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Mémoire

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Master en Mathématiques

Brief History Of Geometry

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Appreciation

We want above all to thank God the almighty for the strength, the will but above all the health he gave us to start and finish this mission.

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Our deepest thanks also go to all the people who have to all the people who have helped and supported us from near and far.

Dedication

We dedicate this work to each member of our respective families.

I dedicate my work:

To the one who gave me the taste for life and the sense of

Responsibility and the One who has always been the source

Inspiring and courageous…my dear Mum.

To my dear sister {AYA} and her daughter {Eline}.

To my cousins {Lina,Yakine,Maram} and my aunts {Radia,Hanan,Yasmina}.

Ghaouti Belkis

I dedicate this modest work

To all the members of my family (my dear parents Abderazek and Boucherab Nadia, the best parents in the world

To my fiance Sofiane

To my dear brothers (Walid and Mohamed) and to my dear sisters: (Dalél , Ghania and Imane).

To my friends and all those who encouraged me (Randa, Meriem)

To those who have been by my side in difficult times.

Alidra Amina

Summary

Since its inception, mathematics has been divided into two main parts: algebra and geometry. In our work, We are interested in the chronology and development of geometry through the ages, and the role of some civilizations that contributed to this development. We also present the most important work in mathematics that made a big leap for geometry, Euclid's book known as "The Elements". Certainly, we cannot forget to mention some of the branches that are attributed to us easy-to-understand branches such as analytic, projective and differential geometry.

Keywords: Euclidian geometry, modern geometry, Geometry in art and science.

Résumé

Depuis leur création, les mathématiques ont été divisées en deux grandes parties : l'algèbre et la géométrie. Dans notre travail, Nous nous intéressons à la chronologie et du développement de la géométrie à travers les âges, et du rôle de certaines civilisations qui ont contribué à ce développement. Nous avons également presenté le travail le plus important en géométrie qui a fait un grand saut en avant, le livre d'Euclide connu sous le nom "Les éléments".Certainement, nous ne pouvons pas oublier mention certaines des branches qui nous sont attribuées. Branches faciles à comprendre telles que la géométrie analytique, projective et différentielle.

Mots clés: Géométrie euclidienne, Géométrie moderne, Géométrie dans l'art et la science.

ملخص:

منذ نشأتها قسمت الرياضيات إلى جزئين رئيسيين: الجبر والهندسة. تحدثنا في عملنا عن التسلسل الزمني وتطور الهندسة عبر العصور ، ودور بعض الحضارات التي ساهمت في هذا التطور. تحدثنا أيضًا عن أهم عمل في الهندسة الذي حقق قفزة كبيرة في الهندسة ، وهو كتاب إقليدس المعروف باسم العناصر. و بالتاكيد لا يمكننا أن ننسى ذكر بعض الفروع التي بالنسبة لنا فروع سهلة الفهم مثل الهندسة التحليلية والإسقاطية والتفاضلية. **الكلمات المفتاحية :** الهندسة الإقليدية ، الهندسة الحديثة ، الهندسة في الفن والعلوم.

Contents

Chapter 1

Introduction

Determining the essence and content of mathematics is undoubtedly difficult. Formal explanations, which are now feasible thanks to the notion of structure in general and other logical concepts, ignore not just historical evolution but also the instinct and experience of the mathematician, who is aware of what is "substantial" and "interesting" and what is not. However, describing what geometry is and how it fits into the history of mathematics is more difficult given the current knowledge of the subject see [**?**].

The general consensus on geometry, as well as its role and significance in mathematics, has not only fluctuated throughout time. Mathematicians have adopted competing stances in their efforts to provide solutions to these concerns as mathematics has become more sophisticated. In ancient societies (such as ancient Egypt, Mesopotamia, India, China, etc.), geometry was primarily thought of as one key mathematics-oriented application among many others, but it later became the main focus of study. It developed into a theory with definitions, axioms, and proofs from the conceptions that resulted from trials and errors of mathematicians.

1.1 Chronology

One of the earliest disciplines is geometry it dates back to thousands of years BC. It has changed over time as a result of how humans have utilized it to design and build buildings, produce tools and machinery, and make structures that are essential for daily life. It is a field that makes use of engineering expertise to address issues and enhance human life in a variety of contexts.

Geometry is considered one of the most important contemporary sciences that cannot be dispensed with in the development of the world. So what is geometry, and how was it discovered? Geometry is the study of shapes and spaces based on a system of rules and ideas used to determine lengths, areas, and volumes. The word geometry is derived from two Greek words: "geo" meaning earth, and "metry" meaning measurement. However, it has been shown that geometry may be utilized to express and develop abstract thoughts and images as well as three-dimensional objects and flat surfaces.

1.1.1 Ancient Ages

There are many civilizations that contributed to geometry in ancient times. We will mention a few of them.

Egyptian civilization: Egypt is regarded as one of the most significant ancient civilizations that pioneered geometry. As it constructed pyramids, classical structures, temples, palaces, banks, canals, bridges, and fortified walls, this civilization made contributions to the advancement of architectural and civil engineering.

Babylonian civilization: This culture is known for its aesthetic and technological advancements. The building and design of Babylonian engineering were renowned for their ingenuity and originality, and they had distinctive characteristics that carried a recognizable mark. They built military forts, dams, water channels, irrigation systems, bridges, and highways in addition to some of the most significant archaeological relics, such as the 90-meter-tall Babylon Tower that was built in the sixth century BC.

Greek civilization: Greek scientists investigated the laws of motion, energy, and torque in the field of mechanical engineering, which made their civilization famous. Levers, springs, and geometric shapes where some sophisticated mechanical devices and instruments were created. Additionally, Greek scientists were excellent in the field of mathematical engineering, where they introduced Several achievements in geometry like Euclidean geometry, projective geometry, and relative geometry. This accomplishment aided in the later development in mathematics.

Indian civilization: The importance og geometry appears in the prominent indian structures and palaces like the Taj Mahal, the Hawa Mahal, and the Palace of the Prince were built utilizing distinctive architectural designs, Indian architecture is also one of the most significant technical achievements in the history of indian civilization, it has built irrigation systems, dams, tunnels, highways, and bridges.

1.1.2 Middle Ages

The most civilization that developed geometry in this era is the arab islamic civilization.

Arabic civilization: One of the sciences that saw advancement in Islamic civilization between the seventh and fifteenth centuries was engineering. Muslims during this time developed advanced geometric methods that had an impact across several sectors. They had scientific and mathematical interests . Muslim scientists collected, and developed significant scientific works from Greek, Indian, Persian, and translated those works from other languages into Arabic. They studied geometric principles in a variety of disciplines, including agriculture, mechanics, and architecture. Muslims pioneered a variety of innovative design and building techniques in the field of architecture. The Grand Mosque in Mecca, the Prophet's Mosque in Medina, the Alhambra Palace in Spain, and the Umayyad Mosque in Damascus are just a few of the well-known structures they constructed that are considered to be among the most significant cultural and historical landmarks in the entire world. Many tools and equipment were created in the field of mechanical engineering and are now employed in a variety of industries, including construction, agriculture, and handicrafts. The wind-powered pump, which first emerged in the ninth century ; the water clock ; the water wheel ; and the air compressor are some of the most best-known examples of these devices.

Al-Khwarizmi is one of the most significant and well-known scientists who contributed to the growth of mathematics during the Islamic civilization. His theories were applied to solve many mathematical problems. On this topic, his work "Al-Jabr" is a crucial resource. **European Renaissance** The European Renaissance was a significant period in the history of geometry, with the fifteenth and sixteenth centuries seeing the greatest growth in the disciplines of mathematics, technology, and engineering. Geometry underwent a transformation from an artisanal skill to a science based on exact mathematical concepts. The Florentine School in Italy served as the birthplace of the engineering movement during the European Renaissance, where new methods for mechanical engineering and drawing were created and applied to the construction of structures including houses, bridges, and machines. Precision, attention to detail, and harmony in design were important aspects of engineering during the Renaissance. During this time, many well-known engineers created brilliant and original works, including the artist and scientist Leonardo da Vinci, who created many of the contemporary engineering equipment and machines we use today. Numerous other engineers, including da Vinci, have made contributions to the growth of engineering, including Blaise Pascal, Galileo Galilei, and Johannes Kepler, who served to create the world and elevate the significance of geometry in daily life.

1.1.3 Modern era

Differential geometry began as the calculus study of curves and surfaces and came to be reformulated by Carl Friedrich Gauss (1777-1855) studying the curvature of surfaces in space. This brings us to the question of the parallel postulate.

Around 1830, János Polyai (1802-1860) and Nikolai Ivanovich Lobachevsky (1792-1856) published detailed works of the geometry in which the parallel postulate was not considered. So that the sum of angles of a triangle is less than two right angles. In fact, They introduced a new geometry without the use of the parallel postulate.

The German mathematician **Bernhard Riemann** is one of the most important figures in the development of **differential geometry**. Riemann proposed the idea of a **Riemannian manifold**, which is a manifold with a metric, in his habilitation speech in 1854. As a result, the theory of **curvature**, which investigates the characteristics of Riemannian manifolds, was created.

The aim of this work is to study the concepts and characteristics of development of geometry from the ancient era to the modern era. Our work is divided into three chapters:

Chapter 1: In the first chapter we talked about geometry in early ages and since it has expanding branches in the ancient, we specifically specialization in Euclidean geometry the mathematician euclid who is known for this famous book the elements Euclid's Elements is by far the most famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical text book. Which includes 13 books.

Chapter 2: In the second chapter we talked about geometry in the middle ages where we are divided into two parts analytic geometry (Coordinate System, Curves) and projective geometry(The axioms of projective geometry, Projective Spaces, Projective Line, Projective Plane, The Theorems of Desargues and Pappus).

