\varepsilon}uv_\varepsilon\varphi''dxdt \tag{4.39}$$

Passing to the limit and making use of (4.20) - (4.22) we get

$$
\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt+\theta\int_0^T\int_\Omega f\Psi\varphi dxdt\\
&=-\theta\int_\Omega u^0\Psi-\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt
\end{aligned}
$$

by integration by parts we get

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt+\theta\int_0^T\int_\Omega f\Psi\varphi dxdt\\
&=-\theta\int_\Omega u^0\Psi+\theta\int_\Omega u\Psi|_0^T dx+\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt\\
&=-\theta\int_\Omega u^0\Psi+\theta\int_\Omega \Psi u(x,0)dxdt+\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt
\end{aligned}$$

we have

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt+\theta\int_0^T\int_\Omega f\Psi\varphi dxdt\\
&=-\theta\int_\Omega u^0\Psi+\theta\int_\Omega \Psi u(0,x)dxdt+\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt.
\end{aligned}\tag{4.40}$$

Combining (4.40) with (4.13) we get

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt+\theta\int_0^T\int_\Omega f\psi\varphi dxdt\\
&+\theta\int_\Omega u^0\Psi dx-\theta\int_\Omega \Psi u(0,x)dxdt-\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt+\theta\int_0^T\int_\Omega u''\Psi\varphi dxdt\\
&+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u+\nabla_y\tilde{u})(\nabla\Psi+\nabla_y\Phi)\varphi dxdydt-\theta\int_0^T\int_\Omega f\psi\varphi dxdt.
\end{aligned}$$

Then we get

$$u(x,0)=u^0.\tag{4.41}$$

4.2 Homogenization of hyperbolic problem in the perforated domain $\Omega_{\varepsilon,\delta}$

In this section, we suppose that $N\geq 3$, and that ε and $\delta=\delta(\varepsilon)$ are such that (3.2) holds, that is, there exists the following limit and is finite

$$k^*=\lim_{\varepsilon\to 0}\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}<+\infty.\tag{4.42}$$

We also denote by $M(\alpha,\beta,\Omega)$ the set of $N\times N$ matrices $A=(a_{ij})_{1\leq i,j\leq N}$ in $(L^\infty(\Omega))^{N\times N}$ such that

(i) $(A(x)\lambda,\lambda)\geq\alpha|\lambda|^2$,

(ii) $|A(x)\lambda| \leq \beta|\lambda|$,

for any $\lambda \in \mathbb{R}^N$ and almost everywhere on Ω, where $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. We want to study the asymptotic behavior as $\varepsilon \rightarrow$, of the problem

$$\begin{cases} u''_{\varepsilon\delta}(x,t) - div(A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)) = f_{\varepsilon,\delta}(x,t) \qquad in \quad \Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,t) = 0 \quad on \quad \partial\Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \quad in \quad \Omega^*_{\varepsilon,\delta}, \end{cases} \tag{4.43}$$

We suppose that the data satisfy the following assumptions

$$\begin{cases} A^\varepsilon \in \ M(\alpha,\beta,\Omega), \\ A^\varepsilon symmetric, \\ f_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta})), \\ u^0_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta}), \\ u^1_{\varepsilon,\delta} \in L^2(\Omega). \end{cases} \tag{4.44}$$

Moreover, we assume that

$$\begin{cases} u^0_{\varepsilon,\delta} \rightharpoonup u^0 \quad weakly \quad in \quad L^2(\Omega), \\ u^1_{\varepsilon,\delta} \rightharpoonup u^1 \quad weakly \quad in \quad L^2(\Omega), \\ f_{\varepsilon,\delta} \rightharpoonup f \quad weakly \quad in \quad L^2(0,T;L^2(\Omega)). \end{cases} \tag{4.45}$$