Chapter 3: The third chapter is devoted to geometry in the modern era and we divide it into two parts, the first is topological manifolds in which we introduce a set of definitions, theories and examples (Topological Space, Hausdorff Space, Manifold...). In the second section we introduce differential manifolds and we give some definitions and concepts (Atlas, Chart, deffeomorphisme, Curvature...).

Chapter 2

Euclidean Geometry

2.1 Postulates and Definitions

One of the key mathematicians in the development of Western mathematics is the well-known ancient Greek mathematician and astronomer Euclid. He is credited with developing many important mathematical ideas as well as the foundational ideas of geometry and arithmetic. He was born in Alexandria, Egypt, around the third century BC. In his well known book "Elements" (see [7, 9]) which is among the most important works in the history of mathematics, he collected and built many fundamental ideas. The thirteen volumes that make up the Elements begin with the basics of geometry such as lines, angles and triangles, then move on to more complex elements such as cones, cylinders, and spheres before ending with arithmetic and number theory. Due to their use in mathematical teaching and research today, it has been translated into many other languages.

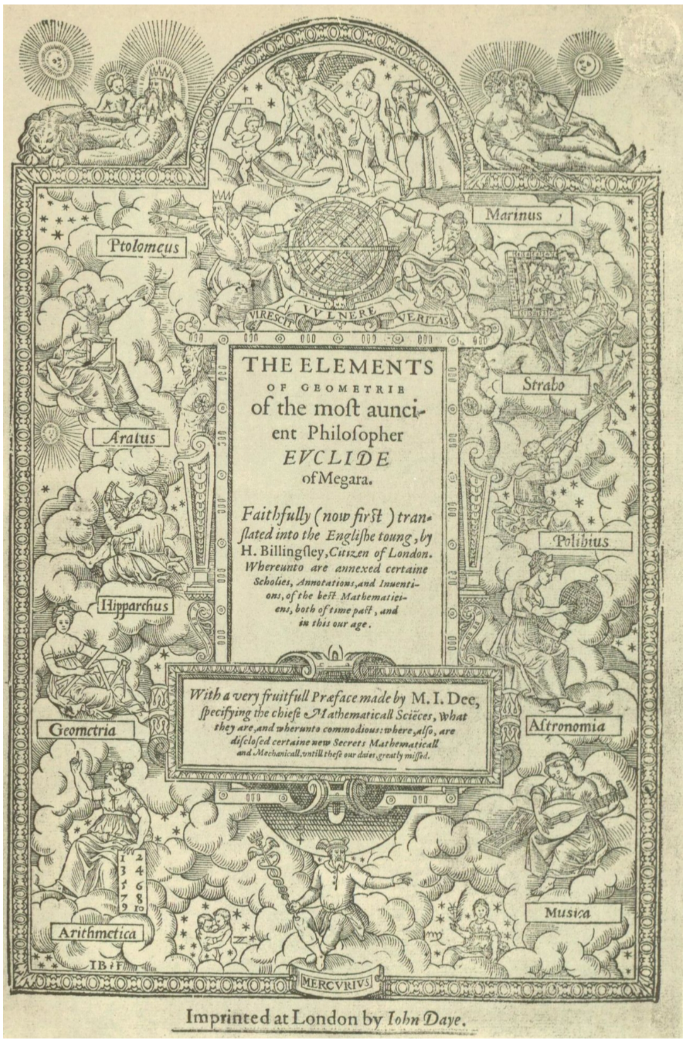

Figure 2.1: Cover of the first English-language edition of The Elements, published in 1570

The basic assumptions on which Euclidean geometry is built are known as Euclid's postulats or Euclid's axioms, and there are five axioms:

⋆ A straight line may be drawn between any two points.

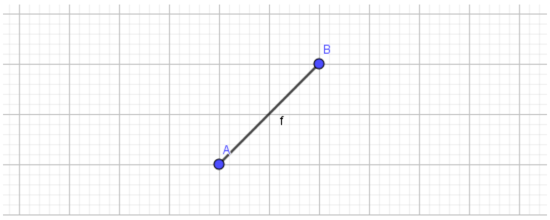

Figure 2.2: First Axiom

⋆ Any terminated straight line may be extended indefinitely.

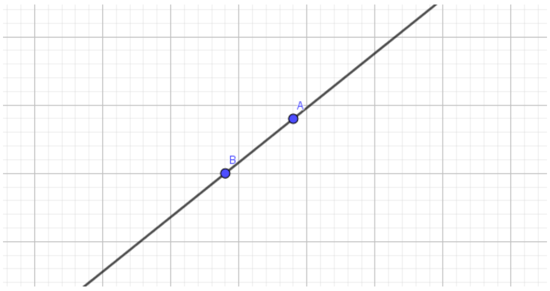

Figure 2.3: Second Axiom

⋆ A circle may be drawn with any given point as center and any given radius..

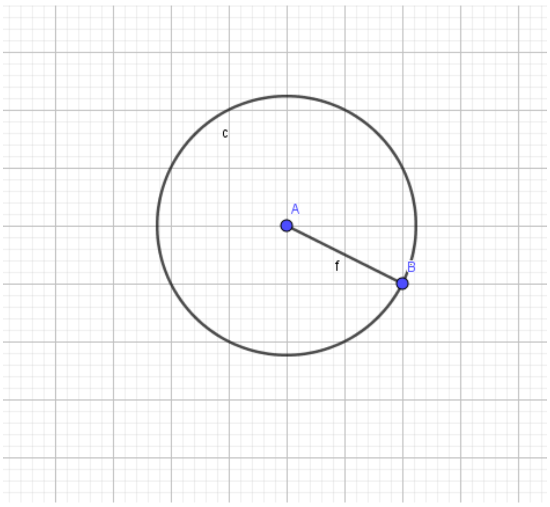

Figure 2.4: Third Axiom

⋆ All right angles are equal.

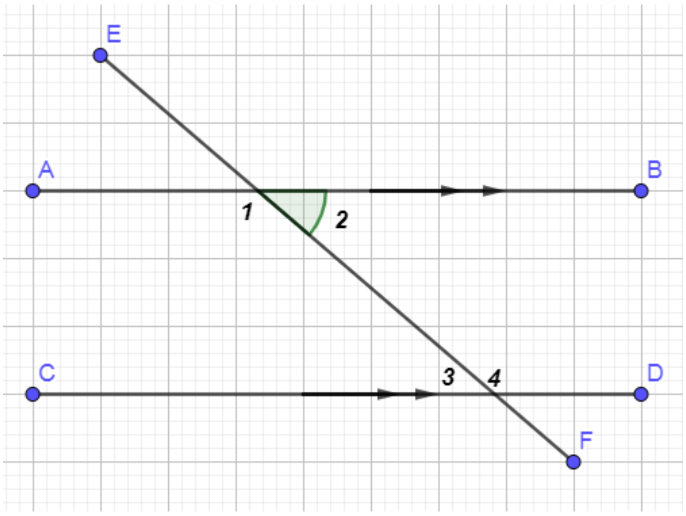

Figure 2.5: Fourth Axiom

⋆ If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

This form of the fifth axiom became known as the parallel postulate or the parallel axiom.

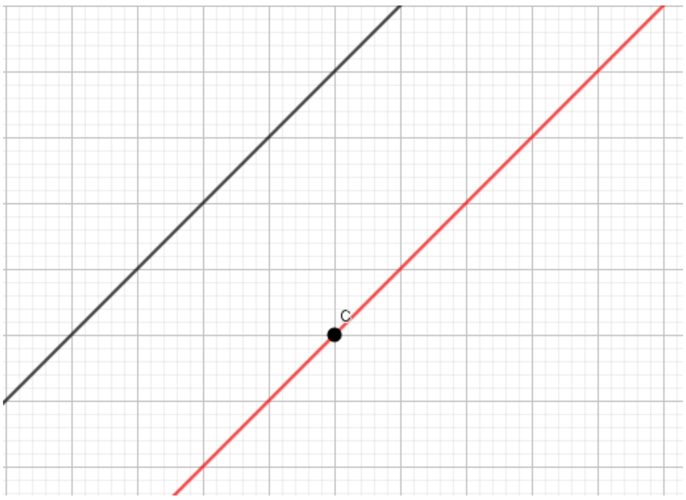

Figure 2.6: Fifth Axiom

2.2 Elements Book

One of the most famous mathematical works ever written is The Thirteen Books of Euclid's Elements. These works, which are regarded as the cornerstones of Greek geometry, have influenced modern mathematics. The texts provide a number of geometrical ideas and precepts that have served as the cornerstone of mathematical instruction since the Middle Ages.

The first book: The first book of Euclid's "Elements" is regarded as one of the most famous works in the history of mathematics and serves as a crucial basis for comprehending classical geometry. It initiates by describing some fundamental geometric concepts and going through various forms including points, lines, straight lines, and angles. The discussion of triangles, polygons, and circles is followed by rules and formulae for determining the areas and side lengths of different forms. In addition, Euclid provides certain geometric measuring instrument.

The book also discusses congruent triangles and equal angles, showing how to apply these ideas to support a variety of geometric conclusions. The study of ratios and proportions and how to utilize them to solve geometry problems is covered in the book's conclusion, along with some useful examples.

Example 1 *: The well-known "Proposition 1", which asserts the following,*

"To draw a straight line from any point to any point."

The proof of this statement, which is based on the first postulate of Euclid's axioms, is a straightforward construction using a straightedge and compass. It reads as follows:

"A straight line may be drawn from any point to any other point."

Since it enables the creation of numerous shapes and figures using straight lines and circles, this proposition serves as the basis for much of Euclid's subsequent work.

Example 2 *Proposition 2 reads as follows:*

"To produce a finite straight line continuously in a straight line."

This claim is proved by extending a specified line segment endlessly in one direction. It is based on Euclid's second postulate, which reads:

"A finite straight line can be continuously extended in a straight line."

The idea of an endless straight line, which is essential to Euclidean geometry, is established by this assertion. Additionally, it promotes the development of ideas like parallel lines and angles, which are *explored in later "Elements" books.*

Example 3 *Proposition 3 reads as follows:*

"To define a circle with a specific center and radius."

"The third postulate of Euclid's axioms serves as the foundation for this argument, which entails drawing two straight lines from the circle's center to its perimeter".

"Any straight line segment can be used as the radius of a circle, with one endpoint serving as the center".

This claim is significant because it introduces the idea of a circle and some of its characteristics, such radius and circumference. It also promotes the growth of other ideas, including tangents, which are examined in later "Elements" books.

The second book: Is a book of "Elements" by the eminent Greek mathematician "Euclid" is regarded as one of the most significant classical sources in the discipline and includes the fundamental ideas of two-dimensional geometry. The book starts out by introducing fundamental concepts in geometry, such as points, lines, and angles, before going into the theoretical underpinnings of geometric theory and their connections. Numerous crucial geometrical tools are present in it, including saws, compasses, circles, right angles, triangles, rectangles, and squares.

Numerous issues relating to geometric forms are also covered in the book, including ratio, proportion, resemblance, and perpendicularity. It offers solutions for geometric issues including "guess and check," "geometric construction," and "cutting and rotating."Euclid's "Elements" book 2, which covers fundamental geometric ideas that may be applied to a variety of geometric problems, is a significant source in the history of classical geometry. It is still used today as one of the core textbooks for geometry instruction all around the world.

The third book: Volume focuses on circles and the methods used to measure them. It provides a collection of essential ideas about circles and angles, including the center point, circumference, arc, central angle, and angles created by the circle. The book also explains how to compute the volumes and areas of several circle-related geometries, including composite areas, arcs, and small and big circles. The book also contains crucial definitions of three-dimensional objects in general as well as geometric forms like triangles and polygons. Because it offers a set of essential ideas that are applied to the study of contemporary geometry and its technical applications in several domains, the book is regarded as one of the key works on classical geometry.

The fourth book: Is a collection of theories and foundations pertaining to triangles and polygons may be found in the fourth volume. The book focuses on topics like calculating various ratios in triangles and polygons as well as figuring out the areas of triangles and polygons. The fourth volume contains 34 propositions, some of which deal with issues like splitting circles and triangles into equal halves, figuring out similar and equal angles in triangles, and figuring out medians, chords, and oblique angles in triangles. Many geometric problems are still solved using the ideas found in the fourth volume, which are still regarded as some of the most significant theories in the discipline of mathematical geometry.

The fifth book: Volume, on ratios and proportions, gives an explanation of each concept as well as instructions on how to measure them in geometry between various numbers. Important geometric ideas like the cut ratio, geometric proportion, and geometric harmony are also included in the book, and Euclid shows how to apply these ideas to solve geometrical problems. Additionally, Euclid discusses proportional addition and subtraction in the fifth book, outlining how to calculate the ratio between various values using parallel and intersecting lines and angles. Euclid can demonstrate the equality of two ratios of quantities using these geometric tools, which helps us comprehend geometric proportion and its applications in geometry and mathematical problems.

The six book: This book examines geometric objects in three dimensions, describing their characteristics and sizes. Euclid makes use of fundamental geometrical ideas, including lines, angles, triangles, polygons, and circles. He focuses on isosceles triangles, right triangles, equilateral triangles, and generic triangles in great detail. A few triangle-related topics, including congruence, similarity, and the Pythagorean theorem, are also explained by Euclid. Additionally, he discusses a few matters pertaining to circles, such as how to calculate a circle's area and circumference, as well as the rules for circle intersections. A few topics related to polygons, prisms, pyramids, cones, and spheres are addressed in the sixth book's conclusion. Given how thoroughly and precisely it presents the principles of this discipline, the sixth volume is regarded as one of the most significant works on classical geometry.

2.3 More on the elements book

In this section we will present some chapters of the famous book Ecluid's elements. Thess books are devoted to the study of numbers more than geometry.

The seventh book: The theory of prime numbers. To start, Euclid defines prime numbers and explains their characteristics. The conventional technique of discovering prime numbers is then described, along with information on how prime numbers are created by multiplying consecutive numbers. Additionally, he describes the characteristics of even and odd prime numbers as well as how prime numbers may be used to represent any number. The notion of proportion and how it may be utilized to determine prime numbers and divide numbers in general is also covered in the book. Algebraic prime theory, which examines the connection between prime numbers and algebra, is the subject on which Euclid closes his book. He describes how determining whether a number is prime or not can be done using algebraic prime theory.

The eighth book: This book is one of the best-known classical works on mathematics, and it focuses on the idea of the "Golden Ratio," also known as the "Continuity of Ratios." A relationship between a group of numbers that are in continuous proportion to one another is represented by this ratio. More specifically, it is the ratio that is equal to the ratio between the bigger amount and the total of the two quantities when the greater of the two quantities is divided by the smaller. This ratio has various characteristics, which the book outlines, including:

A straight line that is split into two halves that are proportionate to one another at a certain point can be used to symbolize the Golden Ratio.

This proportion may be seen in many geometric forms, notably rectangles with this ratio of length to width.

The correlations between sets of numbers that are present in some physical events are examples of how the Golden Ratio may be observed in nature.

The eighth book of Euclid is regarded as one of the most significant works in the history of mathematics and is a key resource for learning ratios, proportions, and fundamental geometry.

The ninth book: Explores the uses of number theory. Numerous significant problems in number theory are covered in this book, including:

Prime numbers: In this book, Euclid describes the characteristics of prime numbers and offers techniques for detecting whether a given number is a prime number or not.

Classification of numbers: Euclid describes a technique for categorizing numbers and examines the significant characteristics of these classes.

Mean proportional: In this book, Euclid examines mean proportional numbers and demonstrates how to compute them using prime numbers.

Fractions: Euclid shows how to deal with fractions and how to turn decimal values into fractions.

Perfect numbers: In this book, Euclid discusses perfect numbers and their characteristics.

This book is a crucial resource for comprehending number theory, a branch of science with applications in many areas, including computer science, symbolic logic, and cryptgraphy.

The tenth book: The idea of incommensurable amounts is covered in Euclid's "Elements" book ten. Its primary goal is to provide the Euclidean theory of three-dimensional geometric forms. It discusses a wide range of geometric shape-related ideas and issues, including symmetry, ratio, and proportion. Euclid explores different ideas and issues pertaining to the theory of three-dimensional geometric forms. The focus of the book is on incommensurable quantities, or those that cannot be determined only by the use of a straight edge and compass, such as volume, angle, and distance. It examines measuring techniques and three-dimensional geometric objects, including the cone, cylinder, and sphere.

The book contains many crucial mathematical ideas, such as the ideas of ratio, proportion, and symmetry, and it offers methods and instruments for resolving many three-dimensional geometry-related issues. Additionally, it offers a few mathematical arguments that rely on visual estimation and are used for the computation of surfaces, volumes, angles, and separations in three-dimensional geometric forms.

One of the foundational texts in classical geometry, Euclid's Tenth Book is crucial for comprehending the mathematical underpinnings of three-dimensional geometric forms.

The eleventh book: The subject of "Intersections of Solid Figures" is covered in Euclid's Eleventh Book of the "Elements" series. Euclid examines the intersections of three-dimensional solid objects in this book; the following subjects are covered:

An explanation of how three-dimensional solid figures connect.

Researching how flat surfaces intersect with one another and with other solid objects. Researching how sharp (curved) surfaces intersect with one another and with other solid forms.

Research on the spherical intersections of other solid figures and spherical surfaces. Researching the ellipsoidal intersections of various solid figures and elliptical surfaces.

One of the most significant books that addressed the study of solid figures and their intersections was book 11, which is part of Euclid's "Elements" collection. Today's Euclidean geometry and mathematics still frequently use it.

The twelfth book: The volume of a cone is one-third that of the cylinder that is identical to it, according to the twelfth book of Euclid's Elements, which goes into great depth on the volumes of cones, pyramids, and cylinders using the exhaustion technique. The book also demonstrates how many pyramids may be built by adding their volumes and how the volume of a sphere is related to the cube of its radius.

The thirteenth book: The platonic solids, which are three-dimensional objects, are discussed in Euclid's thirteenth book of "Elements." These solids are distinguished by a number of distinctive qualities that set each one apart from the others, such as:

Form: a material's geometric shape, which distinguishes Platonic solids from other types of solids, decides whether it falls under this category or not.

Edges: a certain number of edges is a property of platonic solids. Depending on the material, there might be anywhere between three and twenty edges.

Angles: a certain number of angles is a property of platonic solids. Angle count varies from one substance to the next, although it typically runs between four and sixteen.

Platonic solids are distinguished by having a predetermined number of faces. Different materials have different face counts, although they typically have between four and twenty-four faces.

In addition to these features the book also discusses the classification of platonic solids and how they might be used in geometry and architectural projects. The volume also addresses the volume and surface area of these solids as well as other relevant mathematical findings.

Chapter 3

Geometry in middle ages

Geometry witnessed substantial progress during the Middle Ages and had a big impact on how the world changed. Two significant discoveries occurred during this time period that contributed to the widespread applications. The first was the creation of analytical geometry by René Descartes and Pierre de Fermat, which allows for the transformation of geometry problems into mathematical equations. The invention of calculus, integration, and precise quantitative physics was essential for this reason. The second innovation is Gerard Desargo's projective geometry, which is mathematical geometry without measurements or parallel lines and only interested in examining the relationships between geometric points. We will talk about them in this chapter.