The Set

$$\mathcal{W}_{\varepsilon,\delta} = \{v_\varepsilon | v_\varepsilon \in L^2(0,T,V^\varepsilon), \quad v' \in L^2(0,T,L^{(}\Omega^*_\varepsilon))\},$$

is equipped with the norm

$$||v_{\varepsilon,\delta}||_{\mathcal{W}_{\varepsilon,\delta}} = ||v_{\varepsilon,\delta}||_{L^2(0,T,H^1_0(\Omega^*_{\varepsilon,\delta})} + ||v'_{\varepsilon,\delta}||_{L^2(0;L^2(\Omega^*_{\varepsilon,\delta}))} \tag{4.46}$$

4.2.1 variational formulation :

we multiply (4.43) by a test function $\boldsymbol{v}$ where $\boldsymbol{v} \in \boldsymbol{V}^\varepsilon$ we get

$$\int_{\Omega^*_{\varepsilon,\delta}} u''_\varepsilon v dx - \int_{\Omega^*_{\varepsilon,\delta}} (div A^\varepsilon \nabla u_{\varepsilon,\delta}) v dx = \int_{\Omega^*_\varepsilon} f_{\varepsilon,\delta} v dx$$

we integrate on $\boldsymbol{\Omega^*_{\varepsilon,\delta}}$ and by Green's formula we get

$$-\int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_\varepsilon) v dx = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v dx - \int_{\partial\Omega_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} d\Gamma \tag{4.47}$$

$$-\int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_{\varepsilon,\delta}) v dx = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v dx - \int_{\partial\Omega_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} v n_{\varepsilon,\delta} d\Gamma$$

then:

$$-\int_{\Omega^*_{\varepsilon,\delta}} div(A^\varepsilon \nabla u_{\varepsilon,\delta}) v dx = \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v dx \tag{4.48}$$

we replace (4.48)in (4.47) we get

$$\int_{\Omega^*_{\varepsilon,\delta}} u'' v dx - \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v dx = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta} v dx \tag{4.49}$$

The variational formulation of problem (4.43) is: Find $\boldsymbol{u_{\varepsilon,\delta}} \in \boldsymbol{\mathcal{W}_{\varepsilon,\delta}}$ such that for all $\boldsymbol{v} \in \boldsymbol{H^1_0(\Omega\varepsilon, \delta)}$

$$\begin{cases} \int_{\Omega^*_{\varepsilon,\delta}} u'' v dx + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla v dx = \int_{\Omega^*_{\varepsilon,\delta}} f_\varepsilon v dx \quad in \quad \mathcal{D}(0,T), \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \quad in \quad \Omega^*_{\varepsilon,\delta}. \end{cases} \tag{4.50}$$

Classical results [22, 13] provide for every fixed $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ the existence and uniqueness of a solution of problem (4.43) such that

$$u_{\varepsilon,\delta} \in C^0([0,T]; H^1_0(\Omega^*_{\varepsilon\delta}) \cap C^1([0,T]; L^2(\Omega^*_{\varepsilon,\delta})),$$

and satisfies the estimate

$$||u_{\varepsilon,\delta}||_{L^\infty(0,T;L^2(\Omega))} + ||u'_{\varepsilon,\delta}||_{L^2(0,T;H^1_0(\Omega))} \leqslant C, \tag{4.51}$$

where C is independent of ε and δ.

Remark 4.1 *: In the following, we identify functions in $H_0^1(\Omega^*_{\varepsilon,\delta})$ with their zero extension to $H_0^1(\Omega)$ so that we can write (4.51) as*

$$||u_{\varepsilon,\delta}||_{L^\infty(0,T,H_0^1(\Omega^*_{\varepsilon,\delta}))} + ||u'_{\varepsilon,\delta}||_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{4.52}$$

where C is independent of ε and δ.