3.1 Analytical Geometry

3.1.1 Overview

Analytical geometry is the use algebra and calculus to investigate geometrical objects, René Descartes, a French philosopher and mathematician, made the initial discovery of it in the 17th century.

Descartes was interested in figuring out how to mathematically express geometric shapes so that they could be explored and examined using algebraic techniques. He recognized that he could describe geometric objects as equations or collection of equations by employing a coordinate system to assign numerical values to points in space. In order to do this, Descartes developed the idea of a coordinate plane, a two-dimensional plane with two perpendicular axes, typically denoted by the letters x and y. Analytical geometry uses numbers and equations to represent geometric shapes in space. This makes it possible to investigate geometrical characteristics with precision and logic. For instance, a plane, a line, and a sphere can all be described mathematically by a quadratic equation, a linear equation, and a second-degree equation, respectively. Descartes was able to describe geometric shapes as equations by giving coordinates to points in space. The discovery of analytic geometry by Descartes transformed mathematics education and helped Sir Isaac Newton and Gottfried Wilhelm Leibniz to introduce calculus. It also had a significant impact on physics and geometry because it made it possible for researchers and practitioners in these fields to examine and address challenging questions using mathematical techniques.

Definition 1 *Analytic geometry, also known as coordinate geometry, is a branch of mathematics that combines algebraic and geometric concepts. It is a contemporary approach to mathematics that describes geometric places in two and three dimensions using mathematical equations.In numerous subjects, including architecture, space engineering, physics, chemical engineering, industrial engineering, and many more, analytical geometry aids in the application of mathematics.*

3.1.2 Coordinate System

"Abscissa" The horizontal Cartesian coordinate of a point in a two-dimensional space is referred to as the "abscissa" in mathematics. Along the x-axis, it shows how far a point is from the y-axis. In equations or coordinate pairs (x, y), the abscissa is frequently represented as $''x''$.

For example the coordinate pair $(3, 4)$ has an abscissa of 3, meaning that the point is 3 units away from the y-axis.

"Ordinate": The vertical coordinate of a point in a two-dimensional Cartesian coordinate system is referred to as the "ordinate" in mathematics. It displays a point's y-axis-measured separation from the x-axis. In equations or coordinate pairs (x, y), the ordinate is frequently represented by the letter $''y''$.

For example, the ordinate in the coordinate pair $(3, 4)$ indicates that the location is 4 units above the x-axis.

Three separate mathematical coordinate systems are employed to locate points in space: the Cartesian, the Rectangular, and the Polar. Here is a quick breakdown of each system:

The Cartesian system It was so named in honor of mathematician Descartes, who popularized the use of positive integers to express, coordinates in a fashion that would be useful to scientists in the future. This is the most usual system, and it is frequently employed in the natural sciences, engineering, and mathematics. The horizontal axis and the vertical axis, which make up the Cartesian system, cross at a single location known as the origin. A pair of real integers (x, y) are used to calculate the position of the point in space, where x represents the horizontal distance traveled and y represents the vertical distance traveled.

Example 4 *Drawing the point in its position from the plane in the figure where the points have been drawn* $A(2,3), B(3,1), C(1,3), D(-2,-3), E(-3,-1), F(0,1)$ *so* $(1,3)$ *and* $(3,1)$ *in order to determine a specific point* (a,b) *The sequence of the numbers inside the brackets is important because, for instance, the numbers* $(3,1)$ *and* $(1,3)$ *represent two different plane positions.*

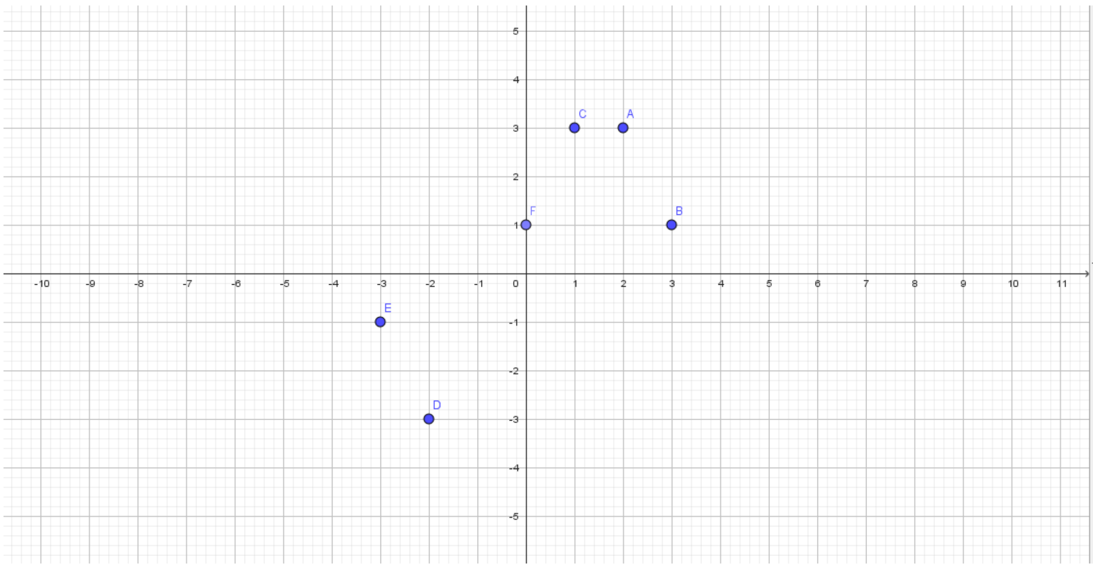

Figure 3.1: Cartesian System

The rectangular system In terms of the perpendicular axes, this system is comparable to the Cartesian system. However, the location of the point in space is determined using a pair of real numbers (x, y), which represent the distances traveled along the horizontal and vertical axes, respectively. The origin of the rectangular system can be found anywhere along the two axes.

The polar system: The polar system uses a pair of real numbers (r, θ), where r is the distance between the origin and the point and θ is the angle indicating the direction of the point from the origin, to identify the positions of points in space.

The angle θ is specified with regard to a certain direction, often using the horizontal axis as a reference axis, and the origin may be found at any point in space. The values of r and θ are used to represent the point in space, and the angle is expressed in radian. The formulas of this system are:

$$x = r\cos\theta$$

and

$$y = r\sin\theta$$

where

$$r = (\sqrt{x^2 + y^2})$$

and

$$\theta = \arctan(y/x).$$

The method by which the position of the point in space is computed varies across coordinate systems in general. Numerous applications in mathematics, science, engineering, and technology may make use of each of these systems.

Example 5 *A coordinate system known as the polar system is used to represent locations in a two-dimensional space. In order to specify a point in the plane, it utilizes two values, often represented as* (r, θ) *.*

The polar system is used in navigation, as one example. Polar coordinates are frequently used to indicate where a ship or aircraft is in relation to the North Pole. In this instance, r stands for the distance between the ship and the North Pole and θ for the ship's orientation with respect to a fixed point, such as true north.

The polar system is also used in mathematics, notably in calculus. Curves in two dimensions are described using polar coordinates. For instance, the curve represented by the equation. $r = 2cos(\theta)$ *generates a circle with a radius of* 1 *and a center at* $(1, 0)$*. A point that travels in a circular route around a fixed point can be described by this equation.*

3.1.3 Curves

Curves are usually defined as a subset of points within the analytic plane. We can also describe it by various equations depending on their specific shape and characteristics. Here are some common types of curves and their definitions and corresponding equations:

Lines and Slopes

Definition 2 *The slope of a line represents the steepness or inclination of the line. It measures how much the line rises or falls for each unit of horizontal distance. In analytic geometry the slope of a nonvertical line that passes through the points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ is :*

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

In analytic geometry, a line is a straight path that extends indefinitely in both directions. It can be defined using various methods, such as the slope-intercept form or the point-slope form. The equation of a straight line is typically given in slope-intercept form as: $y = mx + b$ where m represents the slope of the line and b is the y-intercept (the point where the line intersects the y-axis).

For example the equation $y = 2x + 3$. In this case, the slope of the line is 2, and the y-intercept is 3. The line has a positive slope, indicating that it rises as you move from left to right, and it intersects the y-axis at the point $(0, 3)$.

Example 6 *To find the equation of the line passing through two points $A(2, 3)$ and $B(5, 1)$.*

We have (x_1, y_1) and (x_2, y_2) can be found using the point-slope form: $y - y_1 = m(x - x_1)$ where m is the slope of the line.

First, find the slope m using the formula,

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Substituting the coordinates of points A and B,

$$m = \frac{(1 - 3)}{(5 - 2)} = (\frac{-2}{3}).$$

Now, choose one of the points (A or B) to substitute into the point-slope form. Let's use point A

$$(y - 3) = (\frac{-2}{3})(x - 2)$$

Simplifying

$$3y - 9 = -2x + 4$$

next

$$3y = -2x + 13$$

we get

$$y = \frac{-2}{3}x + \frac{13}{3}.$$

Therefore, the equation of the line passing through points A and B is $y = \frac{-2}{3}x + \frac{13}{3}$.

Circles

Definition 3 *In analytic geometry, a circle with center* (h, k) *and radius r can be defined using this equation :*

$$(x-h)^2 + (y-k)^2 = r^2.$$

were:

$(x-h)$ *represents the horizontal distance between the x-coordinate of a point on the circle and the x-coordinate of the center of the circle.*

$(y-k)$ *represents the vertical distance between the y-coordinate of a point on the circle and the y-coordinate of the center of the circle.*

r is the radius of the circle, which is the distance from the center to any point on the circle.

If the center is the origin $(0,0)$ *, the equation is:* $x^2 + y^2 = r^2$

For example the equation of the circle with raduis 2 *and center* $(-2,3)$ *is:* $(x+2)^2 + (y-5)^2 = 4$

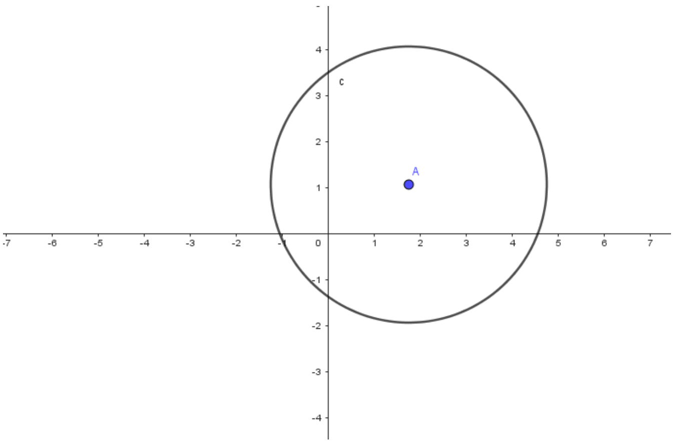

Figure 3.2: Circle

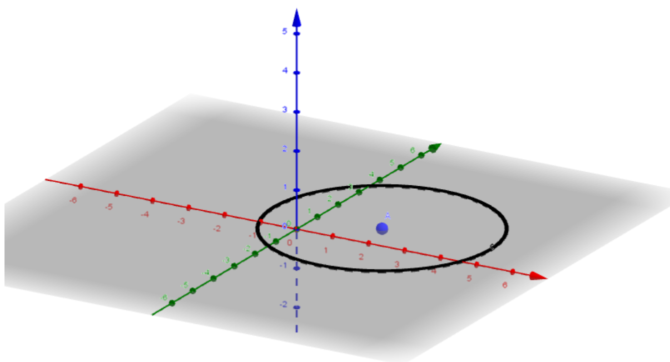

Figure 3.3: Circle 3D

Hyperbola

Definition 4 *In analytic geometry, a hyperbola is a type of conic section that is defined as the set of all points in a plane such that the difference between the distances to two fixed points, called the foci, is* *constant. The shape of a hyperbola is characterized by its two branches that open either horizontally or vertically. It can be defined by this equation:*

$$(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1.$$

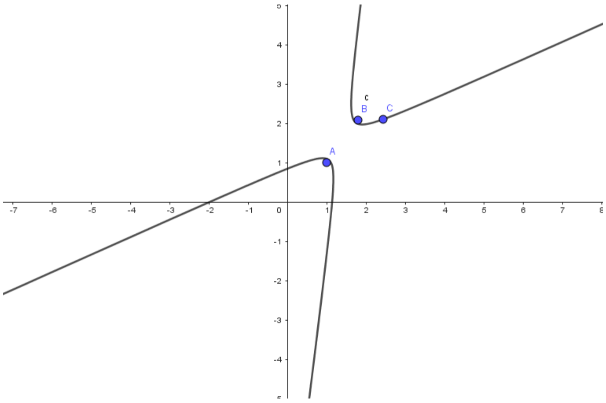

Figure 3.4: HyPerabola

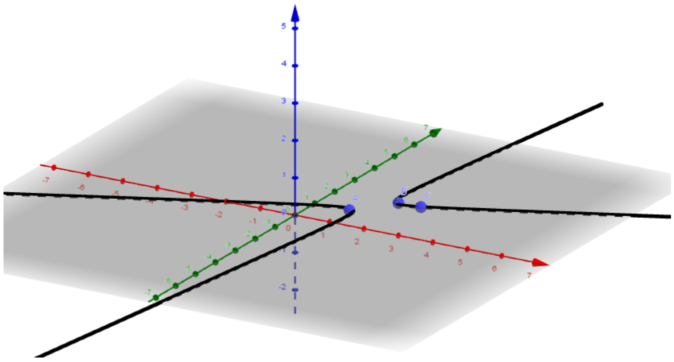

Figure 3.5: Hyperabola 3D

The equation of a hyperbola depends on its orientation, whether it opens horizontally or vertically, and the position of its center such :

Horizontal Hyperbola: The equation of a horizontally oriented hyperbola with its center at the point (h, k) *is given by:* $[\frac{(x-h)^2}{a^2}] - [\frac{(y-k)^2}{b^2}] = 1.$

Vertical Hyperbola: The equation of a vertically oriented hyperbola with its center at the point (h, k) *is given by:* $[\frac{(y-k)^2}{a^2}] - [\frac{(x-h)^2}{b^2}] = 1.$

Parabola

Definition 5 *In analytic geometry, a parabola is a conic section defined as the set of all points in a plane that are equidistant from a fixed point (called the focus) and a fixed line (called the directrix). The shape of a parabola can vary depending on the position of the focus and the directrix. It can be defined by this equation:*

$$(y-k) = a(x-h)^2$$

where (h, k) *is the center of the parabola.*

For example if the center is the origin $(0, 0)$ *the equation will be :* $y = ax^2$ *(a is constant that affect the position of the parabola in the coordinate system).*

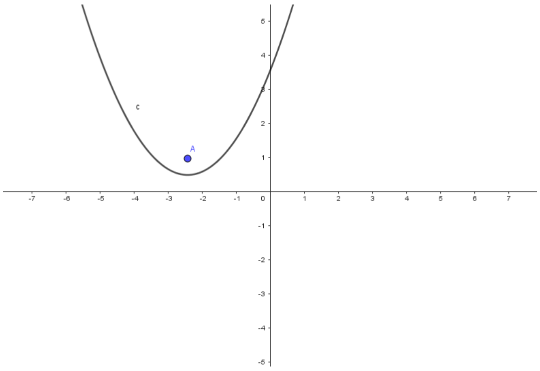

Figure 3.6: Parabola

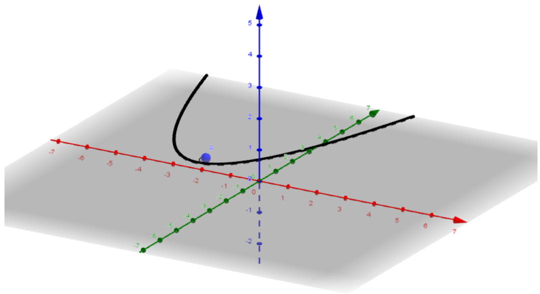

Figure 3.7: Parabola 3D

Ellipse

Definition 6 *In analytic geometry, an ellipse is a curve defined as the set of all points in a plane such that the sum of the distances from any point on the curve to two fixed points, called the foci, is constant. The distance between the foci is the major axis of the ellipse, and the constant sum of distances is equal to the length of the major axis. It can be described by this equation:*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

were $a > b$ and (h, k) is the center.

For example if the center is the origin $(0, 0)$ the equation will be : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

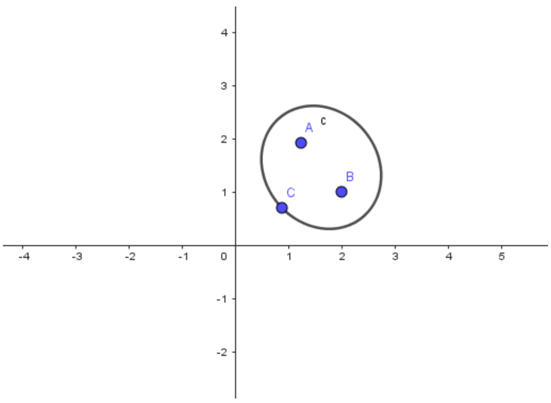

Figure 3.8: Ellipse

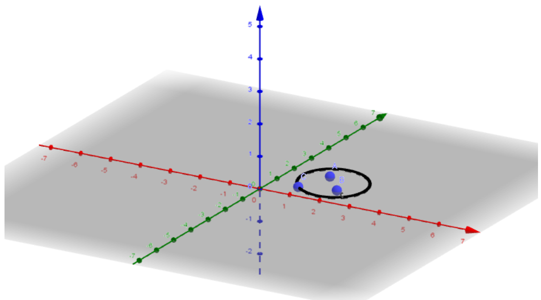

Figure 3.9: Ellipse 3D

3.1.4 Parametric Equation

More generally we define any curve by a parametric equation as follow.

Definition 7 *Suppose that t is a number in an interval I. A plane curve is the set of ordered pairs (x, y) where:*

$$x = f(t), y = g(t), \tag{3.1}$$

for t in the interval I. The variable t is called a parameter, and the equations (3.1) are called parametric equations for the curve.

3.2 Projective Geometry

The first geometrical features of a projective nature were found around the 3rd century by Pappus of Alexandria. It arose in the 17th century as a powerful mathematical tool for representing and evaluating perspective drawings in art and architecture. However, the formal development of projective geometry as a distinct area of mathematics happened in the 19th century, primarily via the efforts of mathematicians such as Jean-Victor Poncelet, August Ferdinand Möbius, and Julius Plücker.

Basic principle of projective geometry is that it deals with the qualities of geometric forms that stay unaltered even when exposed to projection or perspective changes. These transformations involve projecting points from three-dimensional space onto a two-dimensional plane. The primary objects of study in projective geometry are points, lines, and planes. However, unlike Euclidean geometry, projective geometry places points at infinity on the same footing as finite points. This notion leads to various fascinating features and theorems that are specific to projective geometry. One of the fundamental notions in projective geometry is duality. In projective duality, points and lines switch roles, allowing assertions about lines to be turned into statements about points and vice versa. This duality plays a vital role in many geometric transformations and proofs within projective geometry (see [2]).

3.2.1 The axioms of projective geometry

We are now going to introduce the axioms of projective geometry:

Axiom 1 (line axiom): For any two distinct points P and Q there is exactly one line that is incident with P and Q. This line is denoted by (PQ).