We adapt here for the evolution problem some arguments introduced in [9]. Let us introduce the functional space

$$K_B = \left\{\Phi \in L^2(0,T;L^{2*}(R^N)) : \nabla\Phi \in L^2(0,T;L^2(R^N)), \Phi \quad is \quad constant \quad on \quad B\right\}. \tag{4.53}$$

We also need the following lemmas from [9] in order to pass to the limit in equation(4.50)

Lemma 4.1 *([9]) Let $N \geqslant 3$. Then for every δ_0 the set*

$$\cup_{0<\delta<\delta_0}\left\{\phi \in H^1_{per}(Y) : \phi = 0 \quad on \quad \delta B\right\},$$

is dense in $H^1_{per}(Y)$.

Lemma 4.2 *([6]) Let $v \in \mathcal{D}(\mathbb{R}^N) \cap K_B$ $(i.e, v = v(B)$ is constant on B) and set*

$$w_{\varepsilon,\delta} = v(B) - v\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad for \quad x \in \mathbb{R}^{\mathrm{N}}.$$

Then

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \quad weakly \quad in \quad H^1(\Omega) \tag{4.54}$$

Remark 4.2 *: (1) From definition of $w_{\varepsilon,\delta}$ above one has*

$$\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})(x,z) = v(B) - v(z) \quad in \quad \hat{\Omega} \times \frac{1}{\delta},$$

and consequently (see [9]),

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})(x,z)) = \frac{1}{\varepsilon\delta}\left(\nabla_z\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\right) = -\frac{1}{\varepsilon\delta}\nabla_z v \quad in \quad \hat{\Omega} \times \frac{1}{\delta}. \tag{4.55}$$

(2) Let $\{w_{\varepsilon,\delta}\}$ be a sequence satisfying (4.54). We have,

$$\mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \to v(B) \text{ strongly in } L^2(\Omega \times Y). \tag{4.56}$$

Indeed, it was shown in [9] that $\{w_{\varepsilon,\delta}\}$ is bounded in $H^1(\Omega)$ so that together with (4.54) and the Rellich compactness theorem, one has $w_{\varepsilon,\delta} \to v(B)$ strongly in $L^2(\Omega)$; that is,

$$||w_{\varepsilon,\delta} - v(B)||_{L^2(\Omega)} \to 0.$$

We state now a homogenization theorem for system (4.43)

Theorem 4.2 *Under assumptions (4.44) and (4.45), suppose that as $\varepsilon \to 0$, there is a matrix field A such that*

$$\mathcal{T}_\varepsilon(A_\varepsilon)(x,y) \to A(x,y) \text{ a.e. in } \Omega \times Y, \tag{4.57}$$

and as both $\varepsilon, \delta \to 0$, there exists a matrix field A^0 such that

$$\mathcal{T}_{\varepsilon,\delta}(A_\varepsilon)(x,z) \to A^0(x,z) \text{ a.e. in } \Omega \times (\mathbb{R}^N \setminus B). \tag{4.58}$$

Let $u_{\varepsilon,\delta}$ be the solution of (4.50). Then there exists u in $L^\infty(0,T;H^1_0(\Omega))$ and $\hat{u}$ in $L^\infty(0,T;L^2(\Omega;H^1_{per}(Y)))$ such that

$$\begin{aligned}
&(i) u_{\varepsilon,\delta} \rightharpoonup u \quad weakly^* in \quad L^\infty(0,T;H^1_0(\Omega)),\\
&(ii) u'_{\varepsilon,\delta} \rightharpoonup u' \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega)),\\
&(iii) \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \rightharpoonup u \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega;H^1(Y))),\\
&(iv) \mathcal{T}_\varepsilon(u'_{\varepsilon,\delta}) \rightharpoonup u' \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega \times Y)),\\
&(v) \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \hat{u} \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega \times Y)).
\end{aligned} \tag{4.59}$$

and $U \in L^2(0,T;L^2_{lOC}(\mathbb{R}^N))$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \text{ weakly in } L^2(0,T;L^2(\Omega;L^2_{\text{loc}}(\mathbb{R}^N))), \tag{4.60}$$

with U vanishing on $\Omega \times B \times]0,T[$ and $U - k^* u \in L^2(0,T,L^2(\Omega;K_B))$ k_b begin defined by(4.53) The couple $(u,\hat{u})$ satisfies the limit equation

$$\int_Y A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla_y \phi dy = 0. \tag{4.61}$$

For a.e. $x \in \Omega$ a.e. $t \in]0,T[$ and for $\phi \in H^1_{per}$ while the function U obeys

$$\int_{R^N \setminus B} A^0(x,z)(\nabla_z U(x,z,t) + \nabla_z v(z)dz = 0. \tag{4.62}$$

For a.e. $x \in \Omega$ a.e. $t \in]0,T[$ and for all $v \in K_B$ with $v_B = 0$.

The ordered triplet $(u,\hat{u},U)$ satisfies the limit equation

$$\begin{aligned}\langle u''(\cdot,t),\psi\rangle_{(H^1_0(\Omega))',H^1_0(\Omega)} &+ \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\,dx\,dy \\ &- k^* \int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B \psi(x)\,dx\,d\sigma_z \\ &= \int_\Omega f(x,t)\psi(x)\,dx, \quad \text{for a.e. } t \in]0,T[\text{ and for all } \psi \in H^1_0(\Omega),\end{aligned} \tag{4.63}$$

with

$$u(x,0) = u_0, \quad u'(x,0) = u_1 \text{ in } \Omega,$$

where ν_B is the inward normal to ∂B and $d\sigma_z$ its surface measure.

In what follows, we will use the notation $m_Y(\cdot)$ for the average over Y defined as

$$m_Y(v) = \frac{1}{|Y|}\int_Y v(y)\,dy, \quad \forall v \in L^1(Y).$$

The result below describes now the homogenized problem in the variable (x,t) in $\Omega\times]0,T[$. To this aim, let us consider the correctors $\hat{\chi}_j$, $j = 1,\dots,N$ solutions of the cell problem; they are the same for domains without holes (see [1, 13])

$$\hat{\chi}_j \in L^\infty(\Omega; H^1_{\text{per}}(Y)),$$

,

$$\int_Y A\nabla(\hat{\chi}_j - y_j)\nabla\phi\,dy = 0 \quad \text{a.e.} \quad x \in \Omega, \forall\phi \in H^1_{\text{per}}(Y),$$

$$m_Y(\hat{\chi}_j) = 0,$$

where $\boldsymbol{A}$ is given by (4.57).

We consider also the cell problem corresponding to the holes $\boldsymbol{B}$ defining the corrector $\boldsymbol{\theta}$ for small holes, introduced in [9],

$$\begin{gathered}
\boldsymbol{\theta} \in L^\infty(\Omega; K_B), \quad \boldsymbol{\theta}(x, B) \equiv 1, \\
\int_{\mathbb{R}^N \setminus B} \tilde{A}_0(x, z) \nabla_z \theta(x, z) \nabla_z \Psi(z)\, dz = 0. \\
\text{a.e. for } x \in \Omega,\ \forall \Psi \in K_B \text{ with } \Psi(B) = 0.
\end{gathered} \tag{4.64}$$

Proof of Theorem We prove the results in several steps.

Step 1. The existence of $\boldsymbol{u} \in L^\infty(0, T; H_0^1(\Omega))$ such that up to subsequences, convergences (4.59))(i)-(ii) hold, follows from estimate (4.51) while the existence of $\hat{\boldsymbol{u}} \in L^\infty(0, T; L^2(\Omega; H^1_{per}(Y)))$ and such that convergences (4.59)(iii)-(v) hold, (see also Remark 4.1).