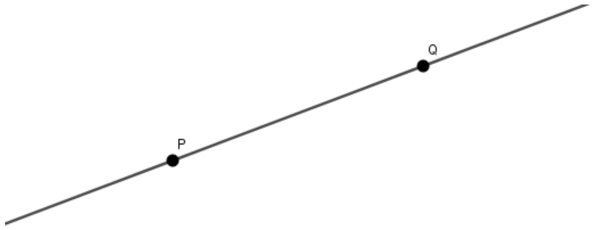

Figure 3.10: Line Axiom

Axiom 2 (Veblen-Young): Let $A, B, C,$ and D be four points such that (AB) intersects the line (CD). Then (AC) also intersects the line (BD).

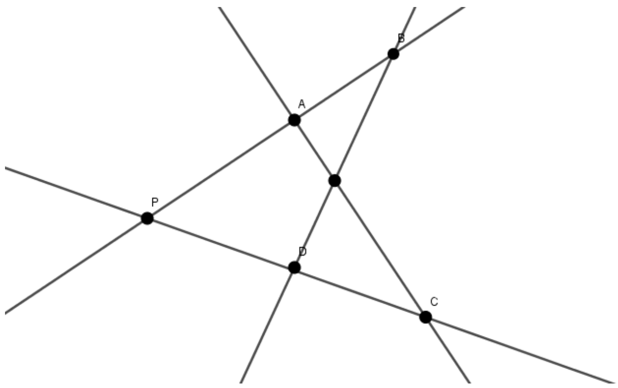

Figure 3.11: Veblen-Young

Axiom 3: Any line is incident with at least three point.

Axiom 4: There are at least two lines.

3.2.2 Projective Spaces

In this section, formal definitions and properties of projective spaces are given, regardless of the dimension.

Definition 8 *The real projective space $\mathbb{P}^n$ of dimension n, associated to R^{n+1} is the set of rays of $\mathbb{R}^{n+1}$. An element of $\mathbb{P}^n$ is called a point and a set of linearly independent (respectively dependent) points of $\mathbb{P}^n$ is defined by a set of linearly independent (respectively dependent) rays.*

Definition 9 *Projective transformation (homography) is a matrix M of dimension* $(n+1) \times (n+1)$ *such that* $det(M) \neq 0$*, or equivalently non-singular, defines a linear transformation from* $\mathbb{P}^n$ *to itself.*

Theorem 1 *[4] Consider m points of* $\mathbb{P}^n$ *that are linearly independent with* $m < n$*. The set of points in* $\mathbb{P}^n$ *that are linearly dependent on these m points form a projective space of dimension* $(m-1)$*. When this dimension is equal to* $1, 2$ *and* $(n-1)$*, this space is called line, plane and hyperplane respectively. The set of subspaces of* $\mathbb{P}^n$ *with the same dimension is also a projective space.*

3.2.3 Projective Line

The projective line $\mathbb{P}^1$, as will be shown in more detail in this subsection, it is the completion of the affine line with a specific projective point, the point at infinity. The projective line can be used to clearly and intuitively illustrate projective concepts like the cross-ratio.

Proposition 1 *A projective line travels across any two separate points in a projective space.*

The cross-ratio

also known as the double ratio, is a number that is retained via projective translation and is the fundamental invariant of $\mathbb{P}^1$. It is the rigid transformation's projective counterpart to the Euclidean distance.

3.2.4 Projective Plane

The projective plane $\mathbb{P}^2$ is the image plane of a 3D world projection, and interactions between images of the same 3D scene can be characterized by projective transformations, which gives it significance in the field of visual computing. Perspective deformations of the circle can be characterized with projective transformations of P^2, this idea is given first by Desargues in the 17th century.

Proposition 2 *Projective plane contains a unique location where two separate lines intersect.*

Points and Lines

A point in P^2 is represented by 3 homogeneous coordinates $[x_0, x_1, x_2]^t$ defined up to scale factor. Consider 2 points A and B of $\mathbb{P}^2$ and the line going through them. A third point C belongs to this line only if the coordinates of A, B and C are linearly dependent.

Theorem 2 *(Duality) Any definition, characteristic, or theorem that holds true for a projective space's points also holds true for the hyperplanes.*

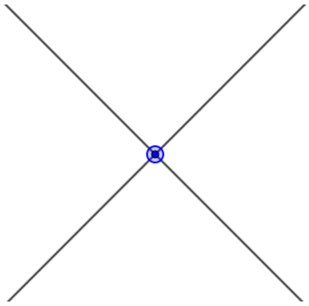

Figure 3.12: 2 lines define a point

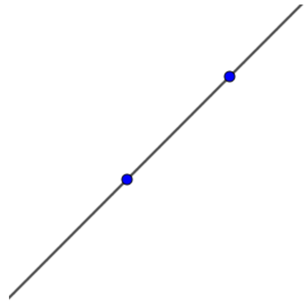

Figure 3.13: 2 points define a line

3.2.5 The Theorems of Desargues and Pappus

[1] Now, we discuss the theorem of Desargues (**Girard Desargues** $(1591 - 1661)$ and the theorem of Pappus (**Pappus Alexandria**, ca.300 A.D.), two crucial configuration theorems. Since Desargues and Pappus examined the theorems on the real plane, where they actually hold, they are referred to as "theorems" while making their assertions. For us, these claims are merely unsupported hypotheses that may or may not hold true in a projective space. We'll give illustrations of both phenomena. You can find other theorems and better explanation in [4].

Definition 10 *Let $\mathbb{P}$ be a projective space. We say that in $\mathbb{P}$* ***the theorem of Desargues*** *holds if the following statement is valid: For any choice $A_1, A_2, A_3, B_1, B_2, B_3$ of points with the properties,*

A_i, B_i are collinear with a point C, $C \neq A_i \neq B_i \neq C$ were $(i = 1, 2, 3)$.

no three of the points C, A_1, A_2, A_3 and no three of the points C, B_1, B_2, B_3 are collinear, we have that the points:

$$P_{12} = A_1A_2 \cap B_1B_2$$

$$P_{23} = A_2A_3 \cap B_2B_3$$

$$P_{31} = A_3A_1 \cap B_3B_1.$$

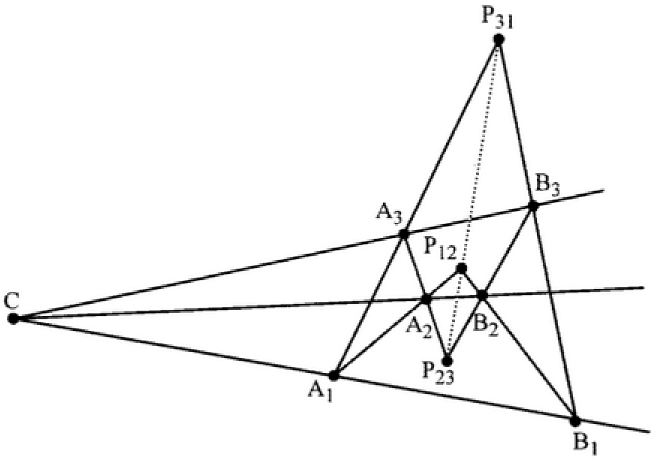

Figure 3.14: The theorem of Desargues

Definition 11 *Let $\mathbb{P}$ be a projective space. We say that in $\mathbb{P}$* ***the theorem of Pappus*** *holds if any two intersecting lines g and h with $g * h$ satisfy the following condition: if $A_1 A_2, A_3$ are distinct points on g and B_1, B_2, B_3 are distinct points on h all different from $g \cap h$ then the points:*

$$Q_{12} = A_1 B_2 \cap B_1 A_2$$

$$Q_{23} = A_2 B_3 \cap B_2 A_3$$

$$Q_{31} = A_3 B_1 \cap B_3 A_1.$$

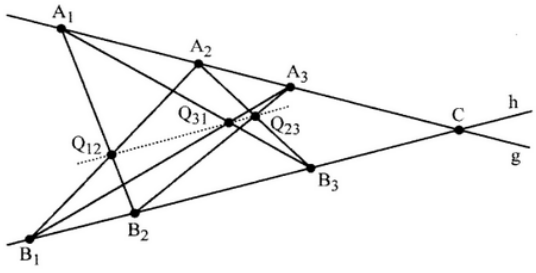

Figure 3.15: The theorem of Pappus

Chapter 4

Modern geometry

The mathematician **Leonhard Euler** made substantial contributions to the study of curves and surfaces in the 17th century. He created the idea of curvature, which assesses the deviation of a curve or surface from a straight or flat surface. In addition, Euler investigated the characteristics of minimum surfaces, which decrease their surface area under specific restrictions.

The idea of a manifold was first developed in the late 19th century by the Italian mathematician **Giuseppe Peano**. A surface may be generalized to a manifold, which enables the study of more intricate things like higher-dimensional spaces. Due to this, Riemannian geometry, which studies the characteristics of spaces with a metric unit for measuring the separation between points, was created.

Élie Cartan, a French mathematician, made substantial contributions to the study of differential geometry in the 20th century. In order to analyze parallel transport and the curvature of a manifold, he invented the idea of a link. As a result, the theories of fiber bundles and gauge theory were developed, both of which have significant physics applications

4.1 Topological Manifolds

A manifold is a **topological space** in mathematics, more specifically in topology, that has a close local resemblance to Euclidean space. a manifold is, more precisely, a **Hausdorff space** (M) in which each point (p) has a neighborhood that is homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$, where n is a constant positive integer known as the manifold's dimension.

In other words, there is an open set (U) containing each point (p) in the manifold (p) and a **homeomorphism** $(\phi : U \Rightarrow V)$ where (V) is an open subset of (R^n). In order to map points in the open set (U) of the manifold to points in the equivalent open set (V)in Euclidean space while maintaining the topological features, one can use the homeomorphism (ϕ).