On the other hand, from (4.52) and Theorem 3.2 there exists a function $\boldsymbol{W}$ in $L^2(0, T; L^2(\Omega; L^2(\mathbb{R}^N)))$ with $\nabla_z W \in L^2(0, T; L^2(\Omega \times \mathbb{R}^N))$ such that (up to a subsequence)

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - \mathcal{M}^1_{\varepsilon Y}(u_{\varepsilon,\delta})) \rightharpoonup W \text{ weakly in } L^2(0, T; L^2(\Omega; L^{2*}(\mathbb{R}^N))). \tag{4.65}$$

Moreover, in view of (3.2), again by Theorem 3.2 there exists $\boldsymbol{U}$ such that (up to a subsequence) (4.60) holds.

Step 2. Let us check the properties of the function $\boldsymbol{U}$. From (i) and (ii) of (4.59) we have by compactness,

$$u_{\varepsilon,\delta} \to u \text{ strongly in } L^2(0, T; L^2(\Omega)), \tag{4.66}$$

so that from (3.2),

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}^1_\varepsilon Y(u_{\varepsilon,\delta})_{\frac{1}{\delta}Y} \to k^* u \text{ strongly in } L^2(0, T; L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^N))). \tag{4.67}$$

Thus, from (4.60), (4.65) and (4.67) we conclude that

$$U = W + k^* u \quad \text{and} \quad \nabla_z U = \nabla_z W.$$

Moreover, by using (3.1) of Theorem 3.2, we have

$$\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\epsilon,\delta}) = \delta_{\varepsilon}^{\frac{N}{2}-1}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \text{ weakly in } L^2(0,T;L^2(\Omega\times\mathbb{R}^N)). \tag{4.68}$$

Also, from Definition 3.3,

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \text{ in } \Omega\times B\times]0,T[,$$

and thus from (4.60) and (4.67),

$$U = u = 0 \text{ in } \Omega\times B\times]0,T[. \tag{4.69}$$

This means that

$$W = U - k^* u \in L^2(0,T;L^2(\Omega;K_B)).$$

Step 3. Let us prove the first limit equation. Let $\psi \in D(\Omega)$ and $\phi \in C^1_{\text{per}}(Y)$ vanishing in a neighborhood of $y=0$, and set :

$$v_\varepsilon(x) = \varepsilon\psi(x)\phi^\varepsilon(x) \quad with \quad \phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right). \tag{4.70}$$

then:

$$\begin{aligned}\nabla v_\varepsilon(x) &= \nabla(\varepsilon\psi(x)\phi^\varepsilon(x)\\ \nabla v_\varepsilon(x) &= \varepsilon\nabla\psi(x)\phi^\varepsilon(x) + \psi(x)\nabla\phi^\varepsilon(x),\end{aligned} \tag{4.71}$$

then

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to \psi\nabla_y\phi \text{ strongly in } L^2(\Omega\times Y). \tag{4.72}$$

Taking v_ε as a test function in (4.50), multiplying by $\varphi \in \mathcal{D}(0,T)$, and integrating over $]0,T[$, we get

$$\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)dxdt + \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A_\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\epsilon(x)\varphi(t)\,dx\,dt$$
$$= \int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)dxdt. \tag{4.73}$$

we have v_ε defined in(4.70) by: $v_\varepsilon(x) = \varepsilon\phi^\varepsilon(x)\psi(x) \quad with \quad \psi^\varepsilon = \left(\frac{x}{\varepsilon}\right)$, then we get

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)(\varepsilon\phi^\varepsilon(x)\psi(x))\varphi''(t)\,dx\,dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt \\
&= \int_0^T \int_{\Omega^*_{\epsilon,\delta}} f_{\varepsilon,\delta}(x,t)(\varepsilon\phi^\varepsilon(x)\psi(x))\varphi(t)\,dx\,dt.
\end{aligned}$$

Not that this equation can be rewritten as

$$\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)(\phi^\varepsilon(x)\psi(x))\varphi''(t)\,dx\,dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt \\
&= \varepsilon \int_0^T \int_{\Omega^*_{\epsilon,\delta}} f_{\varepsilon,\delta}(x,t)\phi^\varepsilon(x)\psi(x))\varphi(t)dxdt.
\end{aligned} \tag{4.74}$$

We first use the unfolding operato $\mathcal{T}_\varepsilon$ to pass to the limit in the second term of the left-hand side of this equation. Using (4.71) and (4.72), we have

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi \quad weakly^* \quad in \quad L^\infty(0,T,L^p(\Omega;W^{1.p}(Y))). \\
&\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to \psi\nabla_y\phi \text{ strongly in } L^2(\Omega\times Y). \\
&\mathcal{T}_\varepsilon(A^\varepsilon) \to A(x,y) \quad a.e, \quad in \quad \Omega\times Y. \\
&\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y\hat{u} \quad weakly^* \quad in \quad L^\infty(0,T;L^2(\Omega\times Y)).
\end{aligned}$$

then we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega\times Y} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt \\
&= \lim_{\varepsilon\to} \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dxdydt \\
&= \lim_{\varepsilon\to 0} \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathcal{T}_\varepsilon(\varphi)dxdydt \\
&= \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u + \nabla_y\hat{u})\psi\nabla_y\phi(y)\varphi(t)dxdydt
\end{aligned} \tag{4.75}$$

On the other hand, the first term on the left-hand side of (4.74) as well as the term on the right-hand side goes to zero as $\varepsilon \to 0$, which implies

$$\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A_\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\epsilon(x)\varphi(t)\,dx\,dt = 0,$$

so that

$$\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u + \nabla_y \hat{u})\psi\nabla_y\phi(y)\varphi(t)dxdydt = 0.$$

By Lemma 4.1, we obtain (4.61) which describes the asymptotic behavior of the problem based on the oscillations in the coefficients of (4.43).

Now, to take into account the effect of the perforations, let us use $\omega_{\varepsilon,\delta}\psi\varphi$ as a test function in (4.51), where $\omega_{\varepsilon,\delta}$ is the function defined in Lemma 4.2 and for $\psi \in \mathcal{D}(\Omega)$ and also $\varphi \in \mathcal{D}(0,T)$ and integrat over $]0,T[$ Thus we have:

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_\varepsilon,\delta} u''_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt \\
&+\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla\omega_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt \\
&+\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,delta}(x,t)\omega_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)dxdt \\
&=\int_0^T \int_{\Omega^*_\varepsilon,\delta} f_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\psi(x)dxdt
\end{aligned} \tag{4.76}$$

For the first term on the left-hand side of this equation, we apply the operator $\mathcal{T}_\varepsilon$.Thus, from Definition 3.3 together with Remark 4.2 and (4.59)(iii), we get

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_\varepsilon,\delta} u''_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt \\
&= \lim_{\varepsilon\to 0}\int_0^T \int_{\Omega\times Y} u_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\psi(x)\varphi''(t)dxdt \\
&= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\omega_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi''(t)dxdydt \\
&= \frac{v(B)}{|Y|}\int_0^T \int_{\Omega\times Y} u(x,t)\psi(x)\varphi''(t)dxdydt.
\end{aligned} \tag{4.77}$$

For the second term on the left-hand side of equation (4.76), we use the operator $\mathcal{T}_{\varepsilon,\delta}$. Then, Remark 3.2, together with (3.2), (4.58), (4.68), (4.69)and Remark 4.2, yield

$$\begin{aligned}
&\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_\varepsilon(x,t)\cdot\nabla\omega_\varepsilon(x)\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega\times\mathbb{R}^N} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\cdot\nabla\omega_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla\omega_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)dxdzdt\\
&=\lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-1/\varepsilon\delta\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi)\phi(t)dxdzdt\\
&=\lim_{\varepsilon\to 0}\left(-\frac{\delta^{\frac{N}{2}-1}}{\varepsilon|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)dxdzdt\right)\\
&=-\frac{K^*}{|Y|}\int_0^T\int_{\Omega\times\mathbb{R}^N} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\
&=-\frac{K^*}{|Y|}\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt
\end{aligned}\tag{4.78}$$

so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega\times Y} A_\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\phi(t)dxdt\\
&=-\frac{k^*}{|Y|}\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt.
\end{aligned}\tag{4.79}$$