The idea of a manifold is essential to topology because it makes it possible to explore geometric and topological aspects in a more organized and understandable way. Manifolds can exist in a variety of dimensions, including zero-dimensional points, one-dimensional curves, two-dimensional surfaces, and higher-dimensional spaces. Each type of manifold has distinct topological properties.

4.1.1 Topological Space

Definition 12 *(Spaces) We establish the following convention for the use of the word space:*

A set together with a topology, a collection of open sets, is called a topological space.

From now on, when we talk about a space, we mean a topological space a set together with a specified topology.

Definition 13 *(Topological space) Let X be a set , A topology is a collection$T \subset \mathbb{P}(x)$ these are the open sets of X .T must satisfy:*

1. $(\phi, X \in T)$ *The empty set ϕ and the space X are both sets in the topology .*
2. *The union of any collection of sets in T is contained in T.*
3. *The intersuction of any finitely many sets on T is also conained in T.*

The pair (X, T) is called a topological space.

We'll find that these areas can actually be fairly pathological, but we'll start with some simple instances first.

This example will make this concept more easy and clear

Example 7 $\mathbb{R}^n$ *with topology induced by any norm.*

(M, T) *with* $T = \{\phi, M\}$ *for any non-empty set M.*

(M, T) *with* $T = \{U \subset M\}$ *for any non-empty set M [all subsets of M are open]*

Example 8 *Consider the metric space* (X, d)*. Then, the set* $T = \{U \subset X : \exists r > 0$ *such that* $B(r, x) \subset U$ *defines a topology on* X *for all* $x \in U$.

Let $X = \{a, b\}$. *A topology* T *must contain and* X. *It may or may not contain* $\{a\}$ *or* $\{b\}$. *If they are both in* T *, this means that the two points are distinct and separate. If neither of them are in* T *, then the points are as close together as they could be (topologically indistinguishable). Finally, one could even declare* $\{a\}$ *to be open and not* $\{b\}$.

Let $X = \{a, b, c\}$ *We start to observe the requirements the axioms place on legitimate topologies. For instance, if both* $\{a\}$ *and* $\{b\}$ *are open, then a and b must also be open. Notice that* $T = \{X, \phi\}$ *is the coarsest topology, and* $T = P(X)$ *is the finest topology, where* $P(X)$ *is the power set of* X.

Definition 14 *A basis of the topology of* (X, T) *is a collection of open sets* $B \subset T$ *iff every open set* $U \in T$ *is the union of members of* B.

Compact Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to cover A if and only if every point of A belongs to at least one of these subsets. In particular, an open cover of X is collection of open sets in X that covers X. If U and V are open covers of some topological space X then V is said to be a subcover of U if and only if every open set belonging to V also belongs to U.

Definition 15 *(Compact spaces; compact subspaces.) We say that a topological space* (X, T) *is compact if every open cover of* X *has a finite subcover. A subset* $A \subseteq X$ *is called compact if it is compact with respect to the subspace topology.*

Proposition 3 *Let* X *be a topological space. A subset* A *of* X *is compact (with respect to the subspace topology on* A*) if and only if, given any collection of open sets in* X *covering* A*, there exists a finite collection* $V_1, V_2, \ldots, V_r$ *of open sets belonging to* U *such that* $A \subset V_1 \subset V_2 \subset \cdots \subset V_r$.

Example 9 *Let* (X, T) *be a finite topological space. Then* X *is compact.*

Let X *be a topological space with the indiscrete topology, Then* X *is compact.*

Connected space

Definition 16 *(Connected spaces) Recall that a topological space X is called connected if X cannot be written as the union of two nonempty disjoint open subsets; or equivalently, if X and ϕ are the only subsets which are both open and closed in X.*

Example If X is not connected, it has subsets $T_\alpha \subset X$ which are both open and closed. Then T_α is called a connected component of X, and X can be considered as the possibly infinite union of its connected components.

Homeomorphism

Definition 17 *In topology, a homeomorphism is an isomorphism between two topological space it is a continuous bijection of on into the other , whose reciprocal is continuous, in this case,the two topological space are said to be homeomorphic.*

Separated Spaces

In mathematics,a topological space is called separated if it contains a countable dense subset.(There is a set with a countable number of elements whose closure is the entire space).

Definition 18 *(Hausdorff Space) Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint . X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. This is why Hausdorff spaces are also called T_2 spaces or separated spaces.*

Proposition 4 *X, Y are topological space such that X is homeomorphism to Y and Y is seperated then X is seperated.*

Definition 19 *Let V be a separated topological space with danombrable base, we say that V is a topological manifold of dimension n if every point of V has a neighborhood homomorphic to (an open set of) R^n or again if V is corvered by open sets homomorphic to R^n.*

Example 10 *$M = R^n$ is a separated topological space we take $U = R^n, \phi = Id_{R^n}$ then (R^n, Id_{R^n}) is an atlas of dimension n is of class C^∞. $Id_{R^n} : R^n \to R^n$ homeomorphism and $R^n \circ (Id_{R^n})^{-1} = Id_{R^n}$ of class C^∞.*

4.1.2 Manifold

A manifold is a geometric object which locally looks like R^n. More precisely, a manifold is a space that, when seen closely enough, resembles a flat Euclidean space, but which may have a more complex global structure. For instance, the surface of a sphere is a two-dimensional manifold even if it seems localized to be a flat plane with a different global structure than a plane. Manifolds can be described mathematically using coordinate systems, which enable us to provide numerical coordinates to each point on the manifold, may be used to characterize manifolds mathematically. Then, using differential calculus, these coordinates may be utilized to define geometric objects like curves, surfaces, and volumes and to investigate their characteristics.

In Einstein's theory of general relativity, manifolds are used to describe the geometry of space-time. They are also a key idea in other branches of mathematics and science where geometric structures play a significant role.

Fundamental of manifolds

The foundational ideas of manifolds include the following:

Dimension: For instance, the dimension of a surface(two-dimensional manifold) is 2.

Continuity: This describes the need that the manifold be linked, and without being able to be divided into distinct pieces.

Differentiability: is the property that allows us to do differential and integral calculus as well as identify the local slope of the manifold at any point on it.

Homogeneity: is the property that makes the manifold look symmetric at any location on it. achieved By giving each point on the manifold the identical geometric features.

These concepts are essential for understanding manifolds and are used in many areas of mathematics and science, including physics, computer graphics, and robotics.

Type of Manifold

Smooth Manifold: A smooth manifold is one that has a smooth structure, which makes it appear to be Euclidean space in the local area. Smooth coordinate charts that enable fluid changes between charts are what define it.

Differentiable Manifold: A differentiable manifold is one in which the functions that connect the charts to one another may be differentiated. It is a generalization of smooth manifolds, which gives one more options for charting.

Riemannian Manifold: A Riemannian manifold is a differentiable manifold that possesses a Riemannian metric that gives an understanding of the separation and angles between tangent vectors. It enables the investigation of geometrical characteristics and curvature.

Topoligical Manifold: A form of manifold known as a topological manifold is one that has no additional smooth or geometric structure and is only described in terms of its topology. Locally, it resembles Euclidean space but is actually a topological space. Topological manifolds can be further divided into groups according to their dimension:

Manifold with zero dimensions: A zero-dimensional manifold is made up of a number of isolated points. Beyond the individual points, it lacks any local organization.

Curve: A one-dimensional manifold is frequently referred to as a curve. Locally, it resembles a line. Circles, loops that are closed, and line segments are some examples. A surface is a two-dimensional manifold, also referred to as a manifold. Locally, it resembles a plane. Examples include projective planes, tori, and spheres.

Manifold of a higher dimension: Higher dimensional manifolds are also possible. Despite the fact that they are frequently hard to visualize, they may be mathematically explained and have a variety of significant applications. An n-demensional topological manifold with boundary is an infinitely countable separated space (countable union of compacts) in which every point has a neighborhood homomorphic to an open set of $]-\infty, 0] \times R^n$ such a homomorphismis called a map.

It's crucial to understand that while topological manifolds give an idea of local structure.

Smooth Manifold

Definition 20 *Let $\mathbb{R}^n$ be an euclidean space.*

A subset $X \subseteq \mathbb{R}^n$ is a k-dimensional smooth manifold if it is locally diffeomorphic to R^k. The latter means that for every point $x \in X$ there is an open subset $V \subset X$ containing x and an open subset $U \subseteq R^k$ such that U and V are diffeomorphic. Note that the number k is the same for all points in X [8].

4.1.3 Klein bottel

The Klein bottle is a mathematical object; it is a non-orientable surface, which means that it cannot be consistently distinguished as having an inside or an outside. The Klein bottle was first introduced by the German mathematician **Felix Klein** in 1882; it is a closed surface that cannot be properly embedded in three-dimensional Euclidean space without self-intersection you can find more in [5, 6]. Klein bottles are frequently visualized as twisted cylinders with self-intersecting loops. You would ultimately reach the "other side" of a Klein bottle if you followed a path down its surface without ever going over an edge or barrier.

The Klein bottle has an intriguing characteristic; that it lacks a clear "inside" and "outside" like a typical bottle. Depending on your vantage position, any point on the surface can be viewed as either inside or outside.

The klein bottel is a closed surface obtained by gluing the two edges of a cylinder in the other direction with respect to the torus. Its visualization is not simple since in space, there is self-intersection. So the klein bottle cannot be immersed in $\mathbb{R}^3$ while it can be in $\mathbb{R}^4$.

This is analogous to the fact that if one draws a node on a leaf, the lines overlap whereas in real life the rope does not self-intersect. We can also obtain a klein bottle by gluing two Möbius strips by their edges. The Klein bottle has neither interior nor exterior, and is thus part of the family non-orientable surfaces, since we cannot tell on which side of the surface is the exterior and the interior. Contrary to what one might think, the word bottle comes from the confusion between the German words flasche (bottle) and flache (surface).