For the third term on the left-hand side of (4.76), we use $\mathcal{T}_\varepsilon$. From Definition 3.3 together with Remark 4.2], (4.58), passing to the limit gives

$$\begin{aligned}
&\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\omega_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)dxdt\\
&=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\omega_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\varphi(t)dxdydt\\
&=\frac{v(B)}{|Y|}\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\hat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt.
\end{aligned}\tag{4.80}$$

For the term on the right-hand side of equation (4.76), we also apply $\mathcal{T}_\epsilon$, Definition 3.3, Remark 4.2, and (4.45)(iii) and passing to the limit, yields

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt \\
&= \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega\times Y} f_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)dxdt \\
&= \lim_{\varepsilon\to 0} \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} \mathcal{T}_\varepsilon(f_{\varepsilon,\delta}) T_\varepsilon(w_{\varepsilon,\delta}) T_\varepsilon(\psi)\phi(t)dxdydt \\
&= \frac{v(B)}{|Y|} \int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)dxdydt.
\end{aligned} \tag{4.81}$$

Thus, combining (4.77)-(4.81), the limit equation of (4.76) is

$$\begin{aligned}
&v(B) \int_0^T \int_{\Omega\times Y} u(x,t)\psi(x)\phi''(t)dxdydt \\
&- k^* \int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\phi(t)dxdzdt \\
&+ v(B) \int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\phi(t)dxdydt \\
&= v(B) \int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\phi(t)dxdydt.
\end{aligned} \tag{4.82}$$

which is true for all $\varphi \in \mathcal{D}(0,T)$, $\psi \in H^1_0(\Omega)$ and $v \in K_B$. So, we obtain (4.62) for $v \in K_B$ such that $v(B) = 0$. If $v(B) \neq 0$, by applying Stokes' formula and (4.62), we have

$$\begin{aligned}
&\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\, dx\, dz\, dt \\
&= v(B) \int_0^T \int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B \psi(x)\varphi(t)\, dx\, d\sigma_z\, dt,
\end{aligned} \tag{4.83}$$

which used in (4.82) gives the first equation of problem (4.63).

Step 4. It remains now to check the limit initial conditions. Let $v_\varepsilon = w_{\varepsilon,\delta}\psi$ where $w_{\varepsilon,\delta}$ is given by Lemma 4.2 and $\psi \in D(\Omega)$. Let $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Take $v_\varepsilon\varphi$ as a test function in (4.51).

$$\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi(t)dxdt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A + \varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)dxdt$$

$$= \int_0^T \int_{\Omega_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x)\varphi(t) dxdt$$

Using the initial condition in (4.43) and by integration by parts of this term $\int_0^T \int_{\Omega_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x)\varphi(t) dxdt$, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x)\varphi(t) dxdt \\
&= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))|_0^T v_\varepsilon(x)\, dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\, dx\, dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,0)v_\varepsilon(x)\, dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)'\, dx\, dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u^1_{\varepsilon,\delta}(x)v_\varepsilon(x)\, dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)'\, dx\, dt.
\end{aligned}$$

In view of (4.77)-(4.81) and (4.45), passing to the limit in this equation yields

$$\begin{aligned}
&v(B)\int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\, dxdydt \\
&+ k^*\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t) dxdzdt \\
&- v(B)\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\varphi(t)\, dx\, dy\, dt \\
&= -v(B)\int_\Omega u^1(x)\psi(x)\, dx - v(B)\int_0^T \int_\Omega u'(x,t)\psi(x)\varphi'(t)\, dx\, dt \\
&= -v(B)\int_\Omega u^1(x)\psi(x)\, dx + v(B)\int_\Omega u'(x,0)\psi(x)\, dx \\
&+ v(B)\int_0^T \int_\Omega u''(x,t), \psi(x)\varphi(t) dxdt
\end{aligned} \tag{4.84}$$

Combining (4.84) with (4.82) yields

$$-\int_\Omega u^1(x)\psi(x)\, dx + \int_\Omega u'(x,0)\psi(x)\, dx = 0, \quad \forall\psi \in \mathcal{D}(\Omega),$$

which implies $u'(x,0) = u^1(x)$.

For the first initial condition, let us now choose $\varphi \in C^\infty([0,T])$ with $\varphi(0) = \varphi(T) = \varphi'(T) = 0$ and $\varphi'(0) = 1$. Let us take again $v_\varepsilon \varphi$ as a test function in (4.43).

$$\begin{aligned}&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) v_\varepsilon(x,z)\varphi(t)dxdt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A_\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x,z)\varphi(t)dxdt\\ &= \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x,z)\varphi(t)dxdt\end{aligned}$$

Using the initial conditions in (4.43) and by integration by parts, we have

$$\begin{aligned}&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x,z)\varphi(t)dxdt\\ &= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))\Big|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi'(t)\,dx\,dt\\ &= -\int_{\Omega^*_{\varepsilon,\delta}} (u_{\varepsilon,\delta}(x,t)\varphi'(t))\Big|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)\,dx\,dt\\ &= -\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,0)v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)dx\,dt\end{aligned}$$

In view of (4.192)-(4.81) and (4.45), passing to the limit in this equation yields

$$\begin{aligned}&v(B)\int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)dxdydt\\ &+ k^*\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)dxdzdt\\ &- v(B)\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla\psi(x)\varphi(t)dxdydt\\ &= -v(B)\int_\Omega u^0(x)\psi(x)\,dx - v(B)\int_0^T \int_\Omega u(x,t)\psi(x)\varphi''(t)dxdt\\ &= -v(B)\int_\Omega u^0(x)\psi(x)dx + v(B)\int_\Omega u(x,0)\psi(x)dx\\ &+ v(B)\int_0^T \int_\Omega u''(x,t), \psi(x)\varphi(t)dxdt\end{aligned} \tag{4.85}$$

combining (4.85) with (4.81) yields:

$$-\int_\Omega u^0(x)\psi(x)dx + \int_\Omega u(x,0)\psi(x)dx = 0, \quad \forall\psi \in \mathcal{D}(\Omega),$$

which implies $u(x,0) = u^0(x)$.

Conclusion in conclusion, we study the homogonenization of the wave equation in the perforated domains Ω_ε and $\Omega_{\varepsilon,\delta}$ by the periodic unfolding methode we get in the domains Ω_ε the homogenazation wav equation give by the flolowing problem

$$\begin{cases} u'' - div(A^{hom}\nabla u) + k^* u = f & in \quad \Omega \times (0,T), \\ u = 0 & in \quad \Omega \times (0,T), \\ u'(x,0) = u^1 & in \quad \Omega, \end{cases}$$

where the homogenized matrix field is

$$A^{\text{hom}} = m_Y \left(a_{ij} + \sum_{k=1}^{N} a_{ik} \frac{\partial \chi_j}{\partial y_k} \right). \tag{4.86}$$

In the domain $\Omega_{\varepsilon,\delta}$ the homogenazation of the wav equation give by the flolowing problem Under assumptions (4.44) and (4.45), $u \in H_0^1(\Omega)$ is the unique solution of the limits problems

$$\begin{cases} u'' - div(A^{hom}\nabla u) + (k^*)\Theta u = f & in \quad \Omega \times (0,T), \\ u = 0 & in \quad \Omega \times (0,T), \\ u'(x,0) = u^1 & in \quad \Omega, \end{cases}$$

where the function Θ is given by

$$\Theta = \int_{\partial B} {}^t A^0 \nabla_z \theta \nu_B \, d\sigma_z. \tag{4.87}$$

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