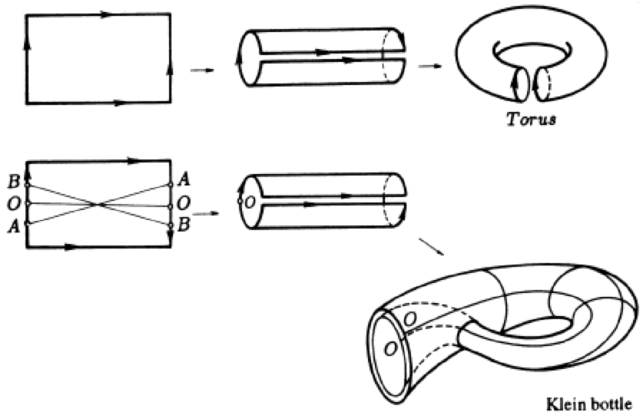

Figure 4.1: Klein bottle

One possible set of parametric equations for the Klein bottle is:

$$\begin{cases} x(u,v) = (R + \cos(\frac{u}{2}) \times \cos(v) - \sin(\frac{u}{2}) \times \sin(2v)) \times \cos(u) \\ y(u,v) = (R + \cos(\frac{u}{2}) \times \cos(v) - \sin(\frac{u}{2}) \times \sin(2v)) \times \sin(u) \\ z(u,v) = \sin(\frac{u}{2}) \times \cos(v) + \cos(\frac{u}{2}) \times \sin(2v). \end{cases} \tag{4.1}$$

In these equations, (u, v) represent parameters that vary over certain ranges. R is a constant that determines the size of the Klein bottle. The values of u and v typically range from 0 to 2Π.

It is significant to remember that the Klein bottle is only parametrically represented by these equations. They explain how three-dimensional coordinates can be assigned to places on the Klein bottle's surface. Other parametrizations and equations can characterize the Klein bottle in addition to the one provided by the equations can be mentioned.

Here's another parametric equation that represents the Klein bottle:

$$\begin{cases} x(u,v) = (2 + \cos(u)) \times \cos(v) \\ y(u,v) = (2 + \cos(u)) \times \sin(v) \\ z(u,v) = \sin(u) \times (1 + \sin(\frac{u}{2}) \times \sin(v) - \cos(\frac{u}{2}) \times \cos(2v)). \end{cases} \tag{4.2}$$

Similarly to the previous equation, u and v are parameters that vary over certain ranges, typically from 0 to 2Π. These formulas translate points on the Klein bottle's surface into three-dimensional coordinates.

Again, it's important to note that the Klein bottle is a non-orientable surface and cannot be embedded in three-dimensional Euclidean space without self-intersection, it should be stressed again. Although the Klein bottle is depicted in these equations, it is not implied that the object can actually exist in our three-dimensional universe.

4.2 Differentiable Manifolds

A differential manifold is a mathematical concept used to describe a space that locally resembles Euclidean space. It is characterized as a topological space that has a number of open subsets known as charts or coordinate neighborhoods, each of which is homomorphic to an open subset of Euclidean space. The transition maps between overlapping charts must also be smooth or differentiable, which means they must have well defined derivatives for every order.

More formally, a differential manifold is a pair (M, A), where M is a topological space and $\mathcal{A}$ is an atlas consisting of a collection of charts $\{(U_i, \varphi_i)\}$ such that: U_i is an open subset of M, for each i. The union of all U_i covers the entire manifold M i.e. $(M = \cup U_i)$.

Each chart (U_i, φ_i) is a homeomorphism from U_i to an open subset of Euclidean space R^n. The homeomorphism function $\varphi_i : U_i \to \varphi_i(U_i)$ is typically referred to as a coordinate map.

For any pair of overlapping charts (U_i, φ_i) and (U_j, φ_j) such that $U_i \cap U_j \neq \emptyset$, the composition

$$(\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j), \tag{4.3}$$

is a smooth or differentiable map. This ensures that the transition from one chart to another is smooth and allows for the concept of differentiability on the manifold.

A differential manifold offers a framework for investigating smooth functions, tangent vectors, and other geometric objects on the manifold by combining numerous charts and associated transition maps. Differential geometry's core concept of the differential manifold serves as the foundation for numerous mathematical and scientific theories, including general relativity.

Remember that a Hausdorff space with a countable basis of open sets, where each point has a neighborhood homeomorphic to an open subset of R^n, is an n-dimensional topological manifold. A parametrization is any pair (U, ϕ) where U is an open subset of R^n and $\phi : U \to \phi(U) \subset M$is a homeomorphism of U to an open subset of M. The set $\phi(U) \subset M$ is referred to as a coordinate neighborhood, and the inverse ϕ^{-1} is known as a coordinate system or chart. We provide calculations for the related coordinate change when two coordinate neighborhoods overlap, The goal is to select a sub-collection of parametrizations so that the coordinate changes are differentiable maps in order to generate differentiable manifolds.

4.2.1 Atlas

Definition 21 *An atlas A on M of demention n and of class $c^k(k\rangle 1)$ is a family of denoted $A = \{(u_i, \phi_i)i \in I\}$ such that the open sets U_i cover M , $M = U_i \in I(U_I, V_i \in I)$.*

Definition 22 *If $B \subset A$ is true for all atlases B on M with $[A] = [B]$, then an atlas A on M is said to be maximum. Atlases form a partially ordered set.*

Theorem 3 *Every atlas is contained in a maximal atlas. If A_1 and A_2 are two maximal atlases on M, such that there exists an atlas B on M with $[A1] = [B]$ and $[A2] = [B]$ then A_1 and A_2 already coincide.*

4.2.2 Chart

Let X be a topological space that is second countable and Hausdorffian.

Definition 23 *A chart is a pair (U, ϕ) where U is an open set in X and $\phi : U \to \mathbb{R}_n$ is homeomorphism onto its image. The components of $\phi = (x_1, x_2, \ldots, x_n)$ are called coordinates. Given two charts (U_1,ϕ_1) and (U_2, ϕ_2) then we get overlap or transition maps*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \tag{4.4}$$

and

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \tag{4.5}$$

Definition 24 *Two charts (U_1, ϕ_1) and (U_2, ϕ_2) are called compatible if the overlap maps are smooth.*

In practice it is useful to consider manifolds with other kinds of regularity. One many consider C^k manifolds where the overlaps are C^k maps with C^k inverses. If we only require the overlap maps to be homeomorphisms we arrive at the notion of a topological manifold. In some very important work of Sullivan one consider Lipschitz, or Quasi-conformal manifolds.

An atlas for X is a (non-redundant) collection A = $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \text{A}\}$ of pair wise compatible charts . Two atlases are equivalent if there their union is an atlas. An atlas A is called maximal if any other atlas compatible with it is contained in it.

4.2.3 Deffeomorphism

In mathematics, a **Deffeomorphism** is an isomorphism of smooth manifolds, it is an invertible function that maps one differentiable manofold to another such that both the function and its inverse are differentiable.

Definition 25 *A smooth map $f : X \to Y$ is called a diffeomorphism if f is one-to-one and onto, and its inverse f^{-1} is smooth as well. We say that X and Y are diffeomorphic if there exists a diffeomorphism $f : X \to Y$.*

4.2.4 Curvature

Curvature is a term used in mathematics and geometry to define a curve or surface's change in direction or geometric structure at a certain location. It is used to gauge how much a certain curve or surface is bending or turning there. The curve's ability to adapt and change direction is measured by its curvature. You can have a flat curve with a modest angle bend at each point, or you might have a convex curve that bends sharply at each point.

Curvature is commonly defined as the rate of change of direction at a location and is proportional to the direction of the centripetal force applied to the curved body. Curvature is divided into two categories: first curvature and second curvature.

First curvature: It represents the rate of change of the tangent's direction to the curve at the point and measures the curvature in a two-dimensional curve.

Second curvature: It represents the rate of change of the second tangent's direction to the surface at the point and quantifies the curvature in a three-dimensional surface.

Curvature is employed in many domains, including mathematics, mechanical engineering, curve theory, physics, computer science, geography, and others. Understanding curvature aids in the study of geometric shapes and various curves, as well as the analysis of their qualities and applications.

Riemannian manifold

let M be a **Riemannian manifold** there is a uniquely determined riemannian connection on M (smooth manifolf + metric tensor = riemannian manifold) [11].

Definition 26 *Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. The Riemann curvature tensor of (M, g) is defined as*

$$R : X(M) \times X(M) \times X(M) \to X(M),$$

$$R(X,Y)Z = \nabla_x \nabla_y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

for all $X, Y, Z \in X(M)$. In the above formula, we understand $\nabla_X \nabla_Y Z$ as $\nabla_X(\nabla_Y Z)$, analogously for X and Y interchanged.

Note 1 *: A pseudo Riemannian manifold with vanishing Riemann curvature tensor is called flat.*

Ricci curvature

Differential geometry's concept of **Ricci curvature** quantifies a Riemannian manifold's inherent curvature. It has the name of the mathematician from Italy, Gregorio Ricci-Curbastro. Simply put, a manifold's Ricci curvature at a certain place refers to how much the manifold curves in various directions at that location. It offers details regarding the growth or decrease of volumes close to the point.

The Riemann curvature tensor, which encompasses all of a manifold's curvature information, is used to define the Ricci curvature mathematically. The Ricci curvature focuses on a particular contraction of the numerous components that constitute the Riemann curvature tensor.

Conclusion

Geometry is an essential pillar in the progress of civilizations. It combines creativity and science. In our work, we touched on specific branches of geometry such as analytical, projective and differential geometry, as geometry includes multiple branches such as algebraic geometry, non-Euclidean geometry, affine geometry and others. It is characterized by the application of mathematics and science to solve problems and improve the quality of life.

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